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The SL_3-module $T(43)$ for $p = 3$.

An appendix to the paper:
Decomposition of tensor products of modular irreducible representations for SL_3
by S.R.Doty and S.Martin.

Let k be an algebraically closed field of characteristic $p = 3$. Following Doty and Martin, we consider rational SL_3-modules with composition factors $L(\lambda)$, where λ is one of the weights $(1,0)$, $(0,5)$, $(5,1)$, $(4,3)$, $(6,2)$. Dealing with a dominant weight (a,b), or the simple module $L(a,b)$, we usually will write just ab. The corresponding Weyl module, dual Weyl module, or tilting module, will be denoted by $\Delta(ab)$, $\nabla(ab)$ and $T(ab)$, respectively.

The paper [DM] by Doty and Martin describes in detail the structure of the modules $\Delta(\lambda), \nabla(\lambda)$ for $\lambda = 10,\ 05,\ 51,\ 43,\ 62$ and also $T(10)$, $T(05)$, $T(51)$ and it provides the factors of a Δ-filtration for $T(43)$. This module $T(43)$ is still quite small (it has length 10), but its structure is not completely obvious at first sight. The main aim of this appendix is to explain the shape of this module.

Let us call a finite set I of dominant weights (or of simple modules) an *ideal* provided for any $\lambda \in I$ all composition factors of $T(\lambda)$ belong to I. The category of modules with all composition factors in an ideal I is a highest weight category with weight set I, thus can be identified with the module category of a basic quasi-hereditary algebra which we denote by $A(I)$. In order to analyse the module $T(43)$, we need to look at the ideal $I = \{10, 05, 51, 43\}$, thus at the algebra $A(10, 05, 51, 43)$.

In order to determine the precise relations for $A(10, 05, 51, 43)$, we will have to look also at the module $T(62)$, see section 4. Note that $\{10,\ 05,\ 51,\ 43,\ 62\}$ is again an ideal, thus we deal with the algebra $A(10, 05, 51, 43, 62)$.

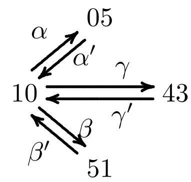
The use of quivers and relations for presenting a basic finite dimensional algebras was initiated by Gabriel around 1970, the text books [ARS] and [ASS] can be used as a reference. The class of quasi-hereditary algebras was introduced by Scott and Cline-Parshall-Scott; for basic properties one may refer to [DR] and [R2]. The author is grateful to S. Doty and R. Farnsteiner for fruitful discussions and helpful suggestions concerning the material presented in the appendix.

1. The main result.

Deviating from [DM], we will consider **right** modules. Thus, given a finite-dimensional algebra A, an indecomposable projective A-module is of the form eA with e a primitive idempotent. The algebras to be considered will be factor algebras of path algebras of quivers and the advantage of looking at right modules will be that in this way we can write the paths in the quiver as going from left to right.

Proposition. *The algebra* $A(10, 05, 51, 43)$ *is isomorphic to the path algebra of the quiver*

$Q = Q(10, 05, 51, 43)$



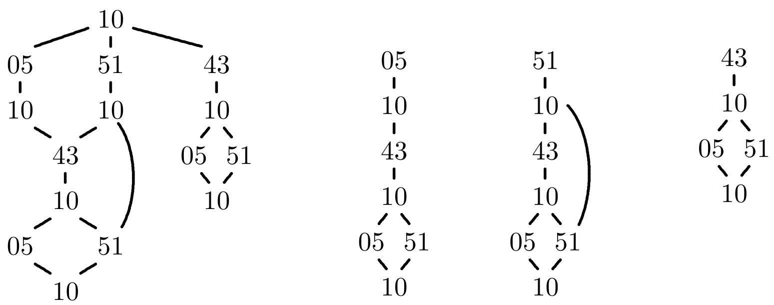
modulo the ideal generated by the following relations

$$\alpha'\alpha = 0, \quad \alpha'\beta = 0, \quad \beta'\alpha = 0, \quad \beta'(1 - \gamma\gamma')\beta = 0,$$
$$\gamma'\gamma = 0, \quad \gamma'(\alpha\alpha' - \beta\beta') = 0, \quad (\alpha\alpha' - \beta\beta')\gamma = 0, \quad \gamma'\alpha\alpha'\gamma = 0.$$

We are going to give some comments before embarking on the proof.

(1) Since the quiver $Q(10, 05, 51, 43)$ is bipartite, say with a $(+)$-vertex 10 and three $(-)$-vertices 05, 51, 43, possible relations between vertices of the same parity involve paths of even lengths, those between vertices with different parity involve paths of odd lengths. Our convention for labelling arrows between a $(+)$-vertex a and a $(-)$-vertex b is the following: we use a greek letter for the arrow $a \to b$ and add a dash for the arrow $b \to a$.

(2) The assertion of the proposition can be visualized by drawing the shape of the indecomposable projective A-modules. The indecomposable projective A-module with top λ will be denoted by $P(\lambda) = e_\lambda A$, where e_λ is the primitive idempotent corresponding to λ, and we will denote the radical of A by J.



These are the coefficient quivers of the indecomposable projective A-modules with respect to suitable bases. In addition, the proposition asserts that all the non-zero coefficents can be chosen to be equal to 1. Note that this means that A has a basis $\mathcal{B}$ which consists of a complete set of primitive and orthogonal idempotents as well as of elements from the radical J, and such that $\mathcal{B}$ is multiplicative (this means: if $u, v \in \mathcal{B}$, then either $uv = 0$ or else $uv \in \mathcal{B}$).

