

Group Based Combinatorial Zeta Functions

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Selberg zeta functions

- $SL_2(\mathbb{R})$ acts on \mathfrak{H} by fractional linear transformation.
- Γ : discrete torsion-free cocompact subgroup of $SL_2(\mathbb{R})$
- $X_\Gamma = \Gamma \backslash \mathfrak{H} = \Gamma \backslash PGL_2(\mathbb{R}) / PO_2(\mathbb{R})$ is a compact Riemann surface with fundamental group $\pi_1(X_\Gamma, pt) \cong \Gamma$.
- Count geodesic cycles on X_Γ up to equivalence, i.e., ignoring the starting point
- A cycle is *primitive* if it is not obtained by repeating a cycle (of shorter length) more than once.
- Equivalence classes $[C]$ of primitive geodesics C are the "primes" of X_Γ .

The Selberg zeta function (1956) counts equiv. classes of geodesic cycles in X_Γ :

$$\begin{aligned} Z(X_\Gamma, s) &= \prod_{[C] \text{ prime}} \prod_{k \geq 0} (1 - e^{-l(C)(s+k)}) \\ &= \prod_{[\gamma] \text{ primitive}} \prod_{k \geq 0} (1 - e^{-l(\gamma)(s+k)}) \quad \text{for } \Re s > 1 \end{aligned}$$

because there is a length preserving bijection between primes $[C]$ and conjugacy classes $[\gamma]$ of primitive elements γ in Γ .

The Laplacian operator Δ acts on $L^2(X_\Gamma)$. It has discrete spectrum

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots .$$

The determinant of the Laplacian $\det(\Delta - s(1 - s))$, defined using the spectral zeta function, is formally equal to

$$\prod_{n \geq 0} (\lambda_n - s(1 - s)).$$

Sarnak, Voros (1987 independently):

$$\det(\Delta - s(1 - s)) = Z(X_\Gamma, s) (e^{E+s(1-s)} \frac{\Gamma_2(s)^2}{\Gamma(s)} (2\pi)^s)^{2g-2},$$

where g is the genus of X_Γ , and E is a constant.

RH: Selberg showed that the nontrivial zeros of $Z(X_\Gamma, s)$ are $\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda_n}$, $n \geq 1$.

The Ihara zeta function of a graph

- Let F be a nonarch. local field with the ring of integers \mathcal{O}_F and q elements in the residue field. Eg. $F = \mathbb{Q}_p, \mathbb{F}_q((t))$.
- Let Γ be a discrete torsion-free cocompact subgroup of $\mathrm{PGL}_2(F)$ and $X_\Gamma = \Gamma \backslash \mathrm{PGL}_2(F) / \mathrm{PGL}_2(\mathcal{O}_F) = \Gamma \backslash \mathcal{T}$.
- Ihara defined the zeta function

$$Z(X_\Gamma, u) = \prod_{[\gamma] \text{ primitive}} \frac{1}{1 - u^{l(\gamma)}}, \quad |u| \ll 1.$$

- Serre observed that X_Γ is a finite $(q+1)$ -regular graph, $Z(X_\Gamma, u)$ counts geodesic cycles in X_Γ , and the definition works for all finite graphs.

- X : connected undirected finite graph
- Primes of X are equivalence classes $[C]$ of primitive geodesic cycles C in X .

The Ihara zeta function of X counts the number $N_n(X)$ of closed geodesic cycles of length n :

$$\begin{aligned}
 Z(X; u) &= \exp \left(\sum_{n \geq 1} \frac{N_n(X)}{n} u^n \right) = \prod_{[C] \text{ prime}} \frac{1}{1 - u^{l(C)}} \\
 &= \prod_{[\gamma] \text{ primitive}} \frac{1}{1 - u^{l(\gamma)}}
 \end{aligned}$$

where γ lies in the fundamental group $\pi_1(X, pt)$ of X .

Properties of zeta functions of regular graphs

Ihara (1968): Let X be a finite $(q+1)$ -regular graph on n vertices. Then its zeta function $Z(X, u)$ is a rational function of the form

$$Z(X; u) = \frac{(1 - u^2)^{\chi(X)}}{\det(I - Au + qu^2I)},$$

where $\chi(X) = \#V - \#E = -n(q-1)/2$ is the Euler characteristic of X and $A = A(X)$ is the adjacency matrix of X .

