Semigroup approach to birth-and-death stochastic dynamics in continuum

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September 23, 2011

Abstract

We describe a general approach to the construction of the state evolution corresponding to the birth-and-death Markov generators of spatial dynamics in $\mathbb{R}^d$. We give the conditions on the birth and death intensities which are sufficient for the existence evolutions as strongly continuous semigroups in proper Banach spaces of correlation functions satisfying Ruelle bounds. The convergence in the Vlasov-type scaling for the corresponding stochastic dynamics is considered.

Key words. C_0-semigroups, continuous systems , Markov evolution, spatial birth-and-death dynamics, correlation functions, evolution equations, Vlasov scaling, Vlasov equation, scaling limits

AMS subject classification.46E30, 47D06, 82C21, 35Q83

1 Introduction

Spatial Markov processes in $\mathbb{R}^d$ may be described as stochastic evolutions of locally finite point configurations. One of the most important classes of such stochastic dynamics is presented by birth-and-death Markov processes in the configuration space Γ over $\mathbb{R}^d$. These are processes in which an infinite number of individuals exist at each instant and the rate at which new individuals appear or old ones disappear depends on the instantaneous configuration of existing individuals [15]. Corresponding Markov generators have a natural heuristic representation in terms of a birth intensity which characterizes an appearing of a new point in $\mathbb{R}^d$ (in the presence of a given configuration) and the death intensity describing the event in which a point from the configuration disappears (depending on the location of the remaining configuration points).

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Spatial birth and death processes in which the birth and death rates depend on the configuration of the system were first studied by C.Preston [26]. His approach considers the solution of the backward Kolmogorov equation under the restriction that only a finite number of individuals are alive at each moment of time. Under certain conditions, the processes exist and are temporally ergodic, that is, there exists a unique stationary distribution. The more general setting only requires that the number of points alive in any compact set remains finite at all times. Next steps in the study of these processes were done by R.Holley and D.Stroock in [15]. They describe in detail an analytic framework for birth-and-death dynamics and analyzed, in particular, the case of birth-and-death in a bounded region.

Stochastic equations for spatial birth and death processes were formulated in [13] using a spatial version of the time-change approach. Later in [14] these processes were represented as solutions to a system of stochastic equations, and conditions for existence and uniqueness of solutions for these equations as well as for the corresponding martingale problems were given. However, in [14], the quite restrictive assumptions on the birth and death rates do not give a possibility to apply these results to several particular models, interesting for applications.

A growing interest in the study of spatial birth-and-death processes that we observe last years is stimulated, between others, by an important role which they play in several applications. For example, in spatial plant ecology a general approach to the so-called individual based models was developed in a series of works, e.g. [2, 3, 4, 25] and references therein. These models are described as birth-and-death Markov processes in the configuration space Γ with specific rates reflecting biological notions of the competition between plants, their establishment and fecundity. Other sources of birth-and-death processes may be found in the mathematical physics. In particular, Glauber-type stochastic dynamics in Γ are properly associated with grand canonical Gibbs measures for classical gases giving a possibility to study these measures as equilibrium states for specific birth-and-death Markov evolutions.

Concerning the study of particular birth-and-death models we shall stress that, on the one hand, for the most of cases appearing in applications, the existence problem for corresponding Markov processes remains an open. In fact, only particular Glauber dynamics and continuous contact model [22] admit a construction of related processes. From the other hand, main questions arising in the statistical description of concrete models are related with the state evolution in course of the stochastic dynamics. A mathematical formulation of these questions may be realized through the forward Kolmogorov equations for probability measures (states) on the configuration space Γ. In the physical literature the latter equations are known as Fokker–Planck equations. However, just the existence of the corresponding Markov process will not give much constructive information about the properties of the state evolution.

An important technical observation concerns a possibility to reformulate the equations for states in terms of time evolutions of corresponding correlation functions, see e.g. [12]. The resulting so-called hierarchical equations may be analyzed by means of pre-dual infinite chains of evolutional equations on Fock-type spaces using proper perturbation techniques for the construction of associated semigroups. This approach was successfully applied to the construction and analysis of the state evolutions for different versions of the Glauber dynamics [19, 11, 6] and for some spatial ecology models [9]. Each of the considered models required an own specific version of the semigroup construction which takes into account particular properties of corresponding birth and death rates.

In the present paper, we develop a general approach to the construction of the state evolution corresponding to the birth-and-death Markov generators. We give conditions on the birth and death intensities which are sufficient for the existence of the corresponding evolutions as strongly continuous semigroups in proper Banach spaces of correlation functions satisfying Ruelle bound. Moreover, we apply this construction to study the convergence of the stochastic dynamics in Γ in the Vlasov-type scaling. This gives a rigorous background for the convergence of the rescaled hierarchical equations to the solution of the limiting Vlasov hierarchy, and for the derivation of a resulting non-linear evolutional equation for the density of the limiting system. We consider some particular birth-and-death models to show how the general conditions proposed in the paper may be verified in applications. These examples include some previously studied dynamics as well as new models from mathematical physics and spatial ecology areas.

The structure of the paper is the following. In Section 2 we give a brief introduction to the configuration space terminology. Subsection 3.1 is devoted to the evolution of the so-called quasi-observables in the Fock-type space which is the pre-dual to the space of correlation functions. We assume that the diagonal part (in a Fock structure of the operator) is dominating. This assumption is illustrated by several examples mentioned above. Under this domination condition we prove the existence of a holomorphic semigroup on the Fock-type space (Theorem 3.2). The dual evolution of correlation functions is considered in Subsection 3.2. This evolution is not generating by a C_0-semigroup in the whole Banach space of correlation functions. In Theorem 3.8 we introduce a proper Banach subspace which is invariant with respect to this evolution. It is worth noting that this subspace does not depend on the generator. The question concerning existence and uniqueness of the solution to the corresponding stationary equation in the space of correlation functions is considered in Subsection 3.3. In Section 4 we discuss the Vlasov-type scaling for birth-and-death stochastic dynamics introduced in [10]. In particular, Theorem 4.4 states strong convergence of rescaled semigroups to the limiting one in the Fock-type space.

2 Basic facts and notation

Let $\mathcal{B}(\mathbb{R}^d)$ be the family of all Borel sets in $\mathbb{R}^d$, $d \geq 1$; $\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ denotes the system of all bounded sets from $\mathcal{B}(\mathbb{R}^d)$.

The configuration space over space $\mathbb{R}^d$ consists of all locally finite subsets (configurations) of $\mathbb{R}^d$. Namely,

$$\Gamma = \Gamma\bigl(\mathbb{R}^d\bigr) := \Bigl\{\gamma \subset \mathbb{R}^d \Bigm| |\gamma_\Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)\Bigr\}. \tag{2.1}$$

Here $|\cdot|$ means the cardinality of a set, and $\gamma_\Lambda := \gamma \cap \Lambda$. The space Γ is equipped with the vague topology, i.e., the weakest topology for which all mappings $\Gamma \ni \gamma \mapsto \sum_{x\in\gamma} f(x) \in \mathbb{R}$ are continuous for any continuous function f on $\mathbb{R}^d$ with compact support. The corresponding Borel σ-algebra $\mathcal{B}(\Gamma)$ is the smallest σ-algebra for which all mappings $\Gamma \ni \gamma \mapsto |\gamma_\Lambda| \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ are measurable for any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, see e.g. [1]. It is worth noting that Γ is a Polish space (see e.g. [18] and references therein).

The space of n-point configurations in $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma^{(n)}(Y) := \Bigl\{\eta \subset Y \Bigm| |\eta| = n\Bigr\}, \quad n \in \mathbb{N}.$$

We set $\Gamma^{(0)}(Y) := \{\emptyset\}$. As a set, $\Gamma^{(n)}(Y)$ may be identified with the symmetrization of $\widetilde{Y^n} = \bigl\{(x_1,\dots,x_n) \in Y^n \bigm| x_k \neq x_l \text{ if } k \neq l\bigr\}$. Hence one can introduce the corresponding Borel σ-algebra, which we denote by $\mathcal{B}\bigl(\Gamma^{(n)}(Y)\bigr)$. The space of finite configurations in $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined as

$$\Gamma_0(Y) := \bigsqcup_{n\in\mathbb{N}_0} \Gamma^{(n)}(Y).$$

This space is equipped with the topology of the disjoint union. Let $\mathcal{B}\bigl(\Gamma_0(Y)\bigr)$ denote the corresponding Borel σ-algebra. In the case of $Y = \mathbb{R}^d$ we will omit the index Y in the previously defined notations. Namely, $\Gamma_0 := \Gamma_0(\mathbb{R}^d)$, $\Gamma^{(n)} := \Gamma^{(n)}(\mathbb{R}^d)$.

The restriction of the Lebesgue product measure $(dx)^n$ to $\bigl(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)})\bigr)$ we denote by $m^{(n)}$. We set $m^{(0)} := \delta_{\{\emptyset\}}$. The Lebesgue–Poisson measure λ on Γ_0 is defined by

$$\lambda := \sum_{n=0}^{\infty} \frac{1}{n!} m^{(n)}. \tag{2.2}$$

For any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ the restriction of λ to $\Gamma(\Lambda) := \Gamma_0(\Lambda)$ will be also denoted by λ. The space $\bigl(\Gamma, \mathcal{B}(\Gamma)\bigr)$ is the projective limit of the family of spaces $\bigl\{\bigl(\Gamma(\Lambda), \mathcal{B}(\Gamma(\Lambda))\bigr)\bigr\}_{\Lambda\in\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)}$. The Poisson measure π on $\bigl(\Gamma, \mathcal{B}(\Gamma)\bigr)$ is given as the projective limit of the family of measures $\{\pi^\Lambda\}_{\Lambda\in\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)}$, where $\pi^\Lambda := e^{-m(\Lambda)}\lambda$ is the probability measure on $\bigl(\Gamma(\Lambda), \mathcal{B}(\Gamma(\Lambda))\bigr)$ and $m(\Lambda)$ is the Lebesgue measure of $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ (see e.g. [1] for details).

A set $M \in \mathcal{B}(\Gamma_0)$ is called bounded if there exists $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $M \subset \bigsqcup_{n=0}^{N} \Gamma^{(n)}(\Lambda)$. The set of bounded measurable functions with bounded support we denote by $B_{\mathrm{bs}}(\Gamma_0)$, i.e., $G \in B_{\mathrm{bs}}(\Gamma_0)$ if $G \restriction_{\Gamma_0\setminus M} = 0$ for some bounded $M \in \mathcal{B}(\Gamma_0)$. Any $\mathcal{B}(\Gamma_0)$-measurable function G on Γ_0, in fact, is defined by a sequence of functions $\bigl\{G^{(n)}\bigr\}_{n\in\mathbb{N}_0}$ where $G^{(n)}$ is a $\mathcal{B}(\Gamma^{(n)})$-measurable function on $\Gamma^{(n)}$. The set of *cylinder functions* on Γ we denote by

$\mathcal{F}_{\rm cyl}(\Gamma)$. Each $F \in \mathcal{F}_{\rm cyl}(\Gamma)$ is characterized by the following relation: $F(\gamma) = F(\gamma_\Lambda)$ for some $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$. Functions on Γ will be called *observables* whereas functions on Γ_0 well be called *quasi-observables*.

