Bessis-Moussa-Villani conjecture and generalized Gaussian random variables

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Abstract

In this paper we give the solution of Bessis-Moussa-Villani conjecture (BMV) conjecture for the generalized Gaussian random variables

\[ G(f) = a(f) + a^*(f) , \]

where \( f \) is in the real Hilbert space \( \mathcal{H} \).

The main examples of generalized Gaussian random variables are q-Gaussian random variables, \((-1 \leq q \leq 1)\), related to q-CCR relation and others commutation relations. We will prove that (BMV) conjecture is true for all operators \( A = G(f), B = G(g) \); i.e. we will show that the function

\[ F(x) = tr(\exp(A + ixB)) \]

is positive definite function on the real line. The case \( q = 0 \) i.e. when \( G(f) \) are the free Gaussian (Wigner) random variables and the operators \( A \) and \( B \) are free with respect to the vacuum trace was proved by M.Fannes and D.Petz [23].

1 Generalized Gaussian Random Variable.

Generalized Gaussian random variables, \( G(f) \) were introduced in our paper with R.Speicher [16], where the main example was coming from the q-CCR relation for \( q \in [-1, 1] : \)

\[ a(f)a^*(g) - qa^*(g)a(f) =< f, g > I, \]

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here $f, g$ are in a real Hilbert space $\mathcal{H}$ and

$$G(f) = a(f) + a^*(f).$$

The others examples of generalized Gaussian random variables were constructed by L. Accardi and M. Bozejko[1], M. Bozejko and M. Guta[9], M. Bozejko and J. Wysoczanski[17, 18], M. Guta and H. Maassen[25, 26], M. Bozejko and H. Yoshida[19], A. Buchholz[20], M. Bozejko M., A. Krystek and L. Wojakowski[10] and recently M. Bozejko M.[8] constructed $q$-Gaussian random variables for $|q| > 1.$

Let $\mathcal{H}$ be a real Hilbert space. A family of self-adjoint operators $G(f) = G(f)^*$, $f \in \mathcal{H}$ is called Generalized Gaussian random variables or Generalized Brownian Motion (GBM), if there exists a state $\varepsilon$ on the von Neumann algebra generated by $G(f)$, $f \in \mathcal{H}$ and a complex valued function $t: \bigcup_{n=1}^{\infty} \mathcal{P}_2(2n) \to \mathbb{C}$, (here $\mathcal{P}_2(2n)$ is the set of 2-partitions of the set $\{1, 2, \ldots , 2n\}$ ), such that the following generalized Wick formula holds:

$$\varepsilon(G(f_1)\ldots G(f_k)) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \sum_{V\in\mathcal{P}_2(2n)} t(V) \prod_{(i,j)\in V} <f_i, f_j> & \text{if } k = 2n. \end{cases}$$

If the dimension of a Hilbert space $\mathcal{H}$ is infinite, then the above definition is equivalent to the following (see F. Lehner-II,[32]):

for each orthogonal linear map $O: \mathcal{H} \to \mathcal{H}$ and $f_i \in \mathcal{H}$ : 

$$\varepsilon(G(f_1)\ldots G(f_k)) = \varepsilon(G(O(f_1))\ldots G(O(f_k))),$$

Typical examples of (GBM) was obtained by R. Speicher and myself[13] in 1991, using $q$-CCR relations for $-1 \leq q \leq 1$, then putting $G(f) = a(f) + a^*(f)$ and knowing that $a(f)\Omega = 0$ we obtain the following Wick formula:

$$< G(f_1)\ldots G(f_{2n})\Omega, \Omega> = \sum_{V\in\mathcal{P}_2(2n)} q^{cr(V)} \prod_{(i,j)\in V} <f_i, f_j>.$$

Here $cr(V)$ is the number of crossing, which is given by the number of pairs of blocks of $V$ which will cross. To obtain the above Wick formula we need a deformed Fock space $\mathcal{F}_q(\mathcal{H}_C)$ constructed by the completion of the free Fock space

$$\mathcal{F}(\mathcal{H}_C) = \mathbb{C}\Omega \oplus \mathcal{H}_C \oplus \ldots$$
by introducing a new scalar product on $\mathcal{H}_C^\otimes n$ as follows:

For $\xi, \eta \in \mathcal{H}_C^\otimes n$ we define a q-deformed scalar product,

$$< \xi, \eta >_q = < P_q^{(n)} \xi, \eta >,$$

where

$$P_q^{(n)} = \sum_{\pi \in S(n)} q^{cr(\pi)} \pi,$$

and where for a permutation $\pi \in S(n)$,

$$cr(\pi) = \sharp \{(i,j) : 1 \leq i \leq j \leq n, \text{ and } \pi(i) > \pi(j)\}.$$

In the construction of $\mathcal{F}_q(\mathcal{H}_C)$ we need the positivity of the operator $P_q^{(n)}$ for $-1 \leq q \leq 1$, which was done by Bozejko and Speicher in the papers [11,12,13].


