A set-valued implicit function theorem under relaxed one-sided Lipschitz conditions.

Wolf-Jürgen Beyn · Janosch Rieger

March 23, 2010

Abstract Implicit function theorems are derived for nonlinear set valued equations that satisfy a relaxed one-sided Lipschitz condition. We discuss a local and a global version and study in detail the continuity properties of the implicit set-valued function. Applications are provided to the Crank-Nicolson scheme for differential inclusions and to the analysis of differential algebraic inclusions.

Keywords set valued implicit function theorem · one-sided Lipschitz condition · differential (algebraic) inclusions

Mathematics Subject Classification (2000) MSC 47H10 · 47N20 · 34A60

1 Introduction

The implicit function theorem is one of the most important tools in analysis. It guarantees that a solution \((p_0, x_0)\) of the equation

\[ F(p, x) = 0 \]

is stable under small perturbations in \(p\) provided that \(F : \mathbb{R}^k \times \mathbb{R}^m \rightarrow \mathbb{R}^m\) is continuous and satisfies a nondegeneracy condition with respect to \(x\) near the known solution \((p_0, x_0)\). Thus, it is a standard ingredient for an abundance of existence and robustness type results.

Supported by CRC 701 'Spectral Structures and Topological Methods in Mathematics', Bielefeld University.

W.-J. Beyn
Fakultät für Mathematik, Universität Bielefeld
Postfach 100131, D-33501 Bielefeld, Germany
Tel.: +49-521-1064798, Fax: +49-521-1066498
E-mail: beyn@math.uni-bielefeld.de

J. Rieger
Fachbereich Mathematik, Universität Frankfurt
Postfach 111932, D-60054 Frankfurt a.M., Germany
E-mail: rieger@math.uni-frankfurt.de
The aim of the present paper is to prove an implicit function theorem for set-valued mappings in the spirit of the classical result: If $F : \mathbb{R}^k \times \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is a set-valued mapping and
\[
0 \in F(p_0, x_0)
\]
for some $p_0 \in \mathbb{R}^k$, $x_0 \in \mathbb{R}^m$, then there exist neighbourhoods $V_1$ and $V_2$ of $p_0$ and $x_0$ such that the set $\{x \in V_2 : 0 \in F(p, x)\}$ can be described as the image $N(p)$ of a set-valued mapping $N : V_1 \rightrightarrows V_2$ with favourable properties. As in the single-valued case, it is necessary to specify matching smoothness and nondegeneracy conditions.

Several generalizations of the implicit function theorem to the set-valued setting have been proposed. These theorems can be divided into two classes.

a) The systematic investigation of optimal control problems with constraints caused a demand for analytical tools designed for nonsmooth optimization. A first generalized implicit function theorem aiming in that direction is presented in [14], which only guarantees the existence of a single-valued resolving function. In [3], a first fully set-valued implicit function theorem is proposed, which requires lower semicontinuity of the mapping $F$ and a technical transversality condition and provides a lower semicontinuous set-valued resolving mapping $N(\cdot)$. The main result of [10] is an implicit function theorem based on traditional set-valued calculus, which allows to express the set derivative of the resolving mapping in terms of the set derivative of the original multifunction. The theorem given in [12] partly generalizes the latter one without providing much information about the nature of $N(\cdot)$, while the main result of [11] emphasizes its continuity properties. Recently, an analysis from a modern point of view and extensions of the classical result from [14] have been given in [8].

b) An implicit function theorem for maximal monotone and uniformly coercive mappings is presented in [1]. It is shown that monotonicity forces the resolving mapping $N(\cdot)$ to be single-valued. Note that the theory of nonlinear maximal monotone operators (see the monograph [16] for a survey) always requires one-sided Lipschitz estimates for all admissible arguments in contrast to the relaxed one-sided Lipschitz conditions in this paper (see below).

Several variants of set-valued implicit function theorems in Banach spaces are proved in [4]. Here, the nondegeneracy assumptions imply that the resolving mapping $N(\cdot)$ is single-valued as well, and in view of Proposition 3.2 in that article, the mapping $F$ is essentially single-valued whenever the resolving mapping $N(\cdot)$ is continuous.

The type of implicit function theorem we develop here is designed for the study of set-valued dynamics and set-valued numerical analysis. We require the mapping $F$ to be convex and compact-valued, continuous in the parameter, and upper semicontinuous and relaxed one-sided Lipschitz (ROSL, see Definition 1) with negative constant in space. Under these assumptions, we can guarantee that the resolving set-valued mapping $N(\cdot)$ is well-defined, compact-valued, and continuous. In addition, we can show that if $F$ is Hölder continuous, then so is $N(\cdot)$.

It was shown in [7] (Theorem 3.2 is of particular importance) that the ROSL condition is one of the most natural and useful stability concepts for set-valued dynamics. In contrast to the stronger one-sided Lipschitz condition (OSL), it allows the mapping...
to be multivalued everywhere and not just on a set of measure zero. Moreover, if the right hand side of a differential inclusion

\[ \dot{x}(t) \in F(t, x(t)) \]

is ROSL, there are typically many solutions, while in the OSL case, it is easy to see that there exists at most one solution.

In Section 2, we prove a local version of the solvability theorem presented in [2] by extending a locally defined ROSL mapping to \( \mathbb{R}^m \) in a suitable manner. This solvability theorem can be regarded as a multivalued version of the uniform monotonicity theorem (see [13], [15]). Thus, the single-valued implicit function theorem given in Section 3 is closely related to the latter result. In Section 4 we propose a global and a local version of our set-valued implicit function theorem, which we apply in Section 5 in order to discuss perturbations of one-sided Lipschitz functions, a set-valued Crank-Nicolson scheme, and differential algebraic inclusions.

