

Algorithms for multiple polylogarithms

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Recall our generic Feynman integral

$$I = \Gamma(\nu - LD/2) \left(\prod_{i=1}^N \int_0^\infty \frac{dx_i x_i^{\nu_i-1}}{\Gamma(\nu_j)} \right) \delta\left(1 - \sum_{i=1}^N x_i\right) \mathcal{U}^{\nu-(L+1)D/2} \mathcal{F}^{-\nu+LD/2}.$$

Our goal: Successively integrate out all Feynman parameters.

Strategy: Build up the result in terms of **iterated integrals**.

A well-understood class of such functions are **multiple polylogarithms** and **multiple zeta values**.

(If these can not be applied, elliptic polylogarithms may be useful. ([Bloch, Vanhove 2013](#), [Adams, CB, Weinzierl 2014, 2015](#), [CB, Schweitzer, Weinzierl 2017](#)))

Here: Assume Feynman integrals which **can** be expressed in terms of multiple polylogarithms and multiple zeta values.

We present **algorithms** ([CB, Brown 2014](#)) and a **computer program MPL** ([CB 2015](#)) for the above strategy.

The origin of polylogarithms

Leibniz, letter to Bernoulli, November 1696



Quaeritur summa horum numerorum $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$ etc. [...]
Unde $\frac{x}{1} + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16}$ etc. = y. Ergo $dy = \frac{\log \cdot 1-x}{x} dx$, seu
$$y = \int \frac{\log \cdot 1-x}{x} dx.$$



Being interested in $\sum_{j=1}^{\infty} \frac{1}{j^2}$, today known as $\zeta(2) = \frac{\pi^2}{6}$ (Euler 1740), Leibniz considered

$$y(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^2}.$$

Comparing with the logarithm $-\ln(1-x) = \sum_{j=1}^{\infty} \frac{x^j}{j}$ he derived

$$\frac{dy}{dx} = -\frac{1}{x} \ln(1-x) \Rightarrow y(x) = -\int_0^x \frac{dx'}{x'} \ln(1-x').$$

These are sum and integral representation of the function today known as **dilogarithm**.

Classical polylogarithms

Definition:

$$\text{Li}_n(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^n} \text{ for } |z| < 1.$$

Special values at $z = 1$ for $n > 1$:

$$\text{Li}_n(1) = \sum_{j=1}^{\infty} \frac{1}{j^n} = \zeta(n) \text{ Riemann's zeta function at integer arguments}$$

Differential equations:

$$\frac{d}{dz} \text{Li}_n(z) = \sum_{j=1}^{\infty} \frac{d}{dx} \frac{z^j}{j^n} = \frac{1}{z} \text{Li}_{n-1}(z) \text{ for } n \geq 2$$

Analytic continuation, following Leibniz:

$$\text{Li}_n(z) = \int_{\gamma} \frac{dx}{x} \text{Li}_{n-1}(x)$$

where γ is a smooth path from 0 to z in $\mathbb{C} \setminus \{0, 1\}$.

\Rightarrow These integrals are multi-valued functions on $\mathbb{C} \setminus \{0, 1\}$.

Explicitly:

$$\operatorname{Li}_1(z) = \int_0^z \frac{dx_1}{1-x_1} = -\ln(1-x),$$

$$\operatorname{Li}_2(z) = \int_0^z \frac{dx_2}{x_2} \int_0^{x_2} \frac{dx_1}{1-x_1},$$

$$\operatorname{Li}_3(z) = \int_0^z \frac{dx_3}{x_3} \int_0^{x_3} \frac{dx_2}{x_2} \int_0^{x_2} \frac{dx_1}{1-x_1},$$

⋮

$$\operatorname{Li}_n(z) = \int_0^z \frac{dx_n}{x_n} \cdots \int_0^{x_3} \frac{dx_2}{x_2} \int_0^{x_2} \frac{dx_1}{1-x_1}.$$

These are **iterated integrals** of the 1-forms in $\left\{ \frac{dx}{x}, \frac{dx}{1-x} \right\}$.

Generalization to **multiple polylogarithms in one variable**:

Definition:

$$\text{Li}_{n_1, \dots, n_r}(z) = \sum_{0 < j_1 < \dots < j_r} \frac{z^{j_r}}{j_1^{n_1} \dots j_r^{n_r}} \text{ for } |z| < 1.$$

Special values at $z = 1$ for $n_r > 1$:

$$\zeta(n_1, \dots, n_r) = \sum_{0 < j_1 < \dots < j_r} \frac{1}{j_1^{n_1} \dots j_r^{n_r}} \text{ multiple zeta values.}$$

Differential equations:

$$\frac{d}{dz} \text{Li}_{n_1, \dots, n_r}(z) = \begin{cases} \frac{1}{z} \text{Li}_{n_1, \dots, n_r-1}(z) & \text{for } n_r > 1, \\ \frac{1}{1-z} \text{Li}_{n_1, \dots, n_{(r-1)}}(z) & \text{for } n_r = 1. \end{cases}$$

\Rightarrow **Iterated integrals** of the 1-forms in $\left\{ \frac{dx}{x}, \frac{dx}{1-x} \right\}$.