There exists mapping from $B_{\rm bs}(\Gamma_0)$ into $\mathcal{F}_{\rm cyl}(\Gamma)$, which plays the key role in our further considerations:

$$KG(\gamma) := \sum_{\eta \Subset \gamma} G(\eta), \quad \gamma \in \Gamma, \tag{2.3}$$

where $G \in B_{\rm bs}(\Gamma_0)$, see e.g. [16, 23, 24]. The summation in (2.3) is taken over all finite subconfigurations $\eta \in \Gamma_0$ of the (infinite) configuration $\gamma \in \Gamma$; we denote this by the symbol, $\eta \Subset \gamma$. The mapping K is linear, positivity preserving, and invertible, with

$$K^{-1}F(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0. \tag{2.4}$$

Set $(K_0 G)(\eta) := (KG)(\eta)$, $\eta \in \Gamma_0$.

The so-called coherent state corresponding to a $\mathcal{B}(\mathbb{R}^d)$-measurable function f is defined by

$$e_\lambda(f, \eta) := \prod_{x \in \eta} f(x), \ \eta \in \Gamma_0 \setminus \{\emptyset\}, \quad e_\lambda(f, \emptyset) := 1.$$

Then

$$(K_0 e_\lambda(f))(\eta) = e_\lambda(f + 1, \eta), \quad \eta \in \Gamma_0 \tag{2.5}$$

and for any $f \in L^1(\mathbb{R}^d, dx)$

$$\int_{\Gamma_0} e_\lambda(f, \eta) d\lambda(\eta) = \exp\Big\{ \int_{\mathbb{R}^d} f(x) dx \Big\}. \tag{2.6}$$

A measure $\mu \in \mathcal{M}^1_{\rm fm}(\Gamma)$ is called locally absolutely continuous w.r.t. (with respect to) the Poisson measure π if for any $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ the projection of μ onto $\Gamma(\Lambda)$ is absolutely continuous w.r.t. the projection of π onto $\Gamma(\Lambda)$. In this case, according to [16], there exists a *correlation functional* $k_\mu : \Gamma_0 \to \mathbb{R}_+$ such that for any $G \in B_{\rm bs}(\Gamma_0)$ the following equality holds

$$\int_\Gamma (KG)(\gamma) d\mu(\gamma) = \int_{\Gamma_0} G(\eta) k_\mu(\eta) d\lambda(\eta). \tag{2.7}$$

The functions $k_\mu^{(n)} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}_+$ given by

$$k_\mu^{(n)}(x_1, \ldots, x_n) := \begin{cases} k_\mu(\{x_1, \ldots, x_n\}), & \text{if } (x_1, \ldots, x_n) \in \widetilde{(\mathbb{R}^d)^n} \\ 0, & \text{otherwise} \end{cases}$$

are called *correlation functions* of the measure μ. Note that $k_\mu^{(0)} = 1$.

Below we would like to mention without proof the partial case of the well-known technical lemma (see e.g. [21]) which plays very important role in our calculations.

Lemma 2.1. *For any measurable function* $H : \Gamma_0 \times \Gamma_0 \times \Gamma_0 \to \mathbb{R}$

$$\int_{\Gamma_0} \sum_{\xi \subset \eta} H(\xi, \eta \setminus \xi, \eta)\, d\lambda(\eta) = \int_{\Gamma_0} \int_{\Gamma_0} H(\xi, \eta, \eta \cup \xi)\, d\lambda(\xi)\, d\lambda(\eta) \tag{2.8}$$

if both sides of the equality make sense.

3 Non-equilibrium evolutions

In birth-and-death dynamics, particles appear and disappear randomly in $\mathbb{R}^d$ according to birth and death rates which depend on the configuration of the whole system. Heuristically, the corresponding Markov generator is described by the following expression

$$\begin{aligned}(LF)(\gamma) :=& \sum_{x \in \gamma} d(x, \gamma \setminus x)\, [F(\gamma \setminus x) - F(\gamma)] \\ &+ \int_{\mathbb{R}^d} b(x, \gamma)\, [F(\gamma \cup x) - F(\gamma)]\, dx,\end{aligned} \tag{3.1}$$

where the coefficient $d(x, \gamma) \geq 0$ represent the rate at which particle of the configuration γ located at x dies or disappears, whereas, for the given configuration γ, the new particle appears at the site x with the rate $b(x, \gamma) \geq 0$.

We always suppose that for all $x \in \mathbb{R}^d$ and a.a. $x' \in \mathbb{R}^d$ the values $d(x, \eta)$ and $b(x', \eta)$ are finite at least for all configurations $\eta \in \Gamma_0$ which do not contain the points x and x'. Here and subsequently, we assume that for a.a. $x \in \mathbb{R}^d$ the functions $d(x, \cdot)$ and $b(x, \cdot)$ are locally integrable, i.e. for all bounded $M \in \mathcal{B}(\Gamma_0)$

$$\int_M \big(d(x, \eta) + b(x, \eta)\big) d\lambda(\eta) < \infty.$$

3.1 Evolutions in the space of quasi-observables

It is very important to emphasize, that, in general, the r.h.s. (right hand side) of (3.1) is not defined for all $\gamma \in \Gamma$ even if $F \in K\big(B_{\mathrm{bs}}(\Gamma_0)\big) \subset \mathcal{F}_{\mathrm{cyl}}(\Gamma)$. Moreover, in this case, $LF \notin \mathcal{F}_{\mathrm{cyl}}(\Gamma)$. However, for concrete birth and death rates we may usually define $(LF)(\eta)$ at least for all $\eta \in \Gamma_0$. Therefore, the expression $K^{-1}LF$ may be defined via (2.4) point-wisely. This fact allows us to consider the so-called K-image of the mapping L. Namely, for $G \in B_{bs}(\Gamma_0)$ we denote $(\hat{L}G)(\eta) := (K^{-1}LKG)(\eta)$, $\eta \in \Gamma_0$. In the next proposition we give the explicit form of this image.

Proposition 3.1. *For any* $G \in B_{bs}(\Gamma_0)$ *the following formula holds*

$$\begin{aligned}(\hat{L}G)(\eta) =& -\sum_{\xi \subset \eta} G(\xi) \sum_{x \in \xi} \big(K_0^{-1} d(x, \cdot \cup \xi \setminus x)\big)(\eta \setminus \xi) \\ &+ \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi \cup x) \big(K_0^{-1} b(x, \cdot \cup \xi)\big)(\eta \setminus \xi) dx, \qquad \eta \in \Gamma_0,\end{aligned} \tag{3.2}$$

provided all terms of the right hand side have sense.

Proof. Following [12], for

$$B_x = K_0^{-1} b(x,\cdot), \quad D_x = K_0^{-1} d(x,\cdot) \tag{3.3}$$

we have

$$(\hat{L}G)(\eta) = -\sum_{x\in\eta} (D_x \star G(\cdot\cup x))(\eta\setminus x) + \int_{\mathbb{R}^d} (B_x \star G(\cdot\cup x))(\eta)dx. \tag{3.4}$$

Here for the given $\mathcal{B}(\Gamma_0)$-measurable functions G_1 and G_2, we define

$$(G_1 \star G_2)(\eta) = \sum_{(\eta_1,\eta_2,\eta_3)\in\mathcal{P}_3(\eta)} G_1(\eta_1\cup\eta_2)G_2(\eta_2\cup\eta_3), \quad \eta\in\Gamma_0, \tag{3.5}$$

where $\mathcal{P}_3(\eta)$ denotes the set of all partitions of η in three parts which may be empty, see [16]. Rewriting (3.5) in the form

$$(G_1 \star G_2)(\eta) = \sum_{\xi\subset\eta} G_1(\xi) \sum_{\zeta\subset\xi} G_2((\eta\setminus\xi)\cup\zeta), \quad \eta\in\Gamma_0, \tag{3.6}$$

we get

$$\begin{aligned}(\hat{L}G)(\eta) = &-\sum_{x\in\eta}\sum_{\xi\subset\eta\setminus x} G(\xi\cup x)\sum_{\zeta\subset\xi} D_x(((\eta\setminus x)\setminus\xi)\cup\zeta)\\ &+\int_{\mathbb{R}^d}\sum_{\xi\subset\eta} G(\xi\cup x)\sum_{\beta\subset\xi} B_x((\eta\setminus\xi)\cup\beta)dx.\end{aligned}$$

Using the fact that for any $\mathcal{B}(\Gamma_0)$-measurable function G

$$(K_0G)(\eta_1\cup\eta_2) = \sum_{\xi\subset\eta_1\cup\eta_2} G(\xi) = \sum_{\xi_1\subset\eta_1}\sum_{\xi_2\subset\eta_2} G(\xi_1\cup\xi_2), \quad \eta_1\cap\eta_2=\emptyset,$$

for $F = K_0G$ we get

$$\left(K_0^{-1}F(\cdot\cup\eta_2)\right)(\xi_1) = \left(K_0G(\xi_1\cup\cdot)\right)(\eta_2), \quad \xi_1\cap\eta_2=\emptyset. \tag{3.7}$$

Now, the simple equality

$$\sum_{x\in\eta}\sum_{\xi\subset\eta\setminus x} h(x,\xi,\eta) = \sum_{\xi\subset\eta}\sum_{x\in\xi} h(x,\xi\setminus x,\eta), \tag{3.8}$$

which holds for any $\mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(\Gamma_0)\times\mathcal{B}(\Gamma_0)$-measurable function h finishes the proposition. □

In general, the r.h.s. of (3.2) may be undefined. For arbitrary and fixed $C > 1$ we consider the functional space

$$\mathcal{L}_C := L^1(\Gamma_0, C^{|\eta|}\lambda(d\eta)). \tag{3.9}$$

Throughout of the whole paper, symbol $\|\cdot\|_C$ stands for the norm of the space (3.9). Now we proceed to study rigorous properties of the operator given by the expression (3.2) in the Banach space $\mathcal{L}_C$.

Remark 3.1. $B_{\rm bs}(\Gamma_0)$ is a dense set in $\mathcal{L}_C$.

Set,

$$D(\eta) := \sum_{x\in\eta} d(x,\eta\setminus x) \geq 0, \quad \eta\in\Gamma_0; \tag{3.10}$$

$$\mathcal{D} := \{G\in\mathcal{L}_C \mid D(\cdot)\, G\in\mathcal{L}_C\}. \tag{3.11}$$

Note that $B_{\rm bs}(\Gamma_0)\subset\mathcal{D}$. In particular, $\mathcal{D}$ is a dense set in $\mathcal{L}_C$.

We will show that $(\hat{L},\mathcal{D})$ given by (3.2), (3.11) generates C_0-semigroup on $\mathcal{L}_C$.