2 Generalized Bessis-Moussa-Villani conjecture.

Let $(\mathcal{A}, \tau)$ be a von Neumann algebra $\mathcal{A}$ with a finite trace $\tau$. We say that generalised (BMV) conjecture holds, if for all $a = a^*, b = b^*$ in $\mathcal{A}$, the function

$$F_{a,b}(x) = \tau(\exp(a + ixb))$$

is positive definite on the real line, i.e. there exists a positive, bounded Borel measure $\mu$ on the real line such that

$$F_{a,b}(x) = \int_{-\infty}^{\infty} e^{ixs} \mu(ds).$$

In the case of the algebra of all complex nxn matrices $\mathcal{A} = M_n(\mathbb{C})$ and the trace $\tau$ is the classical trace, (BMV) conjecture is called $(BMV)_n$.

From the paper of Lieb-Seiringer[33], we know that for a fixed natural number $n$, $(BMV)_n$-condition is equivalent to the following statement:
$(LS)_n$: For each positive definite matrices $A, B \in M_n(C)$, for all natural $m$ and complex $z$, the polynomial

$$L_{A,B,m}(z) = Tr[(A + zB)^m]$$

has only non-negative coefficients.

From the last condition it is easy to show that $(LS)_n$ is true for $n = 2$.

The hint for that result is the following: for two positive definite 2x2-complex matrices there exists a basis, in which that matrices have only non-negative entries.

The case $(BMV)_3$ for 3x3 matrices is STILL OPEN!

See more in the recent paper of Ch.Hillar[?].

In this note we show the generalized (BMV) conjecture for all generalized Gaussian random variables by reducing to the case q-Gaussian, where $q = -1$, i.e. the classical canonical anticommutation relation (CAR), in which we have representation of (CAR) relations using Dirac-Pauli matrices.

3 Pauli-Dirac matrices.

We are looking for 2x2-complex self-adjoint matrices $Q_1, Q_2$ satisfying the following conditions:

(i) $Q_1^2 = Q_1^2 = I, Q_k = Q_k^*, k = 1, 2.$

(ii) $Q_1Q_2 + Q_2Q_1 = O$

and

(iii) $Tr(Q_{j_1}Q_{j_2}...Q_{j_{2n}}) = \sum_{\nu \in P_2(2n)} (-1)^{cr(\nu)} \prod_{(l,m) \in \nu} \delta_{j_l,j_m}.$

One can see that the following matrices satisfying (i)-(iii).

$Q_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

and

$Q_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$
Hence we have the following well know fact (folklore):

**Proposition 3.1.** For all real $s, x$, the function

$$\varphi_s(x) = Tr(exp(s(Q_1 + i x Q_2)))$$

is positive definite on the real line as a function of variable $x$.

Moreover

$$\varphi_s(x) = \sum_{n=0}^{\infty} \frac{(1-x^2)^n s^{2n}}{(2n)!} = \cosh(s\sqrt{1-x^2}).$$

**Proof.**

Let $Z(x) = Q_1 + i x Q_2 = \begin{pmatrix} 0 & 1-x \\ 1+x & 0 \end{pmatrix}.$

Hence

$$Z(x)^{2n} = (1-x^2)^n I, \ Tr(Z(x)^{2n+1}) = 0$$

and

$$Tr(Z(x)^{2n}) = (1-x^2)^n ,$$

where $Tr$ is the normalized trace.

By the result of Lieb-Seiringer [33], we obtain that our function $\varphi_s$ is positive definite.

This finishes the proof.

**Remark.** One can also show that the function $\varphi_s$ is of the following form:

$$\varphi_s(x) = \int_{-\infty}^{\infty} e^{ixs} \mu_s(dx),$$

where the measure

$$\mu_s(dx) = \frac{1}{2} (\delta(s) + \delta(-s)) + f_s(x) dx .$$
The density
\[ f_s(x) = \frac{I_1(\sqrt{s^2 - x^2})}{\sqrt{s^2 - x^2}}. \]

Here the function
\[ I_1(x) = \sum_{k=0}^{\infty} \frac{x^k}{2^k k!(k+1)!} \]
is the modified Bessel function.

