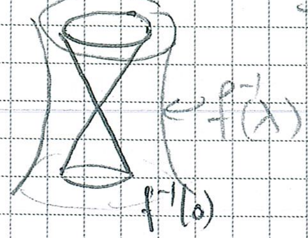
Motivic invariants at infinity of polynomial maps

(joint work with Pierrette Cassou-Noguès)

1) Singularities at infinity

$f(z) = z^n \quad n \geq 2 \qquad f: \begin{matrix} \mathbb{C}^* \\ \downarrow \\ \mathbb{C}^* \end{matrix}$ covering

$f(x,y,z) = x^2+y^2-z^2$ homogeneous polynomial



$f: \begin{matrix} \mathbb{C} \setminus f^{-1}(0) \\ \downarrow \\ \mathbb{C}^* \end{matrix}$ is a locally trivial fibration

Thm (Thom) Let $f \in \mathbb{C}[x_1, \dots, x_d]$

(*) then there is a finite set B $\quad f: \begin{matrix} \mathbb{C}^d \setminus f^{-1}(B) \\ \downarrow \\ \mathbb{C} \setminus B \end{matrix}$

is a loc. trivial fibration

def B_f^{top} = topological bifurcation set
smallest finite set B satisfying (*)

remark $\mathrm{disc}\, f \subseteq B_f^{top}$

but $\underbrace{B_f^{top} \setminus \mathrm{disc}\, f}_{\text{set of ``atypical values'' due to singularities at } \infty \text{ of } f} \neq \emptyset$ in general

(Consequence of the non properness of f)

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Broughton's example $\quad f(x,y) = x(xy-1)$

$\operatorname{disc} f = \emptyset \qquad B_f^{top} = \{0\}$

Rem $f^{-1}(0)$ not connected ; $f^{-1}(\lambda)$ is a graph for any $\lambda \neq 0$.

Study at infinity : Compactification of f in $\mathbb{P}^2 \times \mathbb{C}$

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, hook] \arrow[d, "f"'] & X = \{([x:y:z], a) \mid x(xy - z^2) = a z^3\} \arrow[dl, "\hat{f} = a \text{ 2nd projection}"] \\ \mathbb{C} & \end{tikzcd}$$

$i(x,y) = ([x:y:1], x(xy-1))$

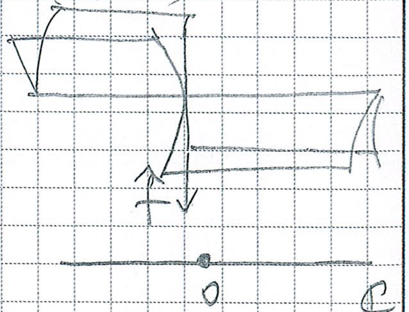
closed subset at ∞ : $X_\infty = X \setminus i(\mathbb{C}^2) = \{([x:y:0], a) \mid x^2 y = 0\}$

In the chart $y \neq 0$ $\quad u = \frac{x}{y} \quad v = \frac{z}{y}$ coordinates

$X_{|y\neq 0} = \{(u,v,a) \in \mathbb{C}^3 \mid u(u - v^2) = a v^3\}$

Singular locus of $X_{|y \neq 0}$ is $\mathbb{C} \cdot (0,0,1)$

Let $g_a(u,v) = u(u - v^2) - a v^3$ $\quad \mathbb{C}(0,0,1)$



- If $a \neq 0$ then the Milnor number at $(0,0)$ is by definition

$$\mu(g_a)(0,0) = \dim \frac{\mathbb{C}\{\{u,v\}\}}{(2u - v^2, -3av^2 - 2uv)} = 2$$

$\rightsquigarrow$ $g_a = 0$ is a cusp around $(0,0)$

- If $a = 0$ then

$$\mu(g_0)(0,0) = \dim \frac{\mathbb{C}\{\{u,v\}\}}{(2u - v^2, -2uv)} = 3$$

There is no equisingularity at ∞ for the value 0.

$\rightarrow$ sing at ∞.

Thm (Hà–Lê) (84)

Euler characteristic

$f \in \mathbb{C}[x,y]$ $\quad B_f^{top} = \{ a \in \mathbb{C} \mid \chi_c(f^{-1}(a)) \neq \chi_c\, f^{-1}(a_{gen}) \}$

More generally

$f \in \mathbb{C}[x_1, \dots, x_d]$ $\quad \tilde{f}(x_0, \dots, x_d)$ associated homogeneous polynomial

$$\begin{array}{ccl} \mathbb{C}^d & \hookrightarrow & X = \{ ([x_0 : \dots : x_d], a) \in \mathbb{P}^d \times \mathbb{A}^1 \mid \tilde{f}(x) = a x_0^D \} \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \hat{f}} & \\ \mathbb{C} & & \end{array}$$

$\deg f = D$

$f = f_D + f_{D-1} + \dots + f_0$

homogeneous

$X_{sing} = A \times \mathbb{C}$ $\quad A = \{ [0 : x_1 : \dots : x_d] \mid \frac{\partial f_D}{\partial x_1} = \dots = \frac{\partial f_D}{\partial x_d} = f_{D-1} = 0 \}$

def f has isolated singularities at ∞ if and only if card A = finite

Thm (Parusiński '95)

If f has isolated sing at ∞

then $B_f^{top} = \{ a \mid \chi_c\, f^{-1}(a) \neq \chi_c\, f^{-1}(a_{gen}) \}$

Q: Computation of $\chi_c(f^{-1}(a))$?