The notation used in this paper is standard: The Euclidean norm is denoted by \( | \cdot | \), closed balls are defined by \( B_R(x) = \{ y \in \mathbb{R}^m : |y - x| \leq R \} \) and \( \| A \| := \sup_{a \in A} |a| \) denotes the maximal norm of the elements of a set \( A \subset \mathbb{R}^m \). The spaces of the nonempty compact and the nonempty convex and compact subsets of \( \mathbb{R}^m \) are denoted by \( C(\mathbb{R}^m) \) and \( CC(\mathbb{R}^m) \), respectively. For \( A, B \in C(\mathbb{R}^m) \), the one-sided and the symmetric Hausdorff distance are given by

\[ \text{dist}(A, B) := \sup_{a \in A} \inf_{b \in B} |a - b| \]

and

\[ \text{dist}_H(A, B) := \max\{\text{dist}(A, B), \text{dist}(B, A)\}. \]

If \( A \subset \mathbb{R}^k \) and \( F: \mathbb{R}^k \rightrightarrows \mathbb{R}^m \) is a set-valued mapping, then \( F(A) := \cup_{a \in A} F(a) \). The map \( F: U \subset \mathbb{R}^m \rightarrow C(\mathbb{R}^m) \) is called upper semicontinuous (usc) at \( x \in U \) if \( \text{dist}(F(x'), F(x)) \rightarrow 0 \) as \( x' \rightarrow x \).

2 A local solvability theorem

The notion of relaxed one-sided Lipschitz (ROSL) set-valued mappings is an important stability criterion for multivalued generators. It generalizes the concepts of Lipschitz continuity and the (strong) one-sided Lipschitz property. A detailed analysis of this property can be found in [6] and several other works of the same author.

**Definition 1** A mapping \( F: U \subset \mathbb{R}^m \rightarrow C(\mathbb{R}^m) \) is called relaxed one-sided Lipschitz with constant \( l \in \mathbb{R} \) if for every \( x, x' \in U \) and \( y \in F(x) \) there exists some \( y' \in F(x') \) such that

\[ \langle y - y', x - x' \rangle \leq l|x - x'|^2. \]

(1)

The following Theorem is a local version of Theorem 2 in [2]. In order to obtain the optimal result, it is necessary to extend the locally defined mapping to the whole space in a suitable way.
Theorem 1 Let $F : B_R(0) \to CC(\mathbb{R}^m)$ be usc and ROSL with a constant $l < 0$ such that $-\frac{1}{l} \text{dist}(0, F(0)) \leq R$. Then the inclusion $0 \in F(x)$ has a solution $\bar{x} \in \mathbb{R}^m$ with 

$$|\bar{x}| \leq \frac{1}{l} \text{dist}(0, F(0)).$$

Proof Let $y_0 \in F(0)$ be the element with minimal norm. We extend $F$ from $B_R(0)$ to $\mathbb{R}^m$ by setting 

$$F(x) := \frac{R - |x|}{R} y_0 + \frac{|x|}{R} F\left(\frac{R}{|x|} x\right)$$

whenever $x \notin B_R(0)$. This extension is usc: In the interior of $B_R(0)$, this is true by assumption. For $|x|, |x'| \geq R$,

$$\text{dist}(F(x'), F(x)) = \text{dist}\left(\frac{R - |x'|}{R} y_0 + \frac{|x'|}{R} F\left(\frac{R}{|x'|} x'\right), \frac{R - |x|}{R} y_0 + \frac{|x|}{R} F\left(\frac{R}{|x|} x\right)\right)$$

$$\leq \left| \frac{R - |x'|}{R} y_0 - \frac{R - |x|}{R} y_0 \right| + \text{dist}\left(\frac{|x'|}{R} F\left(\frac{R}{|x'|} x'\right), \frac{|x|}{R} F\left(\frac{R}{|x|} x\right)\right)$$

$$+ \text{dist}\left(\frac{|x|}{R} F\left(\frac{R}{|x|} x\right), \frac{|x'|}{R} F\left(\frac{R}{|x'|} x'\right)\right)$$

$$\to 0$$

as $x' \to x$,

because $F$ is usc and thus bounded on $B_R(0)$. In particular, the extension is usc on $\partial B_R(0)$, because by assumption, $\text{dist}(F(x'), F(x)) \to 0$ as $x' \to x$ whenever $x \in \partial B_R(0)$ and $x' \in \text{int} B_R(0)$.

The extended $F$ is not necessarily ROSL, but for every $x \in \mathbb{R}^m$, there exists some $y \in F(x)$ such that

$$\langle y - y_0, x \rangle \leq l|x|^2.$$

(2)

Indeed, for $x \in B_R(0)$, this is just the ROSL property, and for $x \notin B_R(0)$, we define

$$y := \frac{R - |x|}{R} y_0 + \frac{|x|}{R} y',$$

where we may choose $y' \in F\left(\frac{R}{|x|} x\right)$ with $\langle y' - y_0, \frac{R}{|x|} x \rangle \leq lR^2$, because $\frac{R}{|x|} x \in B_R(0)$ where $F$ is ROSL. But then,

$$\langle y - y_0, x \rangle = \left(\frac{R - |x|}{R} y_0 + \frac{|x|}{R} y'\right) - y_0, x) = \frac{|x|}{R} \langle y' - y_0, x \rangle$$