Theorem 3.2. *Suppose that there exists $a_1\geq 1$, $a_2>0$ such that for all $\xi\in\Gamma_0$ and a.a. $x\in\mathbb{R}^d$*

$$\sum_{x\in\xi}\int_{\Gamma_0}\left|K_0^{-1}d(x,\cdot\cup\xi\setminus x)\right|(\eta)\, C^{|\eta|}d\lambda(\eta) \leq a_1 D(\xi), \tag{3.12}$$

$$\sum_{x\in\xi}\int_{\Gamma_0}\left|K_0^{-1}b(x,\cdot\cup\xi\setminus x)\right|(\eta)\, C^{|\eta|}d\lambda(\eta) \leq a_2 D(\xi). \tag{3.13}$$

and, moreover,

$$a_1 + \frac{a_2}{C} < \frac{3}{2}. \tag{3.14}$$

Then $(\hat{L},\mathcal{D})$ is the generator of a holomorphic semigroup $\hat{T}(t)$ on $\mathcal{L}_C$.

Proof. Let us consider the multiplication operator $(L_0,\mathcal{D})$ on $\mathcal{L}_C$ given by

$$(L_0 G)(\eta) = -D(\eta)\, G(\eta), \quad G\in\mathcal{D}, \quad \eta\in\Gamma_0. \tag{3.15}$$

We recall that a densely defined closed operators A on $\mathcal{L}_C$ is called sectorial of angle $\omega\in(0,\frac{\pi}{2})$ if its resolvent set $\rho(A)$ contains the sector

$$\mathrm{Sect}\left(\frac{\pi}{2}+\omega\right) := \left\{z\in\mathbb{C}\;\Big|\; |\arg z| < \frac{\pi}{2}+\omega\right\}\setminus\{0\}$$

and for each $\varepsilon\in(0;\omega)$ there exists $M_\varepsilon\geq 1$ such that

$$||R(z,A)|| \leq \frac{M_\varepsilon}{|z|} \tag{3.16}$$

for all $z\neq 0$ with $|\arg z| \leq \dfrac{\pi}{2}+\omega-\varepsilon$. Here and below we will use notation

$$R(z,A) := (z1\!\!1 - A)^{-1}, \quad z\in\rho(A).$$

The set of all sectorial operators of angle $\omega\in(0,\frac{\pi}{2})$ in $\mathcal{L}_C$ we denote by $\mathcal{H}_C(\omega)$. Any $A\in\mathcal{H}_C(\omega)$ is a generator of a bounded semigroup $T(t)$ which is holomorphic in the sector $|\arg t|<\omega$ (see e.g. [5, Theorem II.4.6]). One can prove the following lemma.

Lemma 3.3. *The operator* $(L_0, \mathcal{D})$ *given by* (3.15) *is a generator of a contraction semigroup on* $\mathcal{L}_C$. *Moreover,* $L_0 \in \mathcal{H}_C(\omega)$ *for all* $\omega \in (0, \frac{\pi}{2})$ *and* (3.16) *holds with* $M_\varepsilon = \frac{1}{\cos\omega}$ *for all* $\varepsilon \in (0;\omega)$.

Proof of Lemma 3.3. It is not difficult to show that the densely defined operator L_0 is closed in $\mathcal{L}_C$. Let $0 < \omega < \frac{\pi}{2}$ be arbitrary and fixed. Clear, that for all $z \in \operatorname{Sect}\left(\frac{\pi}{2} + \omega\right)$

$$\left|D(\eta) + z\right| > 0, \quad \eta \in \Gamma_0.$$

Therefore, for any $z \in \operatorname{Sect}\left(\frac{\pi}{2} + \omega\right)$ the inverse operator $R(z, L_0) = (z1\!\!1 - L_0)^{-1}$, the action of which is given by

$$[R(z, L_0)G](\eta) = \frac{1}{D(\eta) + z}\, G(\eta), \tag{3.17}$$

is well defined on the whole space $\mathcal{L}_C$. Moreover,

$$|D(\eta) + z| = \sqrt{(D(\eta) + \operatorname{Re} z)^2 + (\operatorname{Im} z)^2} \geq \begin{cases} |z|, & \text{if } \operatorname{Re} z \geq 0 \\ |\operatorname{Im}\, z|, & \text{if } \operatorname{Re} z < 0 \end{cases},$$

and for any $z \in \operatorname{Sect}\left(\frac{\pi}{2} + \omega\right)$

$$|\operatorname{Im}\, z| = |z||\sin\arg z| \geq |z|\left|\sin\left(\frac{\pi}{2} + \omega\right)\right| = |z|\cos\omega.$$

As a result, for any $z \in \operatorname{Sect}\left(\frac{\pi}{2} + \omega\right)$

$$||R(z, L_0)|| \leq \frac{1}{|z|\cos\omega}, \tag{3.18}$$

that implies the second assertion. Note also that $|D(\eta) + z| \geq \operatorname{Re} z$ for $\operatorname{Re} z > 0$, hence,

$$||R(z, L_0)|| \leq \frac{1}{\operatorname{Re} z}, \tag{3.19}$$

that proves the first statement by the classical Hille–Yosida theorem. □

For any $G \in B_{bs}(\Gamma_0)$ we define

$$\begin{aligned}(L_1 G)(\eta) &:= (\hat{L}G)(\eta) - (L_0 G)(\eta) \\ &= -\sum_{\xi \subsetneq \eta} G(\xi) \sum_{x \in \xi} \left(K_0^{-1} d(x, \cdot \cup \xi \setminus x)\right)(\eta \setminus \xi) \\ &\quad + \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi \cup x)\left(K_0^{-1} b(x, \cdot \cup \xi)\right)(\eta \setminus \xi)\, dx. \end{aligned} \tag{3.20}$$

Next Lemma shows that, under conditions (3.12), (3.13) above, the operator L_1 is relatively bounded by the operator L_0.

Lemma 3.4. *Let* (3.12), (3.13) *hold. Then* $(L_1, \mathcal{D})$ *is a well-defined operator in* $\mathcal{L}_C$ *such that*

$$\|L_1 R(z, L_0)\| \le a_1 - 1 + \frac{a_2}{C}, \quad \operatorname{Re} z > 0 \tag{3.21}$$

and

$$\|L_1 G\| \le \Big(a_1 - 1 + \frac{a_2}{C}\Big) \|L_0 G\|, \quad G \in \mathcal{D}. \tag{3.22}$$

Proof of Lemma 3.4. By Lemma 2.1, we have for any $G \in \mathcal{L}_C$, $\operatorname{Re} z > 0$

$$\begin{aligned}
&\int_{\Gamma_0} \Big| - \sum_{\xi \subsetneq \eta} \frac{1}{z + D(\xi)} G(\xi) \sum_{x \in \xi} \big(K_0^{-1} d(x, \cdot \cup \xi \setminus x)\big)(\eta \setminus \xi) \Big| C^{|\eta|} d\lambda\,(\eta) \\
\le& \int_{\Gamma_0} \sum_{\xi \subsetneq \eta} \frac{1}{|z + D(\xi)|} |G(\xi)| \sum_{x \in \xi} \big|K_0^{-1} d(x, \cdot \cup \xi \setminus x)\big|(\eta \setminus \xi) C^{|\eta|} d\lambda\,(\eta) \\
=& \int_{\Gamma_0} \frac{1}{|z + D(\xi)|} |G(\xi)| \sum_{x \in \xi} \int_{\Gamma_0} \big|K_0^{-1} d(x, \cdot \cup \xi \setminus x)\big|(\eta) C^{|\eta|} d\lambda\,(\eta)\, C^{|\xi|} d\lambda\,(\xi) \\
&- \int_{\Gamma_0} \frac{1}{|z + D(\eta)|} D\,(\eta)\, |G(\eta)|\, C^{|\eta|} d\lambda\,(\eta) \\
\le& (a_1 - 1) \int_{\Gamma_0} \frac{1}{\operatorname{Re} z + D(\eta)} D(\eta) |G(\eta)| C^{|\eta|} d\lambda(\eta) \le (a_1 - 1) \|G\|_C,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Gamma_0} \Big| \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} \frac{1}{z + D(\xi \cup x)} G(\xi \cup x) \big(K_0^{-1} b(x, \cdot \cup \xi)\big)(\eta \setminus \xi) dx \Big| C^{|\eta|} d\lambda\,(\eta) \\
\le& \int_{\Gamma_0} \int_{\Gamma_0} \int_{\mathbb{R}^d} \frac{1}{|z + D(\xi \cup x)|} |G(\xi \cup x)|\, \big|K_0^{-1} b(x, \cdot \cup \xi)\big|\,(\eta) dx C^{|\eta|} C^{|\xi|} d\lambda\,(\xi)\, d\lambda\,(\eta) \\
=& \frac{1}{C} \int_{\Gamma_0} \frac{1}{|z + D(\xi)|} |G(\xi)| \sum_{x \in \xi} \int_{\Gamma_0} \big|K_0^{-1} b(x, \cdot \cup \xi \setminus x)\big|\,(\eta) C^{|\eta|} d\lambda\,(\eta)\, C^{|\xi|} d\lambda\,(\xi) \\
\le& \frac{a_2}{C} \int_{\Gamma_0} \frac{1}{\operatorname{Re} z + D(\xi)} |G(\xi)|\, D(\xi) C^{|\xi|} d\lambda\,(\xi) \le \frac{a_2}{C} \|G\|_C.
\end{aligned}$$

Combining these inequalities we obtain (3.21). The same considerations yield

$$\begin{aligned}
&\int_{\Gamma_0} \Big| - \sum_{\xi \subsetneq \eta} G(\xi) \sum_{x \in \xi} \big(K_0^{-1} d(x, \cdot \cup \xi \setminus x)\big)(\eta \setminus \xi) \Big| C^{|\eta|} d\lambda\,(\eta) \\
&+ \int_{\Gamma_0} \Big| \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi \cup x) \big(K_0^{-1} b(x, \cdot \cup \xi)\big)(\eta \setminus \xi) dx \Big| C^{|\eta|} d\lambda\,(\eta) \\
\le& \Big((a_1 - 1) + \frac{a_2}{C}\Big) \int_{\Gamma_0} |G(\eta)|\, D(\eta) C^{|\eta|} d\lambda\,(\eta),
\end{aligned}$$

that proves (3.22) as well. □

And now we proceed to finish the proof of the Theorem 3.2. Let us set $\theta := a_1 + \frac{a_2}{C} - 1 \in \left(0; \frac{1}{2}\right)$. Then $\frac{\theta}{1-\theta} \in (0;1)$. Let $\omega \in \left(0; \frac{\pi}{2}\right)$ be such that $\cos\omega < \frac{\theta}{1-\theta}$. Then, by the proof of Lemma 3.3, $L_0 \in \mathcal{H}_C(\omega)$ and $||R(z, L_0)|| \le \frac{M}{|z|}$ for all $z \neq 0$ with $|\arg z| \le \dfrac{\pi}{2} + \omega$, where $M := \frac{1}{\cos\omega}$. Then

$$\theta = \frac{1}{1 + \frac{1-\theta}{\theta}} < \frac{1}{1 + \frac{1}{\cos\omega}} = \frac{1}{1+M}.$$

