One-radius results
for supermedian functions on $\mathbb{R}^d$, $d \leq 2$

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Abstract

A classical result states that every lower bounded superharmonic function on $\mathbb{R}^2$ is constant. In this paper the following (stronger) one-circle version is proven. If $f: \mathbb{R}^2 \to (-\infty, \infty]$ is lower semicontinuous, $\liminf_{|x|\to\infty} f(x)/\ln |x| \geq 0$, and, for every $x \in \mathbb{R}^2$, $1/(2\pi) \int_0^{2\pi} f(x + r(x)e^{it}) dt \leq f(x)$, where $r: \mathbb{R}^2 \to (0, \infty)$ is continuous, $\sup_{x \in \mathbb{R}^2} (r(x) - |x|) < \infty$, and $\inf_{x \in \mathbb{R}^2} (r(x) - |x|) = -\infty$, then $f$ is constant.

Moreover, it is shown that, assuming $r \leq c \cdot |\cdot| + M$ on $\mathbb{R}^d$, $d \leq 2$, and taking averages on $\{y \in \mathbb{R}^d: |y - x| \leq r(x)\}$, such a result of Liouville type holds for supermedian functions if and only if $c \leq c_0$, where $c_0 = 1$, if $d = 2$, whereas $2.50 \leq c_0 \leq 2.51$, if $d = 1$.

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1 Introduction and results

It is a well-known fact that every lower bounded superharmonic function on $\mathbb{R}^2$ is constant. We recall that superharmonic functions $u$ on $\mathbb{R}^2$ are lower semicontinuous functions on $\mathbb{R}^2$ such that $u > -\infty$, $u \not\equiv \infty$, and, for every circle $S(x, \rho)$ of center $x \in \mathbb{R}^2$ and radius $\rho > 0$, the average $\sigma_{x, \rho}(u)$ of $u$ on $S(x, \rho)$ is at most $u(x)$. In this note, we shall present the following stronger result (where, as usual, we do not distinguish between $\mathbb{C}$ and $\mathbb{R}^2$).

Theorem 1.1. Let $r$ be a strictly positive real function on $\mathbb{R}^2$ such that

(i) $r$ is continuous,

(ii) $\limsup_{|x| \to \infty} (r(x) - |x|) < \infty$,

(iii) $\liminf_{|x| \to \infty} (r(x) - |x|) = -\infty$.\(^1\)

\(^1\)Having (i), properties (ii),(iii) are equivalent to $\sup_{x \in \mathbb{R}^2} (r(x) - |x|) < \infty$, $\inf_{x \in \mathbb{R}^2} (r(x) - |x|) = -\infty$, respectively.
Let \( f > -\infty \) be a lower semicontinuous numerical function on \( \mathbb{R}^2 \) such that
\[
\liminf_{|x| \to \infty} \frac{f(x)}{|x| \ln |x|} \geq 0
\]
and \( f \) is \((\sigma, r)\)-supermedian, that is,
\[
\sigma_{x,r}(f) := \frac{1}{2\pi} \int_0^{2\pi} f(x + r e^{it}) \, dt \leq f(x) \quad (x \in \mathbb{R}^2).
\]
Then \( f \) is constant.

Remarks 1.2. 1. Obviously, \( r : \mathbb{R}^2 \to (0, \infty) \) has the properties (i) – (iii), if there exists \( L \in (0, 1) \) such that, for all \( x, y \in \mathbb{R}^2 \),
\[
|r(x) - r(y)| \leq L |x - y|.
\]
However, assuming only that \( |r(x) - r(y)| < |x - y| \) (which implies (i) and (ii)), even the conclusion breaks down. Indeed, if \( f := (1 - |x|)^+ \) and \( r := |x| + 2 + (|x| + 1)^{-1} \), then, for all \( x, y \in \mathbb{R}^2 \), \( \sigma_{x,r}(f) = 0 \leq f(x) \) and the inequality \( |r(x) - r(y)| < |x - y| \) holds, since \((|x| + 1)^{-1} > (|y| + 1)^{-1} \) if \( |x| < |y| \). In fact, none of the properties (i), (ii), (iii) may be dropped (see Section 3). Moreover, since the function \(-\ln(|\cdot|^2 + 1)\) is superharmonic, it is clear that (1.1) cannot be replaced by \( \liminf_{|x| \to \infty} f(x)/|x| \ln |x| > -\infty \).

