Cluster-concealed algebras

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Abstract. The cluster-tilted algebras have been introduced by Buan, Marsh and Reiten, they are the endomorphism rings of cluster-tilting objects T in cluster categories; we call such an algebra cluster-concealed in case T is obtained from a preprojective tilting module. For example, all representation-finite cluster-tilted algebras are cluster-concealed. If C is a representation-finite cluster-tilted algebra, then the indecomposable C-modules are shown to be determined by their dimension vectors. For a general cluster-tilted algebra C, we are going to describe the dimension vectors of the indecomposable C-modules in terms of the root system of a quadratic form. The roots may have both positive and negative coordinates and we have to take absolute values.

Let k be an algebraically closed field. For any finite-dimensional k-algebra R, we consider its Grothendieck group $K_0(R)$ of (finitely generated) R-modules modulo exact sequences: it is the free abelian group with basis the set of isomorphism classes of simple R-modules. Using this basis, we identify $K_0(R)$ with $\mathbb{Z}^n$, where n is the number of isomorphism classes of simple R-modules. For any R-module N, we denote by $\mathbf{dim}\, N$ the corresponding element in $K_0(R)$. With respect to our identification $K_0(R) = \mathbb{Z}^n$, the coefficients of $\mathbf{dim}\, N$ are just the Jordan-Hölder multiplicities of N and the set of simple R-modules which occur as composition factors of N will be called the *support* of N and denotes by $\operatorname{supp} N$.

Throughout the paper, A will denote a finite-dimensional hereditary k-algebra. Recall that a k-algebra B is said to be *tilted* provided B is the endomorphism ring of a tilting A-module T, where A is a finite-dimensional hereditary algebra. If B is a tilted algebra, one may consider the corresponding trivial extension algebra $B^c = B \ltimes I$, where I is the B-B-bimodule $I = \operatorname{Ext}^2_B(DB, B)$, with $D = \operatorname{Hom}(-, k)$ the k-duality. The algebras of the form B^c are called the *cluster tilted* algebras; this is not the original definition as given by Buan, Marsh and Reiten [BMR], but it is an equivalent one, due to Zhu [Z] and Assem, Brüstle and Schiffler [ABS]. The definition shows that B is both a subalgebra as well as a factor algebra of B^c, and that the C-modules N with $IN = 0$ are just the B-modules.

Theorem 1. *Let C be a representation-finite cluster-tilted algebra. If N, N' are indecomposable C-modules with $\mathbf{dim}\, N = \mathbf{dim}\, N'$, then N and N' are isomorphic.*

After finishing a first version of this paper, we were informed about a parallel investigation by Geng and Peng [GP] which gives a different proof of this result using mutations.

2000 *Mathematics Subject Classification.* Primary 16D90, 16G60. Secondary: 16G20, 16G70.

The paper by Geng and Peng also outlines the link to cluster algebras and shows in which way Theorem 1 settles a conjecture of Fomin and Zelevinsky concerning cluster variables.

The next result will provide a description of the set of dimension vectors $\mathbf{dim}\, N$ in $K_0(C)$ with N indecomposable. This will be done in a more general setting. Recall that a tilted algebra B is said to be *concealed* provided B is the endomorphism ring of a preprojective tilting A-module. If B is a concealed algebra, then we will say that B^c is a *cluster-concealed* algebra. Of course, representation-finite cluster-tilted algebras are cluster-concealed algebras, but there are also many cluster-concealed algebras which are tame or wild.

A famous theorem of Kac asserts that the dimension vectors of the indecomposable A-modules are just the positive roots of the (Kac-Moody) root system Φ_A in $K_0(A)$ corresponding to the underlying graph of the quiver of A. Note that $q_A(x) \le 1$ for any $x \in \Phi_A$, here q_A is the Euler form on $K_0(A)$ (the definition will be recalled in section 10).

The reason for us to exhibit cluster-tilted algebras as $B^c = B \ltimes I$ is that this allows to identify the Grothendieck groups $K_0(B^c)$ and $K_0(B)$. Let T be a tilting A-module with endomorphism ring B, and let q_B be the Euler form of B on $K_0(B)$. Since we identify the Grothendieck groups $K_0(B) = K_0(B^c)$, we can apply q_B to the dimension vectors of the indecomposable B^c-modules; this is what we will do. On the other hand, consider the tilting functor $G = \operatorname{Hom}_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$. Let $T_1, \dots, T_n$ be indecomposable direct summands of T, one from each isomorphism class. Then $\mathbf{dim}\, T_1, \dots, \mathbf{dim}\, T_n$ is a basis of $K_0(A)$, whereas $\mathbf{dim}\, G(T_1), \dots, \mathbf{dim}\, G(T_n)$ is a basis of $K_0(B)$, and we denote by $g\colon K_0(A) \to K_0(B)$ the linear bijection such that $g(\mathbf{dim}\, T_i) = \mathbf{dim}\, G(T_i)$, for $1 \le i \le n$. We set $\Phi_B = g(\Phi_A)$. If $x \in \Phi_A$, then it is well-known that x or $-x$ belongs to $\mathbb{N}$, but, usually, Φ_B will contain elements for which some coefficients are positive, and some negative. For any element $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$, let $\operatorname{abs} x = (|x_1|, \dots, |x_n|)$. We use this function abs in order to attach to any vector $x \in \Phi_B$ an element in $\mathbb{N}^n$.

Theorem 2. *Let B be a concealed algebra and $C = B^c$ the corresponding cluster-concealed algebra,*

(a) *The dimension vectors of the indecomposable C-modules are precisely the vectors* $\operatorname{abs} x$ *with $x \in \Phi_B$.*

(b) *If Z is an indecomposable C-module, then $q_B(\mathbf{dim}\, N) \le 1$ if and only if Z is a B-module; if N is not a B-module, then $q_B(\mathbf{dim}\, N)$ is an odd integer (greater than 2).*

(c) *If N is an indecomposable C-module which is not a B-module, then* $\operatorname{End}(N) = k$.

The special case when A is of type $\mathbb{A}_n$ has been considered already in the thesis of Parsons [P], using a different approach. As Robert Marsh has pointed out, some further cases have been considered by Parsons and him in this way but this work is still ongoing.

Remarks.

1. The quadratic form q_B. We want to stress that the quadratic form q_B used here in order to deal with B^c-modules depends on the choice of B: it is the Euler form for B, not for B^c (there may not even exist a Euler form for B^c, since usually the global dimension of B^c is infinite). Also, for a given cluster-concealed algebra C, there usually will exist several concealed algebras B with $C = B^c$ and then we will obtain different quadratic forms q_B on $K_0(C)$, see Example 13.2.

2. About the proof. The upshot of our investigation is Proposition 4. A direct proof of this result (as well as a generalization to tilting modules which are neither preprojective nor preinjective) would be of interest. Our proof invokes a second quadratic form r_E which concerns the extension behavior of two torsion pairs. Here, we deal with what Drozd [D] has called E-matrices where E is a bimodule. The preprojectivity of T is used in order to show that the corresponding categories of E-matrices are essentially directed: according to de la Peña and Simson [DP] this then implies that the indecomposable objects correspond bijectively to the positive roots of the corresponding quadratic forms. But actually, these quadratic forms coincide, in this way we obtain the required bijection. We should stress that the equality of the quadratic forms used follows from the fact that the bimodules E which arise can be written as $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G})$ and as $\mathrm{Hom}(\mathcal{G}, \tau\mathcal{F})$, respectively, and, of course, one of the basic results of Auslander-Reiten theory asserts that these bimodules are dual to each other.

A second ingredient of our proof is the following separation property (see section 2): Let T be a preprojective tilting module, and M an indecomposable A-module. Then we show that the B-modules $G(M) = \mathrm{Hom}_A(T, M)$ and $F(M) = \mathrm{Ext}^1(T, M)$ have disjoint supports. This property is the reason for the appearance of absolute values in Theorem 2. In case T has an indecomposable regular direct summand, the separation property no longer holds, see Proposition 7. Thus, one cannot expect that a generalization of the main theorem for arbitrary cluster-tilted algebras will use the vectors $\mathrm{abs}\, x$ with $x \in \Phi_B$.

Invariants such as quadratic forms or root systems have often been used in order to obtain a classification of the indecomposable R-modules, for an algebra R. Usually, one starts to guess all these objects, then one shows that they are pairwise non-isomorphic and that all the indecomposable R-modules have been obtained in this way, and finally, one tries to describe the structure of the module category. In our case of dealing with a cluster-tilted algebra, the procedure is completely reversed: the module category is known from the beginning, but one is lacking sufficient information concerning the modules themselves.

3. The relevance of cluster-concealed algebras. The importance of the concealed algebras should be mentioned here. The tame ones have been classified by Happel and Vossieck [HV], and are used by the Bongartz criterion for determining whether a finite-dimensional algebra is representation-finite or not. The list of all the possible "frames" of tame concealed algebras can be found in several books and papers, the corresponding cluster algebras have been considered by Seven [S], the relationship has been discussed in [BRS]. Wild concealed algebras have been considered by Unger [U].

Acknowledgment. The results presented in the paper have been obtained during a stay at the Newton Institute in Cambridge, February and March 2009. The author is

grateful to the Institute for its hospitality. Also, the author is indebted to Idun Reiten and the referee for a careful reading of a first version.

1. Notation.

If R is a finite-dimensional k-algebra, the modules considered usually will be finite-dimensional left modules, and homomorphisms will be written at the opposite side of the scalars. Let $\operatorname{mod} R$ be the category of R-modules, and $\operatorname{ind} R$ a set of indecomposable R-modules, one from each isomorphism class, or also the corresponding full subcategory of $\operatorname{mod} R$.

We denote by A a finite-dimensional hereditary k-algebra which is connected. Let T be a tilting A-module with endomorphism ring B. As usually, we let

$$\begin{aligned}
\mathcal{F} = \mathcal{F}(T) &= \{M \in \operatorname{ind} A \mid \operatorname{Hom}(T,M) = 0\},\\
\mathcal{G} = \mathcal{G}(T) &= \{M \in \operatorname{ind} A \mid \operatorname{Ext}^1(T,M) = 0\},\\
\mathcal{X} = \mathcal{X}(T) &= \{M \in \operatorname{ind} B \mid T \otimes_B M = 0\},\\
\mathcal{Y} = \mathcal{Y}(T) &= \{M \in \operatorname{ind} B \mid \operatorname{Tor}_1(T,M) = 0\}.
\end{aligned}$$

The pair $(\mathcal{F},\mathcal{G})$ is a torsion pair in $\operatorname{mod} A$. Given an A-module M, we denote its torsion submodule by tM. The pair $(\mathcal{Y},\mathcal{X})$ is a torsion pair in $\operatorname{mod} B$ which is even split.