Formal definition of iterated integrals:

Consider

- a manifold M over a field K ,
- a piecewise smooth path $\gamma : [0, 1] \rightarrow M$,
- $\omega_1, \dots, \omega_r$ smooth, K -valued 1-forms on M with pull-backs $\gamma^*(\omega_j)(t) = f_j(t)dt$.

The *iterated integral* of $\omega_1, \dots, \omega_r$ along γ is defined by

$$\int_{\gamma} \omega_1 \dots \omega_r = \int_{0 \leq t_1 \leq \dots \leq t_r \leq 1} f_r(t_1) dt_1 \dots f_1(t_r) dt_r.$$

We also use the term *iterated integral* for every K -linear combination of these.

Some properties:

- $\int_{\gamma} \omega_1 \dots \omega_r$ is independent of the parametrization of the path γ .
- For $\gamma^{-1}(t) = \gamma(1 - t)$ the reversal of γ :

$$\int_{\gamma^{-1}} \omega_1 \dots \omega_r = (-1)^r \int_{\gamma} \omega_r \dots \omega_1.$$

- Product of two iterated integrals (shuffle product):

$$\int_{\gamma} \omega_1 \dots \omega_r \cdot \int_{\gamma} \omega_{r+1} \dots \omega_{r+s} = \sum_{\sigma \in \Sigma(r,s)} \int_{\gamma} \omega_{\sigma(1)} \dots \omega_{\sigma(r+s)}$$

where

$$\Sigma(r,s) = \{ \sigma \in \text{permutations on } 1, \dots, r+s : \\ \sigma(1) < \dots < \sigma(r) \text{ and } \sigma(r+1) < \dots < \sigma(r+s) \}$$

- For $\alpha\beta$ the path composed of two paths α and β :

$$\int_{\alpha\beta} \omega_1 \dots \omega_r = \sum_{i=0}^r \int_{\alpha} \omega_1 \dots \omega_i \cdot \int_{\beta} \omega_{i+1} \dots \omega_r.$$

Iterated integrals in particle physics:

- Harmonic polylogarithms: $\left\{ \frac{dx}{x}, \frac{dx}{1-x}, \frac{dx}{1+x} \right\}$ (Remiddi, Vermaseren 1999)
- Two-dimensional harmonic polylogarithms: $\left\{ \frac{dx}{x}, \frac{dx}{1-x}, \frac{dx}{x+y}, \frac{dx}{x+y-1} \right\}$ (Gehrmann, Remiddi 2001)
- Cyclotomic harmonic polylogarithms (Ablinger, Blümlein, Schneider '11)
- Hyperlogarithms (a.k.a. Goncharov polylogarithms)

In the context of **Feynman integrals**:

Computer programs by Panzer (HyperInt), Maitre (HPL), Vermaseren (in FORM), Ablinger (HarmonicSums, MultiIntegrate), Vollinga, Weinzierl (in GiNaC), ...

Instructive computations by Duhr, Blümlein, Raab, Schneider, Wissbrock, von Manteuffel, Schlotterer, Broedel, Stieberger, ...

For $\text{Li}_n(z)$ and $\text{Li}_{n_1, \dots, n_r}(z)$ we had the 1-forms $\frac{dx}{x}$, $\frac{dx}{1-x}$ defined on $M = \mathbb{C} \setminus \{0, 1\}$.

Generalization: Consider $M = \mathbb{C} \setminus \Sigma$ with $\Sigma \subset \mathbb{C}$ (including 0) and the corresponding 1-forms $\omega_i = \frac{dx}{x - \sigma_i}$, $\sigma_i \in \Sigma$.

⇒ **Hyperlogarithms** (Poincare, Kummer, Lappo-Danilevsky)

$$L_{\sigma_1, \dots, \sigma_r}(z) = \int_{\gamma} \omega_1 \dots \omega_r = \int_0^z \omega_1 L_{\sigma_2, \dots, \sigma_r}(x)$$

Goncharov (1998): **Multiple polylogarithms** in several variables

$$\text{Li}_{n_1, \dots, n_r}(z_1, \dots, z_r) = \sum_{0 < j_1 < \dots < j_r} \frac{z_1^{j_1} \dots z_r^{j_r}}{j_1^{n_1} \dots j_r^{n_r}} \text{ for } |z_i| < 1$$

can be expressed in terms of hyperlogarithms:

$$(-1)^r \text{Li}_{n_1, \dots, n_r} \left(\frac{\sigma_2}{\sigma_1}, \frac{\sigma_3}{\sigma_2}, \dots, \frac{z}{\sigma_r} \right) = L_{\underbrace{0, \dots, 0}_{n_r - 1 \text{ times}}, \sigma_r, \dots, \underbrace{0, \dots, 0}_{n_1 - 1 \text{ times}}, \sigma_1}(z).$$