$$= \frac{|x|^2}{R^2} \langle y' - y_0, \frac{R}{|x|} x \rangle \leq l|x|^2.$$
Thus, for any \( \varrho > -\frac{1}{l} \text{dist}(0, F(0)) \), \( |x| \leq \varrho \), and \( \alpha \) small enough so that \( 1 + 2\alpha l \geq 0 \), inequality
\[
|z|^2 \leq \varrho^2 + 2\alpha (l \varrho + \text{dist}(0, F(0))) \varrho + \alpha^2 |y|^2 \\
\leq \alpha \left[ 2 \varrho (l \varrho + \text{dist}(0, F(0))) + \alpha |y|^2 + \varrho^2 \right] < 0
\]
holds. As \( F \) is usc,
\[
M_{\varrho} := \sup_{x \in B_{\varrho}(0)} \| F(x) \| < \infty,
\]
and there exists an \( \alpha > 0 \) such that \( |z|^2 \leq \varrho^2 \) follows from (3). This means that for this fixed \( \alpha \),
\[
H(x) := G(x) \cap B_{\varrho}(0) \neq \emptyset \quad \text{for all } x \in B_{\varrho}(0),
\]
and \( H(\cdot) \) is also usc. By the Kakutani Theorem, \( H \) and thus also \( G \) have a fixed point \( x_0 \) in \( B_{\varrho}(0) \), which implies that \( 0 \in F(x_0) \).

In particular, we find elements \( x_n \in B(0, -\frac{1}{l} \text{dist}(0, F(0)) + 1/n) \) for all \( n \in \mathbb{N} \) such that \( 0 \in F(x_n) \). As \( B(0, -\frac{1}{l} \text{dist}(0, F(0)) + 1) \) is compact, there exists a convergent subsequence of \( \{x_n\}_{n \in \mathbb{N}} \) with limit
\[
\bar{x} \in B(0, -\frac{1}{l} \text{dist}(0, F(0))).
\]
Since \( F \) is usc,
\[
0 \in F(\bar{x}).
\]
As \( \bar{x} \in B_R(0) \), the element \( \bar{x} \) is a zero of the original mapping \( F \).

As an immediate consequence, we obtain the following corollary.

**Corollary 1** Let \( x, y \in \mathbb{R}^m \) and let \( F : B_R(x) \rightarrow \mathbb{C}C(\mathbb{R}^m) \) be usc and ROSL with a constant \( l < 0 \) such that \( -\frac{1}{l} \text{dist}(y, F(x)) \leq R \). Then there exists an \( \bar{x} \in \mathbb{R}^m \) such that \( y \in F(\bar{x}) \) and
\[
|x - \bar{x}| \leq -\frac{1}{l} \text{dist}(y, F(x)).
\]

**Proof** Consider the usc mapping \( G : B_R(0) \rightarrow \mathbb{C}C(\mathbb{R}^m) \) given by
\[
G(x) := F(x + x) - y.
\]
It is ROSL in \( B_R(0) \) with the same constant \( l < 0 \), and
\[
-\frac{1}{l} \text{dist}(0, G(0)) = -\frac{1}{l} \text{dist}(y, F(x)) \leq R.
\]
By Theorem 1, there exists a solution \( \tilde{x} \) of \( 0 \in G(\tilde{x}) \) such that
\[
|\tilde{x}| \leq -\frac{1}{l} \text{dist}(0, G(0)),
\]
so that, setting \( \bar{x} := \tilde{x} + x \), we obtain \( y \in F(\bar{x}) \) and
\[
|x - \bar{x}| \leq -\frac{1}{l} \text{dist}(y, F(x)).
\]
3 An implicit function theorem for single-valued OSL functions

Applying Corollary 1 to a single-valued mapping yields a variant of the classical implicit function Theorem. Recall that any single-valued function which is usc and ROSL is continuous and OSL.

**Theorem 2** Let $U_1 \subset \mathbb{R}^k$ and $U_2 \subset \mathbb{R}^m$ be open sets, and let $f : U_1 \times U_2 \to \mathbb{R}^m$ be continuous in the first and continuous and OSL in the second argument with constant $l < 0$. Assume furthermore that there exist $p_0 \in U_1$ and $x_0 \in U_2$ such that $f(p_0, x_0) = 0$. Then there exist neighborhoods $V_1 \subset U_1$ and $V_2 \subset U_2$ with $p_0 \in V_1$ and $x_0 \in V_2$ and a continuous function $g : V_1 \to V_2$ such that

$$f(p, x) = 0 \iff g(p) = x$$

whenever $p \in V_1$ and $x \in V_2$. Moreover, if $f(\cdot, x)$ is $(L, \beta)$-Hölder continuous for any $x \in U_2$, then $g$ is $(-\frac{L}{\beta}, \beta)$-Hölder continuous.

**Proof** Let $R := \sup\{r > 0 : B_r(x_0) \subset U_2\}$, and set $V_2 := \text{int}B_R(x_0)$. As $f(\cdot, x_0)$ is continuous, there exists a neighbourhood $V_1 \subset U_1$ of $p_0$ such that $|f(p, x_0)| < -lR$ for all $p \in V_1$. Thus we can apply Corollary 1 to the mapping $f(p, \cdot)$ for any $p \in V_1$, which yields a point $x_p \in \mathbb{R}^m$ such that $f(p, x_p) = 0$ and

$$|x_p - x_0| \leq \frac{1}{l} |f(p, x_0)| < R.$$

The zero $x_p$ is unique in $\text{int}B_R(x_0)$: Assume that there were two zeroes $x_p$ and $x_p'$ in $\text{int}B_R(x_0)$. Then

$$0 = \langle f(x_p) - f(x_p'), x_p - x_p' \rangle \leq |x_p - x_p'|^2$$

implies that $x_p = x_p'$.