2. In Theorem 1.1, we may just as well assume that \( f \) does not attain the value \( \infty \) (it suffices to consider the functions \( f_n := \min\{f, n\} \), \( n \in \mathbb{N} \), which are \((\sigma, r)\)-supermedian provided \( f \) is \((\sigma, r)\)-supermedian; if these functions are constant, then \( f := \lim_{n \to \infty} f_n \) is constant).

Let us assume, for a moment, that \( f : \mathbb{R}^2 \to \mathbb{R} \) is continuous and \((\sigma, r)\)-median, that is, such that
\[
\sigma_{x,r}(f) = f(x) \quad (x \in \mathbb{R}^2).
\]
P.C. Fenton [2] showed that \( f \) has to be constant provided \( f \) is lower bounded, \( r \) is continuous and, for some \( x_0 \in \mathbb{R}^2 \), the set \( \{x \in \mathbb{R}^2 : r(x) > |x - x_0|\} \) is bounded, a requirement which may be replaced by the weaker property (ii) (see Remark 2.1). If \( f \) is bounded, then (ii) alone (without any further assumption on \( r \)) is sufficient to conclude that \( f \) is constant ([7, Theorem 1.1], cf. also [5]). On the other hand, there exist \( r : \mathbb{R}^2 \to (0, \infty) \) and a continuous \((\sigma, r)\)-median function \( f \) on \( \mathbb{R}^2 \) such that \( r \leq 4(|\cdot| + 1) \) and \( \min f(\mathbb{R}^2) = 0, \max f(\mathbb{R}^2) = 1 \) (see [5, Proposition 6.1] or [7, Section 5]).

An essential step for the strong version [7, Theorem 1.1] of Liouville’s theorem consists in proving that, assuming (ii), every lower semicontinuous \((\sigma, r)\)-supermedian function \( f \geq 0 \) on \( \mathbb{R}^2 \) attains a minimum. It immediately implies that constant functions are the only lower semicontinuous, lower bounded functions \( f \) on \( \mathbb{R}^2 \) which are \((\lambda, r)\)-supermedian, that is, which have the property that, for every \( x \in \mathbb{R}^2 \), the average \( \lambda_{x,r}(f) \) of \( f \) on the (closed) disk \( B(x, r(x)) \) is at most \( f(x) \) (see [7, Corollary 6.1]). We recall that \((\lambda, r)\)-supermedian functions are \((\sigma, \tilde{r})\)-supermedian for some function \( 0 < \tilde{r} \leq r \) (cf. [7, Section 6]). The following result shows that the existence of a minimum fails, if (ii) is replaced by an inequality \( r \leq c |\cdot| + M \), where \( c > 1 \).

Proposition 1.3. Let \( c > 1 \), \( M > 0 \), and \( r(x) := \max\{|x|, M\} \), \( x \in \mathbb{R}^2 \). Then the functions \( r^{-\alpha} \) on \( \mathbb{R}^2 \) are \((\sigma, r)\)-supermedian (and \((\lambda, r)\)-supermedian) provided \( \alpha > 0 \) is sufficiently small.
So, assuming that $r \leq c|\cdot| + M$, where $c, M \in (0, \infty)$, a result of Liouville type for $(\lambda, r)$-supermedian functions on $\mathbb{R}^2$ holds if and only if $c \leq 1$.

On the real line, this will turn out to be strikingly different. By the following proposition, such a result of Liouville type on $\mathbb{R}$ holds if and only if $c \leq c_0$, where $c_0 \in [2.50, 2.51]$ is the unique solution to the equation

$$(t + 1) \ln(t + 1) + (t - 1) \ln(t - 1) = 2t$$

in $(1, \infty)$ (see Section 5).