Mult. polylogs: **several variables** \longleftrightarrow Hyperlogs: **one variable** and several constants

For m variables we consider the closed differential 1-forms

$$\Omega_m = \left\{ \frac{dx_1}{x_1}, \dots, \frac{dx_m}{x_m}, \frac{d\left(\prod_{a \leq i \leq b} x_i\right)}{\prod_{a \leq i \leq b} x_i - 1} \text{ where } 1 \leq a \leq b \leq m \right\}$$

Examples:

$$\begin{aligned}\Omega_1 &= \left\{ \frac{dx_1}{x_1}, \frac{dx_1}{x_1 - 1} \right\}, \\ \Omega_2 &= \left\{ \frac{dx_1}{x_1}, \frac{dx_2}{x_2}, \frac{dx_1}{x_1 - 1}, \frac{dx_2}{x_2 - 1}, \frac{x_1 dx_2 + x_2 dx_1}{x_1 x_2 - 1} \right\}\end{aligned}$$

Difference to hyperlogarithms: In the same variables, the 1-forms for hyperlogarithms are

$$\Omega_m^{\text{Hyp}} = \left\{ \frac{dx_m}{x_m}, \frac{\left(\prod_{a \leq i \leq m-1} x_i\right) dx_m}{\prod_{a \leq i \leq m} x_i - 1} \text{ for } 1 \leq a \leq n \right\}$$

with x_m being the distinguished variable.

The 1-forms in

$$\Omega_m = \left\{ \frac{dx_1}{x_1}, \dots, \frac{dx_m}{x_m}, \frac{d\left(\prod_{a \leq i \leq b} x_i\right)}{\prod_{a \leq i \leq b} x_i - 1} \text{ where } 1 \leq a \leq b \leq m \right\}$$

are defined on m -dimensional moduli spaces of curves of genus zero with $n = m + 3$ ordered, marked points

$$\mathcal{M}_{0,n}(\mathbb{C}) \cong \left\{ (x_1, \dots, x_{n-3}) \in \mathbb{C}^{n-3} \mid \prod_{i \leq k \leq j} x_k \notin \{0, 1\} \text{ for all } 1 \leq i \leq j \leq n-3 \right\}.$$

Examples:

$$\mathcal{M}_{0,4}(\mathbb{C}) \cong \mathbb{C} \setminus \{0, 1\},$$

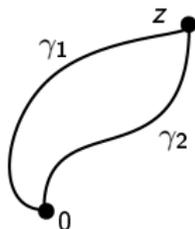
$$\mathcal{M}_{0,5}(\mathbb{C}) \cong \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1, x_2, x_1 x_2 \notin \{0, 1\}\}$$

Additional condition on the iterated integrals:

From Ω_m we construct **homotopy invariant** iterated integrals, i.e.

$$\int_{\gamma_1} \omega_1 \dots \omega_r = \int_{\gamma_2} \omega_1 \dots \omega_r$$

for homotopic paths γ_1, γ_2 .



Such iterated integrals are **well-defined functions of the end-point coordinates**.

Integration map $\mathcal{I}_\gamma : \omega_1 \otimes \dots \otimes \omega_r \equiv [\omega_1 | \dots | \omega_r] \rightarrow \int_\gamma \omega_1 \dots \omega_r$

Question: Which sequences of 1-forms in Ω_m map to homotopy invariant iterated integrals?

Answer: Define an operator D by

$$D([\omega_1|\dots|\omega_r]) = \sum_{i=1}^r [\omega_1|\dots|\omega_{i-1}|d\omega_i|\omega_{i+1}|\dots|\omega_r] + \sum_{i=1}^{r-1} [\omega_1|\dots|\omega_{i-1}|\omega_i \wedge \omega_{i+1}|\dots|\omega_r].$$

A \mathbb{Q} -linear combination of tensor products

$$\xi = \sum_{l=0}^r \sum_{i_1, \dots, i_l} c_{i_1, \dots, i_l} [\omega_{i_1}|\dots|\omega_{i_l}], \quad c_{i_1, \dots, i_l} \in \mathbb{Q}$$

is called **integrable word** if

$$D(\xi) = 0.$$

Theorem (Chen '77): Under certain conditions on Ω the integration map is an **isomorphism** from **integrable words** to **homotopy invariant iterated integrals**.

\Rightarrow After fixing a boundary condition, every integrable word determines an iterated integral. We write

$$[\omega_1|\dots|\omega_r] = \int_{\gamma} \omega_1 \dots \omega_r$$

Example:

$$\text{Li}_3(z) = \int_0^z \frac{dx_3}{x_3} \int_0^{x_3} \frac{dx_2}{x_2} \int_0^{x_2} \frac{dx_1}{1-x_1} = \left[\frac{dx}{x} \middle| \frac{dx}{x} \middle| \frac{dx}{1-x} \right]$$

How do we obtain the integrable words?

