Motivic Milnor fiber at infinity

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Definitions

1) Equivariant Grothendieck group

All algebraic varieties are over $\mathbb{C}$.

Let X be an algebraic variety and G an algebraic group acting on X. We say that the action is good if all G-orbit is contained in an affine open subset of X. In the following we shall take $G = \mathbb{G}_m$ the multiplicative group and assume that the actions are good.

Let S be an algebraic variety.

We denote by

$$\mathrm{Var}_{S\times\mathbb{G}_m}^{\mathbb{G}_m}$$

the category whose objects are

$$X \to S \times \mathbb{G}_m, (p_S, p_{\mathbb{G}_m}), \sigma$$

- where σ is a good action of $\mathbb{G}_m$ on X
- $\exists n, \forall x \in X, \lambda \in \mathbb{G}_m, p_{\mathbb{G}_m}(\sigma(\lambda, x)) = \lambda^n p_{\mathbb{G}_m}(x)$
- $\forall x \in X, \lambda \in \mathbb{G}_m, p_S(\sigma(\lambda, x)) = p_S(x)$

We consider the Grothendieck ring $K_0(\mathrm{Var}^{\mathbb{G}_m}_{S\times\mathbb{G}_m})$.
It is generated by classes $[X \to S \times \mathbb{G}_m, \sigma]$, with

$$[X \to S\times\mathbb{G}_m, \sigma] = [Y \to S\times\mathbb{G}_m, \sigma] + [X\setminus Y \to S\times\mathbb{G}_m, \sigma]$$

for all Y closed in X and invariant by $\mathbb{G}_m$.
The ring operation is given by

$$[X \times_{S\times\mathbb{G}_m} X' \to S \times \mathbb{G}_m, \sigma_X \times \sigma_{X'}] =$$

$$[X \to S \times \mathbb{G}_m, \sigma_X][X' \to S \times \mathbb{G}_m, \sigma_{X'}]$$

with two technical conditions:

$$[X \times \mathbb{A}^n \to S \times \mathbb{G}_m, \sigma] = [X \times \mathbb{A}^n \to S \times \mathbb{G}_m, \sigma']$$

where σ and σ' lift the same action on X on $X \times \mathbb{A}^n$

$$[X \to S \times \mathbb{G}_m, \sigma] = [X \to S \times \mathbb{G}_m, \sigma_k]$$

where for all $k > 0$, $\sigma_k(\lambda, x) = \sigma(\lambda^k, x)$

We denote by

$$\mathbb{L} = [S \times \mathbb{G}_m \times \mathbb{A}^1 \to S \times \mathbb{G}_m, p_{S\times\mathbb{G}_m}, \tau]$$

where $p_{S\times\mathbb{G}_m}$ is the projection on $S \times \mathbb{G}_m$ and $\tau(\lambda,(s,\mu,x)) = (s,\lambda\mu,x)$.

$$\mathcal{M}^{\mathbb{G}_m}_{S\times\mathbb{G}_m} = K_0(\mathrm{Var}^{\mathbb{G}_m}_{S\times\mathbb{G}_m})[\mathbb{L}^{-1}]$$

Let $l : S \to S'$ be a morphism. There exists a morphism $l_!$ (direct image) from $\mathcal{M}_{S\times\mathbb{G}_m}^{\mathbb{G}_m}$ to $\mathcal{M}_{S'\times\mathbb{G}_m}^{\mathbb{G}_m}$ such that

$$l_!([X \to S \times \mathbb{G}_m, (p_S, p_{\mathbb{G}_m}), \sigma]) =$$

$$[X \to S' \times \mathbb{G}_m, (l \circ p_S, p_{\mathbb{G}_m}), \sigma].$$

Let $T \subset S$ and $i : T \to S$ the injection, there exists a morphism (restriction) from $\mathcal{M}_{S\times\mathbb{G}_m}^{\mathbb{G}_m}$ to $\mathcal{M}_{T\times\mathbb{G}_m}^{\mathbb{G}_m}$ such that

$$i^{-1}([X \to S \times \mathbb{G}_m, (p_S, p_{\mathbb{G}_m}), \sigma]) =$$

$$[p_S^{-1}(T) \to T \times \mathbb{G}_m, (p_S, p_{\mathbb{G}_m}), \sigma].$$

2) Arc spaces.

Let X be a $\mathbb{C}$-variety. For any natural number n, we denote by $\mathcal{L}_n(X)$ the *space of* n-*jets* of X. This set is an algebraic variety whose K-rational points, for any field extension $K/\mathbb{C}$ are the $K[t]/t^{n+1}$-rational points of X.

