p-adic String Amplitudes and Local Zeta Functions (joint work with H. Compeán and W. Zúñiga)

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26 de mayo de 2017

Overview

- These connections were established in our paper *Regularization of p-adic String Amplitudes, and Multivariate Local Zeta Functions*, M. Bocardo-Gaspar, H. García-Compeán, W. Zúñiga-Galindo.

Results:

*p-adic string amplitudes are 'essentially' local zeta functions.

* Amplitudes are convergent integrals that admit meromorphic continuations as rational functions i.e. *the problem of regularization of p-adic string amplitudes.*

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- Physicists have discovered that the limit when $p \to 1$ allows to pass from the non-Archimedean world to the Archimedean one.

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- Paper in preparation aims to construct new topological string zeta functions.

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- p-adic string amplitudes.
- Topological string zeta functions.

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Multivariate local zeta functions

Let $f_i : U \subseteq \mathbb{Q}_p^n \to \mathbb{Q}_p$ be non-constant analytic functions, $\Phi : \mathbb{Q}_p^n \longrightarrow \mathbb{C}$ be a locally constant function with compact support on U, and $s_i \in \mathbb{C}$ be complex numbers. The multivariate local zeta function attached to $(\{s_i\}, \{f_i\}, \phi)$ is defined as

$$Z_\phi(s_1, ..., s_l; f_1, ..., f_l) = \int_{\mathbb{Q}_p^n} \phi(x) \prod_{i=1}^{l} |f_i(x)|^{s_i} |dx|.$$

- Loeser studied these integrals and showed that they admit meromorphic continuations as rational functions in the variables p^{-s_i}, $i = 1, ..., l$.

Loeser, F., *Fonctions zêta locales d'Igusa à plusieurs variables, intégration dans les fibres, et discriminants.* Ann. Sc. Ec. Norm. Sup. 22 (1989), no. 3, 435-471.

Local Zeta Functions → **What is the relation?** ← p-adic String Amplitudes

II. p-adic String Amplitudes

- The string theory was motivated by need of understanding aspects of the strong interactions of elementary particles. Strong interactions are described by functions (amplitudes), which satisfy some physical requirements.
- In 1968, G. Veneziano proposed a function for describing the interaction of four particles.
- The generalization of the Veneziano amplitude to case of interaction of N particles:

$$A^{(N)}(\boldsymbol{k}) = \int_{\mathbb{R}^{N-3}} \prod_{i=2}^{N-2} |x_i|_{\mathbb{R}}^{\boldsymbol{k}_1 \boldsymbol{k}_i} |1 - x_i|_{\mathbb{R}}^{\boldsymbol{k}_{N-1} \boldsymbol{k}_i} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_{\mathbb{R}}^{\boldsymbol{k}_i \boldsymbol{k}_j} \prod_{i=2}^{N-2} dx_i.$$

p-adic String Amplitudes

- The p-adic string theory started around 1987 with the work of three groups: Volovich, Freund and Witten, and Frampton and Okada.
- Volovich noted that the integral expression for the Veneziano amplitude of the open string can be generalized to a p-adic integral.

- The p-adic string amplitudes for N particles have the form:

$$\boldsymbol{A}_p^{(N)}(\underline{\boldsymbol{k}}) := \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{\boldsymbol{k}_1\boldsymbol{k}_i} |1-x_i|_p^{\boldsymbol{k}_{N-1}\boldsymbol{k}_i} \prod_{2\leq i<j\leq N-2} |x_i-x_j|_p^{\boldsymbol{k}_i\boldsymbol{k}_j} \prod_{i=2}^{N-2} dx_i, \tag{1}$$

where $\prod_{i=2}^{N-2} dx_i$ is the normalize Haar measure of $\mathbb{Q}_p^{N-3}$, $\underline{\boldsymbol{k}} = (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N)$, $\boldsymbol{k}_i = (k_{0,i}, \ldots, k_{25,i})$, $i = 1, \ldots, N$, $N \geq 4$, (with Minkowski product $\boldsymbol{k}_i\boldsymbol{k}_j = -k_{0,i}k_{0,j} + k_{1,i}k_{1,j} + \cdots + k_{25,i}k_{25,j}$) satisfying

$$\sum_{i=1}^{N} \boldsymbol{k}_i = \boldsymbol{0},\ \boldsymbol{k}_i\boldsymbol{k}_i = 2 \text{ for } i = 1, \ldots, N-1.$$

- Brekke, Freund and Witten (Non-Archimedean string dynamics) estudied these p-adic string amplitudes and work out in explicit forms.

  A central problem is to know whether or not integrals of type (1) converge for some values $\boldsymbol{k}_i\boldsymbol{k}_j \in \mathbb{C}$.