For the convenience of the reader, let us recall the notion of a coefficient quiver (see for example [R3]): By definition, a representation M of a quiver Q over a field k is of the form $M = (M_x; M_\alpha)_{x,\alpha}$; here, for every vertex x of Q, there is given a finite-dimensional k-space M_x, say of dimension d_x, and for every arrow $\alpha : x \to y$, there is given a linear transformation $M_\alpha : M_x \to M_y$. A *basis* $\mathcal{B}$ of M is by definition a subset of the disjoint union of the various k-spaces M_x such that for any vertex x the set $\mathcal{B}_x = \mathcal{B} \cap M_x$ is a basis of M_x. Now assume that there is given a basis $\mathcal{B}$ of M. For any arrow $\alpha : x \to y$, write M_α as a $(d_x \times d_y)$-matrix $M_{\alpha,\mathcal{B}}$ whose rows are indexed by $\mathcal{B}_x$ and whose columns are indexed by $\mathcal{B}_y$. We denote by $M_{\alpha,\mathcal{B}}(b, b')$ the corresponding matrix coefficients, where $b \in \mathcal{B}_x$, $b' \in \mathcal{B}_y$, these matrix coefficients $M_{\alpha,\mathcal{B}}(b, b')$ are defined by $M_\alpha(b) = \sum_{b' \in \mathcal{B}} b'\, M_{\alpha,\mathcal{B}}(b, b')$. By definition, the *coefficient quiver* $\Gamma(M, \mathcal{B})$ of M with respect to $\mathcal{B}$ has the set $\mathcal{B}$ as set of vertices, and there is an arrow (α, b, b') provided $M_{\alpha,\mathcal{B}}(b, b') \neq 0$ (and we call $M_{\alpha,\mathcal{B}}(b, b')$ the corresponding coefficient). If b belongs to $\mathcal{B}_x$, we usually label the vertex b_x by x. If necessary, we label the arrow (α, b, b') by α; but since we only deal with quivers without multiple arrows, the labelling of arrows could be omitted. In all cases considered in the appendix, we can arrange the vertices in such a way that all the arrows point downwards, and then replace arrows by edges. This convention will be used throughout.

Note that there is a long-standing tradition in matrix theory to focus attention to such coefficient quivers (see e.g. [BR]), whereas the representation theory of groups and algebras is quite reluctant to use them.

Looking at the pictures one should be aware that the four upper base elements form a complete set of primitive and orthogonal idempotents, thus these are the generators of the indecomposable projective A-modules. Those directly below generate the radical of A, and they are just the arrows of the quiver (or better: the residue classes of the arrows in the factor algebra of the path algebra modulo the relations). Of course, on the left we see $P(10)$, then $P(05)$ and $P(51)$, and finally, on the right, $P(43)$.

(3) The strange relation $\beta'(1 - \gamma\gamma')\beta = 0$ leads to the curved edge in $P(51)$ as well as in $P(10)$. Note that the submodule lattice of $P(51)$ would not at all be changed when deleting this extra line — but its effect would be seen in $P(10)$. Namely, without this extra line, the socle of $P(10)$ would be of length 3 (namely, top $\mathrm{rad}^2 P(10)$ is the direct sum of three copies of 10, and the two copies displayed in the left part are both mapped under γ to 43, thus there is a diagonal which is mapped under γ to zero; without the curved line, this diagonal would belong to the socle), whereas the socle of $P(10)$ is of length 2.

(4) Looking at the first four relations presented above, one could have the feeling of a certain asymmetry concerning the role of $P(05)$ and $P(51)$, or also of the role of 05 and 51 as composition factors of the radical of $P(51)$. But such a feeling is misleading as will be seen in the proof. The pretended lack of symmetry concerns also our display of $T(43)$. Sections 7 and 8 will be devoted to a detailed analysis of the module $T(43)$ in order to focus the attention to its hidden symmetries.

(5) Note that all the tilting A-modules are local (and also colocal):

$$\begin{array}{rcl} T(10) &=& P(10)/(\alpha A+\beta A+\gamma A) \\ T(05) &=& P(10)/(\beta A+\gamma A), \\ T(51) &=& P(10)/(\alpha A+\gamma A), \\ T(43) &=& P(10)/\gamma A. \end{array}$$

As we have mentioned, sections 7 and 8 will discus in more detail the module $T(43)$.

(6) A further comment: One may be surprised to see that one can find relations which are not complicated at all: many are monomials, the remaining ones are differences of monomials, always using paths of length at most 4.

2. Preliminaries on algebras and the presentation of algebras using quivers and relations.

Let t be a natural number. Recall that the zero module has Loewy length 0 and that a module M is said to have *Loewy length at most* t with $t \geq 1$, provided it has a submodule M' of Loewy length at most $t-1$ such that M/M' is semisimple. Given a module M, we denote by $\mathrm{soc}_t M$ the maximal submodule of Loewy length at most t, and by $\mathrm{top}^t M$ the maximal factor module of Loewy length t. Of course, we write $\mathrm{soc} = \mathrm{soc}_1$ and $\mathrm{top} = \mathrm{top}^1$, but also $\mathrm{top}^t M = M/\mathrm{rad}^t M$.

Let A be a finite-dimensional basic algebra with radical J and quiver Q. Let us assume that Q has no multiple arrows (which is the case for all the quivers considered here). For any arrow $\zeta : i \to j$ in Q, we choose an element $\eta(\zeta) \in e_iJe_j \setminus e_iJ^2e_j$; the set of elements $\eta(\zeta)$ will be called a *generator choice* for A. In this way, we obtain a surjective algebra homomorphisms

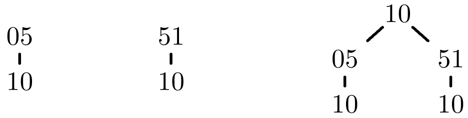
$$\eta : kQ \to A$$

If ρ is the kernel of η, then $\rho = \bigoplus_{ij} e_i\rho e_j$, and we call a generating set for ρ consisting of elements in $\bigcup_{ij} e_i\rho e_j$ a *set of relations* for A. We are looking for a generator choice for the algebra $A(10, 05, 51, 43)$ which allows to see clearly the structure of $T(43)$. Usually, we will write ζ instead of $\eta(\zeta)$ and hope this will not produce confusion. If $\zeta \in e_iJe_j \setminus e_iJ^2e_j$ belongs to a generator choice, we obviously may replace it by any element of the form $c\zeta + d$ with $0 \neq c \in k$ and $d \in e_iJ^2e_j$ and obtain a new generator choice.