Tilting theory asserts that the functor $G = \operatorname{Hom}(T,-)$ gives an equivalence $\mathcal{G}(T) \to \mathcal{Y}(T)$ and that the functor $F = \operatorname{Ext}^1(T,-)$ gives an equivalence $\mathcal{F}(T) \to \mathcal{X}(T)$. We should stress that for any A-module M

$$\begin{aligned}
G(M) &= \operatorname{Hom}(T,M) = \operatorname{Hom}(T,tM) = G(tM),\\
F(M) &= \operatorname{Ext}^1(T,M) = \operatorname{Ext}^1(T,M/tM) = F(M/tM).
\end{aligned}$$

In this paper, the main interest will concern

$$\mathcal{M} = \mathcal{M}(T) = \{M \in \operatorname{ind} A \mid \operatorname{Hom}(T,M) \neq 0,\ \operatorname{Ext}^1(T,M) \neq 0\},$$

as well as

$$\mathcal{N} = \mathcal{N}(B) = \{N \in \operatorname{ind} B^c \mid IN \neq 0\},$$

these are the indecomposable B^c-modules which are not B-modules.

2. $\mathcal{M}(T)$ for T preprojective.

Recall the following: Given a B-module N, its *support* is the set of isomorphism classes of the simple B-modules which occur as composition factors of N. The indecomposable projective B-modules are of the form $G(T_i)$, thus the simple B-modules are indexed by the natural numbers $1, 2, \ldots, n$. Tilting theory shows that for any A-module M, the support

$\operatorname{supp} G(M)$ of $G(M)$ is the set of indices i such that $\operatorname{Hom}(T_i, M) \neq 0$, and that the support $\operatorname{supp} F(M)$ of $F(M)$ is the set of indices i such that $\operatorname{Ext}^1(T_i, M) \neq 0$.

Lemma 1. *Let M, M' be indecomposable A-modules. Assume that there is an index i in the intersection of $\operatorname{supp} F(M')$ and $\operatorname{supp} G(M)$ such that T_i is preprojective or preinjective. Then $\operatorname{supp} F(M)$ and $\operatorname{supp} G(M')$ do not intersect.*

Proof: Since $i \in \operatorname{supp} F(M') \cap \operatorname{supp} G(M)$, we have

$$\operatorname{Ext}^1(T_i, M') \neq 0 \quad \text{and} \quad \operatorname{Hom}(T_i, M) \neq 0.$$

Assume that there is j in the intersection of $\operatorname{supp} F(M)$ and $\operatorname{supp} G(M')$, thus

$$\operatorname{Ext}^1(T_j, M) \neq 0 \quad \text{and} \quad \operatorname{Hom}(T_j, M') \neq 0.$$

Note that $\operatorname{Ext}^1(T_i, M') = D\operatorname{Hom}(M', \tau T_i)$ and $\operatorname{Ext}^1(T_j, M) = D\operatorname{Hom}(M, \tau T_j)$. Thus, we obtain a proper cycle

$$T_i \preccurlyeq M \preccurlyeq \tau T_j \prec T_j \preccurlyeq M' \preccurlyeq \tau T_i \prec T_i$$

whereas T_i is preprojective or preinjective, thus directing.

The case $M = M'$ yields the following corollary:

Corollary. *Let M be an indecomposable A-module. Assume that there is an index i in the intersection of $\operatorname{supp} F(M')$ and $\operatorname{supp} G(M)$, then T_i is neither preprojective nor preinjective.*

Thus we have the following

Separation Property. *If T is a preprojective tilting module, and M is indecomposable, then the supports of $F(M)$ and $G(M)$ are disjoint.*

For the remainder of this section, we assume that T is a preprojective tilting module. Let $\mathcal{D}$ be the set of predecessors of the modules τT_i, where T_i is an indecomposable direct summand of T.

Lemma 2. *We have $\mathcal{F} \subseteq \mathcal{D}$.*

Proof: Let X be in $\mathcal{F}$. Since $F\colon \mathcal{F} \to \mathcal{X}$ is an equivalence, we have $\operatorname{Ext}^1(T, X) = FX \neq 0$, thus $0 \neq \operatorname{Ext}^1(T_i, X) = D\operatorname{Hom}(X, \tau T_i)$ for some i. This means that X is a predecessor of τT_i, thus is in $\mathcal{D}$.

Lemma 3. *If M belongs to $\mathcal{M}$, then any indecomposable direct summand of M, tM, M/tM belongs to $\mathcal{D}$.*

Proof: For any module M, the indecomposable direct summands of M/tM belong to $\mathcal{F}$, thus to $\mathcal{D}$, according to Lemma 2. For $M \in \mathcal{M}$, the factor module M/tM is non-zero, thus M has an indecomposable factor module M' which belongs to $\mathcal{F}$, thus also M belongs to $\mathcal{D}$. But then also any indecomposable summand of any submodule of M belongs to $\mathcal{D}$. In particular, any indecomposable summand of tM belongs to $\mathcal{D}$.

Lemma 4. *If $M \in \mathcal{M}$, then* $\operatorname{Hom}(M/tM, tM) = 0$ *and both tM and M/tM have no self-extensions.*

Proof: Since M is indecomposable and preprojective, its endomorphism ring is k. Any non-zero homomorphism $M/tM \to tM$ would yield a non-zero nilpotent endomorphism of M, which is impossible.

Let M' be an indecomposable direct summand of tM. Then we get the following exact sequence

$$\operatorname{Hom}(M', M/tM) \to \operatorname{Ext}^1(M', tM) \to \operatorname{Ext}^1(M', M).$$

Here, the first term is zero, since M' is torsion, and M/tM torsion-free. Also, the last term is zero, since M' is a predecessor of M, thus there cannot be a proper path from M to M'. Thus $\operatorname{Ext}^1(M', tM) = 0$ and therefore $\operatorname{Ext}^1(tM, tM) = 0$.

Similarly, let M' be an indecomposable direct summand of M/tM. There is the following exact sequence

$$\operatorname{Hom}(tM, M') \to \operatorname{Ext}^1(M/tM, M') \to \operatorname{Ext}^1(M, M').$$

Again, the first term is zero, since tM is torsion and M' is torsion-free. The last term is zero, since M' is a successor of M, thus there cannot be a proper path from M' to M. Therefore, $\operatorname{Ext}^1(M/tM, M') = 0$, thus $\operatorname{Ext}^1(M/tM, M/tM) = 0$.

3. The matrix category of a bimodule.

Given two additive categories $\mathcal{A}$ and $\mathcal{B}$, an $\mathcal{A}$-$\mathcal{B}$-bimodule ${}_{\mathcal{A}}E_{\mathcal{B}}$ is by definition a bilinear functor $\mathcal{A}^{\text{op}} \times \mathcal{B} \to \operatorname{mod} k$. Given such an $\mathcal{A}$-$\mathcal{B}$-bimodule $E = {}_{\mathcal{A}}E_{\mathcal{B}}$, let $\operatorname{Mat} E$ be the category of *E-matrices* as introduced by Drozd [D]: its objects are triples (A, B, m), where A is an object of $\mathcal{A}$, B is an object of $\mathcal{B}$ and $m \in E(A, B)$. A morphism $(A, B, m) \to (A', B', m')$ is a pair (α, β), where $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ are morphisms in $\mathcal{A}$, and $\mathcal{B}$ respectively, such that $m\beta = \alpha m'$.

Given a bimodule ${}_{\mathcal{A}}E_{\mathcal{B}}$, one may introduce a quadratic form r_E as follows: it is defined on the direct sum of the Grothendieck groups $K(\mathcal{A}, \oplus)$ and $K(\mathcal{B}, \oplus)$. If X is an object in $\mathcal{A}$, and Y an object in $\mathcal{B}$, then one calls the pair $(X, Y) \in K(\mathcal{A}, \oplus) \oplus K(\mathcal{B}, \oplus)$ a *coordinate vector* and one puts

$$r_E((X, Y)) = \dim \operatorname{End}_{\mathcal{A}}(X) + \dim \operatorname{End}_{\mathcal{B}}(Y) - \dim E(X, Y),$$

and this extends in a unique way to a quadratic form on $K(\mathcal{A}, \oplus) \oplus K(\mathcal{B}, \oplus)$. This quadratic form can be presented by drawing a graph with two kinds of edges, say solid ones and dotted ones.

We recall the following: Assume there is given a free abelian group K with a fixed basis $\mathcal{B}$ and a quadratic form q on K with integer values such that $q(b) = 1$ for all $b \in \mathcal{B}$, then we draw the following graph: its vertices are the elements of $\mathcal{B}$; and for $b \neq b'$ in $\mathcal{B}$, we consider $c_{bb'} = q(b + b')$. If $c_{bb'}$ is negative, then we draw $-c_{bb'}$ solid edges between b and b', otherwise we draw $c_{bb'}$ dotted edges between b and b'. In the case of the quadratic form r_E on $K = K(\mathcal{A}, \oplus) \oplus K(\mathcal{B}, \oplus)$, we take as $\mathcal{B}$ the set of indecomposble objects in $\mathcal{A}$ and $\mathcal{B}$ and observe that the required condition $r_E(b) = 1$ for $b \in \mathcal{B}$ is satisfied. Thus, in our case the vertices are the isomorphism classes of the indecomposable objects in $\mathcal{A}$ and $\mathcal{B}$, there are solid edges between vertices for $\mathcal{A}$ and $\mathcal{B}$, and there are dotted edges between the vertices for $\mathcal{A}$ as well as between the vertices for $\mathcal{B}$; in this way, the graph is *bipartite.* The number of edges is as follows: for indecomposable objects X, X' both in $\mathcal{A}$, or both in $\mathcal{B}$, the number of dotted edges is $\dim \operatorname{rad}(X, X') + \dim \operatorname{rad}(X', X)$, whereas the number of solid edges between an isomorphism class A in $\mathcal{A}$ and an isomorphism class B in $\mathcal{B}$ is $\dim E(A, B)$.

A Krull-Remak-Schmidt category with finitely many isomorphism classes of indecomposable objects is said to be *directed*, provided the endomorphism rings of all the indecomposable objects are division rings and there is a total ordering $\prec$ on the set of isomorphism classes of indecomposable objects such that $\operatorname{Hom}(M, M') = 0$ for $M' \prec M$.

There is the following important result of de la Peña and Simson ([DS], Propositions 1.1, 4.2 and 4.13): *If the category* $\operatorname{Mat} E$ *is directed, then* r_E *is weakly positive, and the use of coordinate vectors provides a bijection between the indecomposable objects in* $\operatorname{Mat} E$ *and the positive roots of* r_E.