**Proposition 1.4.** 1. Let $r : \mathbb{R} \to (0, \infty)$ and $M > 0$ such that

$$r(x) \leq c_0|x| + M \quad \text{for all } x \in \mathbb{R} \setminus [-M, M].$$

Then every lower semicontinuous $(\lambda, r)$-supermedian function $f > -\infty$ on $\mathbb{R}$ which is lower bounded (or satisfies $\liminf_{|x| \to \infty} f(x)/\ln |x| \geq 0$) is constant.

2. If, however, $c > c_0$, $M > 0$, and $r := \max(c|\cdot|, M)$, then the function $r^{-\alpha}$ is $(\lambda, r)$-supermedian provided $\alpha > 0$ is sufficiently small.

**Remark 1.5.** As in the 2-dimensional case, the condition $\lim\inf_{|x| \to \infty} f(x)/\ln |x| \geq 0$ cannot be replaced by $\lim\inf_{|x| \to \infty} f(x)/\ln |x| > -\infty$. Indeed, let $r(x) := c_0(|x| + 1) + f(x) := -\ln(|x| + 1)$, $x \in \mathbb{R}$. We recall that $\int_0^t f(s) \, ds = t - (t + 1) \ln(t + 1)$, $t \geq 0$. If $x \geq 0$, then

$$a := (c_0 + 1)(x + 1) = x + r(x) + 1,$$

$$b := (c_0 - 1)(x + 1) < r(x) - x + 1 = |x - r(x)| + 1,$$

and hence

$$\lambda_{x,r(x)}(f) < 1 - \frac{1}{2c_0(x + 1)}(a \ln a + b \ln b) = f(x).$$

By symmetry, $\lambda_{x,r(x)}(f) < f(x)$ holds as well if $x < 0$.

Finally, let us recall that every bounded harmonic function on $\mathbb{R}^d$, $d \geq 1$, is constant. Hence the results of Liouville type, which we discussed until now, are special cases of one-radius results for harmonic functions on open sets $U$. The trivial requirement that $r$ be at most the distance to $U^c$, if $U \neq \mathbb{R}^d$, implies the existence of a real $M > 0$ such that

$$r \leq |\cdot| + M$$

(which may justify considering (1.5) as a natural assumption on $r$ in the case $U = \mathbb{R}^d$).

One radius-results for harmonic functions have a long history (see the survey papers [11, 3] and the references therein). If $U$ is an arbitrary open set in $\mathbb{R}^d$, then every continuous $(\lambda, r)$-median functions $f : U \to \mathbb{R}$ admitting a (sub)harmonic minorant and a (super)harmonic majorant is harmonic (provided (1.5) holds, if $U = \mathbb{R}^d$).

Let us now return to continuous bounded $(\sigma, r)$-median functions. If $d = 1$, the corresponding result fails almost trivially both for $\mathbb{R}$ and $(-1, 1)$ (in the first case consider $f(x) := \sin x$ and $r(x) := 2\pi$, for the interval see e.g. [1, Section IV.3, cf. also [9]). We already mentioned the positive result for $U = \mathbb{R}^2$. On the unit disk, however, there exists a continuous function $0 \leq f \leq 1$ having the one-circle property which is not harmonic (see [8] and [6]). The corresponding general problems for $\mathbb{R}^d$, $d \geq 3$, are unsolved, both for the open unit ball and the entire space (if, however, $r$ is Lipschitz with constant $L \in (0, 1)$, the answer is positive [10, Theorem 2]).
2 Proof of Theorem 1.1

Let \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) be lower semicontinuous such that (1.1) holds and \( f \) is \((\sigma, r)\)-supermedian, where \( r: \mathbb{R}^2 \rightarrow (0, \infty) \) satisfies (i), (ii), and (iii). For all \( x \in \mathbb{R}^2 \) and \( \rho > 0 \), let

\[
B(x, \rho) := \{ y \in \mathbb{R}^2 : |y - x| \leq \rho \} \quad \text{and} \quad S(x, \rho) := \{ y \in \mathbb{R}^2 : |y - x| = \rho \}.
\]