Hence, by (3.22) and the proof of [5, Theorem III.2.10], we have that $(\hat{L} = L_0 + L_1, \mathcal{D})$ is a generator of holomorphic semigroup on $\mathcal{L}_C$. $\square$

Remark 3.2. By (3.10), the estimates (3.12), (3.13) are satisfied if

$$\int_{\Gamma_0} \left|K_0^{-1} d\left(x, \cdot \cup \xi\right)\right| (\eta)\, C^{|\eta|} d\lambda\left(\eta\right) \le a_1 d\left(x, \xi\right), \tag{3.23}$$

$$\int_{\Gamma_0} \left|K_0^{-1} b\left(x, \cdot \cup \xi\right)\right| (\eta)\, C^{|\eta|} d\lambda\left(\eta\right) \le a_2 d\left(x, \xi\right). \tag{3.24}$$

Example 1. (Glauber-type dynamics in continuum). Let L be given by (3.1) with

$$d(x, \gamma \setminus x) = \exp\Big\{s \sum_{y \in \gamma\setminus x} \phi(x-y)\Big\}, \qquad x \in \gamma,\ \gamma \in \Gamma, \tag{3.25}$$

$$b(x, \gamma) = z \exp\Big\{(s-1) \sum_{y \in \gamma} \phi(x-y)\Big\}, \qquad x \in \mathbb{R}^d \setminus \gamma,\ \gamma \in \Gamma, \tag{3.26}$$

where $\phi : \mathbb{R}^d \to \mathbb{R}_+$ is a pair potential, $\phi(-x) = \phi(x)$, $z > 0$ is an activity parameter and $s \in [0;1]$. For any $s \in [0;1]$ the operator L is well defined and, moreover, symmetric in the space $L^2(\Gamma, \mu)$, where μ is a Gibbs measure, given by the pair potential ϕ and activity parameter z (see e.g. [20] and references therein). This gives possibility to study the corresponding semigroup in $L^2(\Gamma, \mu)$. In the case $s = 0$, the corresponding dynamics was also studied in another Banach spaces, see e.g. [19, 11, 6]. Below we show that one of the main result of the paper stated in Theorem 3.2 can be applied to the case of arbitrary $s \in [0;1]$. Set

$$\beta_\tau := \int_{\mathbb{R}^d} \left|e^{\tau\phi(x)} - 1\right| dx \in [0;\infty], \quad \tau \in [-1;1]. \tag{3.27}$$

Let s be arbitrary and fixed. Suppose that $\beta_s < \infty$, $\beta_{s-1} < \infty$. Then, by (3.25), (2.5), and (2.6)

$$K_0^{-1} d\left(x, \cdot \cup \xi\right)(\eta) = d(x,\xi)\, e_\lambda(e^{s\phi(x-\cdot)} - 1, \eta),$$

$$\int_{\Gamma_0} \left|K_0^{-1} d\left(x, \cdot \cup \xi\right)\right| (\eta)\, C^{|\eta|} d\lambda\left(\eta\right) = d(x,\xi) e^{C\beta_s},$$

and, analogously,

$$\int_{\Gamma_0} \left|K_0^{-1} b\,(x, \cdot \cup \xi)\right|(\eta)\, C^{|\eta|} d\lambda\,(\eta) = b(x,\xi) e^{C\beta_{s-1}} \le z d(x,\xi) e^{C\beta_{s-1}},$$

since $\phi \ge 0$. Therefore, to apply Theorem 3.2 we should assume additionally that

$$e^{C\beta_s} + \frac{z}{C} e^{C\beta_{s-1}} < \frac{3}{2}. \tag{3.28}$$

In particular, for $s = 0$ we obtain the condition (cf. [19])

$$\frac{z}{C} e^{C\beta_{-1}} < \frac{1}{2}. \tag{3.29}$$

Example 2. (Bolker–Dieckman–Law–Pacala (BDLP) model) This example describes the model of plant ecology, see [9] and references therein. Let L be given by (3.1) with

$$d(x, \gamma \setminus x) = m + \varkappa^- \sum_{y \in \gamma \setminus x} a^-(x-y), \qquad x \in \gamma,\ \gamma \in \Gamma, \tag{3.30}$$

$$b(x,\gamma) = \varkappa^+ \sum_{y \in \gamma} a^+(x-y), \qquad x \in \mathbb{R}^d \setminus \gamma,\ \gamma \in \Gamma, \tag{3.31}$$

where $m > 0$, $\varkappa^\pm \ge 0$, $0 \le a^\pm \in L^1(\mathbb{R}^d, dx) \cap L^\infty(\mathbb{R}^d, dx)$, $\int_{\mathbb{R}^d} a^\pm(x)dx = 1$. Then

$$K_0^{-1} d\,(x, \cdot \cup \xi)\,(\eta) = d(x,\xi) 0^{|\eta|} + \varkappa^- 1\!\!1_{\Gamma^{(1)}}(\eta) \sum_{y \in \eta} a^-(x-y),$$

$$\int_{\Gamma_0} \left|K_0^{-1} d\,(x, \cdot \cup \xi)\right|(\eta)\, C^{|\eta|} d\lambda\,(\eta) = d(x,\xi) + C\varkappa^-,$$

and, analogously,

$$\int_{\Gamma_0} \left|K_0^{-1} b\,(x, \cdot \cup \xi)\right|(\eta)\, C^{|\eta|} d\lambda\,(\eta) = b(x,\xi) + C\varkappa^+.$$

Therefore, if we suppose, for example, that (cf. [9])

$$4\varkappa^- C < m \tag{3.32}$$

$$4\varkappa^+ a^+(x) \le C\varkappa^- a^-(x), \quad x \in \mathbb{R}^d, \tag{3.33}$$

then there exists $\delta > 0$ such that

$$d(x,\xi) + C\varkappa^- \le d(x,\xi) + \frac{m}{4+\delta} \le \Big(1 + \frac{1}{4+\delta}\Big) d(x,\xi)$$

and

$$b(x,\xi) + C\varkappa^+ \le \frac{C}{4} \varkappa^- \sum_{y \in \xi} a^-(x-y) + \frac{Cm}{16} < \frac{C}{4} d(x,\xi),$$

since $4\varkappa^+ \le C\varkappa^- < \frac{m}{4}$. The last bound we get integrating both sides of (3.33) over $\mathbb{R}^d$.

Hence, (3.12), (3.13) hold and

$$a_1 + \frac{a_2}{C} = 1 + \frac{1}{4+\delta} + \frac{1}{4} < \frac{3}{2}.$$

3.2 Evolutions in the space of correlation functions

We denote $d\lambda_C := C^{|\cdot|} d\lambda$; and the dual space $(\mathcal{L}_C)' = \big(L^1(\Gamma_0, d\lambda_C)\big)' = L^\infty(\Gamma_0, d\lambda_C)$. The space $(\mathcal{L}_C)'$ is isometrically isomorphic to the Banach space

$$\mathcal{K}_C := \Big\{ k : \Gamma_0 \to \mathbb{R} \;\Big|\; k \cdot C^{-|\cdot|} \in L^\infty(\Gamma_0, \lambda) \Big\}$$

with the norm

$$\|k\|_{\mathcal{K}_C} := \|C^{-|\cdot|} k(\cdot)\|_{L^\infty(\Gamma_0,\lambda)},$$

where the isomorphism is given by the isometry R_C

$$(\mathcal{L}_C)' \ni k \longmapsto R_C k := k \cdot C^{|\cdot|} \in \mathcal{K}_C. \tag{3.34}$$

In fact, one may consider the duality between the Banach spaces $\mathcal{L}_C$ and $\mathcal{K}_C$ given by the following expression

$$\langle\!\langle G, k \rangle\!\rangle := \int_{\Gamma_0} G \cdot k \, d\lambda, \quad G \in \mathcal{L}_C, \; k \in \mathcal{K}_C \tag{3.35}$$

with $|\langle\!\langle G, k \rangle\!\rangle| \le \|G\|_C \cdot \|k\|_{\mathcal{K}_C}$. It is clear that $k \in \mathcal{K}_C$ implies

$$|k(\eta)| \le \|k\|_{\mathcal{K}_C} C^{|\eta|} \qquad \text{for } \lambda\text{-a.a. } \eta \in \Gamma_0.$$

Let $\big(\hat{L}', \mathrm{Dom}(\hat{L}')\big)$ be an operator in $(\mathcal{L}_C)'$ which is dual to the closed operator $\big(\hat{L}, \mathcal{D}\big)$. We consider also its image on $\mathcal{K}_C$ under the isometry R_C. Namely, let $\hat{L}^* = R_C \hat{L}' R_{C^{-1}}$ with the domain $\mathrm{Dom}(\hat{L}^*) = R_C \mathrm{Dom}(\hat{L}')$.

Remark 3.3. Similarly, one can consider the adjoint semigroup $\hat{T}'(t)$ in $(\mathcal{L}_C)'$ and its image $\hat{T}^*(t)$ in $\mathcal{K}_C$. From general theory (see e.g. [5]) the last semigroup will be weak*-continuous, weak*-differentiable at 0 and $\hat{L}^*$ will be weak*-generator of $\hat{T}^*(t)$. Therefore, one has an evolution in the space of correlation functions. In fact, we have a solution to the evolution equation (3.43) below, in a weak*-sense. This subsection is devoted to the study of a strong solution to this equation.