The eigenvalues of A are real:

$$q + 1 = \lambda_1 > \lambda_2 \geq \cdots \lambda_n \geq -(q + 1).$$

Hence

$$\det(I - Au + qu^2I) = \prod_{1 \leq j \leq n} (1 - \lambda_j u + qu^2).$$

Connection between zeta functions of graphs and curves

The zeta function of a smooth irred. proj. curve V over \mathbb{F}_q counts $V(\mathbb{F}_{q^n})$. It is a rational function

$$Z(V, u) = \frac{P(V, u)}{(1 - u)(1 - qu)}.$$

The modular curve $X_0(N) = \Gamma_0(N) \backslash \mathfrak{H}^*$ is defined over \mathbb{Q} , and has good reduction at $p \nmid N$. When N is a prime $\equiv 1 \pmod{12}$, for each $p \neq N$, there is a subgroup Γ_N of $PGL_2(\mathbb{Q}_p)$ such that the zeta of the graph X_{Γ_N} is closely related to the zeta of $X_0(N)_{/\mathbb{F}_p}$, namely

$$\frac{\det(I - A(X_{\Gamma_N})u + pu^2I)}{1 - (p + 1)u + pu^2} = P(X_0(N)_{/\mathbb{F}_p}, u).$$

Riemann Hypothesis and Ramanujan graphs

- The trivial eigenvalues of X are $\pm(q + 1)$, of multiplicity ≤ 1 . The nontrivial eigenvalues λ satisfy $q + 1 > \lambda > -(q + 1)$.
- $Z(X, u)$ satisfies RH if the poles of $Z(X, u)$ from nontrivial eigenvalues of X all have the same absolute value $q^{-1/2}$ iff all nontrivial eigenvalues λ satisfy the bound

$$|\lambda| \leq 2\sqrt{q}.$$

Such a graph is called a *Ramanujan graph*.

- $Z(X, u)$ satisfies RH if and only if X is a Ramanujan graph.
- $[-2\sqrt{q}, 2\sqrt{q}]$ is the spectrum of the $(q + 1)$ -regular tree, the universal cover of X .
- Ramanujan graphs are spectrally extremal by Alon-Boppana.

Hashimoto's expression

Endow two orientations on each edge of a finite graph X . The neighbors of $u \rightarrow v$ are the directed edges $v \rightarrow w$ with $w \neq u$.

Associate the (directed) edge adjacency matrix T .

Hashimoto (1989): $N_n(X) = \text{Tr } T^n$ so that

$$Z(X, u) = \frac{1}{\det(I - Tu)}.$$

Combine both expressions for $Z(X, u)$ to get the identity

$$Z(X, u) = \frac{(1 - u^2)\chi(X)}{\det(I - Au + qu^2I)} = \frac{1}{\det(I - Tu)}.$$

This is the discrete analog of the relation between quantum and classical resonances.

Artin L -functions for graphs

For a finite-dimensional unitary representation ρ of $\pi_1(X, pt)$, we associate the Artin L -function

$$L(X, \rho, u) = \prod_{[\gamma] \text{ primitive}} \frac{1}{\det(I - \rho(\gamma)u^{l(\gamma)})}.$$

Ihara, Hashimoto, Stark-Terras: $L(X, \rho, u)^{-1}$ is a polynomial. Further, there are matrices A_ρ and T_ρ such that

$$L(X, \rho, u) = \frac{(1 - u^2)^{\chi(X) \deg \rho}}{\det(I - A_\rho u + qu^2 I)} = \frac{1}{\det(I - T_\rho u)}.$$

Artin L -functions for Riemann surfaces

Back to the Riemann surface $X_\Gamma = \Gamma \backslash \mathfrak{H}$ considered by Selberg.

For a finite-dim'l unitary representation ρ of Γ , Selberg defined

$$Z(X_\Gamma, \rho, s) = \prod_{[\gamma] \text{ primitive}} \prod_{k \geq 0} \det(1 - \rho(\gamma) e^{-l(\gamma)(s+k)}).$$

This is an Artin L -function. As Pohl explained, for Γ cofinite, it can be expressed in terms of a transfer operator

$$Z(X_\Gamma, \rho, s) = \det(I - \mathcal{L}_{s, \rho}),$$

which is parallel to the Hashimoto expression. The Ihara (spectral) expression for $Z(X_\Gamma, \rho, s)$ is known only for ρ trivial.