Definitions:

- $V(\Omega_m^{\text{Hyp}})$: the \mathbb{Q} -vector space of hyperlogarithms with 1-forms in Ω_m^{Hyp} .
- \mathcal{A}_m : the \mathbb{Q} -vector space spanned by Ω_m .
- $V(\Omega_m)$: the \mathbb{Q} -vector space of homotopy invariant iterated integrals of differential 1-forms in \mathcal{A}_m (normalized and regularized and at a tangential basepoint).

The **symbol map** is the unique linear map

$$\Psi : V(\Omega_m^{\text{Hyp}}) \rightarrow V(\Omega_m).$$

satisfying

$$(id \otimes \Psi) \circ \nabla_{\mathcal{T}} = d \circ \Psi$$

(where d is differentiation and $\nabla_{\mathcal{T}}$ is total differentiation).

Explicit constructions for Ψ are given in [CB](#), [Brown 2012](#), [2014](#).

Example:

$$\Psi \left(\left[\frac{x_2 d(x_3)}{1-x_2 x_3} \middle| \frac{dx_3}{x_3} \right] \right) = \left[\frac{d(x_2 x_3)}{1-x_2 x_3} \middle| \frac{dx_3}{x_3} \right] - \left[\frac{d(x_2)}{x_2} \middle| \frac{d(x_2 x_3)}{1-x_2 x_3} \right]$$

with $\left[\frac{x_2 d(x_3)}{1-x_2 x_3} \middle| \frac{dx_3}{x_3} \right] \in V(\Omega_m^{\text{Hyp}})$ and $\left[\frac{d(x_2 x_3)}{1-x_2 x_3} \middle| \frac{dx_3}{x_3} \right] - \left[\frac{d(x_2)}{x_2} \middle| \frac{d(x_2 x_3)}{1-x_2 x_3} \right] \in V(\Omega_m)$

In MPL:

>MPLCoordinates(x,3):

>MPLSymbolMap(bar((x[2]*d(x[3]))/(1-x[2]*x[3]),d(x[3])/x[3]));

$$\begin{aligned} & \text{bar}(d(x[3])/x[3], (x[3] * d(x[2]) + x[2] * d(x[3]))/(1 - x[2] * x[3])) + \\ & \text{bar}(d(x[2])/x[2], (x[3] * d(x[2]) + x[2] * d(x[3]))/(1 - x[2] * x[3])) \end{aligned}$$

The resulting **vectorspace** $V(\Omega_m)$ of **iterated integrals** has the following properties (Brown '05):

- $V(\Omega_m)$ includes the multiple polylogarithms $\text{Li}_{n_1, \dots, n_r}(z_1, \dots, z_r)$.
- Functional relations turn into algebraic identities (cf. literature on the “symbol”).
- $V(\Omega_m)$ has a unique basis.
- $V(\Omega_m)$ is **closed** under taking **primitives**.
- **Limits** at 0 and 1 are combinations of these functions with **multiple zeta values**.

⇒ Explicit **integration algorithms** based on these functions (CB, Brown 2014)
implemented in **MPL**

The **basis** of $V(\Omega_m)$ up to a chosen weight can be constructed by MPL.

Example:

>MPLBasis(y,2,2);