Thm (Artal-Bartolo, Luengo, Melle) (Loop)

$f \in \mathbb{C}[x_1, \dots, x_d]$ with isolated sing in $\mathbb{C}^d$

i) $\chi_c\left(f^{-1}(a_{gen})\right) = 1 + (-1)^{d-1}\left(\mu(f) + \lambda(f)\right)$

ii) $\forall a \in \mathbb{C}$

$$\chi_c\left(f^{-1}(a)\right) = \chi_c\left(f^{-1}(a_{gen})\right) + (-1)^{d-1}\left(\mu_a(f) + \lambda_a(f)\right)$$

where $\mu_a(f)$ is the sum of Milnor numbers of critical points of $f^{-1}(a)$

$\lambda_a(f)$ measures "the lack of equisingularity at ∞ for the fiber $f^{-1}(a)$"

ex $d=2$ $\lambda_a(f) = \mu(g_a) - \mu(g_{a_{gen}})$

for $d \geq 3$ AB-L-M use a stratified version of Milnor number given by Parusinski

Topologically if f has isolated singularities at ∞ then Parusinski proved that the generic fiber of f is a bouquet of $\mu + \lambda$ spheres and we go from the generic fiber to the special fiber by contracting spheres (vanishing cycles)

2 The motivic point of view

a) Motivic measure

$X \in Var_{\mathbb{C}}$ alg variety $/\mathbb{C}$

$\mathcal{L}(X)$ = arc space of X $\qquad \mathcal{L}(X)(\mathbb{C}) = X(\mathbb{C}[[t]])$

ex $\quad X : \begin{cases} P_1(x_1, \dots, x_d) = 0 \\ \vdots \\ P_\ell(x_1, \dots, x_d) = 0 \end{cases}$

$$\mathcal{L}(X) = \{ x(t) \in \mathbb{C}[[t]]^d \mid \forall i \;\; P_i(x(t)) = 0 \}$$

Problem $\quad \mathbb{C}[[t]]$ with the t-adic valuation is not locally compact. There is no Haar measure with real values!

(Kontsevich 95) Constructed an integration theory on $\mathbb{C}[[t]]$ by analogy with the p-adic $\int$

Idea $\quad p = \text{card } \mathbb{F}_p = \text{card } \mathbb{Z}_p/(p)$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

$$\mathbb{L} := [\mathbb{A}^1_{\mathbb{C}}] \qquad \mathbb{C} = \mathbb{C}[[t]]/(t)$$

isomorphism class of alg variety

$\longrightarrow$ Grothendieck ring of varieties

$K_0(\mathrm{Var}_{\mathbb{C}})$: Grothendieck ring of varieties

abelian group : generated by symbols $[X]$

relations:

1) $X \simeq Y \Rightarrow [X] = [Y]$

2) $F \subset X$ Z-closed $\Rightarrow [X] = [F] + [X \setminus F]$

ring structure $[X \times Y] = [X] \cdot [Y]$

ex $[\mathbb{P}^1] = [\mathbb{A}^1] + [\{*\}] = \mathbb{L} + 1$

$[y^2 = x^3] = \mathbb{L}$

Universal property

$$\begin{tikzcd} \mathrm{Var}_{\mathbb{C}} \arrow[r, "\lambda"] \arrow[d] & R \\ K_0(\mathrm{Var}_{\mathbb{C}}) \arrow[ur, dashed, "\exists!\, \tilde{\lambda}"'] & \end{tikzcd}$$

additive and mult inv

ring morphism

=: realization map

Consequence $[X] = [Y] \Rightarrow \lambda(X) = \lambda(Y)$

$[X]$ contains all the add and mult inv of X

$\rightsquigarrow$ "motive" of X

ex Kontsevich X, Y CY bir $\Rightarrow [X] = [Y]$

$\mathcal{M} = K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}] \quad \longleftrightarrow \quad \mathbb{Z}[\frac{1}{p}]$ p-adic $\int$

$\hat{\mathcal{M}} \quad (\mathbb{L}^{-n} \longrightarrow 0) \quad \longleftrightarrow \quad \mathbb{R}$

completion

Motivic measure $X \in Var_{\mathbb{C}}$ $\dim X = d$

$$mes(\mathcal{L}(X)) = \lim_{n \to \infty} \frac{[\pi_n \mathcal{L}(X)]}{\mathbb{L}^{nd}} \in \hat{\mathcal{M}}$$

$$\pi_n \mathcal{L}(X) = \{ \varphi \bmod t^{n+1} \mid \varphi \in \mathcal{L}(X) \}$$

ex: If X is smooth then $mes\, \mathcal{L}(X) = [X]$.