There are canonical morphisms $\mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$.

The *arc space* of X, denoted by $\mathcal{L}(X)$, is the projective limit of this system. This set is a $\mathbb{C}$-scheme and we denote by $\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$ the canonical morphisms.

For a non zero element ϕ in $\mathbb{C}[[t]]$ or $\mathbb{C}[t]/t^{n+1}$, we denote by $\operatorname{ord}\phi$ the valuation of ϕ and by $\overline{ac}(\phi)$ its first non zero coefficient. The group $\mathbb{G}_m$ acts on $\mathcal{L}(X)$ by $\sigma(\lambda, \phi)(t) = \phi(\lambda t)$.

a) The motivic Zeta function.

Let X be a smooth variety and $f : X \mapsto \mathbb{A}^1$ a morphism. Let

$$X_0(f) = \{x \in X | f(x) = 0\}$$

and for $n \geq 1$

$$X_n(f) = \{\phi \in \mathcal{L}(X) | \text{ord}\, f(\phi) = n\}$$

We can consider

$$[X_n(f) \to X_0(f) \times \mathbb{G}_m, p, \sigma] \in \mathcal{M}^{\mathbb{G}_m}_{X_0(f)\times\mathbb{G}_m}$$

where σ the standard action on $\mathcal{L}(X)$ and $p = (p_{X_0(f)}, p_{\mathbb{G}_m})(\phi) = (\phi(0), \overline{ac}(f(\phi)))$.

We denote by $X_n^m(f)$, the image by π_m of $X_n(f)$. We have for $m \geq n$

$$[X_n^m(f)]\mathbb{L}^{-md} = [X_n^n(f)]\mathbb{L}^{-nd}$$

And call this the *motivic measure* of $X_n(f)$. (Kontsevitch) We define (Denef Loeser)

$$Z_f(T) = \sum_{n \geq 1} mes X_n(f) T^n \in \mathcal{M}_{X_0(f) \times \mathbb{G}_m}^{\mathbb{G}_m}[[T]].$$

b) The modified Zeta function.

Let Z be a smooth variety, and U a dense open subset of Z, let F be its complement and let $f : Z \to \mathbb{A}^1$ be a morphism. Let n and δ be two positive integers, we consider the arc space

$$Z_n^\delta(f) := \{\varphi \in \mathcal{L}(Z) \mid \text{ord}\, f(\varphi) = n, \text{ord}\, \varphi^* \mathcal{I}_F \le n\delta\}$$

endowed with the arrow "origin, angular component" and the standard action of $\mathbb{G}_m$ on arcs. Then, we consider the modified motivic zeta function (Guibert, Loeser, Merle)

$$Z_{f,U}^\delta(T) := \sum_{n \ge 1} \text{mes}(Z_n^\delta(f)) T^n \in \mathcal{M}_{Z_0(f) \times \mathbb{G}_m}^{\mathbb{G}_m}[[T]].$$

It is proven that there exists an integer δ_0 such that for all integer $\delta \geq \delta_0$, the series $Z^{\delta}_{f,U}(T)$ is rational and its limit when T goes to infinity is independent of δ. We will denote by $\mathcal{S}_{f,U}$ the limit $-\lim_{T\to\infty} Z^{\delta}_{f,U}(T)$.

c) The motivic Milnor fiber at infinity.

Let f be a polynomial in $\mathbb{C}[x,y]$. A *compactification* of f is a data $(X,i,\widehat{f})$ with X an algebraic variety, $\widehat{f}$ a proper map and i an open dominant immersion, such that the following diagram is commutative

$$\begin{array}{ccc} \mathbb{A}^2 & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widehat{f}} \\ \mathbb{A}^1 & \xrightarrow[j]{} & \mathbb{P}^1 \end{array},$$

where j is the following open dominant immersion

$$\begin{array}{rccc} j & : & \mathbb{A}^1 & \rightarrow & \mathbb{P}^1 \\ & & a & \mapsto & [1:a] \end{array}$$

With these notations, we denote by $1/\widehat{f}$ the extension of $1/f$ on $X \setminus \widehat{f}^{-1}(0)$.