$$\begin{aligned}
 & \left[\left[\text{bar} \left(\frac{d(y_2)}{y_2} \right), \text{bar} \left(\frac{d(y_2)}{1-y_2} \right), \text{bar} \left(\frac{y_2 d(y_1) + y_1 d(y_2)}{1-y_1 y_2} \right), \text{bar} \left(\frac{d(y_1)}{y_1} \right), \text{bar} \left(\frac{d(y_1)}{1-y_1} \right) \right], \right. \\
 & \quad \left[\text{bar} \left(\frac{d(y_2)}{y_2}, \frac{d(y_2)}{y_2} \right), \text{bar} \left(\frac{d(y_2)}{y_2}, \frac{d(y_2)}{1-y_2} \right), \text{bar} \left(\frac{d(y_2)}{1-y_2}, \frac{d(y_2)}{y_2} \right), \right. \\
 & \quad \left. \text{bar} \left(\frac{d(y_2)}{y_2}, \frac{y_2 d(y_1) + y_1 d(y_2)}{1-y_1 y_2} \right) + \text{bar} \left(\frac{d(y_1)}{y_1}, \frac{y_2 d(y_1) + y_1 d(y_2)}{1-y_1 y_2} \right), \right. \\
 & \quad \text{bar} \left(\frac{y_2 d(y_1) + y_1 d(y_2)}{1-y_1 y_2}, \frac{d(y_2)}{y_2} \right) - \text{bar} \left(\frac{d(y_1)}{y_1}, \frac{y_2 d(y_1) + y_1 d(y_2)}{1-y_1 y_2} \right), \text{bar} \left(\frac{d(y_2)}{1-y_2}, \frac{d(y_2)}{1-y_2} \right), \\
 & \quad \text{bar} \left(\frac{d(y_2)}{1-y_2}, \frac{y_2 d(y_1) + y_1 d(y_2)}{1-y_1 y_2} \right) - \text{bar} \left(\frac{d(y_1)}{1-y_1}, \frac{y_2 d(y_1) + y_1 d(y_2)}{1-y_1 y_2} \right) + \text{bar} \left(\frac{d(y_1)}{1-y_1}, \frac{d(y_2)}{1-y_2} \right) \\
 & \quad - \text{bar} \left(\frac{d(y_1)}{y_1}, \frac{y_2 d(y_1) + y_1 d(y_2)}{1-y_1 y_2} \right), \text{bar} \left(\frac{y_2 d(y_1) + y_1 d(y_2)}{1-y_1 y_2}, \frac{d(y_2)}{1-y_2} \right) \\
 & \quad + \text{bar} \left(\frac{d(y_1)}{1-y_1}, \frac{y_2 d(y_1) + y_1 d(y_2)}{1-y_1 y_2} \right) - \text{bar} \left(\frac{d(y_1)}{1-y_1}, \frac{d(y_2)}{1-y_2} \right) \\
 & \quad \left. \left. + \text{bar} \left(\frac{d(y_1)}{y_1}, \frac{y_2 d(y_1) + y_1 d(y_2)}{1-y_1 y_2} \right), \text{bar} \left(\frac{y_2 d(y_1) + y_1 d(y_2)}{1-y_1 y_2}, \frac{y_2 d(y_1) + y_1 d(y_2)}{1-y_1 y_2} \right), \dots \right] \right]
 \end{aligned}$$

Remarks on the algorithms:

- They operate **only on words** $[\omega_1 | \dots | \omega_r]$ (implicitly using Chen's theorem).
- They make use of an isomorphism of vectorspaces
$$V(\Omega_m) \cong V(\Omega_{m-1}) \otimes V(\Omega_m^{\text{Hyp}}).$$
- A technical backbone are Arnold's equations of the type

$$\omega_i \wedge \omega_j = \sum_k c_k \alpha_k \wedge \beta_k$$

with $\omega_i, \omega_j, \alpha_k$ in (a lifted version of) Ω_m^{Hyp} and β_k in Ω_{m-1} .

- \Rightarrow The symbol map and the algorithm for primitives construct integrable words (i.e. members of $V(\Omega_m)$) **recursively**.

Algorithm for limits $\lim_{x_i \rightarrow u} f$ with $u \in \{0, 1\}$:

- expand $f \in V(\Omega_m)$ at $x_i = u$
- our regularization: the limit is the coefficient of $\ln(x_i - u)^0$ (i.e. throw away logarithmic divergences)
- evaluation: recursively reduce to $V(\Omega_1)$, i.e. multiple polylogarithms in one variable
- here:
 - limits at 0 vanish in our normalization,
 - limits at 1 are **multiple zeta values**

\Rightarrow All results for definite integrals will be \mathcal{Z} -linear combinations of functions in $V(\Omega_m)$
where \mathcal{Z} is the \mathbb{Q} -vectorspace of multiple zeta values.

Related theorem of [Brown \(2005\)](#): All periods of the moduli spaces $\mathcal{M}_{0,n}$ are in \mathcal{Z} .

MPL computes (multiple) integrals of the **cubical type**:

$$I = \int_0^1 dx_m \frac{q}{\left(\prod_j p_j^{a_j}\right)} f$$

where

$$f \in V(\Omega_m),$$

$$p_j \in \{x_m, 1 - x_m, 1 - x_{m-1}x_m, \dots, 1 - x_1 \cdots x_m\},$$

q some polynomial, $a_j \in \mathbb{N}$.

Applications: periods of $\mathcal{M}_{0,n}$, expansion of hypergeometric functions, irrationality statements, Feynman integrals, ...

Example (from Beukers' proof on the irrationality of $\zeta(3)$):

$$g = \frac{x_1^4 (1-x_1)^4 x_2^9 (1-x_2)^4 x_3^4 (1-x_3)^4}{(1-x_1x_2)^5 (1-x_2x_3)^5}$$

With the MPL command `MPLCubicalIntegrate(g,x[3],3)` we compute

$$\int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 g = -\frac{11424695}{144} + 66002\zeta(3).$$

Using `MPLCubicalIntegrate(g,x[3],2)` we obtain a result of the form

$$\int_0^1 dx_2 \int_0^1 dx_3 g = c_1 + c_2 \underbrace{\left[\frac{dx_1}{1-x_1} \right]}_{=-\ln(1-x_1)} + c_3 \underbrace{\left[\frac{dx_1}{1-x_1} \middle| \frac{dx_1}{1-x_1} \right]}_{=\frac{1}{2} \ln^2(1-x_1)} + c_4 \underbrace{\left[\frac{dx_1}{x_1} \middle| \frac{dx_1}{1-x_1} \right]}_{=\text{Li}_2(x_1)}$$