3. The algebra $B = A(10, 05, 51)$.

Consider a quasi-hereditary algebra B with quiver being the full subquiver of $Q(10, 05, 51, 43)$ with vertices $10,\ 05,\ 51$ and with ordering $10 < 05$, $10 < 51$.

It is well-known (and easy to see) that B is uniquely determined by these data. The indecomposable projectives have the following shape



What we display are the again coefficient quivers of the indecomposable projective B-modules considered as representations of kQ with respect to a suitable basis.

We see that the algebra B is of Loewy length 3 and that it can be described by the relations:

$$\alpha'\alpha = \alpha'\beta = \beta'\alpha = \beta'\beta = 0.$$

Of course, $\Delta(10) = \nabla(10) = 10$; and the modules $\Delta(05)$, $\Delta(51)$, $\nabla(05)$ and $\nabla(51)$ are serial of length 2, always with 10 as one of the composition factors. This means that the structure of the modules $\Delta(\lambda)$, $\nabla(\lambda)$, for $\lambda = 10,\ 05\ 51$ can be read off from the quiver (but, of course, conversely, the quiver was obtained from the knowledge of the corresponding Δ- and ∇-modules).

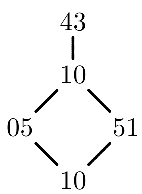
Note that $T(05)$ is the only indecomposable module with a Δ-filtration with factors $\Delta(10)$ and $\Delta(05)$, since $\mathrm{Ext}^1(\Delta(10), \Delta(05)) = k$. Similarly, $T(51)$ is the only indecomposable module with a Δ-filtration with factors $\Delta(10)$ and $\Delta(51)$.

Let us remark that the structure of the module category mod B is well-known: using covering theory, one observes that mod B is obtained from the category of representations of the affine quiver of type $\widetilde{A}_{22}$ with a unique sink and a unique source by identifying the simple projective module with the simple injective module. In mod B, there is a family of homogeneous tubes indexed by $k \setminus \{0\}$, the modules on the boundary are of length 4 with top and socle equal to 10 and with rad/soc $= 05 \oplus 51$. We will call these modules the *homogeneous B-modules of length* 4. (The representation theory of affine quivers can be found for example in [R1] and [SS]; from covering theory, we need only the process of removing a node, see [M].)

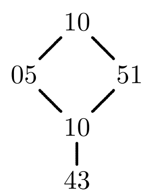
4. The modules rad $\Delta(43)$ **and** $\nabla(43)$/soc **are isomorphic.**

We will use the following information concerning the modules $\Delta(43)$ and $\nabla(43)$, see [DM]. Both rad $\Delta(43)$ and $\nabla(43)$/soc are homogeneous B-modules of length 4, thus the modules $\Delta(43)$ and $\nabla(43)$ have the following shape

$\Delta(43)$



$\nabla(43)$



Here, we have drawn again coefficient quivers with respect to suitable bases. But note that we do not (yet) claim that all the non-zero coefficients can be chosen to be equal to 1.

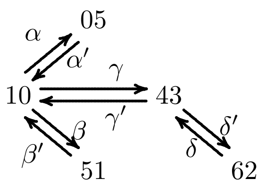
In order to show the assertion in the title, we have to expand our considerations taking into account also the weight 62. The existence of an isomorphism in question will be obtained by looking at the tilting module $T(62)$.

In dealing with a tilting module $T(\mu)$, there is a unique submodule isomorphic to $\Delta(\mu)$, and a unique factor module isomorphic to $\nabla(\mu)$. Let $R(\mu) = \operatorname{rad} \Delta(\mu)$ and let $Q(\mu)$ be the kernel of the canonical map $\pi : T(\mu) \to \nabla(\mu)/\mathrm{soc}$. Note that $\Delta(\mu) \subseteq Q(\mu)$ (namely, if $\pi(\Delta(\mu))$ would not be zero, then it would be a submodule of $\nabla(\mu)/\mathrm{soc}$ with top equal to μ; however $\nabla(\mu)/\mathrm{soc}$ has no composition factor of the form μ). It follows that $R(\mu) \subset Q(\mu)$ and we call $C(\mu) = Q(\mu)/R(\mu)$ the *core* of the tilting module $T(\mu)$. Also, we see that $\mu = \Delta(\mu)/R(\mu)$ is a simple submodule of $C(\mu)$. In fact, *μ is a direct summand of $C(\mu)$.* Namely, there is $U \subset T(\mu)$ with $T(\mu)/U = \nabla(\mu)$. Then $U \subset Q(\mu)$ and $Q(\mu)/U = \mu$. Since $R(\mu) \subset Q(\mu)$ and $R(\mu)$ has no composition factor of the form μ, it follows that $R(\mu) \subseteq U$. Altogether, we see that $U + \Delta(\mu) = Q(\mu)$ and $U \cap \Delta(\mu) = R(\mu)$. Thus $Q(\mu)/R(\mu) = U/R(\mu) \oplus \Delta(\mu)/R(\mu) = U/R(\mu) \oplus \mu$.