By (ii), there is a real \( M > 0 \) such that

\[
r(x) \leq |x| + M \quad \text{for all} \quad x \in B(0, M)^c.
\]

If \( f \) is lower bounded, then, by [5, Proposition 2.1] (see also [7]),

\[
(2.1) \quad f(x_0) \leq f \quad \text{for some} \quad x_0 \in B(0, M + 2).
\]

In fact, a short look at the proof for (2.1) reveals that it is valid as well under our weaker assumption (1.1). So we may suppose without loss of generality that \( f(0) = 0 \) and \( f \geq 0 \) (we can replace \( f \) by the function \( x \mapsto f(x_0 + x) - f(x_0) \)). Since \( f \) is lower semicontinuous, we know, by (1.2), that

\[
(2.2) \quad f = 0 \quad \text{on} \quad S(x, r(x)), \quad \text{whenever} \quad x \in \mathbb{R}^2 \quad \text{such that} \quad f(x) = 0.
\]

Let us use the technique developed in [2]. We define an increasing sequence \((\alpha_n)\) of continuous real functions on the unit circle \( S := S(0, 1) \) by \( \alpha_0 := r(0) \) and

\[
\alpha_n(u) := \alpha_{n-1}(u) + r(\alpha_{n-1}(u))u \quad (n \in \mathbb{N}).
\]

By induction, we conclude from (2.2) that, for all \( u \in S \) and \( n \in \mathbb{N} \),

\[
(2.3) \quad f(\alpha_n(u)) = 0.
\]

Since \( r \) is continuous and strictly positive, we obtain immediately that \( \lim \alpha_n = \infty \). So there exists \( n \in \mathbb{N} \) such that

\[
\alpha_n > M \quad \text{on} \quad S.
\]

For the moment, let us fix \( u \in S \). We claim that

\[
(2.4) \quad f(\alpha u) = 0 \quad \text{for every} \quad \alpha \geq \alpha_n(u).
\]

Indeed, suppose that (2.4) does not hold. Then there exists a maximal real \( a \) such that \( a \geq \alpha_n(u) \) and \( f(\alpha u) = 0 \) for every \( \alpha \in [\alpha_n(u), a] \). We may join the points \( y_0 := au \) and \( y_1 := -\alpha_n(-u)u \) continuously by an arc \( \gamma : [0, 1] \rightarrow \mathbb{R}^2 \) contained in the set

\[
\{ \alpha u : a \geq \alpha \geq \alpha_n(u) \} \cup \{ \alpha_n(v) v : v \in S \}.
\]

In particular, \( f \circ \gamma = 0 \) by (2.3). Fix \( 0 < \beta \leq r(y_0) \) and let \( z := y_0 + \beta u \) so that \( |z - y_0| \leq r(y_0) \). Clearly, \( |z - y_1| \geq r(y_1) \), since the origin is contained in the line segment from \( y_1 \) to \( y_0 \), \( r(y_1) \leq |y_1| + M \), and \( a + \beta \geq \alpha_n(u) > M \). By continuity of \( r \), there exists \( s \in [0, 1] \) such that \( |z - \gamma(s)| = r(\gamma(s)) \). Since \( f(\gamma(s)) = 0 \), we conclude by (2.2) that \( f(z) = 0 \). Thus \( a \geq a + r(y_0) \), a contradiction proving (2.4).

4
Fixing $R > 0$ such that $\alpha_n \leq R$ on $S$, we therefore know that $f = 0$ on $\mathbb{R}^2 \setminus B(0, R)$. Since $\liminf_{|x| \to \infty} (r(x) - |x|) = -\infty$, there exists $x \in \mathbb{R}^2$ such that $|x| - r(x) > R$, and hence

$$f = 0 \quad \text{on } B(x, r(x)).$$

Suppose that there is a point $y \in \mathbb{R}^2$ such that $f(y) > 0$. We define

$$t := \sup \{s \in [0, 1]: f(sx + (1 - s)y) > 0\} \quad \text{and} \quad z := tx + (1 - t)y.$$ 