More generally in order to study singularities and monodromy invariants we work with

$Var_{\mathbb{C}}^{\hat{\mu}}$: $X^{\circlearrowright \hat{\mu}}$ $\hat{\mu}$ = group of roots of unity

$K_0(Var_{\mathbb{C}}^{\hat{\mu}})$, $\mathcal{M}^{\hat{\mu}}$ ...

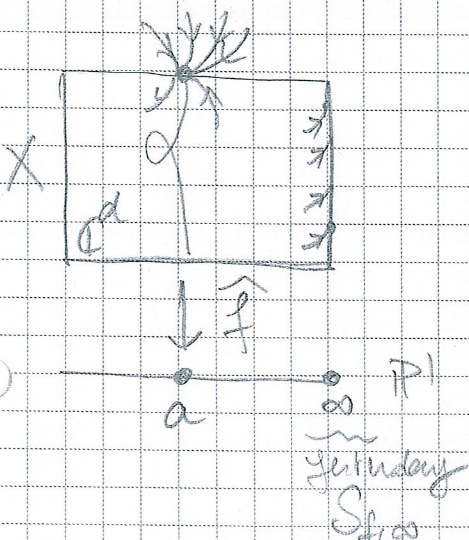
b) Motivic zeta fct at infinity for a value a

Let $f \in \mathbb{C}[x_1, \dots, x_d]$

$$\begin{array}{ccc} \mathbb{C}^d & \longrightarrow & X \text{ compactification} \\ {\scriptstyle f}\downarrow & & \downarrow {\scriptstyle \hat{f}} \text{ proper} \\ \mathbb{C} & \longrightarrow & \mathbb{P}^1 \end{array}$$

$X_\infty = X \setminus i(\mathbb{C}^d)$

Fix a value a of f and use Denef–Loeser technics (Guibert–Loeser–Merle)



$$X_{n,a}^{\delta,\infty} = \left\{ \varphi \in \mathcal{L}(X) \;\middle|\; \begin{array}{l} \varphi(0) \in X_\infty \\ \hat{f}(\varphi) = a + 1t^n + \cdots \\ \operatorname{ord} \varphi^*(\mathcal{I}_{X_\infty}) \leq n\delta \end{array} \right\}$$

for $n \geq 1$, $\delta \geq 1$

$\hat{\mu}$ action

$\lambda . \varphi(t) = \varphi(\lambda t)$

Contact order of φ along X_∞ $:= \min_{g \in \mathcal{I}_{X_\infty}} \operatorname{ord}\, g \circ \varphi$

Motivic zeta function at ∞ for the value a

$$Z^{\delta,\infty}_{f,a}(T) = \sum_{n\geq 1} \mathrm{mes}\left(X^{\delta,\infty}_{n,a}\right) T^n$$

By DL / GLM $\forall \delta \gg 1$ $Z^{\delta,\infty}_{f,a}$ is rational

and $S^{\infty}_{f,a} = -\lim_{T\to\infty} Z^{\infty,\delta}_{f,a}(T)$

does not depend on δ

Thm (R) 1) $S^{\infty}_{f,a}$ does not depend on the compactification X

2) $\{a \in \mathbb{C} \mid S^{\infty}_{f,a} \neq 0\}$ is finite

def $B_f^{mot} = \mathrm{disc}\, f \cup \{a \mid S^{\infty}_{f,a} \neq 0\}$

Motivic bifurcation set

Thm (CN-R) 1) $\tilde{\chi}\left(S^{\infty}_{f,a}\right) = \chi\left(f^{-1}(a_{gen})\right) - \chi_c\left(f^{-1}(a)\right)$, $a \notin \mathrm{disc}\, f$

2) If f has isolated sing in $\mathbb{C}^d$

then $(-1)^{d-1} \chi_a(f) = \chi_c\left(S^{\infty}_{f,a}\right)$

3) Furthermore if $B_f^{top} = \{a \mid \chi_c(f^{-1}(a)) \neq \chi_c f^{-1}(a_{gen})\}$

then $B_f^{top} \subseteq B_f^{mot}$

ex curves, f with isolated sing at ∞

In the case of curves: Algorithm to compute $S^{\infty}_{f,a}$ in terms of iterated Newton polygons of f.

Broughton $S^{\infty}_{f,0} = [x(xy-1)=1 \cap \mathbb{G}_m^2]$; $S^{\infty}_{f,a\neq 0} = 0$