The module $\Delta(62)$ is serial with going down factors 62, 43, 10, 51, and the module $\nabla(62)$ is serial with going down factors 51, 10, 43, 62, see [DM], 5.1.15. Also we will use that $T(62)$ has Δ-factors $\Delta(51)$, $\Delta(43)$, $\Delta(62)$, each with multiplicity one (and thus ∇-factors $\nabla(62)$, $\nabla(43)$, $\nabla(51)$). To get the Δ-factors of $T(62)$, one has to use [DM], (2.4.2) along with the known structure of the Deltas (this requires a small calculation, which is left to the reader.)

The quiver of $Q(10, 05, 51, 43, 62)$ of $A(10, 05, 51, 43, 62)$ is

$Q(10, 05, 51, 43, 62)$



with ordering $10 < 05 < 43 < 62$, and $10 < 51 < 43$.

Lemma 1. *The core of $T(62)$ is of the form* $\operatorname{rad} \Delta(43) \oplus 62$ *as well as of the form* $\nabla(43)/\mathrm{soc} \oplus 62$.

Corollary. *The modules* $\operatorname{rad} \Delta(43)$ *and* $\nabla(43)/\mathrm{soc}$ *are isomorphic.*

Note that it is quite unusual that the modules $\operatorname{rad} \Delta(\lambda)$ and $\nabla(\lambda)/\mathrm{soc}$ are isomorphic, for a weight λ.

Proof of Lemma 1. Let $T_1 \subset T_2 \subset T(62)$ be a filtration with factors

$$T_1 = \Delta(62), \quad T_2/T_1 = \Delta(43), \quad T(62)/T_2 = \Delta(51).$$

Now $R(62) = \text{rad}\ \Delta(62) \subset T_1 \subset T_2$, thus we may look at the factor module $T_2/R(62)$ and the exact sequence

$$0 \to 62 \to T_2/R(62) \to \Delta(43) \to 0$$

(with $62 = T_1/R(62)$). We consider the submodule $N = \text{rad}\ \Delta(43)$ of $\Delta(43)$, with factor module $\Delta(43)/N = 43$. We have $\text{Ext}^1(N, 62) = 0$, since $\text{Ext}^1(S, 62) = 0$ for all the composition factors S of N. This implies that there is an exact sequence

$$0 \to N \oplus 62 \to T_2/R(62) \to 43 \to 0.$$

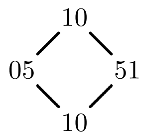
Thus, there is a submodule $U \subset T_2$ with $R(62) \subset U$ such that $U/R(62)$ is isomorphic to $N \oplus 62$ and T_2/U is isomorphic to 43. Since $T(62)/T_2 = \Delta(51)$ is of length 2, we see that $T(62)/U$ is of length 3.

Now consider the canonical map $\pi : T(62) \to \nabla(62)/\text{soc}$. This map vanishes on $R(62)$, thus induces a map $\pi' : T(62)/R(62) \to \nabla(62)/\text{soc}$. Let us look at the submodule $U/R(62)$ of $T(62)/R(62)$. Since the socle of $\nabla(62)/\text{soc}$ is equal to 43, and $U/R(62) = N \oplus 62$ has no composition factor of the form 43, we see that $U/R(62)$ is contained in the kernel of π', and therefore U is contained in the kernel of π.

By definition, the kernel of the canonical map $\pi : T(62) \to \nabla(62)/\text{soc}$ is $Q(62)$, thus we have shown that $U \subseteq Q(62)$. But $T(62)/U$ is of length 3 as is $T(62)/Q(62)$, thus $U = Q(62)$. But this means that $Q(62)/R(62) = U/R(62) = N \oplus 62 = \text{rad}\ \Delta(62) \oplus 62$.

The dual arguments show that $Q(62)/R(62) = \nabla(62)/\text{soc} \oplus 62$

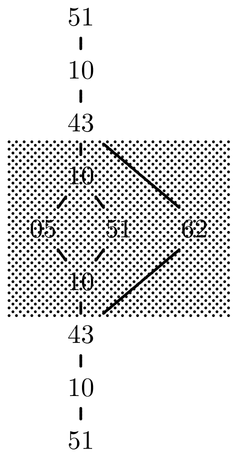
As we have mentioned, the module $N = \text{rad}\ \Delta(43)$ is a B-module, where $B = A(10, 05, 51)$. This algebra B has been discussed in section 3. The coefficient quiver of N is



Now, choosing a suitable basis of N, we can assume that at least 3 of the non-zero coefficients are equal to 1 and we look at the remaining coefficient, say that for the arrow α. It will be a non-zero scalar c in k. Recall that we have started with a particular generator choice for the algebra B which we can change. If we replace the element $\alpha \in J$ by $\frac{1}{c}\alpha$, then the coefficients needed for N will all be equal to 1.

Remark. Extending the analysis of the Δ- and the ∇-filtrations of $T(43)$, one can show that $T(62)$ is the indecomposable projective $A(10, 05, 51, 43, 62)$-module with top 51 (as well as the indecomposable injective $A(10, 05, 51, 43, 62)$-module with socle 51). As Doty has pointed out, the last assertion follows also from Theorem 5.1 of the DeVisscher-Donkin paper [DD] (that result is based on their Conjecture 5.2 holding, but it is proved in Section 7 of the same paper that the conjecture holds for GL(3); hence it holds also for SL(3)).

Let us add without proof that in this way one may show that the module $T(62)$ has a coefficient quiver of the form



the shaded part being the core of $T(62)$.

5. The module $T(43)$**.**

Lemma 2. *We have* top $T(43) = 10 = \operatorname{soc} T(43)$.

Proof: We use that $T(43)$ has Δ-factors $\Delta(10)$, $\Delta(05)$, $\Delta(51)$, $\Delta(43)$ in order to show that top $T(43) = 10$. Since top $T(43)$ is isomorphic to a submodule of the direct sum of the tops of the Δ-factors, it follows that top $T(43)$ is multiplicity free. Since $T(43)$ maps onto $\nabla(43)$, the only composition factor 43 cannot belong to the top.