Distribution and density of primes in a graph

Let $\alpha : Y \rightarrow X$ be a finite unramified Galois cover of X with Galois group G . Each prime $[C]$ of $X \rightarrow$ primitive $[\gamma_C]$ in $\pi_1(X, pt) \rightarrow Frob_{[C]} := [\overline{\gamma_C}]$ in G . Given a conjugacy class \mathcal{C} of G , let

$$S_{\mathcal{C}} = \{\text{primes } [C] \text{ of } X : [\overline{\gamma_C}] = \mathcal{C}\}.$$

Assume $\pi_1(X, pt)$ has rank ≥ 2 .

Cebotarev Density Theorem [Hashimoto, Stark-Terras] The $Frob_{[C]}$ are uniformly distributed in G w.r.t. Dirichlet density, i.e., for each conj. class \mathcal{C} of G ,

$$\frac{\sum_{[C] \in S_{\mathcal{C}}} u^{l(C)}}{\sum_{[C] \text{ prime}} u^{l(C)}} \rightarrow \frac{|\mathcal{C}|}{|G|} \quad \text{as } u \rightarrow \frac{1}{\lambda(X)}^-.$$

Here $\frac{1}{\lambda(X)}$ is the radius of convergence of $Z(X, u)$.

Theorem [Huang-L] *The natural density for $S_{\mathcal{C}}$ given by*

$$\lim_{x \rightarrow \infty} \frac{|\{[C] \in S_{\mathcal{C}} : l(C) \leq x\}|}{|\{\text{primes } [C] : l(C) \leq x\}|}$$

exists (and hence equal to $\frac{|\mathcal{C}|}{|G|}$) if and only if $\delta(Y) = \delta(X)$.

Here $\delta(X) = \gcd_{\text{primes } [C]} l(C)$, which corresponds to topological entropy for manifolds.

Stark-Terras: $\delta(Y) = \delta(X)$ or $2\delta(X)$ and both cases do occur.

The zeta functions of Y and X are related by

$$Z(Y, u) = Z(X, u) \prod_{\text{nontrivial } \rho \in \text{Irr}(G)} L(X, \rho, s)^{\deg \rho}.$$

The proofs of both theorems use the analytic behavior of $L(X, \rho, u)$.

Hashimoto showed:

(1) If ρ has degree ≥ 2 , then $L(X, \rho, u)$ is holo. on the closed disk $|u| \leq 1/\lambda(X)$;

(2) If ρ has degree 1, then $L(X, \rho, u)$ is holo. on the open disk $|u| < 1/\lambda(X)$, $Z(X, u)$ has a simple pole at $u = 1/\lambda(X)$, and other $L(X, \rho, u)$ are holo. there.

(1) & (2) imply Chebotarev in Dirichlet density.

Have $\lambda(X) = \lambda(Y)$. By rescaling, may assume $\delta(X) = 1$.

Chebotarev in natural density holds iff $L(X, \rho, u)$ are holo. on the circle $|u| = 1/\lambda(X)$ for all nontrivial ρ , which is equiv. to $\delta(Y) = \delta(X)$ by using Prime Geodesic Theorem for graphs and Tauberian theorem.

The Bruhat-Tits building \mathcal{B}_3 attached to $\mathbf{SL}_3(F)$

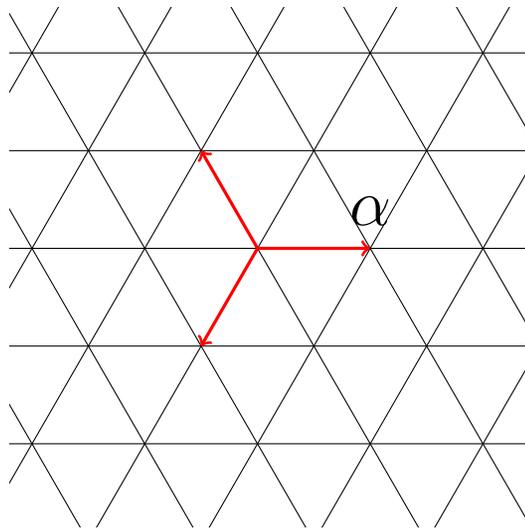
- The vertices of the building \mathcal{B}_3 of $SL_3(F)$ are equivalence classes of rank 3 lattices (i.e. \mathcal{O}_F -modules) in F^3 .
- 3 vertices form a chamber (i.e., a triangle) if they are represented by lattices L_1, L_2, L_3 satisfying