Let $(X, i, \widehat{f})$ be a compactification. Let consider $\delta > 0$. For any integer $n \in \mathbb{N}^*$, we consider

$$X^{\delta}_{n,\mathbb{A}^2}(1/\widehat{f}) = \left\{ \varphi \in \mathcal{L}(X) \;\middle|\; \begin{array}{l} \text{ord } \varphi^* \mathcal{I}_{X \setminus i(\mathbb{A}^2)} \leq n\delta \\ \text{ord } \frac{1}{\widehat{f}}(\varphi(t)) = n \end{array} \right\}$$

As $n \geq 1$, for any arc φ in X_n^δ, $\varphi(0)$ belongs to $\widehat{f}^{-1}(\infty)$. So, we have a canonical map

$$\begin{array}{ccc} X_{n,\mathbb{A}^2}^\delta(1/\widehat{f}) & \to & \widehat{f}^{-1}(\infty) \times \mathbb{G}_m \\ \varphi & \mapsto & \left(\varphi(0), \overline{\mathsf{ac}}\, \frac{1}{\widehat{f}}(\varphi(t))\right). \end{array}$$

and a $\mathbb{G}_m$-action given by $(\lambda, \varphi)(t)$ equal to $\varphi(\lambda t)$. In particular, the motivic measure of $X_{n,\mathbb{A}^2}^\delta(1/\widehat{f})$ belongs to $\mathcal{M}_{\widehat{f}^{-1}(\infty)\times\mathbb{G}_m}^{\mathbb{G}_m}$.

$$Z_{\frac{1}{\widehat{f}},\mathbb{A}^2}^\delta(T) = \sum_{n\geq 1} \mathrm{mes}(X_{n,\mathbb{A}^2}^\delta(1/\widehat{f}))T^n \in \mathcal{M}_{\widehat{f}^{-1}(\infty)\times\mathbb{G}_m}^{\mathbb{G}_m}[[T]$$

Theorem 1 *Let $(X, i, \widehat{f})$ be a compactification of f and $\delta > 0$. The zeta function $Z^{\delta}_{\frac{1}{\widehat{f}}, \mathbb{A}^2_{\mathbb{C}}}(T)$ is rational for δ large enough and has a limit when T goes to infinity. We denote*

$$S_{\frac{1}{\widehat{f}}}(\mathbb{A}^2_{\mathbb{C}}) = -\lim_{T \to \infty} Z^{\delta}_{\frac{1}{\widehat{f}}, \mathbb{A}^2_{\mathbb{C}}}(T) \in \mathcal{M}^{\mathbb{G}_m}_{\widehat{f}^{-1}(\infty) \times \mathbb{G}_m}.$$

$$S_{f,\infty}(\mathbb{A}^2_{\mathbb{C}}) = \widehat{f}_! S_{\frac{1}{\widehat{f}}}(\mathbb{A}^2_{\mathbb{C}}) \in \mathcal{M}^{\mathbb{G}_m}_{\{\infty\} \times \mathbb{G}_m}$$

does not depend on the chosen compactification and is called motivic Milnor fiber at infinity *of f.*

(Raibaut, Matsui, Takeuchi)

Our aim is to compute $S_{f,\infty}(\mathbb{A}^2_{\mathbb{C}})$

Michel Raibaut has computed $S_{f,\infty}(\mathbb{A}^2_{\mathbb{C}})$ in the case where f is non degenerate for its Newton polygon. We want to generalize these results for all polynomials in two variables.

Newton algorithm

1) Newton polygon at infinity.

Let $f(x,y)=\sum c_{a,b}x^a y^b + c_{0,0}$ with $c_{0,0}$ generic. Let

$$\text{Supp} f = \{(a,b)\in\mathbb{N}^2 | c_{a,b}\neq 0\}$$

Let $\mathcal{N}_\infty(f)$ be the set of compact faces of the convex hull of Supp$f$ and $\mathcal{N}^0_\infty(f)$ the set of faces of $\mathcal{N}_\infty(f)$ which do not contain the origin.

Write $p\alpha+q\beta = N$, $(|p|,|q|) = 1$, the equation of a face γ of dimension 1 of $\mathcal{N}^0_\infty(f)$.

$$f_\gamma(x,y) = \sum_{(a,b)\in\gamma} c_{a,b} x^a y^b = x^{a_\gamma} y^{b_\gamma} \prod_{\mu\in R_{\gamma,f}} (x^q - \mu y^p)^{\nu_\mu}$$

We say that f is *non degenerate* if $\nu_\mu = 1$ for all $\gamma \in \mathcal{N}^0_\infty(f)$, all $\mu \in R_{\gamma,f}$.