When dealing with a bimodule $E = {}_{\mathcal{A}}E_{\mathcal{B}}$, we sometimes will write $E = (\mathcal{A}, E, \mathcal{B})$ in order to stress the categories $\mathcal{A}, \mathcal{B}$. As usually, we may factor our the annihilators, thus we may consider $\overline{E} = (\overline{\mathcal{A}}, \overline{E}, \overline{\mathcal{B}})$, where $\overline{\mathcal{A}}$ is obtained from $\mathcal{A}$ by factoring out the ideal of all maps $\alpha \in \mathcal{A}$ such that $E(\alpha, 1_B) = 0$ for all objects B in $\mathcal{B}$, where similarly $\overline{\mathcal{B}}$ is obtained from $\mathcal{B}$ by factoring out the ideal of all maps $\beta \in \mathcal{B}$ such that $E(1_B, \beta) = 0$ for all objects A in $\mathcal{A}$, and $\overline{E}(\overline{\alpha}, \overline{\beta}) = E(\alpha, \beta)$. We say that $E = (\mathcal{A}, E, \mathcal{B})$ is *essentially directed*, provided that $\overline{E} = (\overline{\mathcal{A}}, \overline{E}, \overline{\mathcal{B}})$ is directed.

4. The category $\operatorname{Mat} \operatorname{Ext}^1(\mathcal{F}, \mathcal{G})$ for T preprojective.

The first bimodule to be considered is $\operatorname{Ext}^1(\mathcal{F}, \mathcal{G})$: here, $\mathcal{A} = \mathcal{F}$, $\mathcal{B} = \mathcal{G}$ and the functor is $E = \operatorname{Ext}^1(-, -)$.

Proposition 1. *If M is an A-module, let $\eta(M) = (M/tM, tM; \epsilon)$, where ϵ is the equivalence class of the canonical exact sequence $0 \to tM \to M \to M/tM \to 0$. This defines a functor η:* $\operatorname{mod} A \to \operatorname{Mat} \operatorname{Ext}^1(\mathcal{F}, \mathcal{G})$ *which is full and dense and its kernel is the ideal generated by all maps $\mathcal{F} \to \mathcal{G}$.*

Proof: well-known and easy (see the appendix).

Corollary. *Assume that T is preprojective. The category* $\operatorname{Mat} \operatorname{Ext}^1(\mathcal{F}, \mathcal{G})$ *is essentially directed.*

Proof: The functor η maps $\mathcal{D}$ onto $\operatorname{Mat} \operatorname{Ext}^1(\mathcal{F}, \mathcal{G} \cap \mathcal{D})$. Now, $\mathcal{D}$ is directed, thus also $\operatorname{Mat} \operatorname{Ext}^1(\mathcal{F}, \mathcal{G} \cap \mathcal{D})$ is directed.

5. The bimodule I and the algebra B_2.

We consider the B-B-bimodule $I = \mathrm{Ext}^2_B(DB, B)$. According to [ABS] (see also [R]), this bimodule can be identified with $\mathrm{Ext}^1_A(T, \tau^{-1}T) = F(\tau^{-1}T)$.

Lemma 5. *For $X \in \mathcal{F}$, the B-modules $\mathrm{Hom}_B(I, F(X))$ and $\mathrm{Hom}_A(T, \tau X)$ are isomorphic.*

Proof:

$$\begin{aligned}\mathrm{Hom}_A(T, \tau X) &\cong \mathrm{Hom}_A(\tau^{-1}T, X)\\ &\cong \mathrm{Hom}_B(F(\tau^{-1}T), F(X)) = \mathrm{Hom}_B(I, F(X)).\end{aligned}$$

First, we have used that τ^{-1} and τ are adjoint, and then the fact that F yields a bijection $\mathrm{Hom}_A(M, X) \to \mathrm{Hom}_B(F(M), F(X))$, for any A-module M, since X belongs to $\mathcal{F}$ (note that this bijection for $M = \tau^{-1}T$ is B-linear, since $\tau^{-1}T$ is an A-B-bimodule).

Lemma 6. *For any B-module N, the module $I \otimes_B N$ belongs to $\mathrm{add}\,\mathcal{X}$, and the module $\mathrm{Hom}_B(I, N)$ to $\mathrm{add}\,\mathcal{Y}$.*

Proof: Note that $I = \mathrm{Ext}^1(T, \tau^{-1}T)$ belongs to $\mathrm{add}\,\mathcal{X}$. Let $p\colon B^t \to N$ be a free cover of N, then $I \otimes p\colon I^t = I \otimes_B B^t \to I \otimes_B N$ is surjective. Thus, with I also $I \otimes_B N$ belongs to $\mathrm{add}\,\mathcal{X}$.

In order to show that $\mathrm{Hom}_B(I, N)$ belongs to $\mathrm{add}\,\mathcal{Y}$, decompose $N = N' \oplus N''$ with $N' \in \mathrm{add}\,\mathcal{X}$ and $N'' \in \mathrm{add}\,\mathcal{Y}$. Then $\mathrm{Hom}_B(I, N) = \mathrm{Hom}_B(I, N')$, thus we can assume that $N \in \mathrm{add}\,\mathcal{X}$. The previous lemma asserts that $\mathrm{Hom}_B(I, N)$ is isomorphic (as a B-module) to $\mathrm{Hom}_A(T, \tau M)$, with $M \in \mathrm{add}\,\mathcal{F}$. But $\mathrm{Hom}_A(T, \tau M) = \mathrm{Hom}_A(T, t(\tau M))$ belongs to $\mathrm{add}\,\mathcal{Y}$.

Remark. Lemma 6 implies the (well-known) fact that $I \otimes_B I = 0$. Namely, consider the adjoint map $\delta\colon I \to \mathrm{Hom}_B(I, I \times I)$ of the identity map. We know that $I \in \mathrm{add}\,\mathcal{X}$, whereas $\mathrm{Hom}_B(I, I \times I)$ belongs to $\mathrm{add}\,\mathcal{Y}$. Thus $\delta = 0$, and therefore $I \otimes I = 0$. — Note that $I \otimes I = 0$ means that the trivial extension $B^c = B \ltimes I$ can be considered also as the tensor algebra of the B-B-bimodule I.

Let us consider the matrix ring $B_2 = \begin{bmatrix} B & I \\ 0 & B \end{bmatrix}$. The B_2-modules can be written in the form

$$(N_0, N_1; \gamma\colon I \otimes_B N_1 \to N_0)$$

where N_0, N_1 are B-modules and γ is a B-homomorphism; in terms of matrices, we write N in the form $\begin{bmatrix} N_0 \\ N_1 \end{bmatrix}$ (and then we can use matrix multiplication, taking into account the map γ).

Consider the subring $B^2 = B \times B = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$ of B_2. We will identify $\mathrm{mod}\,B$ with the B_2-modules of the form $(N, 0; 0)$, thus with those annihilated by the idempotent $e_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

We also consider a second embedding functor $\operatorname{mod} B \to \operatorname{mod} B_2$; it sends the B-module N to $N[1] = (0, N; 0)$; the B_2-modules of the form $N[1]$ are just the B_2-modules annihilated by the idempotent $e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (the reason for writing $N[1]$ will become clear when we deal with B_∞).

Let $\widetilde{\mathcal{N}}$ denote the indecomposable B_2-modules which are not B^2-modules, thus those indecomposable B_2-modules N with $IN \neq 0$ (again, we take just one module from each isomorphism class).

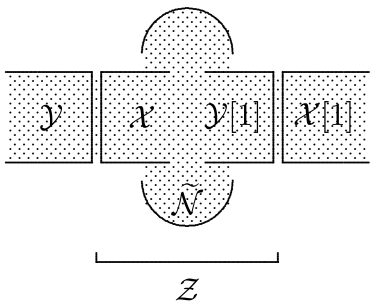
Lemma 7. *For any B_2-module $N = (N_0, N_1; \gamma)$ in $\widetilde{\mathcal{N}}$, we have $N_0 \in \operatorname{add} \mathcal{X}$ and $N_1 \in \operatorname{add} \mathcal{Y}$.*

Note that this means: *There exists an exact sequence* $0 \to X \to N \to Y[1] \to 0$ *with* $X \in \operatorname{add} \mathcal{X}$ *and* $Y \in \operatorname{add} \mathcal{Y}$, namely the sequence $(N_0, 0; 0) \to N \to (0; N_1; 0)$.

Proof: We have shown in Lemma 6 that $I \otimes_B N_1$ belongs to $\operatorname{add} \mathcal{X}$. Decompose $N_0 = X \oplus N_0'$ with X in $\operatorname{add} \mathcal{X}$ and $N_0' \in \operatorname{add} \mathcal{Y}$. Since $\operatorname{Hom}(\mathcal{X}, N_0') = 0$, we see that we can split off $(N_0', 0; 0)$ from N, thus $N_0' = 0$ and therefore $N_0 = X$ belongs to $\operatorname{add} \mathcal{X}$.

Instead of looking at the homomorphism $\gamma\colon I \otimes_B N_1 \to N_0$, we also may consider the adjoint map $\gamma'\colon N_1 \to \operatorname{Hom}_B(I, N_0)$. Since $\operatorname{Hom}_B(I, N_0)$ belongs to $\operatorname{add} \mathcal{Y}$, any direct summand of N_1 which belongs to $\operatorname{add} \mathcal{X}$ has to lie in the kernel of γ', thus can be split off. This shows that N_1 belongs to $\operatorname{add} \mathcal{Y}$.

We may illustrate the structure of $\operatorname{mod} B_2$ in the following way:



Here, $\mathcal{Z}$ denotes the indecomposable B_2-modules N with an exact sequence $0 \to X \to N \to Y[1] \to 0$ with $X \in \operatorname{add} \mathcal{X}$ and $Y \in \operatorname{add} \mathcal{Y}$. Note that $\mathcal{Z}$ consists of $\mathcal{X}, \mathcal{Y}[1]$ and $\widetilde{\mathcal{N}}$.

6. The equivalence of $\operatorname{Mat} \operatorname{Hom}(\mathcal{G}, \tau\mathcal{F})$ and $\mathcal{Z}$.

The second bimodule to be considered is $\operatorname{Hom}(\mathcal{G}, \tau\mathcal{F})$. Here, $\mathcal{A} = \mathcal{G}$, $\mathcal{B} = \mathcal{F}$ and $E(X, Y) = \operatorname{Hom}(X, \tau Y)$.

Proposition 2. *There is an equivalence of categories*

$$\eta\colon \operatorname{Mat} \operatorname{Hom}(\mathcal{G}, \tau\mathcal{F}) \longrightarrow \mathcal{Z}$$

such that $\eta(M_1, M_2, \phi) = (G(M_1), F(M_2), \phi')$ for some ϕ'.