Actually, it is $N = T(43)/\operatorname{rad} \Delta(43)$ which maps onto $\nabla(43)$, and $\nabla(43)$ maps onto $\nabla(05)$ which is serial with top 10 and socle 05; this shows that the only composition factor of the form 05 of N does not belong to top N. Now 05 is not in top N and not in top rad $\Delta(43)$, thus not in top $T(43)$. Similarly, 51 is not in top $T(43)$. It follows that top $T(43) = 10$.

Note that the ∇-factors of $T(43)$ are $\nabla(10)$, $\nabla(05)$, $\nabla(51)$, $\nabla(43)$. Namely, $T(43)$ maps onto $\nabla(43)$, say with kernel N'. The number of composition factors of N' of the form $05, 51, 10$ is $1, 1, 3$, respectively. Since N' has a ∇-filtration, its ∇-factors have to be $\nabla(05), \nabla(51)$ and $\nabla(10)$, each with multiplicity one. In the same way, as we have seen that $T(43)$ has simple top 10, we now see that it also has simple socle 10.

Let us add also the following remark:

Remark *The module* $T(43)$ *is a faithful* A*-module.*

Proof: First of all, we show that the modules $T(05)$ and $T(51)$ are both isomorphic to factor modules (and to submodules) of $T(43)$. The Δ-filtration of $T(43)$ shows that $T(43)$ has a factor module with factors $\Delta(10)$ and $\Delta(05)$. Since this factor module is indecomposable, it follows that it is $T(05)$. Similarly, $T(51)$ is a factor module of $T(43)$. (And dually, $T(05)$ and $T(51)$ are also submodules of $T(43)$). Of course, also $T(10)$ is a factor module and a submodule of $T(43)$. It follows that $T(43)$ is faithful, since the direct sum of all tilting modules is always a faithful module (it is a "tilting" module in the sense used in [R2]).

6. Algebras with quiver $Q(10, 05, 51, 43)$.

Let us assume that we deal with a quasi-hereditary algebra A with quiver $Q(10, 05, 51, 43)$, with ordering $10 < 05 < 43$ and $10 < 51 < 43$ and such that $\operatorname{rad}\Delta(43)$ and $\nabla(43)/\operatorname{soc}$ both are homogeneous B-modules of length 4.

Since we know the composition factors of all the A-modules $\nabla(\lambda)$, we can use the reciprocity law in order to see that the indecomposable projective modules have the following Δ-factors (going downwards)

$P(43)$	$\Delta(43)$
$P(05)$	$\Delta(05) \mid \Delta(43)$
$P(51)$	$\Delta(51) \mid \Delta(43)$
$P(10)$	$\Delta(10) \mid \Delta(05) \oplus \Delta(51) \mid \Delta(43) \oplus \Delta(43).$

We see: Since the Loewy length of these factors of $P(10)$ are $1, 2, 4$, the Loewy length of $P(10)$ can be at most 7. Of course, the Loewy length of $P(43) = \Delta(43)$ is 4 and that of $P(05)$ and $P(51)$ is at most 6. It follows that $J^7 = 0$.

Our aim is to contruct a presentation of A by the quiver Q and suitable relations. As we have mentioned, for any arrow $\alpha : i \to j$ in Q we choose an element in $e_iJe_j \setminus e_iJ^2e_j$ which we denote again by α, in order to obtain a surjective algebra homomorphisms

$$\eta : kQ \to A.$$

Since $J^7 = 0$, we see that *all paths of length 7 in the quiver are zero when considered as elements of A.*

Lemma 3. *Any generator choice for A satisfies the conditions*

$$\alpha'\alpha, \alpha'\beta, \beta'\alpha, \beta'\beta \ \in J^4,$$

$$\gamma'\gamma = 0, \quad \gamma'(\alpha\alpha' - c_0\beta\beta') = 0, \quad (\alpha\alpha' - c_1\beta\beta')\gamma = 0, \quad \gamma'\alpha\alpha'\gamma = 0.$$

for some non-zero scalars $c_0, c_1 \in k$.

Proof. The algebra B considered in section 3 is the factor algebra of A modulo the ideal generated by e_{43}. Since we know that the paths $\alpha'\alpha,\ \alpha'\beta,\ \beta'\alpha,\ \beta'\beta$

are zero in B, they belong to J^4 (any path between vertices of the form 05 and 51 which goes through 43 has length at least 4):

$$\alpha'\alpha, \alpha'\beta, \beta'\alpha, \beta'\beta \ \in J^4.$$

Since $e_{43}Je_{43} = 0$, we have

$$\gamma'\gamma = 0.$$

Also, the shape of $P(43)$ shows that $e_{43}J^3e_{10}$ is one-dimensional, and that the paths $\gamma'\alpha\alpha'$ and $\gamma'\beta\beta'$ both are non-zero, thus they are scalar multiples of each other. Thus, we can assume that

$$\gamma'(\alpha\alpha' - c_0\beta\beta') = 0,$$

with some non-zero scalar c_0. Dually, we have

$$(\alpha\alpha' - c_1\beta\beta')\gamma = 0$$

with some non-zero scalar c_1. (Later, we will use the fact that the modules rad $\Delta(43)$ and $\nabla(43)/\mathrm{soc}$ are isomorphic, then we can assume that $c_0 = c_1$; also, we will replace one of the arrows $\alpha, \alpha', \beta, \beta'$ by a non-zero scalar multiples, in order to change the coefficient c_0 to 1).

Since $P(43) = \Delta(43)$ is of Loewy length 4, we see that $\gamma' J^3 = 0$, in particular we have

$$\gamma'\alpha\alpha'\gamma = 0$$

(and also that $\gamma'\alpha\alpha'\alpha$ and $\gamma'\alpha\alpha'\beta$ are zero.)

