Periods and Feynman Integrals

- an overview -

Christian Bogner

(Humboldt-Universität zu Berlin)

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Motivation:

Feynman integrals are important functions for our understanding of particle physics.

Zeta values and **multiple zeta values** arise very frequently in their computation.

Reasons to study this correspondence:

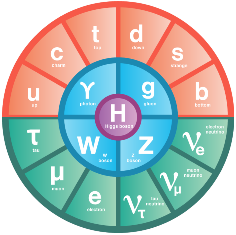
For the Physicist: Improvement of **computational methods** with insights from algebraic geometry and number theory

For the Mathematician: Particle physics as a source of interesting **examples and applications**

Outlook:

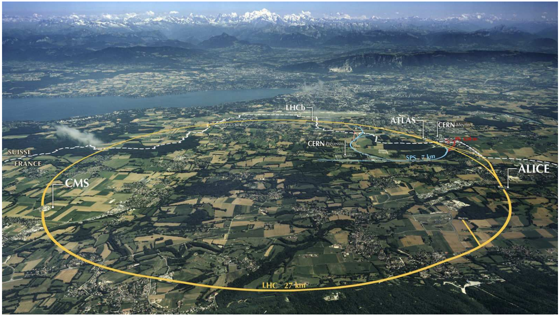
- Introduction to Feynman integrals
- Feynman parameters and graph polynomials
- Periods
- Towards a classification

The **Standard Model** is a **quantum field theory**, describing the known elementary building blocks of nature and their interactions (except for gravity).

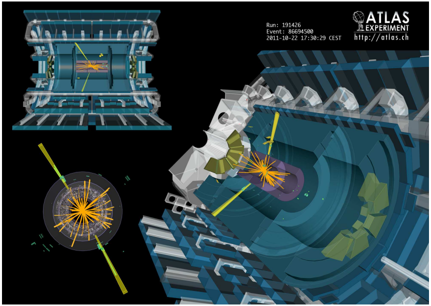


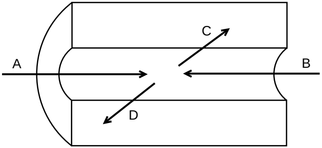
It is developed and tested with the help of **scattering experiments**.

Rutherford's laboratory (1905)



Large Hadron Collider since 2009 (picture: M. Brice, CERN)

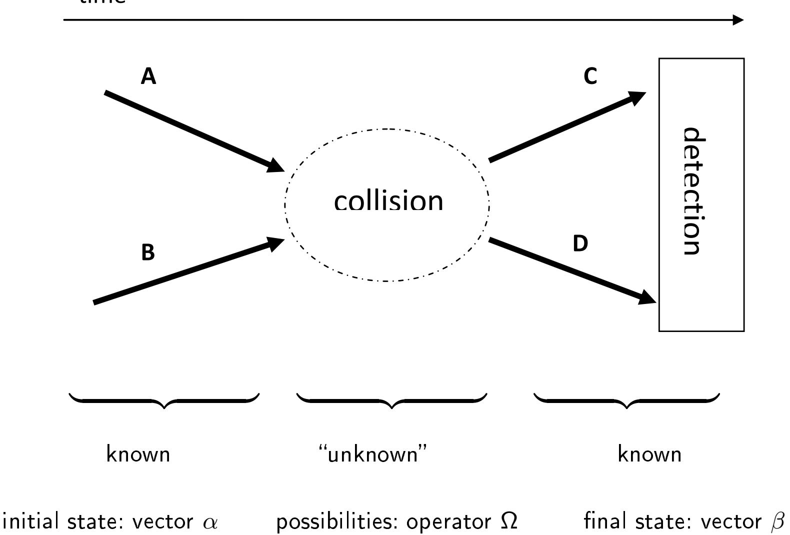




The **probability** (cross-section) for a process like $A\,B \to C\,D$ (e.g. $e^+e^- \to \mu^+\mu^-$)