Proof: Let (M_1, M_2, ϕ) be given, with $M_1 \in \mathcal{G}, M_2 \in \mathcal{F}, \phi \in \mathrm{Hom}(M_1, \tau M_2)$. Applying $G = \mathrm{Hom}_A(T, -)$ to ϕ, we obtain

$$\begin{aligned} G(\phi)\colon G(M_1) \to G(\tau M_2) &= \mathrm{Hom}_A(T, \tau M_2) \\ &\cong \mathrm{Hom}_A(\tau^{-1}T, M_2) \\ &\cong \mathrm{Hom}_B(F(\tau^{-1}T), F(M_2)) = \mathrm{Hom}_B(I, F(M_2)). \end{aligned}$$

First, we have used again that τ^{-1} and τ are adjoint, and then the fact that F yields a bijection $\mathrm{Hom}_A(X, M_2) \to \mathrm{Hom}_B(F(X), F(M_2))$, for any A-module X, since M_2 belongs to $\mathcal{F}$ (note that this bijection is B-linear, since $\tau^{-1}T$ is an A-B^{op}-bimodule). The required map ϕ' is the adjoint of the map $G(M_1) \to \mathrm{Hom}_B(I, F(M_2))$.

For the converse, we only have to observe that G yields a bijection $\mathrm{Hom}_A(M_1, X) \to \mathrm{Hom}_B(G(M_1), G(X))$ for any A-module X, since M_1 belongs to $\mathcal{G}$.

7. The algebras B^c and B_∞.

We also consider the $(\mathbb{Z} \times \mathbb{Z})$-matrix ring

$$B_\infty = \begin{bmatrix} \ddots & \ddots & & & \\ & B & I & & \\ & & B & I & \\ & & & B & \ddots \\ & & & & \ddots \end{bmatrix}$$

of $(\mathbb{Z} \times \mathbb{Z})$-matrices with only finitely many non-zero entries: on the main diagonal, there are copies of B, above the main diagonal, there are copies of I (since $I \otimes I = 0$, we do not have to worry about multiplying elements from different copies of I). Note that here we deal with a ring without identity element, but at least it has sufficiently many idempotents. The B_∞-modules are of the form $(N_i, \gamma_i)_i$, indexed over $\mathbb{Z}$, with B-modules N_i and B-linear maps $\gamma_i\colon I \otimes_B N_i \to N_{i-1}$. Note that B_∞ is locally bounded (this means that any simple B_∞-modules has a projective cover and an injective envelope, both being of finite length).

We will consider $\mathrm{mod}\, B_2$ as the full subcategory of $\mathrm{mod}\, B_\infty$ with objects (N_i, γ_i) where $N_i = 0$ for $i \notin \{0, 1\}$. In this way, we consider $\mathcal{Z}$ as a fixed subcategory of $\mathrm{mod}\, B_\infty$. Also, we define for any $t \in \mathbb{Z}$ a shift functor $[t]\colon \mathrm{mod}\, B_\infty \to \mathrm{mod}\, B_\infty$ by $N[t] = (N_{i-t}, \gamma_{i-t})_i$, for $N = (N_i, \gamma_i)_i$.

Proposition 3. *The indecomposable B_∞-modules of finite length are of the form $N[t]$ with $N \in \mathcal{Z}$ and $t \in \mathbb{Z}$, and N, t are uniquely determined.*

Proof: Let $N = (N_i, \gamma_i)_i$ be a B_∞-module of finite length. Decompose $N_i = X_i \oplus Y_i$ with $X_i \in \mathcal{X}$ and $Y_i \in \mathcal{Y}$, for all $i \in \mathbb{Z}$. The discussion of the B_2-modules shows that

γ_i vanishes on $I \otimes X_i$ and maps into Y_{i-1}, thus we see that there is the following direct decomposition of B_∞-modules $M^{(t)} = (M_i^{(t)}, \gamma_i^{(t)})_i$ where

$$M_i^{(t)} = \begin{cases} X_{t-1} & \text{for} \quad i = t-1 \\ Y_t & \quad i = t \\ 0 & \quad \text{otherwise} \end{cases}$$

and where $\gamma_t^{(t)}$ is given by γ_t. Obviously, $M^{(t)}$ belongs to $\mathcal{Z}[t]$. As a consequence, if N is an indecomposable B_∞-module of finite length, then it belongs to $\mathcal{Z}[t]$ for some $t \in \mathbb{Z}$.

Recall that a locally bounded ring R is said to be *locally support-finite* provided for any simple R-module S there exists a finite set of simple R-modules $\mathcal{S}(S)$ with the following property: if M is an indecomposable R-module of finite length which has S as a composition factor, then any composition factor of M belongs to $\mathcal{S}(S)$.

Corollary. *The algebra B_∞ is locally support-finite.*

Proof. Let S be a simple B_∞-module, say $S = (S_i, 0)_i$ with $S_i = 0$ for $i \neq t$, and assume that $N = (N_i, \gamma_i)_i$ is an indecomposable B_∞-module of finite length with composition factor S. Then N belongs to $\mathcal{Z}$ or to $\mathcal{Z}[-1]$. This shows that there are only finitely many simple B^c-modules which can occur as composition factors of N.

Our interest lies in the cluster-tilted algebra $B^c = B \ltimes I$. Obviously, the algebra B_∞ is a Galois covering of B^c with Galois group $\mathbb{Z}$ given by the shift functors $[t]$ with $t \in \mathbb{Z}$. The covering functor $\pi\colon \operatorname{mod} B_\infty \to \operatorname{mod} B^c$ sends $(N_i, \gamma_i)_i$ to $(\bigoplus_i N_i, \gamma)$ with γ being given by the γ_i.

According to Dowbor-Lenzing-Skowronski [DLS], the Proposition 3 and its Corollary have the following consequences:

Corollary. *The covering functor π is dense and induces a bijection between $\mathcal{Z}$ and $\operatorname{mod} B^c$.*

Of course, this bijection yields a bijection from $\widetilde{\mathcal{N}}$ onto $\mathcal{N}(B)$.

Besides the covering functor $\pi\colon \operatorname{mod} B_\infty \to \operatorname{mod} B^c$ itself, we also may look at its restriction to $\operatorname{mod} B_2$. Note that the subring of B_2 of all matrices of the form $\begin{bmatrix} b & x \\ 0 & b \end{bmatrix}$ with $b \in B$ and $x \in I$ is just B^c, and this inclusion gives rise to the restriction of the covering functor

$$\operatorname{mod} B_2 \ \subset \ \operatorname{mod} B_\infty \xrightarrow{\pi} \operatorname{mod} B^c.$$

The original definition of a cluster-tilted algebra C as introduced by Buan-Marsh-Reiten [BMR] implies that the module category $\operatorname{mod} C$ is a factor category of a cluster category $\mathcal{C}_A$. Namely, one starts with the derived category $D^b(\operatorname{mod} A)$, say with shift functor Σ and Auslander-Reiten translation τ, and considers the orbit category $\mathcal{C}_A =$

$D^b(\operatorname{mod} A)/\sigma$ with $\sigma = \Sigma\tau^{-1}$. Then, one takes the factor category $\mathcal{C}_A/\langle\tau T\rangle$ (here, $\langle\tau T\rangle$ is the ideal of all maps which factor through $\operatorname{add}\tau T$). It turns out that T, considered as an object of $\mathcal{C}_A/\langle\tau T\rangle$ is a progenerator, and its endomorphism ring is B^c, thus one can identify

$$\operatorname{mod} B^c = \mathcal{C}_A/\langle\tau T\rangle.$$

We may change the procedure slightly: Starting with the derived category $D^b(\operatorname{mod} A)$, we now first want to factor out the ideal $\langle\sigma^z(\tau T) \mid z \in \mathbb{Z}\rangle$ and only in the second step form the orbit category with respect to the action of σ. We can make the identification

$$\operatorname{mod} B_\infty = D^b(\operatorname{mod} A)/\langle\sigma^z(\tau T) \mid z \in \mathbb{Z}\rangle$$

so that the shift functor $M \mapsto M[1]$ on the left coincides with the operation of σ on the right. In particular, the covering functor

$$\pi\colon \operatorname{mod} B_\infty \to \operatorname{mod} B_\infty/[1] = \operatorname{mod} B^c$$

is nothing else than forming the orbit category $\left(D^b(\operatorname{mod} A)/\langle\sigma^z(\tau T) \mid z \in \mathbb{Z}\rangle\right)/\sigma$.

Let us remark that the importance of B_2 and B_∞ for dealing with a cluster-tilted algebra B^c has been stressed already in [R].

8. The category $\operatorname{Mat}\operatorname{Hom}(\mathcal{G}, \tau\mathcal{F})$ for T preprojective.

In case T is preprojective, we can improve the assertion of Lemma 7.

Lemma 8. *Assume that T is preprojective. For any B_2-module $N = (N_0, N_1, \gamma)$ in $\widetilde{\mathcal{N}}$ we have $N_0 \in \operatorname{add}\mathcal{X}$ and $N_1 \in \operatorname{add} G(\mathcal{G} \cap \mathcal{D})$.*

Proof: According to Lemma 7, we know that $N_0 \in \operatorname{add}\mathcal{X}$ and $N_1 \in \operatorname{add}\mathcal{Y}$, thus $N_0 = F(M_0)$ for some $M_0 \in \operatorname{add}\mathcal{F}$ and $N_1 = G(M_1)$ for some $M_1 \in \operatorname{add}\mathcal{G}$. Instead of looking at $\gamma\colon I \otimes N_1 \to N_0$ we look again at the adjoint map

$$\gamma'\colon N_1 = G(M_1) \longrightarrow \operatorname{Hom}_B(I, N_0) = \operatorname{Hom}_B(I, F(M_0))$$

Using Lemma 5, we see that

$$\operatorname{Hom}_B(I, F(M_0)) = \operatorname{Hom}_A(T, \tau M_0) = \operatorname{Hom}_A(T, t\tau M_0) = G(t\tau M_0).$$

Since G is an equivalence between $\mathcal{G}$ and $\mathcal{Y}$, there is a homomorphism $f\colon M_1 \to t\tau M_0$ such that $\gamma' = G(f)$.

Now we use that N belongs to $\widetilde{\mathcal{N}}$. If M_1' is an indecomposable direct summand of M_1, then there must exist an indecomposable direct summand M_0' of M_0 such that $\operatorname{Hom}(M_1', t\tau M_0') \neq 0$. Note that M_0' belongs to $\mathcal{F}$, thus to $\mathcal{D}$, according to Lemma 2. But then also M_1' belongs to $\mathcal{D}$, since by definition, $\mathcal{D}$ is closed under predecessors. This shows that M_1' belongs to $\mathcal{G} \cap \mathcal{D}$, thus N_1 belongs to $\operatorname{add} G(\mathcal{G} \cap \mathcal{D})$.