Proposition 3.5. *Let* (3.12), (3.13) *be satisfied. Suppose that there exists* $A > 0$, $N \in \mathbb{N}_0$, $\nu \ge 1$ *such that for* $\xi \in \Gamma_0$ *and* $x \notin \xi$

$$d\,(x, \xi) \le A(1 + |\xi|)^N \nu^{|\xi|}, \tag{3.36}$$

Then for any $\alpha \in \big(0; \frac{1}{\nu}\big)$

$$\mathcal{K}_{\alpha C} \subset \mathrm{Dom}(\hat{L}^*). \tag{3.37}$$

Proof. In order to show (3.37) it is enough to verify that for any $k \in \mathcal{K}_{\alpha C}$ there exists $k^* \in \mathcal{K}_C$ such that for any $G \in \mathrm{Dom}(\hat{L})$

$$\left\langle\!\left\langle \hat{L}G,\, k \right\rangle\!\right\rangle = \langle\!\langle G,\, k^* \rangle\!\rangle\,. \tag{3.38}$$

According to [12], (3.38) is valid for any $k \in \mathcal{K}_{\alpha C}$ with $k^* = \hat{L}^* k$, where

$$\begin{aligned}(\hat{L}^* k)(\eta) = &- \int_{\Gamma_0} k(\zeta \cup \eta) \sum_{x\in\eta} \sum_{\xi\subset\eta\setminus x} D_x(\zeta\cup\xi) d\lambda(\zeta)\\ &+ \int_{\Gamma_0} \sum_{x\in\eta} k(\zeta\cup(\eta\setminus x)) \sum_{\xi\subset\eta\setminus x} B_x(\zeta\cup\xi)d\lambda(\zeta),\end{aligned}$$

provided $k^* \in \mathcal{K}_C$. Using (3.7), one can rewrite the last expression

$$\begin{aligned}(\hat{L}^* k)(\eta) = &- \sum_{x\in\eta} \int_{\Gamma_0} k(\zeta\cup\eta)\big(K_0^{-1} d(x,\cdot\cup\eta\setminus x)\big)(\zeta)d\lambda(\zeta)\\ &+ \sum_{x\in\eta} \int_{\Gamma_0} k(\zeta\cup(\eta\setminus x))\big(K_0^{-1} b(x,\cdot\cup\eta\setminus x)\big)(\zeta)d\lambda(\zeta).\end{aligned}$$

Then, by (3.12), (3.13), and (3.36),

$$\begin{aligned}&C^{-|\eta|}\left|(\hat{L}^* k)(\eta)\right|\\ \le& C^{-|\eta|} \sum_{x\in\eta} \int_{\Gamma_0} |k(\zeta\cup\eta)|\left|K_0^{-1} d(x,\cdot\cup\eta\setminus x)\right|(\zeta)d\lambda(\zeta)\\ &+ C^{-|\eta|} \sum_{x\in\eta} \int_{\Gamma_0} |k(\zeta\cup(\eta\setminus x))|\left|K_0^{-1} d(x,\cdot\cup\eta\setminus x)\right|(\zeta)d\lambda(\zeta)\\ \le& \|k\|_{\mathcal{K}_{\alpha C}} \alpha^{|\eta|} \sum_{x\in\eta} \int_{\Gamma_0} (\alpha C)^{|\zeta|}\left|K_0^{-1} d(x,\cdot\cup\eta\setminus x)\right|(\zeta)d\lambda(\zeta)\\ &+ \frac{1}{\alpha C} \|k\|_{\mathcal{K}_{\alpha C}} \alpha^{|\eta|} \sum_{x\in\eta} \int_{\Gamma_0} (\alpha C)^{|\zeta|}\left|K_0^{-1} d(x,\cdot\cup\eta\setminus x)\right|(\zeta)d\lambda(\zeta)\\ \le& \|k\|_{\mathcal{K}_{\alpha C}} \left(a_1 + \frac{a_2}{\alpha C}\right) \alpha^{|\eta|} \sum_{x\in\eta} d\,(x,\eta\setminus x)\\ \le& A\,\|k\|_{\mathcal{K}_{\alpha C}} \left(a_1 + \frac{a_2}{\alpha C}\right) \alpha^{|\eta|}(1+|\eta|)^{N+1}\nu^{|\eta|-1}.\end{aligned}$$

Using elementary inequality

$$(1+t)^b a^t \le \frac{1}{a}\left(\frac{b}{-e\ln a}\right)^b, \quad b\ge 1,\ a\in(0;1)\,,\ t\ge 0, \tag{3.39}$$

we have for $\alpha\nu < 1$

$$\operatorname*{ess\,sup}_{\eta\in\Gamma_0} C^{-|\eta|}\left|(\hat{L}^* k)(\eta)\right| \le \|k\|_{\mathcal{K}_{\alpha C}} \left(a_1 + \frac{a_2}{\alpha C}\right) \frac{A}{\alpha\nu^2}\left(\frac{N+1}{-e\ln(\alpha\nu)}\right)^{N+1} < \infty. \ \square$$

Lemma 3.6. *Let* (3.36) *holds. We define for any* $\alpha \in (0;1)$

$$\mathcal{D}_\alpha := \{G \in \mathcal{L}_{\alpha C} \mid D(\cdot)\, G \in \mathcal{L}_{\alpha C}\}.$$

Then for any $\alpha \in (0; \frac{1}{\nu})$

$$\mathcal{D} \subset \mathcal{L}_C \subset \mathcal{D}_\alpha \subset \mathcal{L}_{\alpha C} \tag{3.40}$$

Proof. The first and last inclusions are obvious. To prove the second one, we use (3.36), (3.39) and obtain for any $G \in \mathcal{L}_C$

$$\begin{aligned}\int_{\Gamma_0} D(\eta)\, |G(\eta)|\, (\alpha C)^{|\eta|}\, d\lambda(\eta) &\leq \int_{\Gamma_0} \alpha^{|\eta|} \sum_{x\in\eta} A(1+|\eta|)^N \nu^{|\eta|-1}\, |G(\eta)|\, C^{|\eta|} d\lambda(\eta) \\ &\leq \text{const} \int_{\Gamma_0} |G(\eta)|\, C^{|\eta|} d\lambda(\eta) < \infty.\end{aligned}$$

□

Proposition 3.7. *Let* (3.12), (3.13), *and* (3.36) *hold with*

$$a_1 + \frac{a_2}{\alpha C} < \frac{3}{2} \tag{3.41}$$

for some $\alpha \in (0;1)$. *Then* $(\hat{L}, \mathcal{D}_\alpha)$ *is a generator of a holomorphic semigroup* $\hat{T}_\alpha(t)$ *on* $\mathcal{L}_{\alpha C}$.

Proof. The proof is similar to the proof of Theorem 3.2, taking into account that bounds (3.13), (3.12) imply the same bounds for αC instead of C. Note also that (3.41) is stronger than (3.14). □

Under conditions (3.12), (3.13), (3.41), and (3.36), we consider the adjoint semigroup $\hat{T}'(t)$ in $(\mathcal{L}_C)'$ and its image $\hat{T}^*(t)$ in $\mathcal{K}_C$. By e.g. [5, Subsection II.2.6], the restriction $\hat{T}^\odot(t)$ of the semigroup $\hat{T}^*(t)$ onto its invariant Banach subspace $\overline{\mathrm{Dom}(\hat{L}^*)}$ (here and below all closures are in the norm of the space $\mathcal{K}_C$) is a strongly continuous semigroup. Moreover, its generator $\hat{L}^\odot$ will be a part of $\hat{L}^*$, namely,

$$\mathrm{Dom}(\hat{L}^\odot) = \Big\{k \in \mathrm{Dom}(\hat{L}^*) \,\Big|\, \hat{L}^* k \in \overline{\mathrm{Dom}(\hat{L}^*)}\Big\}$$

and $\hat{L}^* k = \hat{L}^\odot k$ for any $k \in \mathrm{Dom}(\hat{L}^\odot)$.

Theorem 3.8. *Let* (3.12), (3.13), *and* (3.36) *hold with*

$$1 \leq \nu < \frac{C}{a_2}\left(\frac{3}{2} - a_1\right). \tag{3.42}$$

Then for any $\alpha \in \left(\dfrac{a_2}{C\left(\frac{3}{2} - a_1\right)}; \dfrac{1}{\nu}\right)$ *the set* $\overline{\mathcal{K}_{\alpha C}}$ *is a* $\hat{T}^\odot(t)$*-invariant Banach subspace of* $\mathcal{K}_C$.

Proof. First of all note that the condition on α implies (3.41). Next, we prove that $\hat{T}_\alpha(t)\,G = \hat{T}(t)\,G$ for any $G \in \mathcal{L}_C \subset \mathcal{L}_{\alpha C}$. Let $\hat{L}_\alpha = (\hat{L}, \mathcal{D}_\alpha)$ is the operator in $\mathcal{L}_{\alpha C}$. There exists $\omega > 0$ such that $(\omega; +\infty) \subset \rho(\hat{L}) \cap \rho(\hat{L}_\alpha)$, see e.g. [5, Section III.2]. For some fixed $z \in (\omega; +\infty)$ we denote by $R(z, \hat{L}) = \bigl(z1\!\!1 - \hat{L}\bigr)^{-1}$ the resolvent of $(\hat{L}, \mathcal{D})$ in $\mathcal{L}_C$ and by $R(z, \hat{L}_\alpha) = \bigl(z1\!\!1 - \hat{L}_\alpha\bigr)^{-1}$ the resolvent of $\hat{L}_\alpha$ in $\mathcal{L}_{\alpha C}$. Then for any $G \in \mathcal{L}_C$ we have $R(z, \hat{L})G \in \mathcal{D} \subset \mathcal{D}_\alpha$ and

$$R(z, \hat{L})G - R(z, \hat{L}_\alpha)G = R(z, \hat{L}_\alpha)\bigl((z1\!\!1 - \hat{L}_\alpha) - \bigl(z1\!\!1 - \hat{L}\bigr)\bigr)R(z, \hat{L})G = 0,$$

since $\hat{L}_\alpha = \hat{L}$ on $\mathcal{D}$. As a result, $\hat{T}_\alpha(t)\,G = \hat{T}(t)\,G$ on $\mathcal{L}_C$.

Note that for any $G \in \mathcal{L}_C \subset \mathcal{L}_{\alpha C}$ and for any $k \in \mathcal{K}_{\alpha C} \subset \mathcal{K}_C$ we have $\hat{T}_\alpha(t)G \in \mathcal{L}_{\alpha C}$ and

$$\Bigl\langle\!\Bigl\langle \hat{T}_\alpha(t)G, k \Bigr\rangle\!\Bigr\rangle = \Bigl\langle\!\Bigl\langle G, \hat{T}^*_\alpha(t)k \Bigr\rangle\!\Bigr\rangle,$$

where, by the same construction as before, $\hat{T}^*_\alpha(t)k \in \mathcal{K}_{\alpha C}$. But $G \in \mathcal{L}_C$, $k \in \mathcal{K}_C$ implies

$$\Bigl\langle\!\Bigl\langle \hat{T}_\alpha(t)G, k \Bigr\rangle\!\Bigr\rangle = \Bigl\langle\!\Bigl\langle \hat{T}(t)G, k \Bigr\rangle\!\Bigr\rangle = \Bigl\langle\!\Bigl\langle G, \hat{T}^*(t)k \Bigr\rangle\!\Bigr\rangle.$$

Hence, $\hat{T}^*(t)k = \hat{T}^*_\alpha(t)k \in \mathcal{K}_{\alpha C}$ that proves the statement due to continuity of the family $\hat{T}^*(t)$. □

Therefore, one can consider the restriction $\hat{T}^{\odot\alpha}$ of the semigroup $\hat{T}^{\odot}$ onto $\overline{\mathcal{K}_{\alpha C}}$. It will be strongly continuous semigroup with the generator $\hat{L}^{\odot\alpha}$ which is a restriction of $\hat{L}^{\odot}$ onto $\overline{\mathcal{K}_{\alpha C}}$ (see e.g. [5, Subsection II.2.3]). Hence, we have the strong solution (in the sense of the norm in $\mathcal{K}_C$) to the evolution equation

$$\frac{\partial}{\partial t} k_t = \hat{L}^* k_t \tag{3.43}$$

on the subspace $\mathcal{K}_{\alpha C}$.