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p-adic String Amplitudes

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$$Z_\phi(\underline{\boldsymbol{k}}) = \int_{\mathbb{Q}_p^{N-3}} \phi(x) \prod_{i=2}^{N-2} |x_i|_p^{\boldsymbol{k}_1 \boldsymbol{k}_i} |1 - x_i|_p^{\boldsymbol{k}_{N-1} \boldsymbol{k}_i} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{\boldsymbol{k}_i \boldsymbol{k}_j} \prod_{i=2}^{N-2} dx_i.$$

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Consider the following integral:

$$Z(s) = \int_{\mathbb{Q}_p} |x|_p^s \, dx.$$

Assume that this integral exists for some $s_0 \in \mathbb{R}$. Then

$$J_0(s_0) = \int_{\mathbb{Z}_p} |x|_p^{s_0} \, dx \text{ and } J_1(s_0) = \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} |x|_p^{s_0} \, dx$$

exist. But

$$J_0(s_0) = \frac{1 - p^{-1}}{1 - p^{-1-s_0}}.$$

converges for $s_0 > -1$,

$$J_1(s_0) = \sum_{j=1}^{\infty} \int_{p^{-j}\mathbb{Z}_p^{\times}} |x|_p^{s_0} \, dx = \sum_{j=1}^{\infty} p^{j(1+s_0)} < \infty$$

converges for $s_0 < -1$.

Regularization of p-adic String Amplitudes

Let $N \geq 4$ and $s_{ij} \in \mathbb{C}$ with $s_{ij} = s_{ji}$ for $1 \leq i < j \leq N-1$. We define the *p-adic open string N-point zeta function* as

$$Z^{(N)}(\underline{s}) = \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i=2}^{N-2} dx_i, \tag{2}$$

where $\underline{s} = (s_{ij}) \in \mathbb{C}^D$, $\prod_{i=2}^{N-2} dx_i$ is the normalized Haar measure of $\mathbb{Q}_p^{N-3}$, $D = (N-4)(N-3)/2 + 2(N-3)$.

We use an approach inspired in the calculations presented in

Brekke Lee, Freund, Peter G. O., Olson Mark, Witten Edward, *Non-Archimedean string dynamics, Nuclear Phys. B 302 (1988), no. 3, 365-402.*

and in the Igusa's p-adic stationary phase formula.

We define for $I \subseteq T = \{2, ..., N-2\}$, *the sector attached to* I as

$$Sect(I) = \left\{ (x_2, \ldots, x_{N-2}) \in \mathbb{Q}_p^{N-3}; |x_i|_p \leq 1 \ \Leftrightarrow i \in I \right\}$$

$$\Rightarrow \mathbb{Q}_p^{N-3} = \sqcup_{I \subseteq T} Sect(I)$$

and

$$Z^{(N)}(\underline{s}; I) := \int_{Sect(I)} \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i=2}^{N-2} dx_i .$$

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Then

$$Z^{(N)}(\underline{s}) = \sum_{I \subseteq T} Z^{(N)}(\underline{s}; I) .$$

$$Z^{(N)}(\underline{s}; I) = Z^{(N)}(\underline{s}; I, 0)\, Z^{(N)}(\underline{s}; I, 1).$$

$$Z^{(N)}(\underline{s}; I, 0) = \int_{\mathbb{Z}_p^{|I|}} \prod_{i\in I} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} \prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I}} |x_i - x_j|_p^{s_{ij}} \prod_{i\in I} dx_i$$

$$Z^{(N)}(\underline{s}; I, 1) = p^{M(\underline{s})} \int_{\mathbb{Z}_p^{|T\setminus I|}} \frac{\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in T\setminus I}} |y_i - y_j|_p^{s_{ij}}}{\prod_{i\in T\setminus I} |y_i|_p^{2+s_{1i}+s_{(N-1)i}+\sum_{2\leq j\leq N-2, j\neq i} s_{ij}}} \prod_{i\in T\setminus I} dx_i$$

where

$$M(\underline{s}) = |T\setminus I|_p + \sum_{i\in T\setminus I} (s_{1i} + s_{(N-1)i}) + \sum_{\substack{2\leq i<j\leq N-2\\ i\in T\setminus I, j\in T}} s_{ij} + \sum_{\substack{2\leq i<j\leq N-2\\ i\in I, j\in T\setminus I}} s_{ij}.$$

- **Problem**: $Z^{(N)}(\underline{s})$ has a meromorphic continuation to the whole $\mathbb{C}^D$ as a rational function in the variables $p^{s_{ij}}$.
- We showed that the functions $Z^{(N)}(\underline{s}; I, 0)$ and $Z^{(N)}(\underline{s}; I, 1)$

  - Meromorphic continuations to the whole $\mathbb{C}^D$ as rational functions in the variables $p^{-s_{ij}}$.

  - Holomorphic on certain domain.

  - The intersection of all these domains contains an open and connected subset of $\mathbb{C}^D$.
- The algorithms compute recursively the integrals.
- These results are still valid if we replace $\mathbb{Q}_p$ by $\mathbb{F}_q((t))$, the field of formal Laurent series over a finite field $\mathbb{F}_q$.