Corollary. *Let T be preprojective. Then* $\operatorname{Mat}\operatorname{Hom}(\mathcal{G}\cap\mathcal{D},\tau\mathcal{F})$ *is directed.*

Proof: Under the equivalence η, the matrix category $\operatorname{Mat}\operatorname{Hom}(\mathcal{G}\cap\mathcal{D},\tau\mathcal{F})$ is mapped to a subcategory $\mathcal{Z}'$ of $\mathcal{Z}$. We claim that $\mathcal{Z}'$ is directed. This is clear in case A is representation finite, since in this case we deal with a subcategory of a factor category of $D^b(\operatorname{mod} A)$, and $D^b(\operatorname{mod} A)$ is directed, thus also $\mathcal{Z}'$ is directed. Thus, we can assume that A is representation infinite (and connected). Let us recall the structure of the categories $\operatorname{mod} A$ and $D^b(\operatorname{mod} A)$. The category $\operatorname{mod} A$ decomposes into three parts: the preprojectives $\mathcal{P}$, the regular modules $\mathcal{R}$ and the preinjectives $\mathcal{Q}$. Looking at $D^b(\operatorname{mod} A)$, the subcategories $\Sigma^z(\mathcal{Q})$ and $\Sigma^{z+1}(\mathcal{P})$ combine to form a transjective component, and any finite subcategory of such a component is directed. But $\mathcal{Z}'$ is a factor category of a finite subcategory of the transjective component with the objects $\mathcal{Q}$ and $\Sigma(\mathcal{P})$, thus $\mathcal{Z}'$ is directed.

9. The bijection between $\mathcal{M}(T)$ and $\mathcal{N}(B)$.

Proposition 4. *Let T be preprojective. There is a bijection $\iota\colon \operatorname{ind} A \to \operatorname{ind} B^c$, such that for $M \in \operatorname{ind} A$, the restriction of $\iota(M)$ to B is $G(M)\oplus F(M)$.*

Remark. Note that for any A-module M, we have

$$G(M) = G(tM) \quad \text{and} \quad F(M) = F(M/tM).$$

Thus we could write $\iota(M) = G(tM)\oplus F(M/tM)$. This would stress that we deal with the middle terms of the exact sequences

$$0 \to tM \to M \to M/tM \to 0,$$
$$0 \to F(M/tM) \to \iota(M) \to G(tM) \to 0.$$

Proof: For M in $\mathcal{F}$, let $\iota(M) = F(M)$; for M in $\mathcal{G}$, let $\iota(M) = G(M)$; thus, it remains to consider M in $\mathcal{M}(T)$.

We consider the categories $\mathcal{A} = \mathcal{F}$, $\mathcal{B} = \mathcal{G}\cap\mathcal{D}$ and the bimodule $E = \operatorname{Ext}^1(\mathcal{F},\mathcal{G}\cap\mathcal{D})$. The quadratic form r_E is defined on

$$K = K(\mathcal{F},\oplus)\oplus K(\mathcal{G}\cap\mathcal{D},\oplus)$$

We also may consider the bimodule $E' = \operatorname{Hom}(\mathcal{G}\cap\mathcal{D},\tau\mathcal{F})$ with quadratic form $r_{E'}$ on K. According to Auslander-Reiten, the bimodules E and E' are dual to each other, thus the quadratic forms r_E and $r_{E'}$ coincide.

The indecomposable objects both in $\mathcal{M}(T)$ and in $\mathcal{N}(B)$ correspond bijectively to the positive non-simple roots of the quadratic form $r_E = r_{E'}$, according to the theorem of de la Peña and Simson quoted in section 3. This completes the proof.

Remark. As we see, a key ingredient of the proof is the duality of the bimodules $\operatorname{Ext}^1(\mathcal{F},\mathcal{G})$ and as $\operatorname{Hom}(\mathcal{G},\tau\mathcal{F})$, which is one of the basic results of Auslander-Reiten theory, since we have to deal with the matrices over these bimodules. (Note that we could write $\overline{\mathrm{Hom}}$ instead of Hom, since the only maps from injective modules to $\tau\mathcal{F}$ are the zero maps.)

Corollary. *Let T be preprojective. Let $Z \in \mathcal{N}(B)$. Then* $\mathrm{End}(Z) = k$.

Proof. We can write $Z = \pi(N)$ for some indecomposable B_2-module $N = (N_1, N_2, \gamma)$. Now, $\mathrm{End}_B(N) = k$. Also, $\mathrm{Hom}(N_1, N_2) = 0$ according to the separation property. Thus $\mathrm{End}(Z) = k$.

Observe that this is the assertion (c) of Theorem 2.

10. The quadratic form q_B.

Given a finite-dimensional algebra R of finite global dimension, we denote by $\langle -, - \rangle$ the bilinear form on $K_0(R)$ with

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, M' \rangle = \sum_{t \geq 0} (-1)^t \dim_k \mathrm{Ext}^t_R(M, M')$$

for all R-modules M, M', and we write $q_B(x) = \langle x, x \rangle$ for $x \in K_0(R)$; in this way, we obtain a quadratic form which is called the *Euler form.*

Let us return to the hereditary algebra A with tilting module T and $B = \mathrm{End}(T)$. Recall that we have denoted by $T_1, \ldots, T_n$ indecomposable direct summands of T, one from each isomorphism class and $g\colon K_0(A) \to K_0(B)$ was defined to be the linear bijection such that $g(\mathbf{dim}\, T_i) = \mathbf{dim}\, G(T_i)$, for $1 \leq i \leq n$.

Addendum to Proposition 4. *We have* $\mathbf{dim}\, \iota(M) = \mathrm{abs}\, g(\mathbf{dim}\, M)$.

Proof. Since tM belongs to $\mathcal{G}$, we have $\mathbf{dim}\, G(M) = \mathbf{dim}\, G(tM) = g(\mathbf{dim}\, tM)$. Since M/tM belongs to $\mathcal{F}$, we have $\mathbf{dim}\, F(M) = \mathbf{dim}\, F(M/tM) = -g(\mathbf{dim}\, M/tM)$. Altogether, it follows from $\mathbf{dim}\, M = \mathbf{dim}\, tM + \mathbf{dim}\, M/tM$ that

$$g(\mathbf{dim}\, M) = g(\mathbf{dim}\, tM) + g(\mathbf{dim}\, M/tM) = g(\mathbf{dim}\, G(M)) - g(\mathbf{dim}\, F(M)).$$

The separation property now implies that

$$\mathrm{abs}\, g(\mathbf{dim}\, M) = g(\mathbf{dim}\, G(M)) + g(\mathbf{dim}\, F(M)).$$

Lemma 9. Let $M \in \mathcal{M}(T)$. Then

$$q_A(\mathbf{dim}\, tM) = \dim \mathrm{End}(tM), \quad q_A(\mathbf{dim}\, M/tM) = \dim \mathrm{End}(M/tM).$$

Proof: Since A is hereditary, $q_A(\mathbf{dim}\, X) = \dim \mathrm{End}(X) - \dim \mathrm{Ext}^1(X, X)$ for any A-module X. According to Lemma 4, both tM and M/tM are modules without self-extensions.

We have denoted by $T_1, \dots, T_n$ indecomposable direct summands of T, one from each isomorphism class. If we define

$$g\colon K_0(A) \to K_0(B) \quad \text{by} \quad g(x) = (\langle t_i, x\rangle)_i,$$

then $g(\mathbf{dim}\, T_i) = \mathbf{dim}\, G(T_i)$, for $1 \le i \le n$, thus g is the linear bijection between $K_0(A)$ and $K_0(B)$ mentioned in the introduction.

Proposition 5. *Let $M \in \mathcal{M}(T)$.*

$$q_B(\operatorname{abs} g(\mathbf{dim}\, M)) = 2(\dim \operatorname{End}(tM) + \dim \operatorname{End}(M/tM)) - 1.$$

Proof: Let $x = \dim M$. Since M is preprojective, $q_A(x) = 1$. Write $y = \mathbf{dim}\, tM$, and $z = \mathbf{dim}\, M/tM$. Since $x = y + z$ and g is linear, $g(x) = g(y) + g(z)$. Now $g(y)$ is positive and $g(z)$ is negative. Since the support of $F(M)$ and $G(M)$ are disjoint (the separation property), we see that $\operatorname{abs} g(x) = g(y) - g(z)$. Thus

$$\begin{aligned} q_B(\operatorname{abs} g(x)) &= q_B(g(y) - g(z)) \\ &= q_B(g(y)) + q_B(g(z)) - 2(g(y), g(z))_B \\ &= q_A(y) + q_A(z) - 2(y, z)_A. \end{aligned}$$

On the other hand,

$$1 = q_A(x) = q_A(y + z) = q_A(y) + q_A(z) + 2(y, z)_A.$$

If we add the two equalities, we obtain

$$q_B(\operatorname{abs} g(x)) + 1 = 2(q_A(y) + q_A(z)).$$

Now, we apply Lemma 9 in order to see that

$$q_B(\operatorname{abs} g(x)) + 1 = 2(\dim \operatorname{End}(tM) + \dim \operatorname{End}(M/tM)).$$

This completes the proof.

Corollary. *If $M \in \mathcal{M}(T)$, then $q_B(\operatorname{abs} g(\mathbf{dim}\, M))$ is an odd integer greater than 2.*

This shows the assertion (b) of Theorem 2.

Proof: For $M \in \mathcal{M}(T)$, both modules tM and M/tM are non-zero, and therefore $\dim \operatorname{End}(tM) \ge 1$ and $\dim \operatorname{End}(M/tM) \ge 1$.

Corollary. *If $M \in \mathcal{M}(T)$, then $q_B(\operatorname{abs} g(\mathbf{dim}\, M)) = 3$ if and only if tM and M/tM are both indecomposable.*

We will discuss this situation in the next section.

11. The mixed pairs.

In order to determine the quadratic form r_E, one needs to know the pairs (X, Y) of indecomposable A-modules with $X \in \mathcal{F}$ and $Y \in \mathcal{G}$ such that $\operatorname{Ext}^1(X, Y) \neq 0$.

We call (X, Y) a *mixed pair* provided X is an indecomposable module in $\mathcal{F}(T)$, whereas Y is an indecomposable module in $\mathcal{G}(T)$, and finally $\operatorname{Ext}^1(X, Y) \neq 0$.