Let us define $g : V_1 \to V_2$ by $g(p) := x_p$. It is continuous, because for any $p, p' \in V_1$,

$$|g(p) - g(p')| \leq \frac{1}{L} |f(p', g(p))| \to 0 \text{ as } p' \to p$$

by Corollary 1 and continuity of $f(\cdot, g(p))$. Moreover, if $f(\cdot, g(p))$ is $(L, \beta)$-Hölder continuous, the same reasoning yields

$$|g(p) - g(p')| \leq \frac{1}{L} |f(p', g(p)) - f(p, g(p))| \leq \frac{L}{l} |p - p'|^\beta,$$

so that $g$ is $(-\frac{L}{\beta}, \beta)$-Hölder continuous.

Note that if $U_2 = \mathbb{R}^m$, then $V_2 = \mathbb{R}^m$ according to the definition of $R$ in the proof.

**Remark 1** This theorem is closely related to the uniform monotonicity theorem, see e.g. [13], [15], which plays a major role in the verification of the solvability of implicit Runge-Kutta methods for ODEs. In Section 5, we will use the set-valued implicit function theorem presented below in order to prove the solvability of a Crank-Nicolson scheme for differential inclusions and to discuss differential algebraic inclusions.

**Remark 2** If a mapping $f : \mathbb{R}^m \to \mathbb{R}^m$ is OSL with constant $l < 0$ and continuously differentiable at $x$, then its differential is invertible and the classical implicit function theorem applies.
4 Implicit function theorems for set-valued ROSL maps

The following implicit function theorem is based on Corollary 1. The existence statement guarantees solvability of the implicit inclusion, while the defect estimate yields the smoothness properties of the resolving mapping $N(\cdot)$.

**Theorem 3** Let $F : \mathbb{R}^k \times \mathbb{R}^m \to \mathcal{C}(\mathbb{R}^m)$ be uniformly continuous in the first argument in the sense that

$\text{dist}(F(p, x), F(p', x)) \leq \omega(|p - p'|) \forall x \in \mathbb{R}^m$

with $\omega(\tau) \to 0$ as $\tau \to 0$ and use and ROSL with a constant $l < 0$ in the second argument. Then there exists a continuous mapping $N(\cdot) : \mathbb{R}^k \to \mathcal{C}(\mathbb{R}^m)$ such that

$0 \in F(p, x) \iff x \in N(p),$

and

$\|N(p)\| \leq -\frac{1}{l} \|F(p, 0)\|.$

For the diameter of the images, the implicit estimate

$\text{diam}\, N(p) \leq -\frac{1}{l} \sup_{x \in N(p)} \text{diam}\, F(x)$

holds. Moreover, if $F$ is uniformly $(L, \beta)$-Hölder continuous in the sense that

$\text{dist}(F(p, x), F(p', x)) \leq L|p - p'|^\beta \forall p, p' \in \mathbb{R}^k, x \in \mathbb{R}^m,$

then $N(\cdot)$ is $(-\frac{L}{l}, \beta)$-Hölder continuous.

**Proof** Define $N(p) := \{x \in \mathbb{R}^m : 0 \in F(p, x)\}$. It follows immediately from Corollary 1 that $N(p) \neq \emptyset$ for all $p \in \mathbb{R}^k$.

Let $\{x_n\}_{n \in \mathbb{N}}$ be a convergent sequence in $N(p)$ with $\lim_{n \to \infty} x_n = x$. Then

$\text{dist}(0, F(p, x)) \leq \text{dist}(F(p, x_n), F(p, x)) \to 0$ as $n \to \infty$

by (upper semi-)continuity of $F(p, \cdot)$. Since $F(p, x)$ is compact, $0 \in F(p, x)$ and $N(p)$ is closed.

Let $x \in N(p)$. Then $0 \in F(p, x)$, and the ROSL condition implies that there exists some $y \in F(p, 0)$ such that

$-|y|x \leq -(y, x) = (y - 0, 0 - x) \leq l|x|^2,$

and hence

$\|N(p)\| \leq -\frac{1}{l} \|F(p, 0)\|.$

In order to estimate $\text{diam}\, N(p)$, consider $x, x' \in N(p)$. As $0 \in F(p, x')$, there exists some $y \in F(p, x)$ satisfying

$-|y| \cdot |x' - x| \leq (0 - y, x' - x) \leq l|x' - x|^2,$

so that

$|x - x'| \leq -\frac{|y|}{l} \leq -\frac{1}{l} \text{diam}\, F(x)$
and hence

\[ \text{diam } N(p) \leq -\frac{1}{l} \text{diam } F(x). \]

If \( x \in N(p) \), Corollary 1 guarantees the existence of some \( x' \in N(p') \) such that

\[ |x - x'| \leq -\frac{1}{l} \text{dist}(0, F(p', x)) \leq -\frac{1}{l} \text{dist}(F(p, x), F(p', x)). \]

Thus,

\[ \text{dist}(N(p), N(p')) \leq -\frac{1}{l} \sup_{x \in N(p)} \text{dist}(F(p, x), F(p', x)) \leq -\frac{1}{l} \omega(|p - p'|) \]

ensures continuity of \( N \). If, in addition, \( F \) is \((L, \beta)\)-Hölder continuous in the first variable, then

\[ \text{dist}(N(p), N(p')) \leq -\frac{1}{l} \sup_{x \in N(p)} \text{dist}(F(p, x), F(p', x)) \leq -\frac{L}{l} |p - p'|^\beta, \]

so that \( N \) is \((-\frac{L}{l}, \beta)\)-Hölder continuous.