$$L_1 \supsetneq L_2 \supsetneq L_3 \supsetneq \varpi L_1.$$

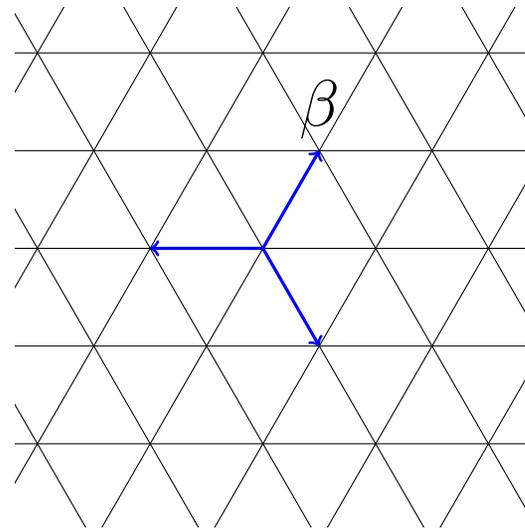
- The group $PGL_3(F)$ acts transitively on vertices and preserves adjacency.
- Each vertex has two types of neighbors/out edges, described by two Hecke operators A_1, A_2 .
- Each edge has a direction of type 1, its opposite has type 2;

adjacency of type 1 (resp. type 2) edges given by the parahoric operator L_E (resp. L_E^t).

- The Iwahori-Hecke operator L_B describes adjacency of directed chambers.



type 1 edges



type 2 edges

Zeta functions for finite quotients of \mathcal{B}_3

Take a discrete torsion-free cocompact subgroup Γ of $PGL_3(F)$ with $\text{ord det } \Gamma \subset 3\mathbb{Z}$. The quotient

$$X_\Gamma = \Gamma \backslash PGL_3(F) / PGL_3(\mathcal{O}_F) = \Gamma \backslash \mathcal{B}_3$$

is a finite 2-dimensional complex.

The zeta function of X_Γ counts the number $N_n(X_\Gamma)$ of geodesic cycles of length n **contained in the 1-skeleton of X_Γ** :

$$Z(X_\Gamma, u) = \exp\left(\sum_{n \geq 1} \frac{N_n(X_\Gamma) u^n}{n}\right) = \prod_{1 \leq i \leq 2} \prod_{[C_i]} \frac{1}{1 - u^{l(C_i)i}},$$

where $[C_i]$ runs through primes using only type- i edges.

Zeta identity for X_Γ from \mathcal{B}_3

Theorem [Kang-L. 2014, Kang-L-Wang 2010, Kang-L. 2015]

$Z(X_\Gamma, u)$ is a rational function given by

$$\begin{aligned} Z(X_\Gamma, u) &= \frac{1}{\det(I - L_E u)} \cdot \frac{1}{\det(I - L_E^t u^2)} \\ &= \frac{(1 - u^3) \chi(X_\Gamma)}{\det(I - A_1 u + q A_2 u^2 - q^3 u^3 I) \det(I + L_B u)}, \end{aligned}$$

where $\chi(X_\Gamma) = \#V - \#E + \#C$ is the Euler characteristic of X_Γ .

Remarks. (1) $\det(I - A_1 u + q A_2 u^2 - q^3 u^3 I)$ is a Langlands L -function.

(2) The zeta identity is reminiscent of the zeta function of a surface over a finite field.

(3) There are moduli surfaces Y and complexes X_{Γ_Y} such that the factor of $Z(Y, u)$ from H^2 agrees with the factor of $Z(X_{\Gamma_Y}, u)$ involving Hecke operators with trivial eigenvalues removed.

(4) X_{Γ} is called a Ramanujan complex
iff the nontrivial eigenvalues of A_1 and A_2 on X_{Γ} fall in the spectrum of A_1 and A_2 on \mathcal{B}_3
iff the nontrivial zeros of $\det(I - A_1u + qA_2u^2 - q^3u^3I)$ have absolute value q^{-1}
iff the nontrivial rep'ns of $PGL_3(F)$ in $L^2(\Gamma \backslash PGL_3(F) / PGL_3(\mathcal{O}_F))$ satisfy the Ramanujan conjecture.