Proposition 6. *Let T be preprojective. Then any mixed pair (X, Y) is an orthogonal exceptional pair consisting of preprojective A-modules such that $\operatorname{Ext}^1(X, Y)$ is one-dimensional. Let M be the middle term of a non-split exact sequence $0 \to Y \to M \to X \to 0$, then M is indecomposable and preprojective, and $\operatorname{Ext}^1(X, M) = 0$ and $\operatorname{Ext}^1(M, Y) = 0$.*

Proof: Let $0 \to Y \to M \to X \to 0$ be a non-split exact sequence. Since $\operatorname{Hom}(Y, X) = 0$, it follows that M is indecomposable. Since M is a predecessor of X, we see that M is preprojective. Since Y is a proper predecessor of X, it follows that $\operatorname{Hom}(X, Y) = 0$ and $\operatorname{Ext}^1(Y, X) = 0$, thus (X, Y) is an orthogonal and exceptional pair. The full subcategory $\mathcal{C}$ of modules with a filtration with factors X and Y consists of X, Y and some modules in $\mathcal{M}(T)$, thus it is of finite type, therefore $\operatorname{Ext}^1(X, Y)$ is at most one-dimensional, thus one-dimensional.

It is well-known (and easy to see) that $\mathcal{C}$ is equivalent to the category of representations of the quiver of Dynkin type $\mathbb{A}_2$, and M, considered as an object of $\mathcal{C}$ is both projective and injective. In particular, we have $\operatorname{Ext}^1(X, M) = 0$ and $\operatorname{Ext}^1(M, Y) = 0$.

Remark. There is the following converse: *If (X, Y) is an orthogonal exceptional pair consisting of preprojective A-modules, then the dimension of $\operatorname{Ext}^1(X, Y)$ is at most 1. If $\dim_k \operatorname{Ext}^1(X, Y) = 1$, then there is a preprojective tilting module T with $X \in \mathcal{F}(T)$ and $Y \in \mathcal{G}(T)$* (and therefore (X, Y) is a mixed pair for T).

Proof. First, if $\dim_k \operatorname{Ext}^1(X, Y) \geq 2$, then there are infinitely many indecomposable A-modules M which have a submodule M' which is a direct sum of copies of Y such that M/M' is a direct sum of copies of X, and all these modules are predecessors of X, this is impossible. This shows that $\dim_k \operatorname{Ext}^1(X, Y) \leq 1$.

Second, assume that $\dim_k \operatorname{Ext}^1(X, Y) = 1$. We claim that $\tau^{-1}X \oplus Y$ is a partial tilting module. Since Y is a predecessor of X, it is also a predecessor of $\tau^{-1}X$, therefore $\operatorname{Ext}^1(Y, \tau^{-1}X) = 0$. On the other hand, also $\operatorname{Ext}^1(\tau^{-1}X, Y) = D\operatorname{Hom}(Y, X) = 0$. The Bongartz completion T of this partial tilting module is a preprojective tilting module and $X \in \mathcal{F}(T)$ and $Y \in \mathcal{G}(T)$.

12. Proof of Theorem 1.

A quadratic form q defined on $\mathbb{Z}^n$ with values in $\mathbb{Z}$ is said to be an *integral* form.

Proposition 7. *Let q be an integral quadratic form on $\mathbb{Z}^n$ which is positive definite. If $x, x' \in \mathbb{Z}^n$ satisfy $q(x) = 1 = q(x')$ and $\operatorname{abs} x = \operatorname{abs} x'$, then $x = \pm x'$.*

Proof. Let $y \in \mathbb{Z}^n$ be defined by $y_i = x_i$ provided $x_i = x'_i$ and $y_i = 0$ otherwise. Let $z = x - y$, thus $x = y + z$ and $x' = y - z$. Let $(-, -)$ be the bilinear form (with values in

$\frac{1}{2}\mathbb{Z}$) corresponding to q. Then

$$1 = q(x) = q(y+z) = q(y) + q(z) + 2(y,z),$$
$$1 = q(x') = q(y-z) = q(y) + q(z) - 2(y,z).$$

shows that $(y,z) = 0$, thus

$$1 = q(x) = q(y) + q(z).$$

Since we assume that q takes values in $\mathbb{Z}$ and since q is positive definite, it follows that $y = 0$ or $z = 0$. If $z = 0$, then $x = x'$. If $y = 0$, then $x = -x'$

Proof of Theorem 1. Let C be representation-finite and cluster-tilted, say $C = B^c$ with $B = \mathrm{End}_A(T)$, where T is a tilting A-module and A is hereditary and representation-finite. Let N, N' be indecomposable C-modules with $\mathbf{dim}\, N = \mathbf{dim}\, N'$. Proposition 4 provides indecomposable A-modules M, M' such that the restriction of $\iota(M)$ to B is N, and the restriction of $\iota(M')$ to B is N'. Let $x = \mathbf{dim}\, M, x' = \mathbf{dim}\, M'$. Then $\mathbf{dim}\, N = \mathrm{abs}\, g(x)$, and $\mathbf{dim}\, N' = \mathrm{abs}\, g(x')$ according to the addendum to Proposition 4. Now, $q_B(g(x)) = q_A(x) = 1$, and also $q_B(g(x')) = q_A(x') = 1$. With q_A also q_B is positive definite. Thus we can apply Proposition 7 in order to see that $g(x) = \pm g(x')$ and therefore $x = \pm x'$. However, both x, x' are non-negative vectors, thus $x = x'$ and therefore M, M' are isomorphic (since any real root module is determined by its dimension vector). Since ι is a bijection of isomorphism classes, it follows that N, N' are isomorphic.

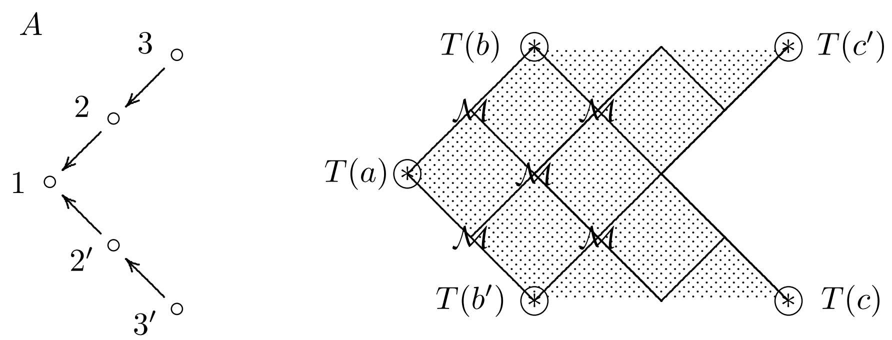
13. Examples.

13.1. Let us exhibit one example in detail. In particular, we will see that the categories $\mathcal{M}(T)$ and $\mathcal{N}(B)$ can be quite different!

Consider the algebra $A = \mathbb{T}_{33}$; this is the path algebra of the quiver Q of type $\mathbb{A}_5$, with a unique sink and indecomposable projective modules of length at most 3. We label the vertices as exhibited on the left. To the right, we present the Auslander-Reiten quiver and mark a tilting module using $*$: it consists of the indecomposable projective modules of length 1 and 3 and the simple injective modules:

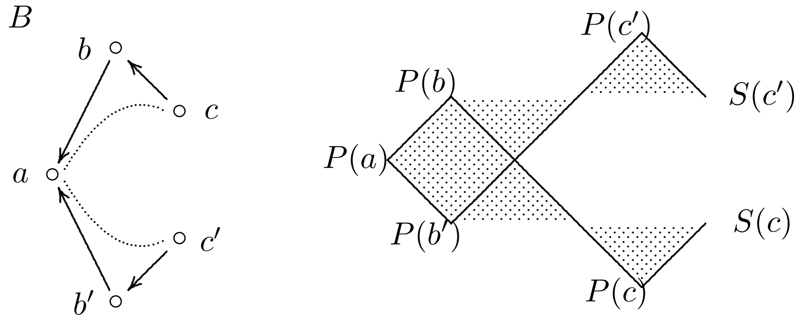
$$T(a) = P(1), \quad T(b) = P(3), \quad T(b') = P(3'), \quad T(c) = I(3), \quad T(c') = I(3').$$

The class $\mathcal{G}$ of indecomposable torsion modules consists of the modules $T(a), T(b), T(b')$ and the five indecomposable injective modules, the class $\mathcal{F}$ of indecomposable torsionfree modules consists of the two modules $\tau I(3)$ and $\tau I(3')$.



The positions of the five mixed modules are denoted by $\mathcal{M}$, in a later table we will provide more information on these modules.

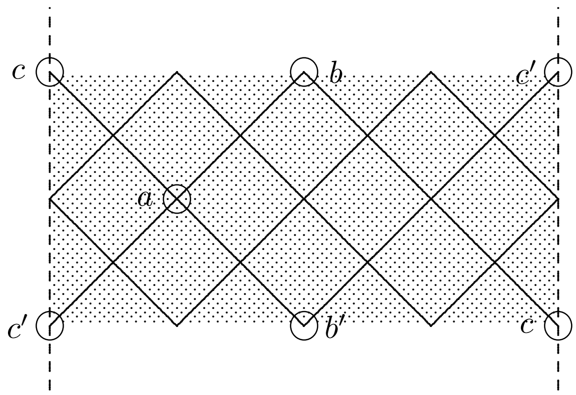
Here is the quiver with relations for the algebra $B = \mathrm{End}(_A T)$, as well as the Auslander-Reiten quiver of $\mathrm{mod}\, B$.



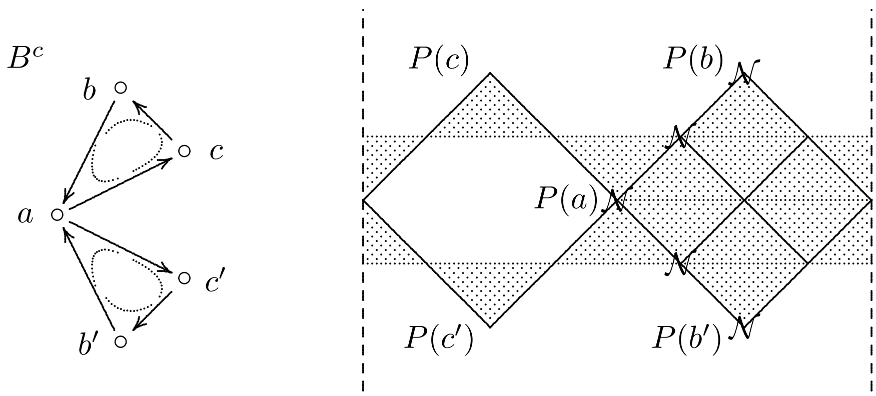
The A-modules in $\mathcal{G}$ and $\mathcal{F}$, and the corresponding B-modules obtained by applying the functors G and F, respectively, are as follows:

$P(1)$	$P(3)$	$P(3')$	$I(1)$	$I(2)$	$I(2')$	$I(3)$	$I(3')$	$\tau I(3)$	$\tau I(3')$
$P(a)$	$P(b)$	$P(b')$	$I(a)$	$S(b)$	$S(b')$	$P(c)$	$P(c')$	$S(c)$	$S(c')$

Next, we present the shape of the cluster category with circles showing the direct summands of T, or better, just their labels (these modules are now considered as objects of $\mathcal{C}_A$), always, the dashed lines have to be identified in order to form Moebius strips:

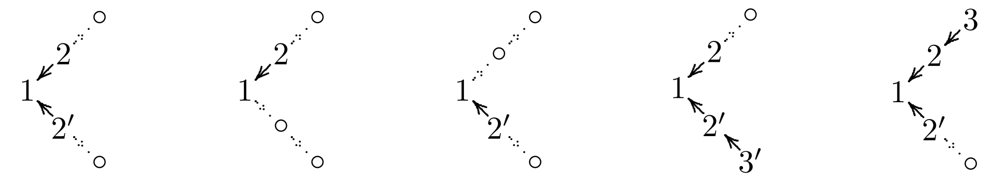


Now, we present first the quiver with relations for the algebra B^c, as well as the Auslander-Reiten quiver of $\mathrm{mod}\, B^c$. The positions of the five mixed modules are denoted by $\mathcal{N}$.