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Conditions of convergence

(C1) $|J| + \sum_{i\in J}(Re(s_{1i}) + Re(s_{(N-1)i})) + \sum_{\substack{2\leq i<j\leq N-2\\ i\in J}} Re(s_{ij}) + \sum_{\substack{2\leq i<j\leq N-2\\ i\in T\setminus J, j\in J}} Re(s_{ij}) < 0$

(C2) $|K| - 1 + \sum_{\substack{2\leq i<j\leq N-2\\ i,i\in K}} Re(s_{ij}) > 0$

(C3) $1 + Re(s_{ij}) > 0$

(C4) $|M| + \sum_{i\in M} Re(s_{ti}) + \sum_{\substack{2\leq i<j\leq N-2\\ i,j\in M}} Re(s_{ij}) > 0,\ t \in \{1, N-1\}$

where J,K, $M \subseteq T$, $i,j \in T \bigcup \{1, N-1\}$.

Main Results [Bocardo-Gaspar, García-Compeán, Zúñiga-Galindo]

Theorem (1), Bocardo-Gaspar, García-Compeán, Zúñiga-Galindo)

(1) The p-adic open string N-point zeta function, $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$*, gives rise to a holomorphic function on* $H(\mathbb{C})$*, which contains an open and connected subset of* $\mathbb{C}^D$*. Furthermore,* $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$ *admits an analytic continuation to* $\mathbb{C}^D$*, denoted also as* $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$*, as a rational function in the variables* $p^{-s_{ij}}, i,j \in \{1,\ldots,N-1\}$*. The real parts of the poles of* $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$ *belong to a finite union of hyperplanes, the equations of these hyperplanes have the form C1-C4 with the symbols '<', '>' replaced by '='. (2) If* $\underline{\boldsymbol{s}} = (s_{ij})_{ij} \in \mathbb{C}^D$*, with* $Re(s_{ij}) \geq 0$ $i,j \in \{1,\ldots,N-1\}$*, then* $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}) = +\infty$*.*

Regularization of p-adic String Amplitudes

- **Regularization:**

  We take the p-adic N-points $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$ as regularizations of the amplitudes $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}})$ and define

$$\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}}) = \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})\,|_{s_{ij}=\boldsymbol{k}_i\boldsymbol{k}_j} \text{ with } i \in \{1, N-1\}, \, j \in T \text{ or } i, j \in T.$$

  By Theorem 1), $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}})$ are well-defined rational functions of the variables $p^{-s_{ij}}$, $i, j \in \{1, \ldots, N-1\}$, which agree with the original p-adic amplitudes when they converge.

Parte II

Topological string zeta functions

Recall that

$$Z^{(N)}(\underline{s}) = \sum_{I \subseteq T} Z^{(N)}(\underline{s}; I, 0)\, Z^{(N)}(\underline{s}; I, 1).$$

- The integrals $Z^{(N)}(\underline{s}; I, 0)$, $Z^{(N)}(\underline{s}; I, 1)$ are local zeta functions.
- As a consequence of our method if we replace $\mathbb{Q}_p$ by K_e the unique unramified extension of $\mathbb{Q}_p$ of degree e we obtain similar explicit formulas for $Z^{(N)}(\underline{s}; I, 0, K_e)$, $Z^{(N)}(\underline{s}; I, 1, K_e)$ in the variables $p^{-es_{ij}}$. By using the theory of Denef and Loeser, we obtain

$$Z^{(N)}_{top}(\underline{s}; I, 0) = \lim_{e \to 0} Z^{(N)}(\underline{s}; I, 0, K_e)$$

$$Z^{(N)}_{top}(\underline{s}; I, 1) = \lim_{e \to 0} Z^{(N)}(\underline{s}; I, 1, K_e)$$

Denef and F. Loeser, Caractéristiques d'Euler Poincaré, fonctions zêta locales et modifications analytiques J. Amer. Math. Soc. 5 (1992), no. 4, 705720.

- We define the N-points topological string amplitude as

$$\boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}}) := \sum_{I \subseteq T} \boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}}; I, 0)\, \boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}}; I, 1)\,.$$

- and the topological string amplitude as

$$\boldsymbol{A}_{top}^{(N)}(\boldsymbol{k}) = \boldsymbol{Z}_{top}^{(N)}(\boldsymbol{s})|_{s_{ij}=\boldsymbol{k}_i \boldsymbol{k}_j}$$

with $i \in \{1, .., N-1\}$, $j \in T$ or $i, j \in T$ where $T = \{e, ..., N-2\}$, which are rational functions of the variables $k_i k_j$, $k_i k_j$, $i, j \in \{1, ...N-1\}$.

$$\begin{tikzcd}
\boxed{\text{p-adic string amplitudes.}} \arrow[r] & \boxed{\text{Local zeta functions.}} \arrow[r] \arrow[d] & \boxed{\text{string amplitudes.}} \\
& \mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{C}((t)) &
\end{tikzcd}$$

Thank you for your attention.