We have seen in the proof that $\gamma' J^3 = 0$, since $\Delta(43)$ is of Loewy length 4. Dually, since $\nabla(43)$ is of Loewy length 4, we have $J^3\gamma = 0$.

Lemma 4. *A factor algebra of the path algebra of the quiver $Q(10, 05, 51, 43)$ satisfying the relations exhibited in Lemma 3 is generated as a k-space by the elements*

Q_0	10, 05, 51, 43,
Q_1	$\alpha,\ \beta,\ \gamma,\ \alpha',\ \beta',\ \gamma'$,
Q_2	$\alpha\alpha',\ \beta\beta',\ \gamma\gamma',\ \alpha'\gamma,\ \beta'\gamma,\ \gamma'\alpha,\ \gamma'\beta$,
Q_3	$\alpha\alpha'\gamma,\ \gamma\gamma'\alpha,\ \gamma\gamma'\beta,\ \alpha'\gamma\gamma',\ \beta'\gamma\gamma',\ \gamma'\alpha\alpha'$,
Q_4	$\alpha\alpha'\gamma\gamma',\ \gamma\gamma'\alpha\alpha',\ \alpha'\gamma\gamma'\alpha,\ \alpha'\gamma\gamma'\beta,\ \beta'\gamma\gamma'\alpha,\ \beta'\gamma\gamma'\beta$,
Q_5	$\alpha\alpha'\gamma\gamma'\alpha,\ \alpha\alpha'\gamma\gamma'\beta,\ \alpha'\gamma\gamma'\alpha\alpha',\ \beta'\gamma\gamma'\beta\beta'$,
Q_6	$\alpha\alpha'\gamma\gamma'\alpha\alpha'$,

thus is of dimension at most 34.

Proof: One shows inductively that the elements listed as Q_i generate the factor space J^i/J^{i+1}. This is obvious for $i = 0, 1, 2$, since here we have listed all the paths of length i. For $i = 3$, the missing paths of length 3 are

$$\alpha\alpha'\alpha,\ \alpha\alpha'\beta,\ \beta\beta'\alpha,\ \beta\beta'\beta,\ \gamma\gamma'\gamma,$$

as well as

$$\beta\beta'\gamma,\ \gamma'\beta\beta'.$$

By assumption, the first five belong to J^4, whereas the last two are equal to a non-zero multiple of $\alpha\alpha'\gamma$ and $\gamma'\alpha\alpha'$, respectively.

Next, consider $i \geq 4$. We have to take the paths in Q_{i-1} and multiply them from the right by the arrows and see what happens. For $i = 4$, the missing paths are $\gamma\gamma'\beta\beta'$ (it is a multiple of $\gamma\gamma'\alpha\alpha'$), the paths $\alpha'\gamma\gamma'\gamma$ and $\beta'\gamma\gamma'\gamma$ (both involve $\gamma'\gamma$) as well as the right multiples of $\gamma'\alpha\alpha'$ (all belong to J^5).

In the same way, we deal with the cases $i = 5, 6, 7$. In particular, for $i = 7$, we see that $J^7 = J^8$, and therefore $J^7 = 0$. This shows that we have obtained a generating set of the algebra as a k-space.

7. The algebra $A = A(10, 05, 51, 43)$.

Now, let $A = A(10, 05, 51, 43)$.

Lemma 5. *For any generator choice of elements of A, the paths listed in Lemma 4 form a basis of A.*

Proof: Lemma 3 asserts that we can apply Lemma 4. On the other hand, we know that $\dim A = 34$, since we know the dimension of the indecomposable projective A-modules.

Lemma 6. *The socle of $P(10)$ has length 2.*

Proof. Since $\Delta(43) \oplus \Delta(43)$ is a submodule of $P(10)$, the length of the socle of $P(10)$ is at least 2.

According to Lemma 2, the top of $T(43)$ is equal to 10, thus we see that $T(43)$ is a factor module of $P(10)$, say $T(43) = P(10)/W$ for some submodule W of $P(10)$. The subcategory of modules with a Δ-filtration is closed under kernels of surjective maps [R2], thus W has a Δ-filtration. But W has a composition factor of the form 43, and is of length 5, thus W is isomorphic to $\Delta(43)$ and therefore has simple socle. Quoting again Lemma 2, we know that also $T(43)$ has simple socle, thus the length of the socle of $P(10)$ is at most 2.

Proof of the proposition. Assume that there is given a generator choice for A. Then $\alpha'\alpha$ belongs to J^4, thus to $e_{05}J^4e_{05}$. The basis of A exhibited in Lemma 4 shows that $e_{05}J^4e_{05}$ is generated by $\alpha'\gamma\gamma'\alpha$, thus we see that $\alpha'\alpha$ has to be a multiple of $\alpha'\gamma\gamma'\alpha$. In the same way, we consider also the elements $\alpha'\beta,\ \beta'\alpha,\ \beta'\beta$ and obtain scalars $c_{aa}, c_{ab}, c_{ba}, c_{bb}$ (some could be zero) such that

$$\begin{aligned}
\alpha'\alpha &= c_{aa}\,\alpha'\gamma\gamma'\alpha,\\
\alpha'\beta &= c_{ab}\,\alpha'\gamma\gamma'\beta,\\
\beta'\alpha &= c_{ba}\,\beta'\gamma\gamma'\alpha,\\
\beta'\beta &= c_{bb}\,\beta'\gamma\gamma'\beta.
\end{aligned}$$

We show that we can achieve that three of these coefficients are zero: Let

$$\begin{aligned} \alpha_0' &= \alpha'(1 - c_{aa}\gamma\gamma'), \\ \beta_0' &= \beta'(1 - c_{ba}\gamma\gamma'), \\ \beta_0 &= (1 - (c_{ab} - c_{aa})\gamma\gamma')\beta, \end{aligned}$$

Then

$$\begin{aligned} \alpha_0'\alpha &= \alpha'(1 - c_{aa}\gamma\gamma')\alpha = 0, \\ \beta_0'\alpha &= \beta'(1 - c_{ba}\gamma\gamma')\alpha = 0, \end{aligned}$$

and

$$\begin{aligned} \alpha_0'\beta_0 &= \alpha'(1 - c_{aa}\gamma\gamma')(1 - (c_{ab} - c_{aa})\gamma\gamma')\beta \\ &= \alpha'(1 - c_{aa}\gamma\gamma' - (c_{ab} - c_{aa})\gamma\gamma')\beta \\ &= \alpha'(1 - c_{ab}\gamma\gamma')\beta \quad = \quad 0. \end{aligned}$$

In the last calculation, we have deleted the summand in rad^6, since actually $\gamma'\gamma = 0$.