Ramanujan complexes are spectrally extremal since an analog of the Alon-Boppana type theorem holds, proved by Li in 2004.

Artin L -functions for $X_\Gamma = \Gamma \backslash \mathcal{B}_3$

Let ρ be a finite-dimensional representation of Γ .

Define the Artin L -function

$$L(X_\Gamma, \rho, u) = \prod_{[C] \text{ prime}} \frac{1}{\det(I - \rho(\text{Frob}_{[C]})u^{l_A(C)})}.$$

Theorem [Kang-L 2015] There are operators $A_1(\rho)$, $A_2(\rho)$, $L_E(\rho)$, and $L_B(\rho)$ such that

$$\begin{aligned} L(X_\Gamma, \rho, u) &= \frac{(1 - u^3)\chi(X_\Gamma) \deg \rho}{\det(I - A_1(\rho)u + qA_2(\rho)u^2 - q^3u^3I) \det(I + L_B(\rho)u)} \\ &= \frac{1}{\det(I - L_E(\rho)u) \det(I - L_E(\rho)^t u^2)}. \end{aligned}$$

The building Δ_4 associated to $Sp_4(F)$

- Three kinds of vertices on the building Δ_4 of $Sp_4(F)$: primitive special, nonprimitive special, and non-special.
 $PGSp_4(F)$ acts transitively on special vertices.
- The special vertices $\leftrightarrow PGSp_4(\mathcal{O}_F)$ -cosets; two vertex adjacency operators A_1 and A_2 on special vertices.
- Two kinds of edges:
 - (1) type 1 edges between primitive and non-primitive special vertices $\leftrightarrow E_1$ (Siegel congruence subgroup)-cosets;
operator L_{E_1} describes adjacency among type 1 edges.
 - (2) type 2 edges between special and non-special vertices $\leftrightarrow E_2$ (Klingen congruence subgroup)-cosets;
operator L_{E_2} describes adjacency among type 2 edges.