Since $r(z) > 0$, there exists a point $\tilde{y} \in [y, z]$ such that $|\tilde{y} - z| < r(z)$ and $f(\tilde{y}) > 0$. By (2.6), $|\tilde{y} - x| > r(x)$. By continuity of $r$, we conclude that there exists $\tilde{z} \in (z, x)$ such that $|\tilde{y} - \tilde{z}| = r(\tilde{z})$. So $f(\tilde{z}) > 0$, by (2.2). However, by definition of $z$, $f = 0$ on $(z, x)$. Thus there is no point $y \in \mathbb{R}^2$ such that $f(y) > 0$, that is, $f$ is identically zero, and the proof of Theorem 1.2 is finished.

**Remark 2.1.** If $f$ is even continuous and $(\sigma, r)$-median, then (iii) is not needed to conclude that $f$ is constant.

Indeed, it suffices to observe that (iii) has not been used to obtain that $f = \inf f(\mathbb{R}^2)$ outside a compact set, and hence $\gamma := \sup f(\mathbb{R}^2) < \infty$. Then, just using (i) and (ii), we get as well that $\gamma - f = \inf(\gamma - f)(\mathbb{R}^2)$ outside a compact set, that is, $f = \sup f(\mathbb{R}^2)$ outside a compact set. Thus $\inf f(\mathbb{R}^2) = \sup f(\mathbb{R}^2)$, $f$ is constant.

### 3 Examples

Simple examples show that a continuous bounded $(\sigma, r)$-supermedian function $f$ on $\mathbb{R}^2$ may be non-constant, if any of the properties (i), (ii), or (iii) of $r$ is violated.

1. Let $f := (1 - |x|)^+$, $x \in \mathbb{R}^2$. Taking $r(x) := 3$, if $|x| < 2$, and $r(x) := 1$, if $|x| \geq 2$, we observe that, of course, property (i), that is, the continuity of $r$, cannot be omitted (or replaced by lower semicontinuity). Considering $r := |x| + 2 + (|x| + 1)^{-1}$ we already noted in Remark 1.2.1 that (iii) cannot be dropped (of course, for this purpose, it would be sufficient to take $r(x) := |x| + 2$).

2. Finally, let us prove that the conclusion of Theorem 1.1 fails, if property (ii) is omitted. For $x \in \mathbb{R}^2$, let

$$f(x) := \min \{1, |x_1|^{-1}\} \quad \text{and} \quad r(x) := 6 \max \{1, x_1^2\}.$$ 

Clearly, $0 \leq f \leq 1$, $f$ and $r$ are continuous functions, and $r$ satisfies (iii), since $r(0, t) = 6$ for every $t \in \mathbb{R}$. To prove that (1.2) holds, we fix $x \in \mathbb{R}^2$ and define

$$a := \max \{|x_1|, 1\}, \quad A := \{y \in \mathbb{R}^2: |y_1| \leq 2a\}.$$ 

Then $f(x) = a^{-1}$. We shall see that

$$\sigma_{x, r(x)}(A) \leq a^{-1}/2$$

and hence

$$\sigma_{x, r(x)}(f) \leq \sigma_{x, r(x)}(A) + \sup f(\mathbb{R}^2 \setminus A) \leq a^{-1}/2 + a^{-1}/2 = f(x).$$

5
To prove (3.1) let $\alpha$ denote the maximal angle between the $x_2$-axis and the lines connecting $x$ with one of the four points $y \in S(x, r(x))$ satisfying $|y_1| = 2a$. Then

$$\sigma_{x, r(x)}(A) \leq \frac{4\alpha}{2\pi} = \frac{2}{\pi} \alpha \leq \frac{3a}{r(x)}.$$  

If $|x_1| \leq 1$, then $a = 1$, $r(x) = 6$, and hence $3a/r(x) = 1/2 = 1/(2a)$. If $|x_1| > 1$, then $a = |x_1|$, $r(x) = 6|x_1|^2$, and hence $3a/r(x) = 1/(2|x_1|) = 1/(2a)$. Thus (3.1) holds.