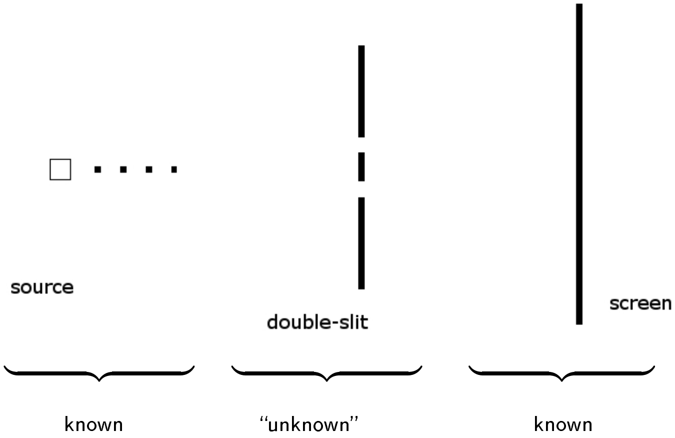
can be measured in the collider **and** derived from a quantum field theory.

$\Rightarrow$ Test of the quantum field theory



Compute physical quantities from a scalar product (amplitude) $G = \langle \beta | \Omega | \alpha \rangle$.

Recall the **double-slit experiment**:



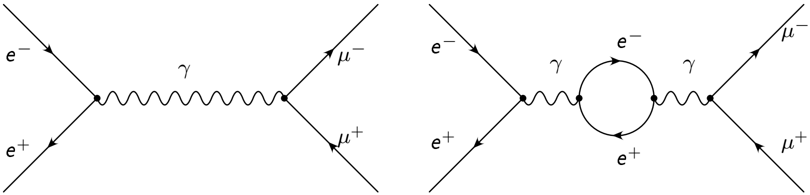
Principle: We have to take **all possibilities** of the unknown region into account.

Example from Quantum Electrodynamics (QED)

Consider the process $e^- + e^+ \to \mu^- + \mu^+$

The amplitude $G = \langle \beta \,|\Omega|\, \alpha \rangle$ is a sum of certain terms, each corresponding to a possible scenario in the unknown region.

For example:

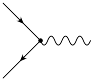


These depictions of the possibilities are called **Feynman graphs.**

Example from Quantum Electrodynamics (QED)

The considered quantum field theory restricts the possible graphs.

In QED we have the building blocks:

- photon-line 
- fermion-line (e.g. $e^+, e^-, \mu^+, \ldots$) 
- interaction vertex 

One derives **Feynman rules** from the theory:

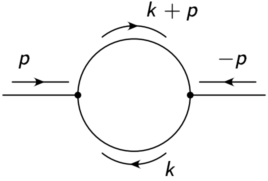
Translation rules from the graphs to the terms in $G = \langle \beta \, |\Omega| \, \alpha \rangle$.

Why do we have to integrate?

Momentum conservation: Impose $\sum_i p_i = 0$ at each vertex.

If we have **loops** (cycles), this leaves some momenta **undetermined**.

Example:



$$I\left(p^2,\, m^2\right) \quad = \quad \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2)\left((k+p)^2 - m^2\right)}$$

with $dk^4 = dk_0 dk_1 dk_2 dk_3$ w.r.t. vectors $k = (k_0, k_1, k_2 k_3)$ in four-dimensional space-time.

principle to sum over "all possibilities" $\Rightarrow$ **integrate** over k

Don't we have to compute infinitely many terms?

The amplitude $G = \langle \beta | \Omega | \alpha \rangle$ (and every resulting observable, e.g. probabilities) can be written as infinite power series

$$G = G_0 + G_1 g + G_2 g^2 + ...$$

in the **coupling constant** g.

- g reflects the typical interaction energy of the theory,
- each vertex of a graph contributes a certain power of g to a term in G.

Idea of **perturbative** Quantum Field Theory:

Assume that g is **small**, compared to all other energies in your process.

$\Rightarrow$ **Truncate the power series** for G at a desired order.

$\Rightarrow$ Consider only **finitely many** Feynman graphs.