The following theorem is a local version of the above. The domains have to be chosen carefully in order to ensure that for all parameters considered, the full sets \( N(p) \) are contained in the specified region. Otherwise we encounter various problems related to the continuity properties of the solution mapping \( N(\cdot) \) due to the fact that its images need not be convex, see Example 10 in [2].

**Theorem 4** Let \( U_1 \subset \mathbb{R}^k \) and \( U_2 \subset \mathbb{R}^m \) be open subsets, and let \( F : U_1 \times U_2 \rightarrow \mathcal{C}(\mathbb{R}^m) \) be uniformly continuous in the first argument in the sense that

\[ \text{dist}(F(p, x), F(p', x)) \leq \omega(|p - p'|) \ \forall p, p' \in U_1, \ x \in U_2 \]

with \( \omega(\tau) \rightarrow 0 \) as \( \tau \rightarrow 0 \) and usc and ROSL with a constant \( l < 0 \) in the second argument. Assume that there exist elements \( p_0 \in U_1 \) and \( x_0 \in U_2 \) such that \( 0 \in F(p_0, x_0) \) and \( \|F(p_0, x_0)\| < -l \text{dist}(x_0, \partial U_2) \).

Then there exist neighbourhoods \( V_1 \subset U_1 \) of \( p_0 \) and \( V_2 \subset U_2 \) of \( x_0 \) and a continuous mapping \( N(\cdot) : V_1 \rightarrow \mathcal{C}(\mathbb{R}^m) \) such that

\[ 0 \in F(p, x), \ p \in V_1, \ x \in V_2 \ \Leftrightarrow \ x \in N(p), \ p \in V_1. \]

For the diameter of the images, the implicit estimate

\[ \text{diam } N(p) \leq -\frac{1}{l} \sup_{x \in N(p)} \text{diam } F(x) \]

holds, and we have

\[ \text{dist}(N(p), x_0) \leq -\frac{1}{l} \|F(p, x_0)\| \]

(5)

for all \( p \in V_1 \). Moreover, if \( F \) is uniformly \((L, \beta)\)-Hölder continuous in the sense that

\[ \text{dist}(F(p, x), F(p', x)) \leq L|p - p'|^\beta \ \forall p, p' \in U_1, \ x \in U_2, \]

then \( N \) is locally \((-\frac{L}{l}, \beta)\)-Hölder continuous, i.e.

\[ \text{dist}(N(p), N(p')) \leq -\frac{L}{l} |p - p'|^\beta \]

for small \( |p - p'| \).
Proof Set \( V_2 := \text{int} B_R(x_0) \), where \( R := \text{dist}(x_0, \partial U_2) \), so that \( V_2 \subset U_2 \).

Properties of the images:
As \( F(\cdot, x_0) \) is usc, there exists a neighbourhood \( V_1 \subset U_1 \) of \( p_0 \) such that \( \|F(p, x_0)\| < -lR \) for all \( p \in V_1 \). Consequently, the images of the mapping \( N : V_1 \to V_2 \) defined by \( N(p) := \{ x \in V_2 : 0 \in F(p, x) \} \) are non-empty by Corollary 1.

Besides, if \( p \in V_1 \) and \( x \in N(p) \), then the ROSL condition implies the existence of some \( y \in F(p, x_0) \) such that

\[
-|y| |x - x_0| \leq (0 - y, x - x_0) \leq l|x - x_0|^2,
\]

and hence

\[
\text{dist}(N(p), x_0) \leq -\frac{1}{l} \|F(p, x_0)\| < R,
\]

so that

\[
\inf_{x \in N(p)} \text{dist}(x, \partial V_2) > 0.
\]

Let \( \{x_n\}_{n \in \mathbb{N}} \) be a convergent sequence in \( N(p) \), \( p \in V_1 \), with \( \lim_{n \to \infty} x_n = x \). Note that (7) ensures that \( x \in V_2 \). Thus,

\[
\text{dist}(0, F(p, x)) \leq \text{dist}(F(p, x_n), F(p, x)) \to 0 \text{ as } n \to \infty
\]

by upper semicontinuity of \( F(p, \cdot) \). Since \( F(p, x) \) is compact, \( 0 \in F(p, x) \). Therefore, \( N(p) \) is closed, and because of (6), \( N(p) \) is compact.

The implicit estimate for the diameter of \( N(p) \) can be obtained precisely as in the proof of Theorem 3, and (5) is proved by the same argument.

Continuity properties of the mapping \( N(\cdot) \):
Let us fix some \( p \in V_1 \). Then

\[
\sup_{x \in N(p)} -\frac{1}{l} \text{dist}(0, F(p', x)) \leq \sup_{x \in N(p)} -\frac{1}{l} \text{dist}(F(p, x), F(p', x))
\]

\[
\leq -\frac{1}{l} \omega(|p - p'|) < \inf_{x \in N(p)} \text{dist}(x, \partial V_2)
\]

for sufficiently small \( |p - p'| \) because of (7). Hence Corollary 1 guarantees that for any \( x \in N(p) \), there exists some \( x' \) such that \( 0 \in F(p', x') \) and

\[
|x - x'| \leq -\frac{1}{l} \omega(|p - p'|),
\]

which is contained in \( V_2 \) by the above reasoning and thus an element of \( N(p') \). It follows that

\[
\text{dist}(N(p), N(p')) \leq -\frac{1}{l} \omega(|p - p'|)
\]

in a neighbourhood of \( p \), and hence \( N(\cdot) \) is lower semicontinuous.