Remark 3.4. Let us clarify the reasons we avoid a construction of this evolution in $\mathcal{K}_C$ directly, via e.g. perturbation techniques. First of all $(L_0, \mathcal{K}_{\alpha C})$ is not closed operator neither in $\mathcal{K}_C$ nor in $\overline{\mathcal{K}_{\alpha C}}$. To make it closed, one can consider the operator L_0 in $\mathcal{K}_C$ on its maximal domain $\mathcal{D}^* := \{G \in \mathcal{K}_C | DG \in \mathcal{K}_C\}$. However, this domain is not dense in $\mathcal{K}_C$. Under condition of Proposition 3.5 one can show that $\mathcal{K}_{\alpha C} \subset \mathcal{D}^*$, but it is not clear whether $\mathcal{D}^* \subset \overline{\mathcal{K}_{\alpha C}}$. Therefore, we are not able to work in the space $\overline{\mathcal{K}_{\alpha C}}$, staying on the operator-dependent space $\overline{\mathcal{D}^*}$. Suppose one can prove estimate like (3.21). Then one can show that $(\hat{L}^*, \mathcal{D}^*)$ will be a generator of a C_0-semigroup $W(t)$ on $\overline{\mathcal{D}^*}$. Even in this case it seems to be very difficult to show that this semigroup will be $\mathcal{K}_{\alpha C}$-invariant.

Example 1 (revisited)**.** To apply Theorem 3.8 to Example 1 it is enough to check (3.36) and (3.42). One has

$$d(x, \xi) = \exp\Bigl\{ s \sum_{y \in \xi} \phi(x - y) \Bigr\} \le \nu^{|\xi|},$$

where $\nu = 1$ for $s = 0$ and $\nu = e^{s\bar{\phi}} \geq 1$, $\bar{\phi} = \max_{x\in\mathbb{R}^d} \phi(x)$ for $s \in (0;1]$ provided ϕ is bounded on $\mathbb{R}^d$. If $s = 0$ then (3.42) is true (whenever condition (3.29) is satisfied). For the bounded ϕ and $s \in (0;1]$ one may rewrite (3.42) in the following form:

$$e^{C\beta_s} + \frac{z}{C} e^{s\bar{\phi}+C\beta_{s-1}} < \frac{3}{2}. \tag{3.44}$$

Note, that (3.44) is the stronger version of condition (3.28).

Example 2 (revisited)**.** According to (3.32)–(3.33),

$$\begin{aligned} d(x,\xi) &= m + \varkappa^- \sum_{y\in\xi} a^-(x-y) \leq m + A^- \varkappa^- |\xi| \\ &< m + A^- \frac{m}{4C} |\xi| < m\Big(1 + \frac{A^-}{4C}\Big)(1+|\xi|), \end{aligned}$$

where $A^- = \|a^-\|_{L^\infty(\mathbb{R}^d)}$. Therefore, (3.36) holds with $\nu = 1$, which makes (3.42) obvious.

3.3 Stationary equation

In this subsection we study the question about stationary solutions to (3.43). For any $s \geq 0$, we consider the following subset of $\mathcal{K}_C$

$$\mathcal{K}_{\alpha C}^{(s)} := \big\{ k \in \mathcal{K}_{\alpha C} \bigm| k(\emptyset) = s \big\}.$$

We define $\widetilde{\mathcal{K}}$ to be the closure of $\mathcal{K}_{\alpha C}^{(0)}$ in the norm of $\mathcal{K}_C$. It is clear that $\widetilde{\mathcal{K}}$ with the norm of $\mathcal{K}_C$ is a Banach space.

Proposition 3.9. *Let* (3.12)*,* (3.13)*, and* (3.36) *be satisfied with*

$$a_1 + \frac{a_2}{C} < 2. \tag{3.45}$$

Assume, additionally, that

$$d(x,\emptyset) > 0, \quad x \in \mathbb{R}^d. \tag{3.46}$$

Then for any $\alpha \in (0; \frac{1}{\nu})$ *the stationary equation*

$$\hat{L}^* k = 0 \tag{3.47}$$

has a unique solution k_{inv} *from* $\mathcal{K}_{\alpha C}^{(1)}$ *which is given by the expression*

$$k_{\mathrm{inv}} = 1^* + (1\!\!1 - S)^{-1} E. \tag{3.48}$$

Here 1^* *denotes the function defined by* $1^*(\eta) = 0^{|\eta|}$*,* $\eta \in \Gamma_0$*, the function* $E \in \mathcal{K}_{\alpha C}^{(0)}$ *is such that*

$$E(\eta) = 1\!\!1_{\Gamma^{(1)}}(\eta) \sum_{x\in\eta} \frac{b(x,\emptyset)}{d(x,\emptyset)}, \quad \eta \in \Gamma_0,$$

and S is a generalized Kirkwood–Salzburg operator on $\tilde{\mathcal{K}}$, given by

$$(Sk)(\eta) = -\frac{1}{D(\eta)}\sum_{x\in\eta}\int_{\Gamma_0\setminus\{\emptyset\}} k(\zeta\cup\eta)(K_0^{-1}d(x,\cdot\cup\eta\setminus x))(\zeta)d\lambda(\zeta) \tag{3.49}$$
$$+\frac{1}{D(\eta)}\sum_{x\in\eta}\int_{\Gamma_0} k(\zeta\cup(\eta\setminus x))(K_0^{-1}b(x,\cdot\cup\eta\setminus x))(\zeta)d\lambda(\zeta),$$

for $\eta\neq\emptyset$ and $(Sk)(\emptyset)=0$. In particular, if $b(x,\emptyset)=0$ for a.a. $x\in\mathbb{R}^d$ then this solution is such that

$$k_{\mathrm{inv}}^{(n)} = 0, \qquad n\geq 1. \tag{3.50}$$

Remark 3.5. It is worth noting that (3.23), (3.24) imply (3.46).

Proof. Suppose that (3.47) holds for some $k\in\mathcal{K}_{\alpha C}^{(1)}$. Then

$$D(\eta)\,k(\eta) = -\sum_{x\in\eta}\int_{\Gamma_0\setminus\{\emptyset\}} k(\zeta\cup\eta)\big(K_0^{-1}d(x,\cdot\cup\eta\setminus x)\big)(\zeta)d\lambda(\zeta)$$
$$+\sum_{x\in\eta}\int_{\Gamma_0} k(\zeta\cup(\eta\setminus x))\big(K_0^{-1}b(x,\cdot\cup\eta\setminus x)\big)(\zeta)d\lambda(\zeta). \tag{3.51}$$

The equality (3.51) is satisfied for any $k\in\mathcal{K}_{\alpha C}^{(1)}$ at the point $\eta=\emptyset$. Using the fact that $D(\emptyset)=0$ one may rewrite (3.51) in terms of the function $\tilde{k}=k-1^*\in\mathcal{K}_{\alpha C}^{(0)}$. Namely,

$$D(\eta)\,\tilde{k}(\eta) = -\sum_{x\in\eta}\int_{\Gamma_0\setminus\{\emptyset\}} \tilde{k}(\zeta\cup\eta)\big(K_0^{-1}d(x,\cdot\cup\eta\setminus x)\big)(\zeta)d\lambda(\zeta)$$
$$+\sum_{x\in\eta}\int_{\Gamma_0} \tilde{k}(\zeta\cup(\eta\setminus x))\big(K_0^{-1}b(x,\cdot\cup\eta\setminus x)\big)(\zeta)d\lambda(\zeta).$$
$$+\sum_{x\in\eta}0^{|\eta\setminus x|}b(x,\eta\setminus x). \tag{3.52}$$

As a result,

$$\tilde{k}(\eta) = (S\tilde{k})(\eta) + E(\eta), \quad \eta\in\Gamma_0.$$

Next, for $\eta\neq\emptyset$

$$C^{-|\eta|}\left|(Sk)(\eta)\right|$$
$$\leq\frac{C^{-|\eta|}}{D(\eta)}\sum_{x\in\eta}\int_{\Gamma_0\setminus\{\emptyset\}}|k(\zeta\cup\eta)|\left|(K_0^{-1}d(x,\cdot\cup\eta\setminus x))(\zeta)\right|d\lambda(\zeta)$$
$$+\frac{C^{-|\eta|}}{D(\eta)}\sum_{x\in\eta}\int_{\Gamma_0}k(\zeta\cup(\eta\setminus x))\left|(K_0^{-1}b(x,\cdot\cup\eta\setminus x))(\zeta)\right|d\lambda(\zeta)$$
$$\leq\frac{\|k\|_{\mathcal{K}_C}}{D(\eta)}\sum_{x\in\eta}\int_{\Gamma_0\setminus\{\emptyset\}}C^{|\zeta|}\left|(K_0^{-1}d(x,\cdot\cup\eta\setminus x))(\zeta)\right|d\lambda(\zeta)$$

$$+\frac{\|k\|_{\mathcal{K}_C}}{D(\eta)}\frac{1}{C}\sum_{x\in\eta}\int_{\Gamma_0}C^{|\zeta|}\left|(K_0^{-1}b(x,\cdot\cup\eta\setminus x))(\zeta)\right|d\lambda(\zeta)$$

$$\leq\frac{\|k\|_{\mathcal{K}_C}}{D(\eta)}D(\eta)\Big(a_1-1+\frac{a_2}{C}\Big)=\Big(a_1-1+\frac{a_2}{C}\Big)\|k\|_{\mathcal{K}_C}.$$

Hence,

$$\|S\|=a_1+\frac{a_2}{C}-1<1$$

in $\widetilde{K}$. This finishes the proof. □

Remark 3.6. The name of the operator (3.49) is motivated by Example 1. Namely, if $s=0$ then the operator (3.49) has form

$$(Sk)(\eta)=\frac{1}{m|\eta|}\sum_{x\in\eta}e_\lambda(e^{-\phi(x-\cdot)},\eta\setminus x)\int_{\Gamma_0}k(\zeta\cup(\eta\setminus x))e_\lambda(e^{-\phi(x-\cdot)}-1,\zeta)d\lambda(\zeta),$$

that is quite similar of the so-called Kirkwood–Salsburg operator known in mathematical physics (see e.g. [27, 17]). For $s=0$ condition (3.45) has form $\frac{z}{C}e^{C\beta_{-1}}<1$ (cf. (3.29)). Under this condition, the stationary solution to (3.47) is a unique and coincides with the correlation function of the Gibbs measure, corresponding to potential ϕ and activity z.

Remark 3.7. It is worth pointing out that $b(x,\emptyset)=0$ in the case of Example 2. Therefore, if we suppose (cf. (3.32), (3.33)) that $2\varkappa^-C<m$ and $2\varkappa^+a^+(x)\leq C\varkappa^-a^-(x)$, for $x\in\mathbb{R}^d$, condition (3.45) will be satisfied. However, the unique solution to (3.47) will be given by (3.50). In the next example we improve this statement.