Consider a Feynman graph with N internal edges and first Betti-number L.

Generic Feynman integral over so-called Feynman parameters (see Nakanishi 1971):

$$I = \Gamma\left(\nu - LD/2\right) \left(\prod_{i=1}^{N} \int_0^\infty \frac{dx_i x_i^{\nu_i - 1}}{\Gamma(\nu_j)} \right) \delta\left(1 - \sum_{i=1}^{N} x_i\right) \mathcal{U}^{\nu-(L+1)D/2} \mathcal{F}^{-\nu+LD/2},$$

with

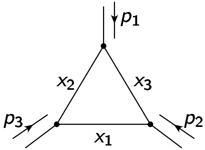
- $\nu_i \in \mathbb{Z}$ labels of the edges, $\nu = \sum_{i=1}^{N} \nu_i$
- D : space-time dimension
- $\mathcal{U}$, $\mathcal{F}$: first and second **Symanzik polynomial**

Construction of Symanzik polynomials:

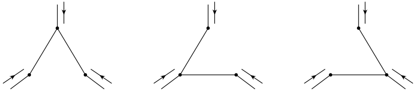
- n-forest: graph without loops (cycles) and n connected components,
- tree: 1-forest,
- spanning forest of a graph G: forest, obtained from G by deletion of edges (without deleting vertices)

Example:

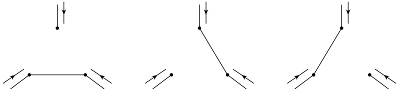
Graph:



Spanning trees:



Spanning 2-forests:



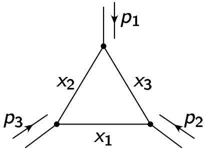
Construction/definition of Symanzik polynomials $\mathcal{U}$ and $\mathcal{F}$ for a Feynman graph G:

$$\mathcal{U} = \sum_{\text{spanning trees } T \text{ of } G} \prod_{\text{edges } \notin T} x_i$$

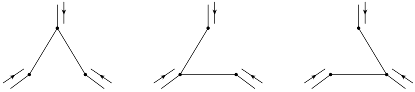
$$\mathcal{F}_0 = - \sum_{\text{spanning 2-forests } (T_1,\, T_2)} \left(\prod_{\text{edges } \notin (T_1,\, T_2)} x_i \right) \left(\sum_{\text{edges } \notin (T_1,\, T_2)} q_i \right)^2 ,$$

$$\mathcal{F} = \mathcal{F}_0 + \mathcal{U} \sum_{i=1}^{N} x_i m_i^2 .$$

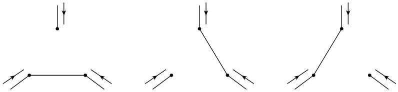
Graph:



Spanning trees:



Spanning 2-forests:



$$\mathcal{U} = x_1 + x_2 + x_3 , \quad \mathcal{F}_0 = -x_1 x_2 p_3^2 - x_2 x_3 p_1^2 - x_1 x_3 p_2^2$$

Properties of Symanyik polynomials:

- They are homogeneous with $\deg(\mathcal{U}) = L$ and $\deg(\mathcal{F}) = L + 1$.

- $\mathcal{U}$ is linear and $\mathcal{F}$ is at most quadratic in each Feynman parameter.

- Let G be a Feynman graph with a regular edge e labeled with Feynman parameter x_e. Let G/e be the graph after contracting $G\backslash e$ the graph after deleting e. Then

$$\begin{aligned} \mathcal{U}(G) &= \mathcal{U}(G/e) + x_e \mathcal{U}(G\backslash e), \\ \mathcal{F}_0(G) &= \mathcal{F}_0(G/e) + x_e \mathcal{F}_0(G\backslash e). \end{aligned}$$

- Using the matrix-tree theorem, the Symanzik polynomials can be obtained from determinants of Laplacian matrices.

  $\Rightarrow$ They satisfy non-trivial Dodgson identities (see CB, Weinzierl 2010 for a review).