On the other hand, inequality (6) and upper semicontinuity of \( F(\cdot, x_0) \) imply that there exist positive constants \( \varepsilon \) and \( \delta \) such that

\[
\text{dist}(N(p'), x_0) \leq -\frac{1}{l} \|F(p', x_0)\| < R - \varepsilon
\]
whenever $p' \in B_2(p)$. In particular,

$$\inf_{x' \in N(p')} \text{dist}(x', \partial V_2) > \varepsilon$$

for all $p' \in B_3(p)$. But then there exists some $\delta' \in (0, \delta]$ such that

$$-\frac{1}{l} \text{dist}(0, F(p, x')) \leq -\frac{1}{l} \text{dist}(F(p', x'), F(p, x')) \leq -\frac{1}{l} \omega(|p - p'|) < \varepsilon$$

for all $p' \in B_U(p)$ and $x' \in N(p')$. Then Corollary 1 applied to $B_\varepsilon(x')$ yields an element $x \in V_2$ with $0 \in F(p, x)$ and

$$|x - x'| < -\frac{1}{l} \omega(|p - p'|).$$

Thus we have

$$\text{dist}(N(p'), N(p)) < -\frac{1}{l} \omega(|p - p'|)$$

for all $p'$ sufficiently close to $p$, and $N(\cdot)$ is upper semicontinuous.

Furthermore, if $F$ is uniformly $(L, \beta)$-Hölder continuous, repeating the above arguments with $L|p - p'|^\beta$ instead of the modulus of continuity $\omega(|p - p'|)$ yields that $N(\cdot)$ is locally $(L, \beta)$-Hölder continuous.

As in the single-valued case, if $U_2 = \mathbb{R}^m$, then $V_2 = \mathbb{R}^m$ according to the definition of $R$ in the proof. In particular, the assumption $\|F(p_0, x_0)\| < -l \text{dist}(x_0, \partial U_2)$ is obsolete, and the mapping $N(\cdot)$ is globally Hölder continuous whenever $F$ is Hölder continuous.

5 Applications

5.1 Perturbations of single-valued OSL functions

Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be continuous and OSL with constant $l < 0$, and assume that $f(0) = 0$. By the Uniform Monotonicity Theorem (see [13],[15]), this zero is unique. We will study set-valued perturbations of $f$ and apply Theorem 4 in order to gain information about the behaviour of their zero sets.

Let $G : \mathbb{R} \times \mathbb{R}^m \to \mathcal{C}(\mathbb{R}^m)$ be a set-valued mapping such that

a) $G(0, x) = 0$ for all $x \in \mathbb{R}^m$,

b) $G(p, \cdot)$ is usc for all $p \in \mathbb{R}$,

c) for all $p \in \mathbb{R}$, the mapping $G(p, \cdot)$ is ROSL with a constant $l_p \in \mathbb{R}$ satisfying $l_p \to 0$ as $p \to 0$, and

d) $\text{dist}(G(p, x), G(p', x)) \leq \omega(|p - p'|) \forall x \in \mathbb{R}^m$, where $\omega(\tau) \to 0$ as $\tau \to 0$.

It is easy to see that each perturbed mapping $x \mapsto f(x) + G(p, x)$ is ROSL with constant $l + l_p$. Thus, the mapping $(p, x) \mapsto f(x) + G(p, x)$ satisfies the assumptions of Theorem 4 with $U_1 := (-p_0, p_0)$ and $U_2 := \mathbb{R}^m$, where $p_0 > 0$ is chosen so small that $l + l_p \leq l_0 < 0$ for all $p \in (-p_0, p_0)$. Thus, there exists a continuous set-valued mapping $N : (-p_0, p_0) \to \mathcal{C}(\mathbb{R}^m)$ such that $0 \in F(p, x)$ is equivalent with $x \in N(p)$. This shows that the set of zeroes of the perturbed mapping $F(p, \cdot)$ deforms continuously into the unique zero of the original mapping $f$ as $p \to 0$. 
5.2 A set-valued Crank-Nicolson method

Consider the ordinary differential equation
\[ \dot{x}(t) = f(x(t)) \text{ for all } t \in [0, T], \quad x(0) = x_0, \] (8)
where \( f : \mathbb{R}^m \to \mathbb{R}^m \). The semi-implicit numerical scheme given by
\[ x_{n+1} = x_n + \frac{h}{2}f(x_n) + \frac{h}{2}f(x_{n+1}), \quad n = 0, \ldots, \lfloor T/h \rfloor, \] (9)
is the classical Crank-Nicolson method, which has particularly favourable stability properties, so that it plays a major role in the treatment of Galerkin approximations of reaction-diffusion equations.

The multivalued analogs of (8) and (9) are the ordinary differential inclusion
\[ \dot{x}(t) \in F(x(t)) \text{ almost everywhere in } [0, T], \quad x(0) = x_0, \] (10)
where \( F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m \) is a set-valued mapping with convex, compact, and nonempty values, and a numerical scheme given by
\[ x_{n+1} \in x_n + \frac{h}{2}F(x_n) + \frac{h}{2}F(x_{n+1}), \quad n = 0, \ldots, \lfloor T/h \rfloor. \] (11)
It is not obvious that relation (11) is well-defined.

In [2], we have analyzed the set-valued implicit Euler scheme
\[ x_{n+1} \in x_n + hF(x_{n+1}), \quad n = 0, \ldots, \lfloor T/h \rfloor \]
in detail. Here, we just want to demonstrate that the elementary properties of the set-valued Crank-Nicolson scheme (11) follow immediately from our implicit function theorems.

**Theorem 5** If \( F : \mathbb{R}^m \to \mathcal{C}(\mathbb{R}^m) \) is usc and ROSL with constant \( l \), then the set-valued Crank-Nicolson scheme defined by (11) has a well-defined solution which depends continuously on \( h \in \left(-\frac{|l|}{2}, \frac{|l|}{2}\right) \). In addition, if \( F \) is uniformly continuous or Lipschitz continuous, then the values of the Crank-Nicolson scheme depend continuously or in a Lipschitz continuous way on \( x \).