This shows that replacing α', β, β' by α_0', β_0, β_0', respectively, we can assume that all the parameters c_{aa}, c_{ab}, c_{ba} are equal to zero.

Thus, we can assume that we deal with the relations:

$$\begin{gathered} \alpha'\alpha = 0, \quad \alpha'\beta = 0, \quad \beta'\alpha = 0, \quad \beta'(1 - c_{bb}\gamma\gamma')\beta = 0, \\ \gamma'\gamma = 0, \quad \gamma'(\alpha\alpha' - c_0\beta\beta') = 0, \quad (\alpha\alpha' - c_1\beta\beta')\gamma = 0, \quad \gamma'\alpha\alpha'\gamma = 0. \end{gathered}$$

Let us show that $c_{bb} \neq 0$. Assume, for the contrary that $c_{bb} = 0$. Then the element $\alpha\alpha' - \beta\beta'$ belongs to the socle of $P(10)$. But of course, also the elements $\alpha\alpha'\gamma\gamma'\alpha\alpha'$ and $\gamma\gamma'\alpha\alpha'$ belong to the socle of $P(10)$, thus the socle of $P(10)$ is of length at least 3. But this contradicts Lemma 6.

We have mentioned already, that the isomorphy of $\mathrm{rad}\,\Delta(43)$ and $\nabla(43)/\mathrm{soc}$ implies that $c_0 = c_1$. Thus we deal with a set of relations

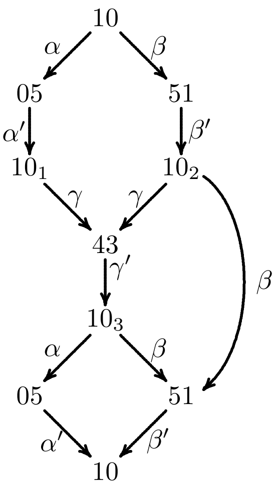
$$\begin{gathered} \alpha'\alpha = 0, \quad \alpha'\beta = 0, \quad \beta'\alpha = 0, \quad \beta'(1 - c'\gamma\gamma')\beta = 0, \\ \gamma'\gamma = 0, \quad \gamma'(\alpha\alpha' - c\beta\beta') = 0, \quad (\alpha\alpha' - c\beta\beta')\gamma = 0, \quad \gamma'\alpha\alpha'\gamma = 0. \end{gathered}$$

with two non-zero scalars c, c'. It remains a last change of the generator choice: Replace say γ by $\frac{1}{c'}\gamma$ and α by $\frac{1}{c}\alpha$. Then we obtain the wanted presentation. This completes the proof of the Proposition.

8. The module $T(43)$.

As we have mentioned, $T(43)$ is a factor module of $P(10)$, namely $T(43) =$

$P(10)/\gamma A$, thus it has the following coefficient quiver:



with all non-zero coefficients being equal to 1.

The picture shows nicely the Δ-filtration of $T(43)$, but, of course, one also wants to see a ∇-filtration. This is the reason why we have labelled the three copies of 10 in the middle (since we exhibit a coefficient quiver, these elements $10_1,\ 10_2,\ 10_3$ are elements of a basis). Consider the subspace

$$V = \langle 10_1, 10_2, 10_3\rangle$$

of $T(43)$ and the elements $x = 10_1 + 10_2 - 10_3$ and $y = 10_1 - 10_3$ of V. One easily sees the following:

The element x lies in the kernel both of β and γ, and it is mapped under α to the composition factor 05 lying in $\mathrm{soc}_2 T(43)$. Thus, it provides an embedding of $\nabla(05)$ into $T(43)/\mathrm{soc}$.

The element y lies in the kernel both of α and γ, and it is mapped under β to the composition factor 51 lying in $\mathrm{soc}_2 T(43)$. Thus, it provides an embedding of $\nabla(51)$ into $T(43)/\mathrm{soc}$.

The sum of the submodules xA and yA is a submodule of $T(43)$ of length 5 with a ∇-filtration with factors going down:

$$\nabla(05) \oplus \nabla(51) \mid \nabla(10).$$

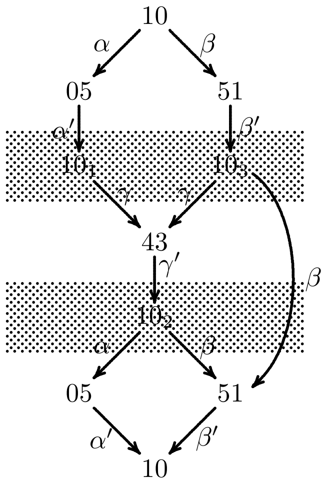
Finally, the factor module $T(43)/(xA + yA)$ is obviously of the form $\nabla(43)$, since its socle is 43 and its length is 5.