Example 3. Let us consider the following natural modification of BDLP-model coming from Example 2: let d be given by (3.30) and

$$b(x,\gamma)=\kappa+\varkappa^+\sum_{y\in\gamma}a^+(x-y),\quad x\in\mathbb{R}^d\setminus\gamma,\ \gamma\in\Gamma,\tag{3.53}$$

where $\varkappa^+,a^+$ are as before and $\kappa>0$. Then, under assumptions

$$2\max\Big\{\varkappa^-C;\frac{2\kappa}{C}\Big\}<m\tag{3.54}$$

and

$$2\varkappa^+a^+(x)\leq C\varkappa^-a^-(x),\quad x\in\mathbb{R}^d,\tag{3.55}$$

we obtain for some $\delta>0$

$$\int_{\Gamma_0}\left|K_0^{-1}d(x,\cdot\cup\xi)\right|(\eta)C^{|\eta|}d\lambda(\eta)=d(x,\xi)+C\varkappa^-\leq\Big(1+\frac{1}{2+\delta}\Big)d(x,\xi)$$

$$\int_{\Gamma_0}\left|K_0^{-1}b(x,\cdot\cup\xi)\right|(\eta)C^{|\eta|}d\lambda(\eta)=b(x,\xi)+C\varkappa^+$$

$$\leq\kappa+\frac{1}{2}C\varkappa^-\sum_{y\in\xi}a^-(x-y)+\frac{m}{4}C<\frac{C}{2}d(x,\xi).$$

The latter inequalities imply (3.45). In this case, $E(\eta)=1\!\!1_{\Gamma^{(1)}}(\eta)\frac{\kappa}{m}$.

Remark 3.8. If $a^+(x) = a^-(x)$, $x \in \mathbb{R}^d$ and $\varkappa^+ = z\varkappa^-$, $\kappa = zm$ for some $z > 0$ then $b(x, \gamma) = zd(x, \gamma)$ and the Poisson measure π_z with the intensity z will be symmetrizing measure for the operator L. In particular, it will be invariant measure. This fact means that its correlation function $k_z(\eta) = z^{|\eta|}$ is a solution to (3.47). Conditions (3.54) and (3.55) in this case are equivalent to $4z < C$ and $2\varkappa^- C < m$. As a result, due to uniqueness of such solution,

$$1^*(\eta) + z(1\!\!1 - S)^{-1} 1\!\!1_{\Gamma^{(1)}}(\eta) = z^{|\eta|}, \quad \eta \in \Gamma_0.$$

4 Scalings

For the reader convenience, we start from the idea of the Vlasov-type scaling. The general scheme for the birth-and-death dynamics as well as for the conservative ones may be found in [10]. The realizations of this approach for the Glauber dynamics (Example 1 with $s = 0$) and for the BDLP dynamics (Example 2) were considered in [8, 7], correspondingly. The idea of the Vlasov-type scaling consists in the following.

We would like to construct some scaling L_ε, $\varepsilon > 0$, of the generator L, such that the following scheme holds. Suppose that we have a semigroup $\hat{U}_\varepsilon(t)$ with the generator $\hat{L}_\varepsilon$ in some $\mathcal{L}_{C_\varepsilon}$, $\varepsilon > 0$. Consider the dual semigroup $\hat{U}_\varepsilon^*(t)$. Let us choose an initial function of the corresponding Cauchy problem with a singularity in ε. Namely, $\varepsilon^{|\eta|} k_0^{(\varepsilon)}(\eta) \sim r_0(\eta)$, $\varepsilon \to 0$, $\eta \in \Gamma_0$ for some function r_0, which is independent of ε. The scaling $L \mapsto L_\varepsilon$ should be chosen in such a way that first of all the corresponding semigroup $\hat{U}_\varepsilon^*(t)$ preserves the order of the singularity:

$$\varepsilon^{|\eta|} (\hat{U}_\varepsilon^*(t) k_0^{(\varepsilon)})(\eta) \sim r_t(\eta), \quad \varepsilon \to 0, \quad \eta \in \Gamma_0, \tag{4.1}$$

and, secondly, the dynamics $r_0 \mapsto r_t$ preserves the Lebesgue–Poisson exponents. Namely, if $r_0(\eta) = e_\lambda(\rho_0, \eta)$ then $r_t(\eta) = e_\lambda(\rho_t, \eta)$. There exists explicit (nonlinear, in general) differential equation for ρ_t:

$$\frac{\partial}{\partial t} \rho_t(x) = \upsilon(\rho_t)(x) \tag{4.2}$$

which will be called the Vlasov-type equation.

Now we explain an informal way to realize such a scheme. Let us consider for any $\varepsilon > 0$ the following mapping (cf. (3.34)) defined for functions on Γ_0

$$(R_\varepsilon r)(\eta) := \varepsilon^{|\eta|} r(\eta). \tag{4.3}$$

This mapping is "self-dual" w.r.t. the duality (3.35), moreover, $R_\varepsilon^{-1} = R_{\varepsilon^{-1}}$. Having $R_\varepsilon k_0^{(\varepsilon)} \sim r_0$, $\varepsilon \to 0$, we need $r_t \sim R_\varepsilon \hat{U}_\varepsilon^*(t) k_0^{(\varepsilon)} \sim R_\varepsilon \hat{U}_\varepsilon^*(t) R_{\varepsilon^{-1}} r_0$, $\varepsilon \to 0$. Therefore, we have to show that for any $t \geq 0$ the operator family $R_\varepsilon \hat{U}_\varepsilon^*(t) R_{\varepsilon^{-1}}$, $\varepsilon > 0$ has limiting (in a proper sense) operator $U(t)$ and

$$U(t) e_\lambda(\rho_0) = e_\lambda(\rho_t). \tag{4.4}$$

But, heuristically, $\hat{U}^*_\varepsilon(t) = \exp\{t\hat{L}^*_\varepsilon\}$ and $R_\varepsilon \hat{U}^*_\varepsilon(t) R_{\varepsilon^{-1}} = \exp\{tR_\varepsilon \hat{L}^*_\varepsilon R_{\varepsilon^{-1}}\}$. Let us consider the "renormalized" operator

$$\hat{L}^*_{\varepsilon,\,\mathrm{ren}} := R_\varepsilon \hat{L}^*_\varepsilon R_{\varepsilon^{-1}}. \tag{4.5}$$

In fact, we need that there exists an operator $\hat{L}^*_V$ such that $\exp\{tR_\varepsilon \hat{L}^*_\varepsilon R_{\varepsilon^{-1}}\} \to \exp\{t\hat{L}^*_V\} =: U(t)$ satisfying (4.4). Therefore, an heuristic way to produce scaling $L \mapsto L_\varepsilon$ is to demand that

$$\lim_{\varepsilon\to 0}\left(\frac{\partial}{\partial t} e_\lambda(\rho_t,\eta) - \hat{L}^*_{\varepsilon,\,\mathrm{ren}} e_\lambda(\rho_t,\eta)\right) = 0, \qquad \eta\in\Gamma_0$$

provided ρ_t satisfies (4.2). The point-wise limit of $\hat{L}^*_{\varepsilon,\,\mathrm{ren}}$ will be natural candidate for $\hat{L}^*_V$.

Note that (4.5) implies informally that $\hat{L}_{\varepsilon,\,\mathrm{ren}} = R_{\varepsilon^{-1}} \hat{L}_\varepsilon R_\varepsilon$. We propose below the scheme to give rigorous meaning to the idea introduced above. We consider, for a proper scaling L_ε, the "renormalized" operator $\hat{L}_{\varepsilon,\,\mathrm{ren}}$ and prove that it is a generator of a strongly continuous contraction semigroup $\hat{U}_{\varepsilon,\,\mathrm{ren}}(t)$ in $\mathcal{L}_C$. Next, we show that the formal limit $\hat{L}_V$ of $\hat{L}_{\varepsilon,\,\mathrm{ren}}$ is a generator of a strongly continuous contraction semigroup $\hat{U}_V(t)$ in $\mathcal{L}_C$. Finally, we prove that $\hat{U}_{\varepsilon,\,\mathrm{ren}}(t) \to \hat{U}_V(t)$ strongly in $\mathcal{L}_C$. This implies weak*-convergence of the dual semigroups $\hat{U}^*_{\varepsilon,\,\mathrm{ren}}(t)$ to $\hat{U}^*_V(t)$. We explain also in which sense $\hat{U}^*_V(t)$ satisfies the properties above.

Let us consider for any $\varepsilon\in(0;1]$ the following scaling of (3.1)

$$\begin{aligned}(L_\varepsilon F)(\gamma) := &\sum_{x\in\gamma} d_\varepsilon(x,\gamma\setminus x)\,[F(\gamma\setminus x) - F(\gamma)]\\ &+\varepsilon^{-1}\int_{\mathbb{R}^d} b_\varepsilon(x,\gamma)\,[F(\gamma\cup x) - F(\gamma)]\,dx,\end{aligned} \tag{4.6}$$

and define the renormalized operator $\hat{L}_{\varepsilon,\mathrm{ren}} := R_{\varepsilon^{-1}} K^{-1} L_\varepsilon K R_\varepsilon$. Using the same arguments as in the proof of Proposition 3.1, we get

$$\begin{aligned}(\hat{L}_{\varepsilon,\mathrm{ren}}G)(\eta) = &-\sum_{\xi\subset\eta} G(\xi)\varepsilon^{-|\eta\setminus\xi|}\sum_{x\in\xi}\big(K_0^{-1}d_\varepsilon(x,\cdot\cup\xi\setminus x)\big)(\eta\setminus\xi)\\ &+\sum_{\xi\subset\eta}\int_{\mathbb{R}^d} G(\xi\cup x)\varepsilon^{-|\eta\setminus\xi|}\big(K_0^{-1}b_\varepsilon(x,\cdot\cup\xi)\big)(\eta\setminus\xi)dx.\end{aligned} \tag{4.7}$$

Below we generalize slightly the previous introduced notations: for $\varepsilon\in(0;1]$, $\alpha\in(0;1)$

$$\begin{aligned} D_\varepsilon(\eta) &:= \sum_{x\in\eta} d_\varepsilon(x,\eta\setminus x);\\ \mathcal{D}^{(\varepsilon)} &:= \big\{G\in\mathcal{L}_C \;\big|\; D^{(\varepsilon)}(\cdot)G\in\mathcal{L}_C\big\};\\ (L_0^{(\varepsilon)}G)(\eta) &:= -D_\varepsilon(\eta)G(\eta), \qquad G\in\mathcal{D}^{(\varepsilon)};\\ (L_1^{(\varepsilon)}G)(\eta) &:= (\hat{L}_{\varepsilon,\mathrm{ren}}G)(\eta) - (L_0^{(\varepsilon)}G)(\eta), \qquad G\in\mathcal{D}^{(\varepsilon)}.\end{aligned}$$

Suppose that there exists $a_1 \geq 1$, $a_2 > 0$, $A > 0$, $N \in \mathbb{N}_0$, $\nu \geq 1$ such that for all $\xi \in \Gamma_0$, for a.a. $x \in \mathbb{R}^d$, and for any $\varepsilon \in (0;1]$

$$\sum_{x\in\xi} \int_{\Gamma_0} \left|K_0^{-1} d_\varepsilon\left(x, \cdot \cup \xi \setminus x\right)\right| (\eta)\, \varepsilon^{-|\eta|} C^{|\eta|} d\lambda\left(\eta\right) \leq a_1 D_\varepsilon\left(\xi\right), \tag{4.8}$$

$$\sum_{x\in\xi} \int_{\Gamma_0} \left|K_0^{-1} b_\varepsilon\left(x, \cdot \cup \xi \setminus x\right)\right| (\eta)\, \varepsilon^{-|\eta|} C^{|\eta|} d\lambda\left(\eta\right) \leq a_2 D_\varepsilon\left(\xi\right), \tag{4.9}$$

$$d_\varepsilon\left(x, \xi\right) \leq A(1+|\xi|)^N \nu^{|\xi|}. \tag{4.10}$$

Without loss of generality we will assume that all constant in (4.8)–(4.10) are the same as before.