Recall our generic Feynman integral

$$I = \Gamma\left(\nu - LD/2\right) \left(\prod_{i=1}^{N} \int_0^\infty \frac{dx_i x_i^{\nu_i - 1}}{\Gamma(\nu_j)} \right) \delta\left(H\right) \mathcal{U}^{\nu-(L+1)D/2} \mathcal{F}^{-\nu+LD/2}$$

where D is the space-time dimension.

Problem: Many Feynman integrals are divergent at $D = 4$.

Idea of **dimensional regularization**: Treat D as a **complex variable**. Then I is a meromorphic function of D.

We are interested in the **Laurent series**:

$$I = \sum_{i=-2L}^{\infty} I_i \epsilon^i,$$

where $D = 4 - 2\epsilon$.

$\Rightarrow$ Computing a Feynman integral now means: computing the coefficients I_i.

Which functions and numbers will we find in these coefficients?

Periods:

Definition (Kontsevich and Zagier 2001):

A *period* is a complex number whose real and imaginary parts are values of absolutely converging integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Classical **example** for a period:

$$\pi = \int \int_{x^2+y^2 \leq 1} dx\, dy$$

Denote the set of periods by $\mathcal{P}$. We have

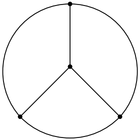
$$\overline{\mathbb{Q}} \subset \mathcal{P} \subset \mathbb{C}.$$

Can we make a general statement about Feynman integrals and periods?

Consider **massless vacuum graphs**: Feynman graphs with no external edges, no particle masses and all $\nu_i = 1$.
Here $\mathcal{F} = 1$ and the integral simplifies to

$$I = \Gamma\left(N - LD/2\right) \left(\prod_{i=1}^{N} \int_0^\infty dx_i \right) \delta\left(1 - \sum_{i=1}^{N} x_i\right) \mathcal{U}^{\nu-(L+1)D/2}$$

Example: 'wheel with three spokes'



First Symanzik polynomial:

$$\begin{aligned} \mathcal{U} &= x_1x_2x_3 + x_3x_5x_6 + x_2x_4x_5 + x_1x_4x_6 \\ &\quad + x_2x_6(x_3+x_1) + x_1x_5(x_2+x_3) + x_3x_4(x_1+x_2) \end{aligned}$$

$L = 3$ loops, $N = 6$ edges, $D = 4 - 2\epsilon$ and propagator powers $\nu_i = 1$ for $i = 1, ..., 6$ we have the Feynman integral

$$\begin{aligned} I(\epsilon) &= \Gamma\left(\epsilon\right) \left(\prod_{i=1}^{6} \int_0^\infty dx_i \right) \delta\left(H\right) \mathcal{U}^{-2+4\epsilon} \\ &= 6\zeta(3)\epsilon^{-1} + \mathcal{O}(\epsilon). \end{aligned}$$

Remark: Bloch, Esnault and Kreimer 2006: The number $6\zeta(3)$ is the **period** of a motive which is constructed using the zero-set of $\mathcal{U}$.

A statement on **all vacuum-graph integrals**

$I = \Gamma\left(N - LD/2\right) \left(\prod_{i=1}^{N} \int_0^\infty dx_i\right) \delta\left(1 - \sum_{i=1}^{N} x_i\right) \mathcal{U}^{\nu-(L+1)D/2}$

Belkale and Brosnan 2003: Up to the Γ-factor, these are of the type

$$J(s) = \int_{\Delta_{N-1}} f^s d^{N-1}x$$

with $f = \mathcal{U}|_{x_N=1-\sum_{i=1}^{N-1} x_i} \in \mathbb{Q}[x_1, ..., x_{N-1}]$ and the standard simplex
$\Delta_{N-1} = \left\{(x_1, ..., x_{N-1}) \in \mathbb{R}^{N-1} | \sum_{i=1}^{N-1} x_i \leq 1\right\}$.
They are Igusa zeta functions.