Remark. In terms of the basis of A presented above, we also can write:

$$x = \alpha\alpha' + \alpha\alpha'\gamma\gamma' - \beta\beta'$$
$$y = \alpha\alpha' - \beta\beta'.$$

9. A further look at the module $T(43)$.

In order to understand the module $T(43)$ better, let us concentrate on the essential part which looks quite strange, namely the three subfactors $10_1,\ 10_2,\ 10_3$ shaded below:



The three elements $10_1,\ 10_2\ 10_3$ are displayed in two layers, namely in the radical layers they belong to. If we consider the position of composition factors of the form 10 in the socle layers, we get a dual configuration, since the subspace inside V generated by the difference $10_1 - 10_3$ lies in the kernel of γ and therefore belongs to $\mathrm{soc}_3 T(43)$.

Let us look at the the space

$$V = 10_1 \oplus 10_2 \oplus 10_3,$$

in more detail, taking into account all the information stored there, namely the endomorphism $\overline{\gamma} = \gamma\gamma'$ as well as the images of the maps to V and the kernels of the maps starting at V. One may be tempted to look at the subspaces

$$\mathrm{Im}(\alpha'),\ \mathrm{Im}(\beta'),\ \mathrm{Ker}(\alpha),\ \mathrm{Ker}(\beta),$$

however, one has to observe that the maps mentioned here are not intrinsically given, but can be replaced by suitable others (as we have done when we were reducing the number of parameters). For example, instead of looking at α', we have to take into account the whole family of maps $\alpha' + c\alpha'\overline{\gamma}$ with $c \in k$. Thus, the intrinsic subspaces to be considered are

$$\begin{aligned} U_1 &= \mathrm{Im}(\alpha') + \mathrm{Im}(\alpha'\overline{\gamma}) = \mathrm{Im}(\alpha') + \mathrm{Im}(\overline{\gamma}), \\ U_2 &= \mathrm{Im}(\beta') + \mathrm{Im}(\overline{\gamma}), \\ U_3 &= \mathrm{Ker}(\alpha) \cap \mathrm{Ker}\overline{\gamma}, \\ U_4 &= \mathrm{Ker}(\beta) \cap \mathrm{Ker}\overline{\gamma}, \end{aligned}$$

as well as $\mathrm{Ker}(\overline{\gamma})$ and $\mathrm{Im}(\overline{\gamma})$. However, since we see that

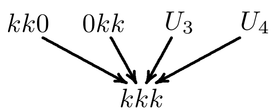
$$\begin{aligned} \mathrm{Ker}(\overline{\gamma}) &= U_3 + U_4, \\ \mathrm{Im}(\overline{\gamma}) &= U_1 \cap U_2, \end{aligned}$$

it is sufficient to consider V with its subspaces $U_1, \dots, U_4$.

This means that we deal with a vector space with four subspaces, thus with a representation of

the 4-subspace quiver with dimension vector

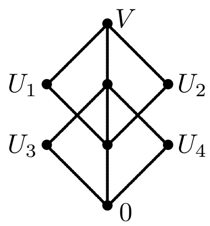
A direct calculation shows that we get the following representation:



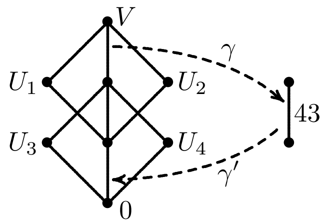
with

$$U_3 = \langle(1,0,-1)\rangle$$
$$U_4 = \langle(1,1,-1)\rangle$$

This is an indecomposable representation of the 4-subspace quiver, it belongs to a tube of rank 2 (and is uniquely determined by its dimension vector). Note that its endomorphism ring is a local ring of dimension 2, with radical being the maps $V/(U_3+U_4) \to U_1 \cap U_2$; and $\overline{\gamma}$ is just such a map. The lattice of subspaces of V generated by the subspaces U_1, U_2, U_3, U_4 looks as follows:



Let us repeat that $\overline{\gamma} = \gamma\gamma'$ maps $V/(U_3+U_4)$ onto $U_1 \cap U_2$, thus we may indicate the operation of γ and γ' as follows:

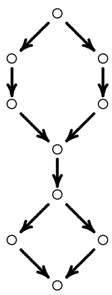


We should stress that the last two pictures show subspace lattices (thus composition factors are drawn as intervals between two bullets), in contrast to the pictures of coefficient quivers, where the composition factors are depicted by their labels (such as 10, 05, 51, ...) and the lines indicate extensions of simple modules.

Note that the core of $T(43)$ is semisimple, namely of the form $10 \oplus 43$, here 10 is just the subfactor $(U_3+U_4)/(U_1 \cap U_2)$.

We hope that the considerations above show well the hidden symmetries of $T(43)$.

Finally, let us remark that the module $T(43)$ has a diagram D in the sense of Alperin (but no strong diagram), namely the following:



(obtained from the coefficient quiver by deleting the by-path β).

10. A related algebra.

We have used the Δ-filtration of $T(43)$ in order to show that $T(43)$ has simple socle, and this implied that the coefficient c_{bb} had to be non-zero. In this way, we have obtained the somewhat strange relations presented in the Proposition. Let us now consider the same quiver $Q(10,05,51,43)$, but with the relations

$$\alpha'\alpha = 0, \quad \alpha'\beta = 0, \quad \beta'\alpha = 0, \quad \beta'\beta = 0,$$
$$\gamma'\gamma = 0, \quad \gamma'(\alpha\alpha' - \beta\beta') = 0, \quad (\alpha\alpha' - \beta\beta')\gamma = 0, \quad \gamma'\alpha\alpha'\gamma = 0.$$

The corresponding algebra A' still is quasi-hereditary, and the Δ-modules and the ∇-modules have the same shape as those for the algebra $A = A(10,05,51,43)$. However, now it turns out that the tilting module for 43 is of length 11, with a Δ-filtration of the form

$$\Delta(10) \oplus \Delta(10) \mid \Delta(05) \oplus \Delta(51) \mid \Delta(43)$$

and a similar ∇-filtration.

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