Back to D -dimensional, scalar L -loop **Feynman integrals**:

$$I(\Lambda) = \frac{\Gamma(\nu - LD/2)}{\prod_{j=1}^N \Gamma(\nu_j)} \int_0^\infty \dots \int_0^\infty \left(\prod_{i=1}^N dx_i x_i^{\nu_i - 1} \right) \delta(H) \frac{\mathcal{U}^{\nu - (L+1)D/2}}{(\mathcal{F}(\Lambda))^{\nu - LD/2}}$$

Problem 1: UV and IR divergences

There are methods (see [Panzer 2014](#), [v.Manteuffel, Panzer, Schabinger 2015](#), [Binoth, Heinrich 2000](#), [Brown, Kreimer 2011](#)) to expand the integral

$$I = \sum_{j=-2L}^{\infty} I_j e^j,$$

in terms of **finite integrals** I_j .

Problem 2: The Symanzik polynomials \mathcal{U} and \mathcal{F} are not of the type $1 - \prod_i x_i$

\Rightarrow By systematic **changes of variables**, MPL maps the integrals I_j to the cubical type, and computes them.

These changes of variables exist under certain **conditions** to the polynomials \mathcal{U} and \mathcal{F} .

Consider a generic integral of the type

$$\int_0^\infty dx_N \dots \int_0^\infty dx_1 \frac{Q \cdot f}{\prod_i P_i}$$

where f is an iterated integral, Q and $P_i \in \mathcal{P}$ are polynomials in Feynman parameters.

Strategy:

- Choose an ordering $x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(N)}$ with σ a permutation on $\{1, \dots, N\}$.
- For each x_i -integration: Change variables to **cubical type** \Rightarrow integrate with above algorithms
- This generates integrands with sets $\mathcal{P}, \mathcal{P}(\sigma(1)), \mathcal{P}(\sigma(1), \sigma(2)), \dots, \mathcal{P}(\sigma(1), \dots, \sigma(N))$.

Condition:

The i -th integration can only be computed if all polynomials in $\mathcal{P}(\sigma(1), \sigma(i-1))$ are linear in $x_{\sigma(i)}$.

Algorithms to check this condition was proposed by Brown (2008, 2009):

They “mimic” the integration, constructing sets of polynomials

$\mathcal{S}\{\sigma(1)\}, \mathcal{S}\{\sigma(1), \sigma(2)\}, \dots, \mathcal{S}\{\sigma(1), \sigma(2), \dots, \sigma(k)\}$ such that

$\mathcal{P}(\sigma(1), \sigma(2), \dots, \sigma(i)) \subseteq \mathcal{S}\{\sigma(1), \sigma(2), \dots, \sigma(i)\}$.

They are implemented in the MPL procedure `MPLPolynomialReduction`.

Fubini algorithm (Brown '08)

- Start with a set of polynomials $S = \{P_1, \dots, P_m\}$ and the ordering $x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(N)}$.
- If all $P_i \in S$ are linear in $x_{\sigma(1)}$ define:
 $S^{(\sigma_1)}$ = irreducible factors of
$$\left\{ \frac{\partial P_i}{\partial x_{\sigma(1)}}, P_i|_{x_{\sigma(1)}=0}, P_j|_{x_{\sigma(1)}=0} \frac{\partial P_i}{\partial x_{\sigma(1)}} - P_i|_{x_{\sigma(1)}=0} \frac{\partial P_j}{\partial x_{\sigma(1)}} \right\}_{1 \leq i < j \leq n}$$
- iterate for a sequence $x_{\sigma(1)}, x_{\sigma(2)}, \dots \Rightarrow S^{(\sigma(1))}, S^{(\sigma(1), \sigma(2))}, \dots$
- take intersections like: $S^{\{\sigma(1), \sigma(2)\}} = S^{(\sigma(1), \sigma(2))} \cap S^{(\sigma(2), \sigma(1))}, \dots$

$$x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)} \Rightarrow S^{\{\sigma(1)\}}, S^{\{\sigma(1), \sigma(2)\}}, \dots, S^{\{\sigma(1), \sigma(2), \dots, \sigma(k)\}}$$

- S is called **linearly reducible** if for all $1 \leq k \leq N$ every polynomial in $S^{\{\sigma(1), \sigma(2), \dots, \sigma(k)\}}$ is linear in $x_{\sigma(k+1)}$
i.e. if the full sequence including $S^{\{\sigma(1), \sigma(2), \dots, \sigma(N)\}}$ is generated.
- We have $\mathcal{P}^{\{\sigma(1), \sigma(2), \dots, \sigma(i)\}} \subseteq S^{\{\sigma(1), \sigma(2), \dots, \sigma(i)\}}$.
- If $S = \{\mathcal{U}_G, \mathcal{F}_G\}$ is linearly reducible we call the **Feynman graph** G linearly reducible.