To (p, q, μ) we associate a *Newton map at infinity*:

$$f_{\sigma_{p,q,\mu}}(v, w) = f(\mu^{q'} v^{-p}, v^{-q}(w+\mu^{p'})) \in \mathbb{C}[v^{-1}, v, w]$$

where $p'p - q'q = 1$ when $p > 0$,

$$f_{\sigma_{p,q,\mu}}(v, w) = f(v^{-p}(w+\mu^{p'}), \mu^{q'} v^{-q}) \in \mathbb{C}[v^{-1}, v, w]$$

when $p \leq 0$.

2) Newton algorithm for $\mathbb{C}[v^{-1}, v, w]$.

Let $f(v,w) = \sum d_{a,b} v^a w^b + d_{0,0} \in \mathbb{C}[v^{-1}, v, w]$ with $d_{0,0}$ generic. We consider $\mathcal{N}_0(f)$, the set of compact faces of the convex hull of

$$\{(a,b) + \mathbb{R}^2_{\geq 0}, (a,b) \in \mathrm{Supp} f\}$$

Let $p\alpha + q\beta = N$, $(p,q) = 1$, the equation of a face γ of dimension 1 of $\mathcal{N}_0(f)$.

$$f_\gamma(x,y) = \sum_{(a,b)\in\gamma} d_{a,b} x^a y^b + d_{0,0}$$

$$= x^{a_\gamma} y^{b_\gamma} \prod_{\mu \in R_{\gamma,f}} (x^q - \mu y^p)^{\nu_\mu}$$

We say that f is *non degenerate* if $\nu_\mu = 1$ for all μ and all $\gamma \in \mathcal{N}_0(f)$, all $\mu \in R_{\gamma,f}$.

To (p, q, μ) we associate a *Newton map at the origin*:

$$f_{\sigma_{p,q,\mu}}(v, w) = f(\mu^{q'} v^p, v^q(w+\mu^{p'})) \in \mathbb{C}[v^{-1}, v, w]$$

where $p'p - q'q = 1$.

Theorem 2 *Let* $f(v,w) = \sum d_{a,b} v^a w^b + d_{0,0} \in \mathbb{C}[v^{-1}, v, w]$ *with* $d_{0,0}$ *generic, then after a finite number of steps, either the Newton polygon at the origin has one face of dimension 0,* $(-M, 0), M \geq 0$*, or one face of dimension 1 containing the origin.*

Computation of $S_{f,\infty}(\mathbb{A}^2_{\mathbb{C}})$

1) Compactification

In the following, we consider the compactification $(X, i, \widehat{f})$ of f with X the set of

$$([x_0 : x_1], [y_0 : y_1], [z_0 : z_1]) \in (\mathbb{P}^1)^3$$

such that

$$z_0 x_0^{d_x} y_0^{d_y} f\left(\frac{x_1}{x_0}, \frac{y_1}{y_0}\right) = z_1 x_0^{d_x} y_0^{d_y}$$

i and j are the following open dominant immersions

$$\begin{array}{rccc} i & : & \mathbb{A}^2 & \to & X \\ & & (x,y) & \mapsto & ([1:x],[1:y],[1:f(x,y)]) \end{array}$$

$$\begin{array}{rccc} j & : & \mathbb{A}^1 & \to & \mathbb{P}^1 \\ & & a & \mapsto & [1:a] \end{array}$$

and $\widehat{f}$ is the following projection which is proper

$$\begin{array}{rccc} \widehat{f} & : & X & \to & \mathbb{P}^1 \\ & & ([x_0:x_1],[y_0:y_1],[z_0:z_1]) & \mapsto & [z_0:z_1]. \end{array}$$

2) First step

From a result of Guibert-Loeser-Merle we can write

$$S_{\frac{1}{\widehat{f}}}(\mathbb{A}^2_{\mathbb{C}}) = S_{\frac{1}{\widehat{f}}}(\mathbb{G}_m^2) + S_{\frac{1}{\widehat{f}}}(\{0\} \times \mathbb{G}_m) +$$

$$S_{\frac{1}{\widehat{f}}}(\mathbb{G}_m \times \{0\}) + S_{\frac{1}{\widehat{f}}}(\{(0,0)\}).$$

We apply $\widehat{f}_!$ and we have

$$S_{f,\infty}(\mathbb{A}^2_{\mathbb{C}}) = S_{f,\infty}(\mathbb{G}_m^2) + S_{f(0,y),\infty} + S_{f(x,0),\infty} + S_{f(0,0),\infty}$$

with $S_{f(x,0),\infty} = \widehat{f}_! S_{\frac{1}{\widehat{f}}}(\mathbb{G}_m \times \{0\})$. If $f(x,0)$ has a degree $l \geq 0$, then we have the equality

$$S_{f(x,0),\infty} = \left[\mathbb{G}_m \to \mathbb{G}_m, 1/(c_{l,0}x^l), \tau\right]$$

with τ the action on $\mathbb{G}_m$, $\tau(\lambda, x) = \lambda^{-1}x$. If $f(x,0)$ is zero then, $S_{f(x,0),\infty} = 0$.