**Proof** Solvability, dependence on \( h \):
Consider some fixed \( x \in \mathbb{R}^m \) and set \( U_1 := (\mathbb{R}^m, \frac{|l|}{2}) \subset \mathbb{R} \) and \( U_2 := B_R(x) \subset \mathbb{R}^m \) for some \( R > 0 \). Then \( G : U_1 \times U_2 \to \mathcal{C}(\mathbb{R}^m) \) defined by
\[ G(h, z) := x + \frac{h}{2}F(x) + \frac{h}{2}F(z) - z \]
satisfies \( G(0, x) = 0 \) and is uniformly Lipschitz continuous in \( h \) and usc and ROSL in \( z \) with constant \( l_G := \frac{1}{2}lh - 1 < 0 \). Thus Theorem 4 guarantees that relation (11) is solvable in the sense that there exist neighbourhoods \( V_1 \subset U_1 \) and \( V_2 \subset U_2 \) of 0 and \( x \) and a Lipschitz continuous mapping \( N_x : V_1 \to \mathcal{C}(\mathbb{R}^m) \) (with Lipschitz constant depending on \( R \)) such that \( N_x(h) \subset V_2 \) for all \( h \in V_1 \) and \( 0 \in G(h, z) \) is equivalent with \( z \in N_x(h) \). In particular, \( N_x(h) \to \{x\} \) as \( h \to 0 \).

In the proof of Theorem 4, the neighbourhood \( V_1 \) is defined by
\[ h\|F(x)\| = \|G(h, x)\| < -l_G R. \]
Thus, if we choose $R > 0$ large enough, we obtain $V_1 = U_1$, so that our numerical scheme is well-defined for all $h \in (-\frac{2}{l}, \frac{2}{l})$.

In fact, our solution $N_x(h)$ is globally unique, because estimate (5) implies that the resolving mapping will not change if we further enlarge $R$.

**Dependence on $x$:**

Now fix $h \in (-\frac{2}{l}, \frac{2}{l})$, assume that $F$ is uniformly continuous, and consider the mapping $H : \mathbb{R}^m \to \mathcal{C}(\mathbb{R}^m)$ given by

$$H(x, z) := x + \frac{h}{2}F(x) + \frac{h}{2}F(z) - z,$$

which is uniformly continuous in $x$ with modulus of continuity independent of $z$ and continuous and ROSL with constant $l_H := \frac{1}{2}lh - 1 < 0$ in $z$. Hence Theorem 3 applies and yields the existence of a continuous resolving mapping $N_h : \mathbb{R}^m \to \mathcal{C}(\mathbb{R}^m)$ which is Lipschitz whenever $F$ is Lipschitz.

Note that $N_x(h) = N_h(x)$ follows from the maximality of the mappings $N_x(\cdot)$ and $N_h(\cdot)$.

Of course, the estimates for the images of $N(\cdot)$ from Theorem 4 apply to $N_x(\cdot)$ for any $x \in \mathbb{R}^m$, so that

$$\text{dist}(N_x(h), x) \leq -\frac{2}{lh - 2}\|G(h, x)\| = -\frac{2h}{lh - 2}\|F(x)\|$$

and

$$\text{diam } N_x(h) \leq -\frac{2}{lh - 2}\sup_{z \in N_{G,x}(h)} \text{diam } G(h, z) \leq -\frac{1}{lh - 2}(\text{diam } F(x) + \sup_{z \in N_{G,x}(h)} \text{diam } F(z)).$$

### 5.3 Differential algebraic inclusions

A thorough treatment of the DAE

$$\dot{x} = f(x, y), \quad 0 = g(x, y) \quad (12)$$

under OSL conditions is given in [9]. The authors require that the map $g$ satisfies a OSL for some $l < 0$ with respect to $y$ and that $f$ is OSL with respect to $x$, and both functions are Lipschitz with respect to the remaining variables. Then the algebraic equation $0 = g(x, y)$ has a unique solution $y = n(x)$ for all $x$, and the authors prove that the resulting differential equation

$$\dot{x} = f(x, n(x))$$

has a OSL right hand side. This leads to estimates of the longterm behavior which can be transferred to discretized versions of (12).

Now let $F : \mathbb{R}^k \times \mathbb{R}^m \to \mathcal{C}(\mathbb{R}^k)$ be continuous and ROSL with constant $l_1 \in \mathbb{R}$ in the first and Lipschitz continuous with constant $L_1 > 0$ in the second argument. Furthermore, let $G : \mathbb{R}^k \times \mathbb{R}^m \to \mathcal{C}(\mathbb{R}^m)$ be Lipschitz continuous with constant...
\[ L_2 > 0 \] in the first and use and ROSL with constant \( l_2 < 0 \) in the second argument. This setting together with the inclusion

\[ \dot{x} \in F(x, y), \quad 0 \in G(x, y) \tag{13} \]

is a straight-forward generalization of system (12).