A closer look would reveal that $f$ is $\tilde{r}$-superharmonic with respect to a continuous function $0 < \tilde{r} \leq r$ satisfying $\tilde{r}(x) = \tilde{r}(|x_1|, 0)$ and

$$(3.2) \quad \lim_{t \to \infty} \frac{\tilde{r}(t, 0)}{t \ln t} = \frac{2}{\pi}.$$  

This follows from the fact that, given $t \geq 1$ and $k \in \mathbb{N}$, the point $x := (t, 0)$ and the set $A := \{y \in \mathbb{R}^2 : |y_1| \leq kt\}$ satisfy

$$\int_{\mathbb{R}^2 \setminus A} f \, d\sigma_{x, \rho} \leq \sup f(\mathbb{R}^2 \setminus A) \leq \frac{1}{k} f(x) \quad \text{for every } \rho > 0,$$

and, for large $\rho$,

$$\int_A f \, d\sigma_{x, \rho} \sim \frac{1}{2\pi \rho} \int_0^{kt} \frac{1}{\tau} \, d\tau = \frac{2 \ln(kt)}{\pi \rho} \sim \frac{2}{\pi} \cdot \frac{t \ln t}{\rho} f(x)$$

(which, incidentally, shows that the limit behavior in (3.2) is optimal for our function $f$).

4 Proof of Proposition 1.3

Let $c > 1$, $M > 0$, and $r := \max\{c \cdot |\cdot|, M\}$ so that, for every $\alpha > 0$,

$$r^{-\alpha}(x) = \min\{|cx|^{-\alpha}, M^{-\alpha}\}, \quad x \in \mathbb{R}^2.$$  

We define

$$I(\alpha) := \frac{1}{2\pi} \int_0^{2\pi} |1 + ce^{it}|^{-\alpha} \, dt, \quad \alpha \geq 0.$$  

Then $I(0) = 1$ and

$$I'(0) = -\frac{1}{2\pi} \int_0^{2\pi} \ln |1 + ce^{it}| \, dt = -\ln c < 0.$$  

So there exists $\alpha_0 > 0$ such that $I < 1$ on $[0, \alpha_0]$. Let us fix $\alpha \in (0, \alpha_0]$ and $x \in \mathbb{R}^2$. If $c|x| > M$, then $r(x) = c|x|$ and hence

$$\sigma_{x, r(x)}(r^{-\alpha}) \leq \sigma_{x, r(x)}(c^{-\alpha}|\cdot|^{-\alpha}) = c^{-\alpha} \frac{1}{2\pi} \int_0^{2\pi} |x + c|x|e^{it}|^{-\alpha} \, dt = |cx|^{-\alpha} I(\alpha) < r^{-\alpha}(x).$$  

If $c|x| \leq M$, then $r(x) = M$, and hence $\sigma_{x, r(x)}(r^{-\alpha}) \leq M^{-\alpha} = r^{-\alpha}(x)$. Thus $r^{-\alpha}$ is $(\sigma, r)$-supermedian.

Since $\lambda_{x, \rho} = 2\rho^{-2} \int_0^{N} \sigma_{x, s} \, ds$ (and $\int_0^{2\pi} \ln |1 + ce^{is}| \, ds = 2\pi \ln 1 = 0$, if $s \in (0, 1)$), we obtain similarly that $r^{-\alpha}$ is $(\lambda, r)$-supermedian provided $\alpha > 0$ is sufficiently small.
5 Proof of Proposition 1.4

Let us define
\[ \psi(t) := (t + 1) \ln(t + 1) + (t - 1) \ln(t - 1) - 2t, \quad 1 < t < \infty. \]

Then \( \psi \) is continuous, \( \lim_{t \to 1} \psi(t) = 2 \ln 2 - 2 < 0 \), \( \lim_{t \to \infty} \psi(t) = \infty \). Moreover,
\[ \psi'(t) = \ln(t + 1) + \ln(t - 1) = \ln(t^2 - 1), \]

hence \( \psi \) is strictly decreasing on \((0, \sqrt{2})\) and strictly increasing on \((\sqrt{2}, \infty)\). So there exists \( c_0 \in (1, \infty) \) such that \( \psi(c_0) = 0 \),
\[ \psi < 0 \text{ on } (1, c_0), \quad \text{and} \quad \psi > 0 \text{ on } (c_0, \infty). \]

In fact, \( 2, 50 < c_0 < 2, 51 \) (since \( \psi(2.50) < 0 \) and \( \psi(2.51) > 0 \)).