Some classes of linearly reducible (massless) Feynman graphs :

- all vacuum graphs with vertex width 3 \Rightarrow corresponding propagator-type graphs (Brown '09)
- all minors of linearly reducible graphs (Brown '09, CB, Lueders, '13)
- all propagator-type graphs with ≤ 4 loops (Panzer '13)
- all graphs with three off-shell legs and ≤ 3 loops (Panzer '14)
- all graphs with vertex width 3 with three off-shell legs (Panzer PhD thesis)
- all ladder-shaped graphs with four off-shell legs (Panzer PhD thesis)

Example 1: Massless one-loop triangle

Feynman integral: $I = \prod_{i=1}^3 \left(\int_0^\infty dx_i \right) \delta(H) \mathcal{U}^{-1+2\epsilon} \mathcal{F}^{-1-\epsilon}$ omitting a trivial factor $\Gamma(1+\epsilon)$

$$\begin{aligned}\mathcal{U} &= x_1 + x_2 + x_3, \\ \mathcal{F} &= -x_1 x_2 p_3^2 - x_2 x_3 p_1^2 - x_1 x_3 p_2^2,\end{aligned}$$

Kinematical invariants: $\frac{p_3^2}{p_2^2} = x_4 x_5$ and $\frac{p_1^2}{p_2^2} = (1-x_4)(1-x_5)$

We consider the momentum space region where $x_4 > 0$, $x_5 > 0$.

Here the integral is finite: $I = I_0 + \epsilon I_1 + \mathcal{O}(\epsilon^2)$

We have to check linear reducibility of $\{\mathcal{U}, \mathcal{F}\}$.

>U:=x[1]+x[2]+x[3]:

>F:=x[1]*x[2]*x[4]*x[5]+x[1]*x[3]+x[2]*x[3]*(1-x[4])*(1-x[5]):

>GenFeyn:=[x[1],x[2],x[3],x[4],x[5]]:

>COMPATIBILITY_GRAPH=true: #This is also the default value.

>MPLPolynomialReduction([U,F],GenFeyn[1..3],GenFeyn);

[[{ }, [x₁ + x₂ + x₃, x₁x₂x₄x₅ + x₁x₃ + x₂x₃(1 - x₄)(1 - x₅)], [{1, 2}]],
[{x₃ }, [x₁ + x₂ - x₂x₅ - x₄x₂ + x₂x₄x₅, x₁ + x₂, x₁ + x₂ - x₂x₅, x₁ + x₂ - x₄x₂],
[{3, 4 }, {1, 3 }, {1, 4 }, {2, 3 }, {2, 4 }]],
[{x₂ }, [x₁x₄x₅ + x₃ - x₃x₅ - x₃x₄ + x₃x₄x₅, x₁ + x₃,
-x₃ + x₁x₅ + x₃x₅, x₁x₄ - x₃ + x₃x₄],
[{3, 4 }, {1, 3 }, {1, 4 }, {2, 3 }, {2, 4 }]],
[{x₁ }, [x₂x₄x₅ + x₃, x₂ + x₃, -1 + x₄, -1 + x₅, x₄x₂ + x₃, x₃ + x₂x₅],
[{3, 4 }, {1, 3 }, {1, 4 }, {2, 3 }, {2, 4 }, {3, 5 }, {4, 5 }, {5, 6 }, {1, 5 },
{1, 6 }, {2, 5 }, {2, 6 }, {3, 6 }]],
[{x₁, x₂ }, [-1 + x₅, -1 + x₄, x₄ - x₅], [{1, 2 }, {1, 3 }, {2, 3 }]],
[{x₁, x₃ }, [-1 + x₅, -1 + x₄, x₄ - x₅], [{1, 2 }, {1, 3 }, {2, 3 }]],
[{x₂, x₃ }, [-1 + x₄, -1 + x₅, x₄ - x₅], [{1, 2 }, {1, 3 }, {2, 3 }]],
[{x₁, x₂, x₃ }, [-1 + x₅, -1 + x₄, x₄ - x₅], [{1, 2 }, {1, 3 }, {2, 3 }]]]

As I is finite, we can expand the integrand to obtain

$$I = I_0 + \epsilon I_1 + \mathcal{O}(\epsilon^2)$$

$$I_0 = \prod_{i=1}^3 \left(\int_0^\infty dx_i \right) \delta(H) \frac{1}{U \tilde{\mathcal{F}}},$$

$$I_1 = \prod_{i=1}^3 \left(\int_0^\infty dx_i \right) \delta(H) \frac{2 \ln(U) - \ln(\tilde{\mathcal{F}})}{U \tilde{\mathcal{F}}}.$$

where $\tilde{\mathcal{F}} = -\frac{\mathcal{F}}{p_2^2}$.