Theorem (Belkale, Brosnan 2003)

For integers s_0, r let

$$J(s) = \sum_{i \geq r} a_i \left(s - s_0\right)^i$$

be the Laurent series expansion of $J(s)$ at s_0. Then the a_i are in $\mathcal{P}$.

$\Rightarrow$ In the **Laurent series of the vacuum-graph integral** $I = \sum_{i=-2L}^{\infty} I_i \epsilon^i$, $D = 4 - 2\epsilon$, the I_i are in $\mathcal{P}$.

We return to the **general case**:

$$I = \Gamma\left(\nu - LD/2\right) \left(\prod_{i=1}^{N} \int_0^\infty \frac{dx_i x_i^{\nu_i - 1}}{\Gamma(\nu_j)} \right) \delta\left(1 - \sum_{i=1}^{N} x_i\right) \mathcal{U}^{\nu-(L+1)D/2} \mathcal{F}^{-\nu+LD/2}.$$

The coefficients of $\mathcal{F}$ are physical parameters (squared particle masses and scalar products of external momenta).

Let us consider these parameters at non-negative rational values.

Theorem (CB, Weinzierl 2009)

In the Laurent series $I = \sum_{i=-2L}^{\infty} I_i \epsilon^i$, $D = 4 - 2\epsilon$, the I_i are periods.

Our proof relies on the **resolution of singularities** according to Hironaka:

- successively decompose and blow-up the integration domain,
- combinatorics is given by Hironaka's polyhedra game,
- winning strategy completes the proof

We have previously applied this concept in a **computer program** for the numerical computation of Feynman integrals. (CB, Weinzierl 2007)

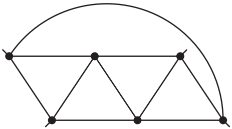
Can we restrict the coefficients in the Laurent series any further?

Let $\mathcal{Z}$ be the $\mathbb{Q}$-vector space of **multiple zeta values**

$$\zeta(n_1, \ldots, n_r) = \sum_{0<j_1<\ldots<j_r} \frac{1}{j_1^{n_1} \ldots j_r^{n_r}},\ n_r > 1.$$

Observation: Many of the known coefficients of **vacuum graph integrals** are in $\mathcal{Z}$.

An all-loop example: The leading coefficient of the n-loop 'zig-zag-graph' integral I_{Z_n} is $4\frac{(2n-2)!}{n!(n-1)!}\left(1 - \frac{1-(-1)^n}{2^{2n-3}}\right)\zeta(2n-3)$. (Brown, Schnetz 2013)



For many further examples see e.g. Broadhurst, Kreimer 1995, 1996 and Schnetz' *Census*.

However, **not every** vacuum graph integral has coefficients in $\mathcal{Z}$. A counter example is known at 10 loops. (Brown, Schnetz 2012, Brown, Doryn 2013)

General case: Feynman integrals with external edges, being functions of physical parameters (masses and momenta)

Observation: Many coefficients are linear combinations of **multiple polylogarithms**

$$\mathrm{Li}_{n_1,\dots,n_r}(z_1,\dots,z_r) = \sum_{0<j_1<\dots<j_r} \frac{z_1^{j_1}\dots z_r^{j_r}}{j_1^{n_1}\dots j_r^{n_r}} \text{ for } |z_i| < 1.$$

Again, this is **not always** the case.

Counter-example: The massive 'sunrise' graph appear now at much lower loop-number. Here we find elliptic generalizations of polylogarithms. (Bloch, Vanhove 2013, Adams, CB, Weinzierl 2014, 2015)

Which Feynman integrals can be expressed in terms of multiple zeta values and multiple polylogarithms?

Assume a finite Feynman integral with Symanzik polynomials $\mathcal{U}, \mathcal{F}$.

Strategy:

- Express multiple polylogarithms and multiple zeta values by an appropriate class of **iterated integrals**.
- Integrate out the Feynman parameters $x_1, ..., x_N$ in some ordering, building up the result as iterated integral.
  (See tomorrow's talk!)