1. Let \( r : \mathbb{R} \to (0, \infty) \) and \( M > 0 \) such that
\[ r(x) \leq c_0 |x| + M \quad \text{for all } x \in \mathbb{R} \setminus [-M, M]. \]

Let \( \varphi := \ln^+ (|x| - M) \) (so that \( \varphi(x) = 0 \), if \(-(M + 1) \leq x \leq M + 1\)). We claim that there exists \( \tilde{M} > 0 \) such that, for every \( x \in \mathbb{R} \setminus [\tilde{M}, -\tilde{M}] \),
\[ \lambda_{x, r(x)}(\varphi) \leq \varphi(x). \tag{5.1} \]

Since \( \lim_{x \to -\infty} (x/M) \ln(1 - (M/x)) = -\ln' 1 = -1 \), there exists \( \tilde{M} \geq 1 + c_0 + 2M \) such that
\[ M \ln(x - M) + c_0 x \ln \frac{x - M}{x} - 1 > 0 \quad \text{for every } x > \tilde{M}. \tag{5.2} \]

For a while, let us fix \( x \in \mathbb{R} \setminus [-\tilde{M}, \tilde{M}] \). To prove (5.1) we may assume, by symmetry, that \( x \) is positive. Let \( y \in (0, c_0 + M) \). If \( x - y \geq M + 1 \), then
\[ \varphi(x - y) + \varphi(x + y) \leq 2 \varphi(x), \tag{5.3} \]

since \( \varphi \) is concave on \((M + 1, \infty)\). If \( M + 1 > x - y \geq -(M + 1) \), then (5.3) holds, since \( \varphi(x - y) = 0 \) and \( x + y - M \leq x(1 + c_0) \leq (x - M)^2 \). Therefore (5.1) holds, if \( x - r(x) \geq -(M + 1) \).

Let us assume next that \( x - r(x) < -(M + 1) \). Then \( t := (r(x) - M)/x \in (1, c_0] \),
\[ \int_0^{r(x) \pm x} \varphi(s) \, ds = \int_{M+1}^{(t \pm 1)x + M} \varphi(s) \, ds \]
\[ = \int_1^{(t \pm 1)x} \ln s \, ds = (t \pm 1)x \ln[(t \pm 1)x] - (t \pm 1)x + 1. \]

Since \( \psi(t) \leq \psi(c_0) = 0 \), we hence see that
\[ \int_{x-r(x)}^{x+r(x)} \varphi(s) \, ds = \psi(t)x + 2tx \ln x + 2 \leq 2tx \ln x + 2, \]
that is, \( r(x)\lambda_{x,r(x)}(\varphi) \leq tx \ln x + 1 \). Thus

\[
r(x)(\varphi(x) - \lambda_{x,r(x)}(\varphi)) \geq (tx + M) \ln(x - M) - (tx \ln x + 1) = M \ln(x - M) + tx \ln \frac{x - M}{x} - 1,
\]

where the right side is positive by (5.2), since \( t \leq c_0 \). This finishes the proof of (5.1).

Now let \( f > -\infty \) be a lower semicontinuous \((\lambda, r)\)-supermedian function on \( \mathbb{R} \) such that \( \liminf_{|x| \to \infty} f(x)/\ln |x| \geq 0 \). To prove that \( f \) is constant, we use ideas from [4] and [7]. There exists \( x_0 \in [-\hat{M}, \hat{M}] \) such that

\[
f(x_0) = \inf f([-\hat{M}, \hat{M}]).
\]

We intend to show that \( f \geq f(x_0) \) on \( \mathbb{R} \). Then the lower semicontinuity of \( f \) and the inequalities \( \lambda_{x,r(x)}(f) \leq f(x) \), \( x \in \mathbb{R} \), will imply that the set \( A := \{ x \in \mathbb{R} : f(x) = f(x_0) \} \) is both closed and open, hence \( A = \mathbb{R} \), \( f = f(x_0) \).