The point $(0,0)$ does not belong to the fiber $\widehat{f}^{-1}(\infty)$ then the motive $S_{f(0,0),\infty}$ is equal to zero, because all the arc spaces used in the corresponding zeta function are empty.

3) Second step

For any $n \geq 1$, for any $\delta \geq 1$, we consider the arc space

$$X^{\delta}_{n,\mathbb{G}_m^2}(1/\widehat{f}) = \left\{ \varphi(t) \in \mathcal{L}(X) \;\middle|\; \begin{array}{l} \text{ord}\, \varphi^* \mathcal{I}_{X \setminus i(\mathbb{G}_m^2)} \leq n\delta \\ \text{ord}\, \frac{1}{\widehat{f}}(\varphi(t)) = n \end{array} \right\}$$

$$X^{\delta}_{n,\mathbb{G}_m^2}(1/\widehat{f}) \to \widehat{f}^{-1}(\infty) \times \mathbb{G}_m : \varphi \mapsto \left(\varphi(0), \overline{\text{ac}}\, \frac{1}{\widehat{f}}(\varphi(t))\right).$$

and a $\mathbb{G}_m$-action given by $(\lambda, \varphi(t))$ equal to $\varphi(\lambda t)$.

For any $n \geq 1$, for any $\delta \geq 1$, there is a bijection between $X^{\delta}_{n,\mathbb{G}_m^2}(1/\widehat{f})$ and the set

$$X^{\delta}_{n,(\alpha,\beta)}(1/f) = \left\{ \left(\frac{P(t)}{t^{\alpha}}, \frac{Q(t)}{t^{\beta}}\right) \;\middle|\; \begin{array}{l} (\alpha, \beta) \in \mathbb{Z}^2 \setminus \mathbb{Z}^2_{\leq 0} \\ |\alpha| + |\beta| \leq n\delta, \\ \text{ord}\, f\left(\frac{P(t)}{t^{\alpha}}, \frac{Q(t)}{t^{\beta}}\right) = -n \end{array} \right\}$$

4) Third step

We consider the linear form $l_{(\alpha,\beta)}$ equal to $((\alpha,\beta) \mid .)$ defined on $\mathbb{R}^2$. The maximum of $l_{(\alpha,\beta)|\mathcal{N}_\infty(f)}$, denoted by $\mathfrak{m}(\alpha,\beta)$, is non negative and obtained on a facet of $\mathcal{N}_\infty(f)$ denoted by $\gamma(\alpha,\beta)$. In particular, $l_{(\alpha,\beta)}$ is constant on $\gamma(\alpha,\beta)$.

For any arc $\varphi = (P(t)/t^\alpha, Q(t)/t^\beta)$, if the origin $\varphi(0)$ belongs to $\widehat{f}^{-1}(\infty)$ then the order ord $1/\widehat{f}(\varphi(t)) \leq \mathfrak{m}(\alpha,\beta)$ is positive and $\mathfrak{m}(\alpha,\beta)$ is positive.

We denote by Ω the open set $\{(\alpha,\beta) \in \mathbb{Z}^2 \mid \mathfrak{m}(\alpha,\beta) > 0\}$. For any (α,β) in Ω, the face $\gamma(\alpha,\beta)$ does not contain 0. For any face γ of $\mathcal{N}_\infty(f)^o$, we denote by C_γ the interior, in its own generated vector space in $\mathbb{R}^2$, of the positive cone generated by the set $\{(\alpha,\beta) \in \Omega \mid \gamma(\alpha,\beta) = \gamma\}$. This set is a polyhedral cone which is rational, convex and relatively open.

We consider the motivic zeta function

$$Z^{\delta}_{1/\hat{f},\mathbb{G}_m^2}(T) = \sum_{n\geq 1} \text{mes}\,(X^{\delta}_{n,\mathbb{G}_m^2}(1/\hat{f}))T^n$$

Let γ be a face in $\mathcal{N}_\infty(f)^o$. For a positive integer δ, we consider the following cones

$$C_\gamma^{\delta,=} = \{(\alpha,\beta) \in C_\gamma \mid |\alpha| + |\beta| \leq \mathfrak{m}(\alpha,\beta)\delta\}$$

and

$$C_\gamma^{\delta,<} = \{n \in \mathbb{N}^*, (\alpha,\beta) \in C_\gamma \mid (|\alpha|+|\beta|)/\delta) \leq n < \mathfrak{m}(\alpha,\beta)\}$$

They are polyhedral rational convex cones.