Computation with MPL:

```
>MPLFeynmanIntegrate(Integrand0,GenFeyn[1..2],GenFeyn):
```

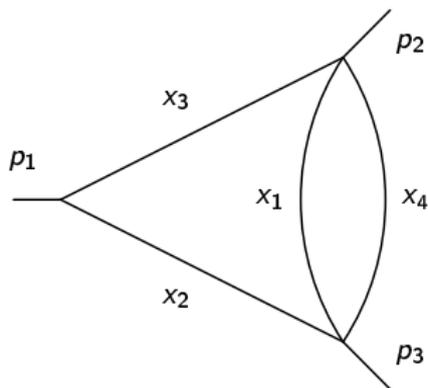
```
>I0:=normal(subs(x[3]=1,%));
```

```
Integration over x[1].
```

```
Integration over x[2].
```

$$I_0 = \frac{1}{-x_5 + x_4} \left(-\text{bar} \left(\frac{d(x_4)}{x_4}, \frac{d(x_4)}{-1 + x_4} \right) - \text{bar} \left(\frac{d(x_4)}{x_4} \right) \text{bar} \left(\frac{d(x_5)}{-1 + x_5} \right) \right. \\ \left. + \text{bar} \left(\frac{d(x_4)}{-1 + x_4}, \frac{d(x_4)}{x_4} \right) + \text{bar} \left(\frac{d(x_4)}{-1 + x_4} \right) \text{bar} \left(\frac{d(x_5)}{x_5} \right) \right. \\ \left. + \text{bar} \left(\frac{d(x_5)}{x_5}, \frac{d(x_5)}{-1 + x_5} \right) - \text{bar} \left(\frac{d(x_5)}{-1 + x_5}, \frac{d(x_5)}{x_5} \right) \right)$$

Example 2: Massless two-loop triangle



Feynman integral: $I = \prod_{i=1}^4 \int_0^\infty dx_i \delta(H) \mathcal{U}^{3\epsilon-2} \mathcal{F}^{-2\epsilon}$ omitting a trivial factor $\Gamma(2\epsilon)$

$$\mathcal{U} = x_1 x_4 + (x_1 + x_4)(x_2 + x_3), \mathcal{F} = -p_1^2 x_2 x_3 (x_1 + x_4) - p_2^2 x_1 x_3 x_4 - p_3^2 x_1 x_2 x_4$$

Kinematical invariants: $\frac{p_2^2}{p_1^2} = (1 + x_5)(1 + x_6)$ and $\frac{p_3^2}{p_1^2} = x_5 x_6$

We consider the momentum space region where $x_5 > 0$, $x_6 > 0$.

With Panzer's method we expand: $I = \frac{1}{\epsilon} I_{-1} + I_0 + \epsilon I_1 + \mathcal{O}(\epsilon^2)$,

Application of MPLPolynomialReduction to \mathcal{U} , \mathcal{F} and of MPLCheckOrder

⇒ allowed order of integrations: x_1, x_4, x_3, x_2

With MPLFeynmanIntegrate we obtain (in agreement with [Chavez, Duhr 2012](#)):

$$l_{-1} = 1,$$

$$l_0 = 5,$$

$$l_1 = \frac{1}{x_6 - x_5} \left(2x_5(1 + x_6) \left(\left[\frac{d(x_6)}{1 + x_6} \right] \left[\frac{d(x_5)}{x_5} \right] + \left[\frac{d(x_5)}{x_5} \middle| \frac{d(x_5)}{1 + x_5} \right] \left[\frac{d(x_5)}{1 + x_5} \middle| \frac{d(x_5)}{x_5} \right] \right) \right. \\ \left. - 2x_6(1 + x_5) \left(\left[\frac{d(x_5)}{1 + x_5} \right] \left[\frac{d(x_6)}{x_6} \right] + \left[\frac{d(x_6)}{x_6} \middle| \frac{d(x_6)}{1 + x_6} \right] + \left[\frac{d(x_6)}{1 + x_6} \middle| \frac{d(x_6)}{x_6} \right] \right) \right) \\ - 3\zeta(2) + 19.$$

Summary of the program MPL

- An implementation of multiple polylogarithms as **homotopy invariant iterated integrals** on $\mathcal{M}_{0,n}$ in cubical coordinates.
- Derivation of the vectorspace $V(\Omega_m)$ of these functions = derivation of all integrable words in Ω_m .
- Multiplication, differentiation, limits at 0 and 1 and primitives of these functions.
- **Multiple zeta values** are obtained in the coefficients.
- Symbolic computation of '**cubical**' **integrals** (arising as periods on $\mathcal{M}_{0,n}$, from irrationality proofs, expanding hypergeometric functions,...)
- Check **linear reducibility** of Symanzik polynomials
- Symbolic computation of a class of **linearly reducible Feynman integrals**.