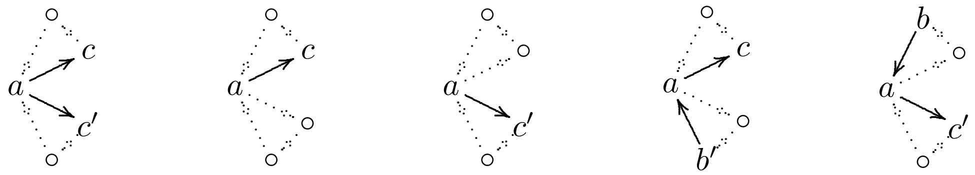
The following table shows the bijection ι between the modules M in $\mathcal{M}(T)$ and the modules $N = \iota(M)$ in $\mathcal{N}(B)$. Below any M we outline its torsion part tM and its torsionfree part M/tM by writing $\frac{M/tM}{tM}$; similarly, under N we show $\frac{N/tN}{tN}$. In the lowest row, one finds the values $q_B(\mathbf{dim}\, N)$.

M



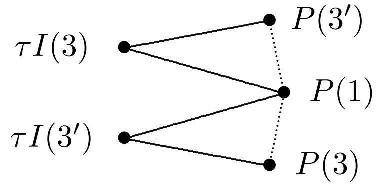
$\frac{\tau I(3)\oplus\tau I(3')}{P(1)}$	$\frac{\tau I(3)}{P(1)}$	$\frac{\tau I(3')}{P(1)}$	$\frac{\tau I(3)}{P(3')}$	$\frac{\tau I(3')}{P(3)}$

N



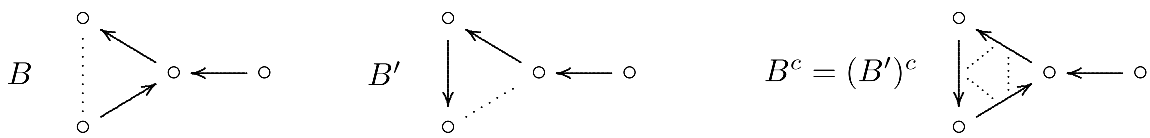
	$\frac{GP(1)}{F\tau I(3)\oplus F\tau I(3')}$	$\frac{GP(1)}{F\tau I(3)}$	$\frac{GP(1)}{F\tau I(3')}$	$\frac{GP(3')}{F\tau I(3)}$	$\frac{GP(3)}{F\tau I(3')}$
$q_B(\mathbf{dim}\, N)$	5	3	3	3	3

Finally, we note that the quadratic form r_E has the following graph:



As usual, we have deleted the isolated vertices (here, a vertex is said to be *isolated* provided it is not the endpoint of any solid edge).

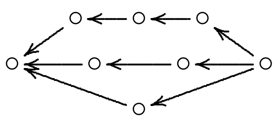
13.2. Next, let us present two non-isomorphic tilted algebras B, B' such that the cluster-tilted algebras B' and $(B')^c$ are isomorphic and representation-finite.



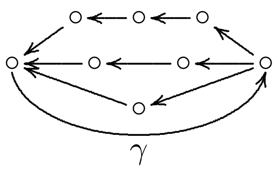
There are 10 isomorphism classes of indecomposable B^c-modules N. The following table lists the values of $q_B(\mathbf{dim}\,N)$ and $q_{B'}(\mathbf{dim}\,N)$ for these modules.

$\mathbf{dim}\,N$	$\begin{smallmatrix}1\\0\end{smallmatrix}00$	$\begin{smallmatrix}0\\1\end{smallmatrix}00$	$\begin{smallmatrix}0\\0\end{smallmatrix}10$	$\begin{smallmatrix}0\\0\end{smallmatrix}01$	$\begin{smallmatrix}1\\1\end{smallmatrix}00$	$\begin{smallmatrix}1\\0\end{smallmatrix}10$	$\begin{smallmatrix}0\\1\end{smallmatrix}10$	$\begin{smallmatrix}0\\0\end{smallmatrix}11$	$\begin{smallmatrix}1\\1\end{smallmatrix}10$	$\begin{smallmatrix}0\\1\end{smallmatrix}11$
$q_B(\mathbf{dim}\,N)$	1	1	1	1	3	1	1	1	1	1
$q_{B'}(\mathbf{dim}\,N)$	1	1	1	1	1	1	3	1	1	3

13.3. Let us consider now canonical algebras. A canonical algebra C is a tilted algebra if and only if it is domestic (thus, the quiver obtained from the quiver of C by deleting the source is a Dynkin diagram), and these algebras are cluster-concealed. For example, let us consider the canonical algebra of type $\mathbb{E}_7$, its quiver has the form



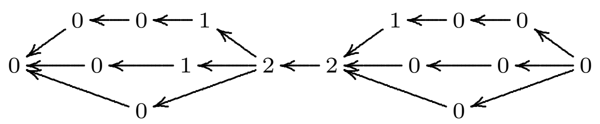
and there is a single relation: the sum of the paths from the source to the sink. The corresponding cluster-tilted algebra has one additional arrow γ:



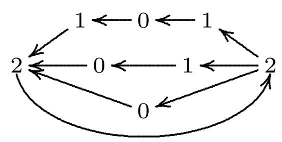
and a lot of zero relations: any path from a vertex x to a vertex y and involving γ is a zero relation, provided the quiver of B contains an arrow $y \to x$.

We consider the indecomposable B_2-module N as well as its image $Z = \pi(N)$ under the covering functor π:

$\mathbf{dim}\,N$

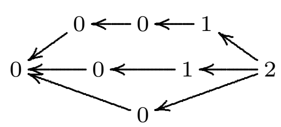


$\mathbf{dim}\,Z$

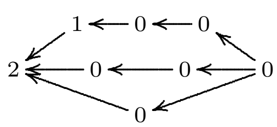


Note that $q_B(\mathbf{dim}\,Z) = 9$. Here, both B-modules N_1, N_2 are decomposable, and it is easy to see that $\dim \operatorname{End}(N_1) = 3$, and $\dim \operatorname{End}(N_2) = 2$; the B-modules N_1, N_2 are the following:

$\mathbf{dim}\,N_1$

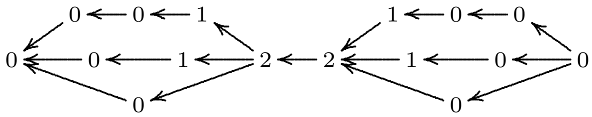


$\mathbf{dim}\,N_2$

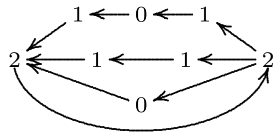


We exhibit another indecomposable B_2-module N' as well as its image $Z = \pi(N)$ under the covering functor π:

dim N'



dim Z'



Note that the composition of the horizontal maps on the left, as well as the composition of the corresponding maps on the right have to be zero. On the right, we see an indecomposable B^c-module Z' such that there is an arrow (namely the one in the center) such that the corresponding vector space map used in Z' is the zero map and neither an epimorphism nor a monomorphism. Here, $q_B(\mathbf{dim}\, Z') = 7$.

14. Tilting modules which are neither preprojective nor preinjective.

The separation property holds only for preprojective (or preinjective) tilting modules, as we are going to show now. As above, let A be a finite-dimensional hereditary k-algebra.

Proposition 8. *Let T_i be an indecomposable regular A-module. Then the component containing T_i contains infinitely many indecomposable modules M such that both* $\mathrm{Hom}(T_i, M) \neq 0$ *and* $\mathrm{Ext}^1(T_i, M) \neq 0$. *If A is wild, then any regular component contains infinitely many indecomposable modules M with this property.*

Proof: In case A is tame, we deal with a stable tube and the stated property is easy to check. Thus, assume that A is wild. Note that a regular component of a wild hereditary algebra is of the form $\mathbb{Z}A_\infty$,

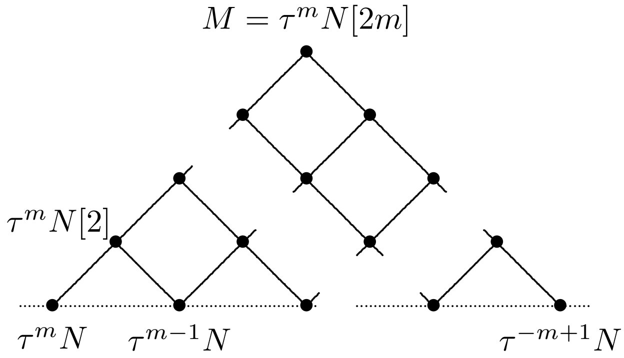
We use the following well-known assertion of Baer [B] and Kerner [K]: If X and Y are indecomposable regular modules, then $\mathrm{Hom}_A(X, \tau^m Y) \neq 0$ and $\mathrm{Hom}(\tau^{-m}Y, X) \neq 0$ for $m \gg 0$. Thus, consider any regular component of the Auslander-Reiten quiver of A and let N be an indecomposable module in this component which is quasi-simple. Then, we have $\mathrm{Hom}(T_i, \tau^m N) \neq 0$ and $\mathrm{Hom}(\tau^{-m}N, T_i) \neq 0$ for $m \gg 0$. Take such a natural number $m \geq 1$ and consider the ray starting in $\tau^m N$

$$\tau^m N = \tau^m N[1] \to \tau^m N[2] \to \cdots \to \tau^m N[t] \to \cdots,$$

it consists of indecomposable modules and irreducible monomorphisms. Let

$$M = \tau^m N[2m].$$

Now M has a filtration going upwards with factors $\tau^m N, \tau^{m-1}N, \ldots, \tau^{-m+1}N$. In particular, it follows that M maps onto $\tau^{-m+1}N$.