With, these notations, for any positive integer δ, the motivic zeta function has the following decomposition

$$Z^{\delta}_{1/\hat{f},\mathbb{G}_m^2}(T) = \sum_{\gamma\in\mathcal{N}_\infty(f)^o} \Big(Z_\gamma^{\delta,=}(T) + Z_\gamma^{\delta,<}(T)\Big)$$

where

$$Z_\gamma^{\delta,=}(T) = \sum_{(\alpha,\beta)\in C_\gamma^{\delta,=}\cap\mathbb{Z}^2} \mathrm{mes}(X_{\mathfrak{m}(\alpha,\beta),(\alpha,\beta)}(1/\hat{f}))T^{\mathfrak{m}(\alpha,\beta)}$$

and

$$Z_\gamma^{\delta,<}(T) = \sum_{(n,(\alpha,\beta))\in C_\gamma^{\delta,<}\cap\mathbb{Z}^3} \mathrm{mes}(X_{n,(\alpha,\beta)}(1/\hat{f}))T^{n}.$$

5) Forth step

Proposition 3 *Let* γ *be a facet in* $\mathcal{N}_\infty(f)^o$. *For* δ *big enough, the formal series* $Z_\gamma^{\delta,=}(T)$ *is rational and admits a limit*

$$-\lim Z_\gamma^{\delta,=}(T) =$$

$$-\chi_c(C_\gamma^{\delta,=})[\mathbb{G}_m^2 \setminus f_\gamma^{-1}(0) \to \mathbb{G}_m, 1/f_\gamma, \sigma_\gamma]$$

with $\chi_c(C_\gamma^{\delta,=})$ *equal to* $(-1)^{\dim\gamma}$ *if* γ *is not contained in a face which contains 0 and otherwise equal to 0. where* σ_γ *is the action of* $\mathbb{G}_m$ *on* $\mathbb{G}_m^2$ *defined by*

$$\sigma_\gamma(\lambda,(x,y)) = (\lambda^{-\alpha}x, \lambda^{-\beta}y).$$

6) Zeta function with a differential.

Let X be a $\mathbb{C}$-variety and $g : X \to \mathbb{A}^1_{\mathbf{k}}$ be a regular map. Let U be a smooth open subvariety of X and F be the closed subset $X \setminus U$. Assume U be dense in X. We assume also X endowed with a differential form ω without poles and with a divisor D included in F as a zero locus.

For any $\delta > 0$, $n \in \mathbb{N}^*$ and $l \in \mathbb{N}^*$, we define

$$X^{\delta}_{n,l}(g,\omega,U) = \left\{ \varphi \in \mathcal{L}(X) \,\middle|\, \begin{array}{c} \text{ord}\, g(\varphi(t)) = n \\ \text{ord}\, \varphi^*(\mathcal{I}_F) \leq n\delta \\ \text{ord}\, \omega\varphi = l \end{array} \right\}.$$

We consider the following motivic zeta function in variables S and T

$$Z^{\delta}(S,T) = \sum_{n \geq 1} \sum_{l \geq 1} \text{mes}\,(X^{\delta}_{n,l}(g,\omega,U))S^l T^n$$

$$\in \mathcal{M}^{\mathbb{G}_m}_{g^{-1}(0)\times\mathbb{G}_m}[[S,T]].$$

Lemma 4 *For any* $\delta > 0$, *the motivic zeta function* $Z^{\delta}(S,T)$ *is rational in the variables* S *and* T. *The evaluation* $Z^{\delta}(\mathbb{L}^{-1},T)$ *is well-defined, and when* T *goes to infinity this series has a limit independant from* δ, *for* δ *large enough. We call the zeta function* $Z^{\delta}(\mathbb{L}^{-1},T)$, motivic zeta function of g relatively to the open set U and the differential form ω *and we denote it by*

$$Z^{\delta}_{g,\omega,U}(T) = \sum_{n\geq 1}\left(\sum_{l\geq 1} \mathit{mes}\,(X^{\delta}_{n,l}(g,\omega,U))\mathbb{L}^{-l}\right)T^{n}$$

We consider also the limit

$$S_{g,\omega,U} = - \lim_{T \to \infty} Z^{\delta}_{g,\omega,U}(T) \in \mathcal{M}^{\mathbb{G}_m}_{g^{-1}(0) \times \mathbb{G}_m}$$

which does not depend on $\delta >> 1$.