Fixing \( \varepsilon > 0 \), it suffices to prove that

\[
\tilde{f} := f + \varepsilon \varphi \geq f(x_0).
\]

Obviously, \( \tilde{f} \) is lower semicontinuous, lower bounded, and \( \lim_{|x| \to \infty} \tilde{f}(x) = \infty \). Therefore \( \tilde{f} \) attains a minimum on \( \mathbb{R} \), and the non-empty set

\[
\tilde{A} := \{ x \in \mathbb{R} : \tilde{f}(x) = \inf \tilde{f}(\mathbb{R}) \}
\]

is closed. Let \( z \in \tilde{A} \) with minimal absolute value. If \( |z| > \hat{M} \), then \( \lambda_{z,r(z)}(\tilde{f}) \leq \tilde{f}(z) \), and hence \( [z - r(z), z + r(z)] \subseteq \tilde{A} \). This is impossible, by our choice of \( z \). Thus \( z \in [-\hat{M}, \hat{M}] \), and \( \tilde{f} \geq \tilde{f}(z) \geq f(z) \geq f(x_0) \).

2. Finally, let \( c > c_0, M > 0 \), and \( r := \max(c, |\cdot|, M) \). We have to show that the function \( r^{-\alpha} \) is \((\lambda, r)\)-supermedian provided \( \alpha > 0 \) is sufficiently small. To that end we define

\[
\Psi(\beta) := (c + 1)^\beta + (c - 1)^\beta - 2\beta c, \quad 0 < \beta < \infty.
\]

Then \( \Psi'(\beta) = (c + 1)^\beta \ln(c + 1) + (c - 1)^\beta \ln(c - 1) - 2c \). In particular, \( \Psi'(1) = \psi(c) > 0 \). So there exists \( \alpha \in (0, 1) \) such that \( \Psi(1 - \alpha) < 0 \).

Let us now fix \( x \in \mathbb{R} \). If \( c|x| \leq M \), then \( r(x) = M \), and hence \( \lambda_{x,r(x)}(r^{-\alpha}) \leq M^{-\alpha} = r^{-\alpha}(x) \). So let us assume that \( c|x| < M \) and hence \( r(x) = c|x| \). Then

\[
\int_{x-r(x)}^{x+r(x)} |s|^{-\alpha} ds = \frac{1}{1-\alpha} |x|^{1-\alpha} ((c + 1)^{1-\alpha} + (c - 1)^{1-\alpha}) \quad \text{and} \quad 2r(x)|x|^{-\alpha} = 2c|x|^{1-\alpha}.
\]

Since \( \Psi(1 - \alpha) < 0 \), we conclude that \( \lambda_{x,r(x)}(|\cdot|^{-\alpha}) \leq |x|^{-\alpha} \) and hence

\[
\lambda_{x,r(x)}(r^{-\alpha}) \leq \lambda_{x,r(x)}(c^{-\alpha} |\cdot|^{-\alpha}) \leq c|x|^{-\alpha} = r^{-\alpha}(x).
\]

Thus \( r^{-\alpha} \) is \((\lambda, r)\)-supermedian.

**Remark 5.1.** If even \( r \leq c \cdot |\cdot| + M \) for some \( c \in (0, c_0) \), then the conclusion in (1) of Proposition 1.4 is still valid, if the condition \( \liminf_{|x| \to \infty} f(x)/\ln |x| \geq 0 \) is replaced by the weaker assumption \( \liminf_{|x| \to \infty} f(x)/\ln |x| > -\infty \). Indeed, by means of the function \( \Psi \), we may then prove that, for some \( \alpha > 0 \), the function \(|\cdot|^\alpha \) (which will replace \( \varphi \)) is \((\lambda, r)\)-supermedian.
References


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