We claim that both $\operatorname{Hom}(T_i, M)$ and $\operatorname{Ext}^1(T_i, M)$ are non-zero. On the one hand, $0 \neq \operatorname{Hom}(T_i, \tau^m N)$ embeds into $\operatorname{Hom}(T_i, M)$, since $\tau^m N$ is a submodule of M. This shows that $\operatorname{Hom}(T_i, M) \neq 0$. On the other hand, we have $0 \neq \operatorname{Hom}(\tau^{-m}N, T_i) \simeq \operatorname{Hom}(\tau^{-m+1}N, \tau T_i)$, since τ^{-1} is left adjoint to τ. Composing a surjective map $M \to \tau^{-m+1}N$ with a non-zero map $\tau^{-m+1}N \to \tau T_i$, we obtain a non-zero map. This shows that $\operatorname{Hom}(M, \tau T_i) \neq 0$, and therefore

$$\operatorname{Ext}^1(T_i, M) \simeq D\operatorname{Hom}(M, \tau T_i) \neq 0.$$

This completes the proof.

Proposition 9. *If T_0 is an indecomposable preprojective module, and T_∞ is an indecomposable preinjective module, then any regular component contains infinitely many indecomposable modules M with* $\operatorname{Hom}(T_0, M) \neq 0$ *and* $\operatorname{Ext}^1(T_\infty, M) \neq 0$.

Proof: The proof is similar to the previous proof. Here, we use that for indecomposable modules P, R, Q with P preprojective, R regular and Q preinjective, $\operatorname{Hom}(P, \tau^m R) \neq 0$ and $\operatorname{Hom}(\tau^{-m}R, Q) \neq 0$ for $m \gg 0$.

Corollary. *Let A be a finite-dimensional hereditary algebra and T a tilting module. The following conditions are equivalent:*
(i) *The tilting module T is neither preprojective nor preinjective.*
(ii) *$\mathcal{M}(T)$ is infinite.*
(iii) *$\mathcal{M}(T)$ contains a regular module.*
(iv) *$\mathcal{M}(T)$ contains indecomposable regular modules of arbitrarily large length.*

Proof: Of course, (iv) implies both (ii) and (iii).

Now assume that (ii) or (iii) holds. If T is preprojective, then we have seen in Lemma 3 that $\mathcal{M}(T)$ is a finite set of preprojective modules; similarly, if T is preinjective, then $\mathcal{M}(T)$ is a finite set of preinjective modules. This contradiction shows that T cannot be preprojective or preinjective, thus (i) holds.

Conversely, let us assume (i). Then either T has an indecomposable summand which is regular, or two indecomposable summands one being preprojective, the other being preinjective. If A is wild, then the previous two propositions show that $\mathcal{M}(T)$ contains infinitely many indecomposable modules which belong to the same regular component. But a regular component cf a hereditary algebra A contains only finitely many indecomposable modules of a given length (for A wild, see for example [Zg]). Actually, one easily observes that the proofs of the two previous propositions yield sequences of indecomposable modules in $\mathcal{M}(T)$ of unbounded length.) This shows (iv) in case A is wild. In case A is tame, the proof is similar.

15. A further remark.

Starting with a finite-dimensional hereditary algebra A and a tilting A-module T, one considers T as an object in $\mathcal{C}_A$ and obtains in this way a so-called cluster-tilting object. However, one knows that not all cluster-tilted objects of $\mathcal{C}_A$ are obtained in this way — it may be necessary to change the orientation of the quiver of A. For the benefit of the reader let us include an easy recipe for finding an orientation such that a given cluster-tilting object can be considered as a module. Of course, we only have to consider the cases when T is not regular.

Proposition 10. *Let T be a cluster-tilting object in a cluster category $\mathcal{C}$. Let $\mathcal{S}$ be a slice in $\mathcal{C}$ such that the sources of $\mathcal{S}$ belong to* $\operatorname{add} T$. *Then no indecomposable direct summand of T belongs to $\tau^{-1}\mathcal{S}$.*

Proof: Assume T' is an indecomposable direct summand of T and belongs to $\tau^{-1}\mathcal{S}$. Then $\tau T' \in \mathcal{S}$. There is a source S in $\mathcal{S}$ with $\operatorname{Hom}(S, \tau T') \neq 0$. Thus $\operatorname{Ext}^1(T', S) = D\operatorname{Hom}(S, \tau T') \neq 0$, thus S cannot be a direct summand of T, in contrast to the assumption.

We can apply the proposition as follows: Let T_1 be any indecomposable direct summand of T which is not regular. Then there is a unique slice in $\mathcal{C}$ such that T_1 is the unique source.

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Appendix: Torsion pairs and matrix categories.

For the convenience of the reader, we add a proof of the result which seems to be folklore. Assertions of this kind can be traced back to the Kiev school of Nazarova, Roiter and Drozd.

Proposition. *Let $(\mathcal{F},\mathcal{G})$ be a torsion pair in the abelian category $\mathcal{A}$. Given $A \in \mathcal{A}$, let ϵ_A be the equivalence class of the canonical exact sequence $0 \to tA \to A \to A/tA \to 0$, and $\eta(A) = (A/tA, tA, \epsilon_A)$. Then this defines a functor $\eta\colon \mathcal{A} \to \operatorname{Mat}\operatorname{Ext}^1(\mathcal{F},\mathcal{G})$ which is full and dense and its kernel is the ideal generated by all maps $\mathcal{F} \to \mathcal{G}$.*

In particular, in case $\mathcal{A}$ is a Krull-Remak-Schmidt category, then we see that the kernel of the functor η lies in the radical of $\mathcal{A}$ and therefore η is a representation equivalence.

Proof: Denote the inclusion maps $tA \to A$ and $tA' \to A'$ by u, u', respectively, and the projection maps $A \to A/tA$ and $A' \to A'/tA'$ by p, p' respectively. Let $\alpha\colon A \to A'$ be a morphism in $\mathcal{A}$. Then α maps tA into tA', thus it induces a map $t\alpha\colon tA \to tA'$ as well as a map $\overline{\alpha}\colon A/tA \to A'/tA'$, thus $u\alpha = (t\alpha)u'$, and $p\overline{\alpha} = \alpha p'$.

Using $t\alpha$, we obtain the following induced exact sequence

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & tA & \xrightarrow{u} & A & \xrightarrow{p} & A/tA & \longrightarrow & 0 \\
 & & {\scriptstyle t\alpha}\big\downarrow & & {\scriptstyle \alpha'}\big\downarrow & & \big\| & & \\
0 & \longrightarrow & tA' & \xrightarrow[v]{} & B & \xrightarrow[q]{} & A/tA & \longrightarrow & 0.
\end{array}
$$

Since $u\alpha = (t\alpha)u'$, there is a map $\alpha''\colon B \to A'$ such that $\alpha = \alpha'\alpha''$ and $v\alpha'' = u'$. It follows that $q\overline{\alpha} = \alpha''p'$, since

$$\alpha'\alpha''p' = \alpha p' = p\overline{\alpha} = \alpha' q\overline{\alpha},$$

and

$$v\alpha''p' = u'p' = 0 = vq\overline{\alpha}.$$

Thus we also have a commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & tA' & \xrightarrow{v} & B & \xrightarrow{q} & A/tA & \longrightarrow & 0 \\
 & & \big\| & & {\scriptstyle \alpha''}\big\downarrow & & {\scriptstyle \overline{\alpha}}\big\downarrow & & \\
0 & \longrightarrow & tA' & \xrightarrow[u']{} & A' & \xrightarrow[p']{} & A'/tA' & \longrightarrow & 0.
\end{array}
$$

The diagram shows that the upper row is induced from the lower one by $\overline{\alpha}$, therefore $t\alpha\epsilon_A = \epsilon_{A'}\overline{\alpha}$. Thus, we see that η is a functor.

First, we show that η is dense: any element $\epsilon \in \operatorname{Ext}^1(F, G)$ is the equivalence class of an exact sequence

$$0 \longrightarrow G \xrightarrow{\mu} A \xrightarrow{\epsilon} F \longrightarrow 0.$$

Here, the image of μ has to be tA, let $u\colon tA \to A$ be the inclusion map and A/tA the canonical projection. We obtain a commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G & \xrightarrow{\mu} & A & \xrightarrow{\epsilon} & F & \longrightarrow & 0 \\
 & & {\scriptstyle \mu'}\big\downarrow & & \big\| & & \big\downarrow{\scriptstyle \epsilon'} & & \\
0 & \longrightarrow & tA & \xrightarrow[v]{} & A & \xrightarrow[q]{} & A/tA & \longrightarrow & 0
\end{array}
$$

where μ' and ϵ' are isomorphisms.

Next, we show that η is full. Let A, A' be objects and assume that there are given maps $\beta\colon tA \to tA'$ and $\gamma\colon A/tA \to A'/tA'$ such that $\beta\epsilon_A = \epsilon_{A'}\gamma$. We obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & tA & \xrightarrow{u} & A & \xrightarrow{p} & A/tA & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \beta'} & & \Big\| & & \\
0 & \longrightarrow & tA' & \xrightarrow[v]{} & B & \xrightarrow[q]{} & A/tA & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow{\scriptstyle \delta} & & \Big\| & & \\
0 & \longrightarrow & tA' & \xrightarrow{v'} & B' & \xrightarrow{q'} & A/tA & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow{\scriptstyle \gamma'} & & \Big\downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & tA' & \xrightarrow[u']{} & A' & \xrightarrow[p']{} & A'/tA' & \longrightarrow & 0,
\end{array}$$

The upper part shows that the second row is induced from the first. the lower part shows that the third row is induced from the forth. The central part means that the two induced sequences are equivalent: Altogether we obtain the map $\alpha = \beta'\delta\gamma' \colon A \to A'$ and we have $t\alpha = \beta$, and $\overline{\alpha} = \gamma$. Thus, $(\beta, \gamma) = \eta(\alpha)$.

It is clear that the maps $\mathcal{F} \to \mathcal{G}$ are in the kernel of η. Conversely, assume that $\alpha \colon A \to A'$ is in the kernel of η, thus $t\alpha = 0$ and $\overline{\alpha} = 0$. Then $\alpha = p\alpha' u'$ for some $\alpha' \colon A/tA \to tA'$, thus α lies in the ideal generated by the maps $\mathcal{F} \to \mathcal{G}$.

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