If $(0,0) \in g^{-1}(0)$ we shall write $(S_{g,\omega,U})_{(0,0)}$ for $i_T^{-1}(S_{g,\omega,U})$ and $T = \{(0,0)\}$.

Proposition 5 *For δ large enough, the motivic zeta function $Z_\gamma^{\delta,<}$ is rational and has a limit independant from δ*

$$-\lim_{T\to\infty} Z_\gamma^{\delta,<}(T) = \sum_{\mu\in R_\gamma} \left(S_{1/f_{\sigma(p,q,\mu)},\omega_{p,q},v\neq 0}\right)_{(0,0)}$$

with the differential form

$$\omega_{p,q}(v,w) = v^{(p+q-1)} dv \wedge dw.$$

We are left studying $\left(S_{1/f,\omega,U}\right)_{(0,0)}$ with

$f \in \mathbb{C}[x^{-1}, x, y]$, w a differential form and U an open dense subset of $\mathbb{A}^2_{\mathbb{C}}$.

7) Newton algorithm: the main ingredient.

We consider an integer $M > 0$ and a rational function f in $\mathbb{C}[x, y, x^{-1}]$ equal to

$$f(x, y) = x^{-M} g(x, y) = \sum_{(a,b)\in\mathbb{Z}\times\mathbb{N}} c_{a,b} x^a y^b + c_{0,0}$$

where g is a polynomial in $\mathbf{k}[x, y]$ not divisible by x.

We consider the case of $f(x,y) = x^{-M} g(x,y)$ with $g(0,0) \neq 0$ and $M > 0$. We have

$$\left(S_{1/f,\omega,x\neq 0}\right)_{(0,0)} = [\mathbb{G}_m \to \mathbb{G}_m, x^M/g(0,0), \sigma_{\mathbb{G}_m}],$$

with $\sigma_{\mathbb{G}_m}$ the action by translation of $\mathbb{G}_m$ defined by $\sigma_{\mathbb{G}_m}(\lambda, x) = \lambda.x$ for any (λ, x) in $\mathbb{G}_m^2$.

We denote by $\mathcal{N}(f)^{<}$ the set of compact faces γ, not contained in $\mathbb{Z} \times \{0\}$ and such that $\mathfrak{m}_{|C_\gamma}$ has only negative values. The one dimensional faces of $\mathcal{N}(f)^{<}$ are the faces supported by lines strictly under 0 namely with equation of type $ap + bq = N$ with (p, q) non negative and $N < 0$. The intersection of two one dimensional faces of $\mathcal{N}(f)^{<}$ belongs to $\mathcal{N}(f)^{<}$

Theorem 6 *Let* $M > 0$ *be an integer and* f *be the rational function* $x^{-M}g(x,y)$, *where* g *is a polynomial in* $\mathbf{k}[x,y]$ *of the form*

$$g(x,y) = cx^l h(x) + yk(x,y)$$

such that $h(0) = 1$, $g(0,0) = 0$ *and* g *is not divisible by* x. *We consider an integer* ν *in* $\mathbb{N}^*$ *and the differential form* $\omega = x^{\nu-1}dx \wedge dy$. *The motivic Milnor fiber of* $1/f$ *in* $(0,0)$ *relatively to the differential form* ω *and the open set* $x \neq 0$ *is :*

$$(S_{1/f,\omega,x\neq 0})_{(0,0)} =$$

$$\sum_{\gamma\in\mathcal{N}(f)^{<}} -(-1)^{\dim(\gamma)}\left[\mathbb{G}_m^2 \setminus (g_\gamma = 0), \frac{x^M}{g_{\gamma(x,y)}}, \sigma_\gamma\right] +$$

$$\epsilon_{l,M}\left[\mathbb{G}_m, cx^{M-l}, \sigma_{(l-M,0)}\right]$$

$$+ \sum_{\gamma\in\mathcal{N}(f)^{<},\dim\gamma=1} \sum_{\mu\in R_\gamma} \left(S_{\frac{1}{f_{\sigma(p,q,\mu)}},\omega_{p,q,\mu},v\neq 0}\right)_{(0,0)},$$

with $\epsilon_{l,M} = 1$ *if* $c \neq 0$ *and* $l < M$ *and* $\epsilon_{l,M} = 0$ *otherwise.*

We are left with the final of Newton algorithm.