For which Feynman integrals can we do this?

There is a sufficient algorithmic criterion: the **linear reducibility** of $\mathcal{U}$ and $\mathcal{F}$. (Brown 2008)

Fubini algorithm (Brown '08)

- Start with a set of polynomials $S = \{P_1, ..., P_m\}$ and the ordering $x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(N)}$.
- If all $P_i \in S$ are linear in $x_{\sigma(1)}$ define:
  $S^{(\sigma_1)}$ = irreducible factors of
  $\left\{ \frac{\partial P_i}{\partial x_{\sigma(1)}}, \; P_i|_{x_{\sigma(1)}=0}, \; P_j|_{x_{\sigma(1)}=0} \frac{\partial P_i}{\partial x_{\sigma(1)}} - P_i|_{x_{\sigma(1)}=0} \frac{\partial P_j}{\partial x_{\sigma(1)}} \right\}_{1 \le i < j \le n}$
- iterate for a sequence $x_{\sigma(1)}, x_{\sigma(2)}, ... \Rightarrow S^{(\sigma(1))}, S^{(\sigma(1),\sigma(2))}, ...$
- take intersections like: $S^{\{\sigma(1),\sigma(2)\}} = S^{(\sigma(1),\sigma(2))} \cap S^{(\sigma(2),\sigma(1))}, \ldots,$
  $S^{\{\sigma(1),\ldots,\sigma(k)\}} = \cap_{1 \le i \le k} S^{\{\sigma(1),\ldots,\hat{\sigma}(i),\ldots,\sigma(k)\}(\sigma(i))}$

$$x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)} \quad \Rightarrow \quad S^{\{\sigma(1)\}}, \; S^{\{\sigma(1),\sigma(2)\}}, \; \ldots, \; S^{\{\sigma(1),\sigma(2),\ldots,\sigma(k)\}}$$

- S is called **linearly reducible** if for all $1 \le k \le N$ every polynomial in $S^{\{\sigma(1),\sigma(2),\ldots,\sigma(k)\}}$ is linear in $x_{\sigma(k+1)}$
  i.e. if the full sequence including $S^{\{\sigma(1),\sigma(2),\ldots,\sigma(N)\}}$ is generated.
- If $S = \{\mathcal{U}_G, \mathcal{F}_G\}$ is linearly reducible we call the **Feynman graph** G linearly reducible.

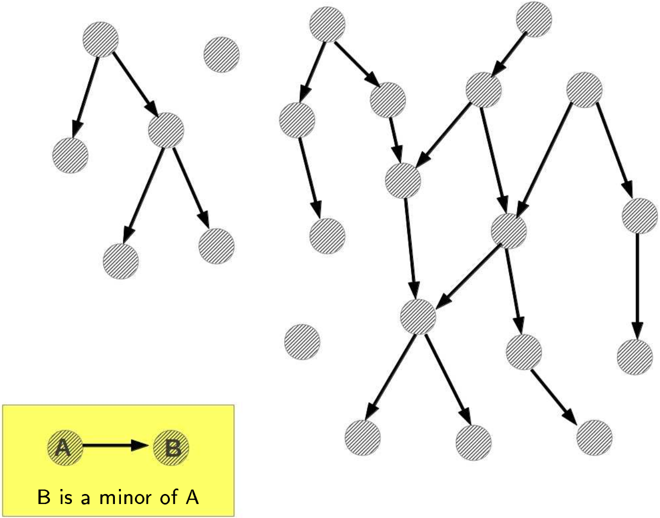
For e an edge of G consider the deletion $(G \backslash e)$ and contraction (G/e) of e