Now we are in position to state and proof our main result:

**Theorem 3.2.** If \( f_1, f_2 \) are in the real Hilbert space \( \mathcal{H} \) and \( G(f_1), G(f_2) \) are generalized Gaussian random variables with respect to the state \( \varepsilon \), then the function
\[ F_G(x) = \varepsilon(\exp(G(f_1) + ixG(f_2))) \]
is positive definite on the real line.

Moreover, if \( < f_1, f_2 > = 0 \) and \( \| f_i \| = 1 \), then
\[ F_G(x) = \int_{-\infty}^{\infty} \varphi_s(x) d\nu_G(ds), \]
where \( \nu_G \) is the probability distribution of the operator \( G(f) \), \( \| f \| = 1 \) with respect to the state \( \varepsilon \),

i.e.
\[ \varepsilon(G(f)^k) = \int_{-\infty}^{\infty} x^k d\nu_G(dx), \]
for all \( k = 0, 1, 2, \ldots \).

**Proof.** In the first part of the proof, we can assume that
\[ \| f_i \| = 1, \text{ and } < f_1, f_2 > = 0. \]

Let us consider the following function on the complex plane \( \mathbb{C} \):
\[ \delta_s(z) = \varepsilon((G(f_1) + zG(f_2))^n), z \in \mathbb{C}. \]
By the very definition we have that \( \delta_{2n+1}(z) = 0 \) and \( \delta_{2n}(z) \) is a polynomial of the degree \( 2n \).
But for real \( x \) we have \( G(f_1) + xG(f_2) = G(f_1 + xf_2) \).

That last fact for the q-Gaussian case follows from the construction, but for generalized Gaussian field it follows from the Theorem of Guta-Maassen, [26], that each generalized Gaussian field is of the following form \( G(f) = a(f) + a^*(f) \), where a creation and an annihilation operators are \( \mathbb{R} - \text{linear} \).

Therefore for real \( x \) we have:

\[
\delta_{2n}(x) = \varepsilon((G(f_1 + xf_2))^{2n}) = \|f_1 + xf_2\|^{2n} \sum_{\mathcal{V} \in \mathcal{P}_2(2n)} t(\mathcal{V}) = (1 + x^2)^n m_{2n}(d\nu_G),
\]

Since

\[
m_{2n}(d\nu_G) = \sum_{\mathcal{V} \in \mathcal{P}_2(2n)} t(\mathcal{V}) = \int_{-\infty}^{\infty} x^k d\nu_G(dx).
\]

By the analytic property of \( \delta_{2n}(z) \), we have that for real \( x \),

\[
\delta_{2n}(ix) = (1 - x^2)^n m_{2n}(\nu_G).
\]

Therefore

\[
F_G(x) = \sum_{n=0}^{\infty} \frac{(1 - x^2)^n}{(2n)!} m_{2n}(\nu_G) = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(1 - x^2)^n}{(2n)!} s^{2n} d\nu_G(ds) = \int_{-\infty}^{\infty} \cosh(s\sqrt{1 - x^2}) d\nu_G(ds) = \int_{-\infty}^{\infty} \varphi_s(x) d\nu_G(ds).
\]

In more general case, if

\[
\|f_i\| = 1, \text{ and } <f_1, f_2> = \alpha.
\]
then as before we get

$$F_G(x) = \int_{-\infty}^{\infty} \cosh(s\sqrt{1-x^2-2ix\alpha})d\nu_G(ds),$$

and this function is positive definite on the real line.
This can be proved as before by reducing to the CAR relation. We are looking for 2x2 complex, self-adjoint matrices

$$R_1 = aQ_1 + bQ_2, R_2 = cQ_1 + dQ_2,$$
a, b, c and d are real and $a^2 + b^2 = 1, c^2 + d^2 = 1,$ and also $ac + bd = \alpha.$

By the same calculations as before we get:

$$\text{Tr}(\exp(R_1 + ixR_2)) = \sum_{n=0}^{\infty} \frac{(1-x^2-2ix\alpha)^n}{(2n)!} = \cosh(\sqrt{1-x^2-2ix\alpha}).$$

So that function is positive definite on the real line by Lieb-Seiringer [33] result.
This finishes the proof of Theorem 3.2.

**Problem 3.3.** The following problem seems to be interesting:

For real s and $-1 \leq \alpha \leq 1$, find explicit the positive measure $\mu_{s,\alpha}$ on the real line such that

$$\cosh(s\sqrt{1-x^2-2ix\alpha}) = \int_{-\infty}^{\infty} e^{ixy} \mu_{s,\alpha}(dy) ?$$

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**References**