In this situation, we can apply Theorem 3, which guarantees the existence of a resolving mapping \( N : \mathbb{R}^k \to \mathcal{C}(\mathbb{R}^m) \) such that \( 0 \in G(x, y) \) is equivalent with \( y \in N(x) \). Since \( x \mapsto G(x, y) \) is Lipschitz with constant \( L_2 \), the mapping \( N(\cdot) \) is Lipschitz with constant \( -\frac{L_2}{l_2} \). Hence, we can rewrite inclusion (13) as

\[ \dot{x} \in F(x, N(x)) =: H(x). \tag{14} \]

We claim that the set-valued mapping \( H : \mathbb{R}^k \to \mathcal{C}(\mathbb{R}^k) \) is ROSL with constant \( l_1 - \frac{L_1 L_2}{l_2} \). Indeed, the Lipschitz continuity of \( F(x, \cdot) \) and compactness of \( N(x) \) ensure that \( H(x) \) is bounded. Let \( (v_i) \subset H(x) \) be a convergent sequence with limit \( v \). As \( v_i \in F(x, N(x)) \), there exist elements \( w_i \in N(x) \) such that \( v_i \in F(x, w_i) \) for all \( i \). By compactness of \( N(x) \), there exists a convergent subsequence (again denoted \( w_i \)) converging to some \( w \in N(x) \). Then

\[ \text{dist}(v_i, F(x, w)) \leq \text{dist}(v_i, F(x, w_i)) + \text{dist}(F(x, w_i), F(x, w)) \leq L_1 |w_i - w| \to 0 \]

shows that \( v \in F(x, w) \subset F(x, N(x)) \), and consequently, \( H(x) \) is closed.

If \( x, x' \in \mathbb{R}^k \) and \( y \in H(x) \) are given, then there exists some \( z \in N(x) \) such that \( y \in F(x, z) \). As \( F(\cdot, z) \) is ROSL, there exists some \( y'' \in F(x', z) \) satisfying

\[ \langle y - y'', x - x' \rangle \leq l_1 |x - x'|^2. \]

Since \( N(\cdot) \) is Lipschitz, there exists some \( z' \in N(x') \) with

\[ |z - z'| \leq \frac{L_2}{l_2} |x - x'|, \]

and by Lipschitz continuity of \( F(x', \cdot) \), there is some \( y' \in F(x', z') \) such that

\[ |y'' - y'| \leq \frac{L_1 L_2}{l_2} |x - x'|. \]

In particular, we have

\[ \langle y - y', x - x' \rangle = \langle y - y'', x - x' \rangle + \langle y'' - y', x - x' \rangle \leq l_1 |x - x'|^2 + |y'' - y'| \cdot |x - x'| \leq l_1 |x - x'|^2 - \frac{L_1 L_2}{l_2} |x - x'|^2 \]

so that \( H \) is ROSL.
Moreover, the set-valued mapping \( H \) is continuous: We have to estimate 
\[
\text{dist}(F(x, N(x)), F(x', N(x'))) \quad \text{for fixed } x \text{ and } x' \text{ in a neighbourhood of } x.
\]
Clearly, 
\[
\text{dist}(F(x, N(x)), F(x', N(x'))) \leq \text{dist}(F(x, N(x)), F(x', N(x))) + \text{dist}(F(x', N(x)), F(x', N(x'))).
\]
As \((x, y) \mapsto F(x, y)\) is Lipschitz continuous in \(y\) with a constant independent of \(x\), we have 
\[
\text{dist}(F(x', N(x)), F(x', N(x'))) \to 0 \text{ as } x' \to x.
\]

On the other hand, 
\[
\text{dist}(F(x, N(x)), F(x', N(x))) \leq \sup_{z \in N(x)} \text{dist}(F(x, z), F(x', z)).
\]
Assume that \(\sup_{z \in N(x)} \text{dist}(F(x, z), F(x', z))\) does not tend to zero as \(x' \to x\). In that case, there exists some \(\varepsilon > 0\) and sequences \(x_n \in \mathbb{R}^m\) and \(z_n \in N(x)\) such that 
\[
|x - x_n| \to 0 \text{ as } n \to \infty \quad \text{and} \quad \text{dist}(F(x, z_n), F(x_n, z_n)) > \varepsilon \text{ for all } n \in \mathbb{N}.
\]
Since \(N(x)\) is compact, we can extract a subsequence (without changing notation) such that \(z_n \to z \in N(x)\). But then, 
\[
\text{dist}(F(x, z_n), F(x_n, z_n)) \\ \leq \text{dist}(F(x, z_n), F(x, z)) + \text{dist}(F(x, z), F(x_n, z)) + \text{dist}(F(x_n, z), F(x_n, z_n)) \\ \leq \varepsilon
\]
for sufficiently large \(n\), because in the first two terms, one of the arguments is fixed, and the third term is small, because \((x, z) \mapsto F(x, z)\) is Lipschitz in \(z\) with a constant independent of \(x\). Thus, we obtain a contradiction, and \(H\) is continuous.

We summarize the above discussion in the following theorem.

**Theorem 6** Let \(F : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k\) be continuous and ROSL with constant \(l_1 \in \mathbb{R}\) in the first and Lipschitz continuous with constant \(L_1 > 0\) in the second argument. Moreover, let \(G : \mathbb{R}^k \times \mathbb{R}^m \to \mathcal{C}(\mathbb{R}^m)\) be a set-valued mapping which is Lipschitz continuous with constant \(L_2 > 0\) in the first and usc and ROSL with constant \(l_2 < 0\) in the second argument.

Then the differential algebraic inclusion (13) can be reformulated as a differential inclusion
\[
\dot{x} \in H(x)
\]
with a continuous right hand side \(H : \mathbb{R}^k \to \mathcal{C}(\mathbb{R}^k)\) which is ROSL with constant \(l_1 - L_1 L_2 l_2\).

It is well-known (see e.g. [5]) that such a differential inclusion has a solution on the unbounded time interval \([0, \infty)\). On every bounded interval \([0, T]\), the set of its solutions is dense in the set of solutions of the convexified problem
\[
\dot{x} \in \text{cov} F(x)
\]
with respect to the supremum norm, which is compact with respect to the same topology.
References