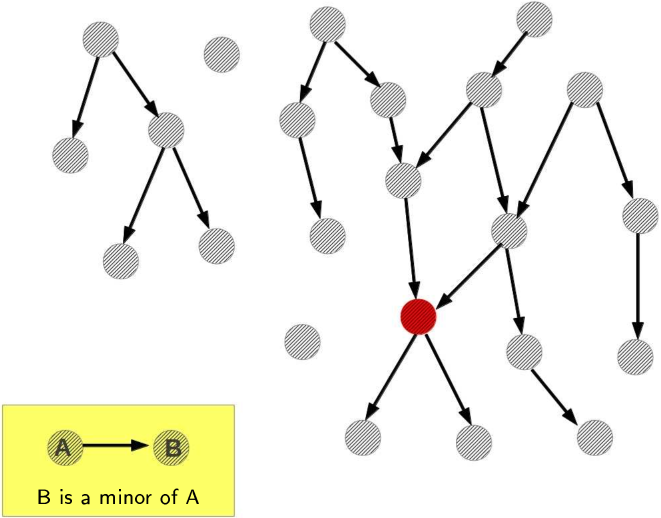
The deletion and contraction of different edges is commutative.

$\Rightarrow$If C, D are disjoint sets of edges of G then $G \backslash D/C$ is a unique graph.
Any such graph is called *minor* of G.

Def.: A set $\mathcal{G}$ of graphs is called *minor-closed* if for each $G \in \mathcal{G}$ all minors belong to $\mathcal{G}$ as well.

Example: The set of all planar graphs is minor-closed.


A
B
B is a minor of A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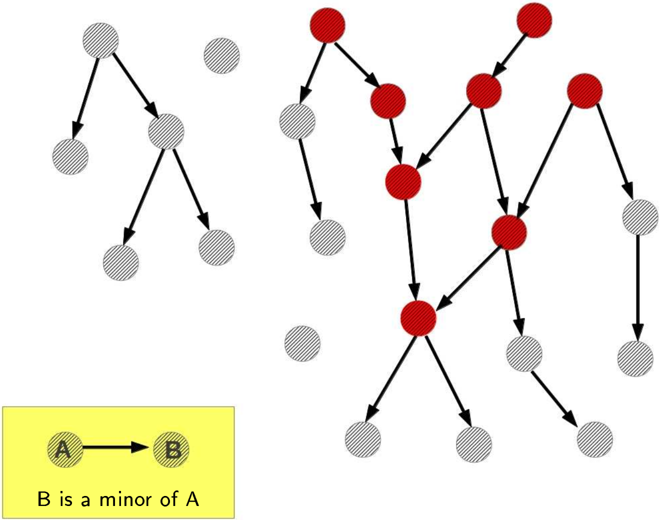

A
B
B is a minor of A

Let $\mathcal{H}$ be a finite set of graphs.

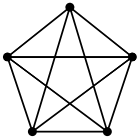
Define $\mathcal{G}_{\mathcal{H}}$ to be the set of graphs **whose minors do not belong** to $\mathcal{H}$.

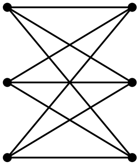
Then the graphs in $\mathcal{H}$ are called *forbidden minors* of $\mathcal{G}_{\mathcal{H}}$. The set $\mathcal{G}_{\mathcal{H}}$ is minor-closed.

Theorem (Robertson and Seymour): Any minor-closed set of graphs is determined by a finite set of forbidden minors.

Example:

The set of planar graphs is the set of all graphs which have neither K_5 nor $K_{3,3}$ as a minor. (Wagner's theorem)





Theorem (CB, Lueders 2013)

The set of linearly reducible Feynman graphs is minor-closed.

For a recent alternative proof see Moore 2017 (masterthesis)

We should search for the forbidden minors!

For an overview of linearly reducible Feynman graphs see Panzer's PhD thesis (2015)

Summary:

- The computation of **Feynman integrals** is crucial for our understanding of elementary particles.
- These integrals arise from the quantum physics principle to 'sum over all possible scenarios in an unknown region'.
- They are computed as **Laurent series** in a regularization parameter ϵ.
- The coefficients in this series are **periods** (or period valued functions).
- In many cases these can be expressed in terms of **multiple zeta values** and **multiple polylogarithms**.
- The **linear reducibility** of the Symanzik polynomials is a sufficient condition.