Application

Let $f(x,y) = \sum c_{a,b}x^a y^b + c_{0,0} \in \mathbb{C}[x,y]$. let $f(x,0) = x^{n_x} + .., f(0,y) = y^{n_y} + ..$ where .. means lower terms. Let $\mathcal{V}(f) = 2S - n_x - n_y$ where S is the surface contained in $\mathcal{N}_\infty(f)$. In 1976, Kouchnirenko proved that if $n_x \neq 0, n_y \neq 0$ (f commode) and if the global Milnor number of f is finite, then

$$\mu(f) \leq \mathcal{V}(f) + 1.$$

If f is non degenerate, we have equality.

In 1996, we proved that if f has only isolated singularities, then

$$\mu(f) + \lambda(f) \leq \mathcal{V}(f) + 1.$$

and if f is non degenerate, we have equality. (The number λ will be defined tomorrow.)

Now we are able to express $\mu(f) + \lambda(f)$ in terms of surfaces of Newton polygons in all cases.

We need to introduce the notion of surface of a Newton polygon with respect to a polynomial. Let $\mathcal{N}$ be a convex polygon in $\mathbb{R}^2_{\geq 0}$ containing the origin and f a polynomial such that $\mathcal{N} = \mathcal{N}_\infty(f)$ Let γ a face of $\mathcal{N}$, let $\mathcal{S}_\gamma$ twice the surface of the triangle defined by the origin and γ, s_γ the number of points with integral coordinates on γ. Denote by $r_{f,\gamma}$ the number of roots of f_γ. Let

$$\mathcal{S}_{\gamma,f} = \frac{\mathcal{S}_\gamma}{s_\gamma - 1} r_{f,\gamma}$$

Let

$$\mathcal{V}_{\mathcal{N},f} = \sum_\gamma \mathcal{S}_{\gamma,f} - n_x - n_y$$

Now consider a Newton polygon $\mathcal{N}_0$ at the origin contained in $\mathbb{R}_{\geq -M} \times \mathbb{R}_{\geq 0}$ containing $(-n, 0)$ as a face, with $n \geq 0$. Let $f \in \mathbb{C}[v^{-1}, v, w]$ such that $\mathcal{N}_0 = \mathcal{N}_0(f)$. Define by induction

$$\tilde{\mathcal{V}}_{\mathcal{N}_0(f),f} = \mathcal{V}_{\mathcal{N}_0(f),f} + \sum_{\sigma_{p,q,\mu}} \tilde{\mathcal{V}}_{\mathcal{N}_0(f_{\sigma_{p,q,\mu}}),f_{\sigma_{p,q,\mu}}}$$

Theorem 7 *Let* $f(x,y) = \sum c_{a,b} x^a y^b + c_{0,0} \in \mathbb{C}[x,y]$*, with isolated singularities. Then*

$$\mu + \lambda = \mathcal{V}_{\mathcal{N}_\infty(f),f} + \sum \tilde{\mathcal{V}}_{\mathcal{N}_0(f_{\sigma_{p,q,\mu}}),f_{\sigma_{p,q,\mu}}} + 1$$

where the summation is taken over all $\sigma_{p,q,\mu}$ *Newton map at infinity associated to the Newton polygon at infinity of* f.

Example:

Consider

$$f(x,y) = x^6y^4+(4x^5+3x^4)y^3+(6x^4+11x^3+3x^2)y^2+ (4x^3+13x^2+2x+1)y+x^2+5x+5+t$$

The Newton polygon has two faces: Clockwise the first one has equation $-x+2y=2$

with face polynomial $y(x^2y+1)^3$ and the second one has equation $x - y = 2$ with face polynomial $x^2(xy + 1)^4$.

We have

$$f_1(v,w) = f(v(w+1), 1/v^2) = v^{-2}(8w^3 - v + \cdots)$$

$$f_2(v,w) = f(1/v, v(w-1)) = v^{-2}(w^4 + 2w^2v + tv^2 + \cdots)$$

We get $\mathcal{V}_{\mathcal{N}_\infty(f),f} = 1$, $\tilde{\mathcal{V}}_{\mathcal{N}_0(f_1),f_1} = 2$,$\tilde{\mathcal{V}}_{\mathcal{N}_0(f_1),f_1} = 0$, Then $\mu + \lambda = 1 + 2 + 1 = 4$.

The ingredients of the proof are the following:

1) $\chi(S_{f,\infty}(\mathbb{A}^2_{\mathbb{C}})) = \mu + \lambda$

2) $\chi([\mathbb{G}_m^2 \setminus f_\gamma^{-1}(0) \to \mathbb{G}_m, 1/f_\gamma, \sigma_\gamma])$ can be computed using Martin-Morales.