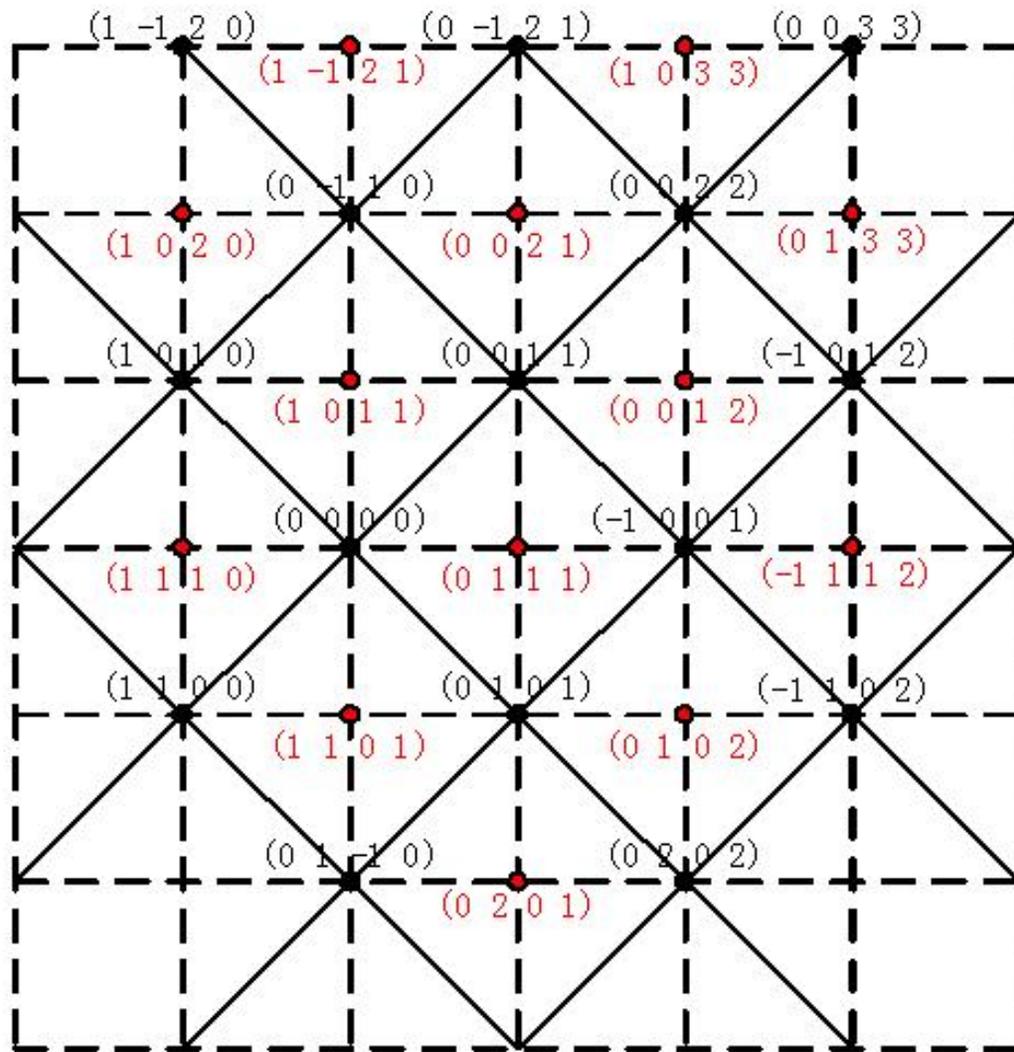


Figure 1: an apartment of Δ_4

- directed chambers $\leftrightarrow I$ (Iwahori subgroup) - cosets; two chamber operators L_I (of type 1) and L'_I (of type 2)

Zeta functions of finite quotients of Δ_4

Let Γ be a discrete torsion-free co-compact subgroup of $PGLSp_4(F)$ preserving the types of the vertices. Let $X_\Gamma = \Gamma \backslash \Delta_4$. Define the zeta function of X_Γ in the same way as the $PGL_3(F)$ case.

Theorem [Fang-L-Wang 2013] *The zeta function $Z(X_\Gamma, u)$ is a rational function with the following two expressions:*

$$\begin{aligned} Z(X_\Gamma, u) &= \frac{(1 - u^2)\chi(X_\Gamma)(1 - q^2u^2)^{-(q^2-1)N_p}}{\det(I - A_1u + qA_2u^2 - q^3A_1u^3 + q^6Iu^4) \det(I - L_Iu)} \\ &= \frac{1}{\det(I - L_{E_1}u) \det(I - L_{E_2}u^2)}, \end{aligned}$$

where $\chi(X_\Gamma)$ is the Euler char. of X_Γ , and N_p is the number of primitive special vertices in X_Γ .

Theorem[Kang-L-Wang]

$$\begin{aligned}
& Z'(X_\Gamma, u) \\
&= \frac{1}{\det(I - L_{E_1}u)^2 \det(I - L_{E_2}u)} \\
&= \frac{(1 - u)^{2\chi(X_\Gamma)} (1 - qu)^{t_1} (1 + qu)^{t_2}}{\det(I - A'_2u + (qA_1^2 - 2q^2A''_2)u^2 - q^4A'_2u^3 + q^8Iu^4) \det(I + L'_Iu)}.
\end{aligned}$$

Here $A'_2 = A_2 - 2q^2I$ and $A''_2 = A_2 - q^2I$, t_1 and t_2 are integers determined by reps occurring in $L^2(\Gamma \backslash PGSp_4(F)/I)$.

The $1/\det$ factor with vertex operators in [FLW] is the Langlands L-function attached to the degree 4 spin repn of \widehat{PGSp}_4 , and that in [KLW] times $(1 - q^2u)^{-2N_p}$ equals the Langlands L-function attached to the degree 5 standard repn of \widehat{PGSp}_4 .

Summary

- The Ihara zeta function for graphs is a p -adic analogue of the Selberg zeta function for compact quotients of $PGL_2(\mathbb{R})$. The Artin L -functions extend the zeta functions to involve representations of the fundamental group.
- For graphs, each zeta and L -functions have Ihara (spectral) and Hashimoto expressions; partial results for manifolds.
- Similar results hold for the zeta and L -functions of finite quotients of the building of $PGL_3(F)$ and two zeta functions for finite quotients of the building of $PGSp_4(F)$.
- What would be their counterparts over \mathbb{R} ?