Proposition 4.1. *1. Let conditions* (4.8) *and* (4.9) *hold with*

$$a_1 + \frac{a_2}{C} < \frac{3}{2}. \tag{4.11}$$

Then, for any $\varepsilon \in (0;1]$, $\left(\hat{L}_{\varepsilon,\mathrm{ren}}, \mathcal{D}^{(\varepsilon)}\right)$*is a generator of the holomorphic semigroup* $\hat{U}_\varepsilon(t)$ *on* $\mathcal{L}_C$*.*

2. Assume, additionally, that (4.10) *is satisfied with*

$$1 \leq \nu < \frac{C}{a_2}\Big(\frac{3}{2} - a_1\Big). \tag{4.12}$$

Then there exists $\alpha_0 \in (0; \frac{1}{\nu})$ *such that for any* $\alpha \in (\alpha_0; \frac{1}{\nu})$ *and for any* $\varepsilon \in (0;1]$ *there exists a strongly continuous semigroup* $\hat{U}_\varepsilon^{\odot\alpha}(t)$ *on the space* $\overline{\mathcal{K}_{\alpha C}}$ *with the generator* $\hat{L}_\varepsilon^{\odot\alpha} = \hat{L}^*_{\varepsilon,\mathrm{ren}}$ *on the domain*

$$\mathrm{Dom}\big(L_\varepsilon^{\odot\alpha}\big) = \big\{k \in \overline{\mathcal{K}_{\alpha C}} \;\big|\; \hat{L}^*_{\varepsilon,\mathrm{ren}} k \in \overline{\mathcal{K}_{\alpha C}}\big\}.$$

Note that, for $k \in \mathcal{K}_{\alpha C}$

$$\begin{aligned}(\hat{L}^*_{\varepsilon,\mathrm{ren}} k)(\eta) = &- \sum_{x\in\eta} \int_{\Gamma_0} k(\xi\cup\eta)\varepsilon^{-|\xi|}\big(K_0^{-1} d_\varepsilon(x, \cdot\cup\eta\setminus x)\big)(\xi) d\lambda(\xi) \\ &+ \sum_{x\in\eta} \int_{\Gamma_0} k(\xi\cup(\eta\setminus x))\varepsilon^{-|\xi|}\big(K_0^{-1} b_\varepsilon(x, \cdot\cup\eta\setminus x)\big)(\xi) d\lambda(\xi).\end{aligned} \tag{4.13}$$

Proof. 1. Identically to the proof of Lemma 3.3 we show that $\big(L_0^{(\varepsilon)}, \mathcal{D}^{(\varepsilon)}\big) \in \mathcal{H}_C(\omega)$ for any $\omega \in (0; \frac{\pi}{2})$. Next, in the same way as in the proof of Lemma 3.4 we prove that, for any $\mathrm{Re}\, z > 0$,

$$\big\|L_1^{(\varepsilon)} R(z, L_0^{(\varepsilon)})\big\| \leq a_1 - 1 + \frac{a_2}{C} < \frac{1}{2}, \tag{4.14}$$

since (3.45) is satisfied. Note also that we may show also another bound (cf. (3.22)):

$$\|L_1^{(\varepsilon)}G\| < \frac{1}{2}\|L_0^{(\varepsilon)}G\|, \quad G \in \mathcal{L}_C. \tag{4.15}$$

Hence, one can prove the statement in the same way as Theorem 3.2.

2. Similarly to Proposition 3.5, we obtain that, under condition (4.10), $\mathcal{K}_{\alpha C} \subset \mathrm{Dom}(\hat{L}^*_{\varepsilon,\mathrm{ren}})$ for any $\alpha \in (0; \frac{1}{\nu})$. Using (4.12), we are able to choose $\theta \in (a_1 + \frac{a_2\nu}{C}; \frac{3}{2})$. Then (3.42) is satisfied, and $\alpha_0 := \frac{a_2}{C(\theta - a_1)} \in (0; \frac{1}{\nu})$. The same considerations as in Theorem 3.8 finish the proof. □

Assumption 4.1. For all $\eta, \xi \in \Gamma_0$ and a.a. $x \in \mathbb{R}^d$ the following limits exist and coincide:

$$\lim_{\varepsilon\to 0} \varepsilon^{-|\eta|}\bigl(K_0^{-1}d_\varepsilon(x, \cdot \cup \xi)\bigr)(\eta) = \lim_{\varepsilon\to 0} \varepsilon^{-|\eta|}\bigl(K_0^{-1}d_\varepsilon(x, \cdot)\bigr)(\eta) =: D_x^V(\eta); \tag{4.16}$$

$$\lim_{\varepsilon\to 0} \varepsilon^{-|\eta|}\bigl(K_0^{-1}b_\varepsilon(x, \cdot \cup \xi)\bigr)(\eta) = \lim_{\varepsilon\to 0} \varepsilon^{-|\eta|}\bigl(K_0^{-1}b_\varepsilon(x, \cdot)\bigr)(\eta) =: B_x^V(\eta). \tag{4.17}$$

We would like to emphasize, that above limits should not depend on ξ. The collection of examples for such d_ε, b_ε can be found in [10]. Note that (4.16), (4.17) imply, in particular,

$$\lim_{\varepsilon\to 0} d_\varepsilon(x,\xi) = D_x^V(\emptyset), \qquad \lim_{\varepsilon\to 0} b_\varepsilon(x,\xi) = B_x^V(\emptyset), \tag{4.18}$$

for all $\xi \in \Gamma_0$ and a.a. $x \in \mathbb{R}^d$.

Combining condition (4.16) with (4.17), we have point-wise limit for $\hat{L}_{\varepsilon,\mathrm{ren}}$:

$$(\hat{L}_V G)(\eta) := -\sum_{\xi\subset\eta} G(\xi) \sum_{x\in\xi} D_x^V(\eta\setminus\xi) + \sum_{\xi\subset\eta}\int_{\mathbb{R}^d} G(\xi\cup x) B_x^V(\eta\setminus\xi)dx. \tag{4.19}$$

Set

$$\begin{aligned}
D_V(\eta) &:= \sum_{x\in\eta} D_x^V(\emptyset);\\
\mathcal{D}^V &:= \bigl\{G \in \mathcal{L}_C \bigm| D_V(\cdot)G \in \mathcal{L}_C\bigr\};\\
(L_0^V G)(\eta) &:= -D_V(\eta)G(\eta), \qquad G \in \mathcal{D}^V;\\
(L_1^V G)(\eta) &:= (\hat{L}_V G)(\eta) - (L_0^V G)(\eta), \qquad G \in \mathcal{D}^V.
\end{aligned}$$

Suppose that for a.a. $x \in \mathbb{R}^d$

$$\int_{\Gamma_0} \bigl|D_x^V(\eta)\bigr| C^{|\eta|} d\lambda(\eta) \le a_1 D_x^V(\emptyset), \tag{4.20}$$

$$\int_{\Gamma_0} \bigl|B_x^V(\eta)\bigr| C^{|\eta|} d\lambda(\eta) \le a_2 D_x^V(\emptyset), \tag{4.21}$$

$$D_x^V(\emptyset) \le A, \tag{4.22}$$

where the constants are the same as before.

Remark 4.1. It is worth pointing out that conditions (4.20)–(4.22), in general, are weaker than (4.8)–(4.10). Indeed, if $b_\varepsilon(x,\gamma) = b'(x,\gamma) + \varepsilon \cdot b''(x,\gamma)$ then (4.21) is an assumption on function b' only, whereas (4.9) requires additional conditions on b''.

Let $c > 0$. We define $\bar{B}_c^\infty$ to be the closed ball of radius c in the Banach space $L^\infty(\mathbb{R}^d)$.

Proposition 4.2. *1. Let conditions* (4.20), (4.21), *and* (4.11) *hold. Then* $\big(\hat{L}_V, \mathcal{D}^V\big)$ *is a generator of the holomorphic semigroup* $\hat{U}_V(t)$ *on* $\mathcal{L}_C$.

2. *Suppose, additionally,* (4.22) *is satisfied. Then, there exists* $\alpha_0 \in (0;1)$ *such that for any* $\alpha \in (\alpha_0;1)$ *there exists a strongly continuous semigroup* $\hat{U}_V^{\odot\alpha}(t)$ *on the space* $\overline{\mathcal{K}_{\alpha C}}$ *with the generator* $\hat{L}_V^{\odot\alpha} = \hat{L}_V^*$,

$$\mathrm{Dom}\big(L_V^{\odot\alpha}\big) = \big\{k \in \overline{\mathcal{K}_{\alpha C}} \bigm| \hat{L}_V^* k \in \overline{\mathcal{K}_{\alpha C}}\big\}.$$

Moreover, for $k \in \mathcal{K}_{\alpha C}$

$$\begin{aligned}(\hat{L}_V^* k)(\eta) = &-\sum_{x\in\eta}\int_{\Gamma_0} k(\xi\cup\eta) D_x^V(\xi)d\lambda(\xi) \\ &+\sum_{x\in\eta}\int_{\Gamma_0} k(\xi\cup(\eta\setminus x)) B_x^V(\xi)d\lambda(\xi).\end{aligned} \tag{4.23}$$

3. *Let* $\alpha \in (\alpha_0;1)$, $\rho_0 \in \bar{B}_{\alpha C}^\infty$. *Then the evolution equation*

$$\begin{cases} \dfrac{\partial}{\partial t} k_t = \hat{L}_V^* k_t \\ k_t\big|_{t=0} = e_\lambda(\rho_0,\eta) \end{cases} \tag{4.24}$$

has a unique solution $k_t = e_\lambda(\rho_t)$ *in* $\overline{\mathcal{K}_{\alpha C}}$ *provided* ρ_t *belongs to* $\bar{B}_{\alpha C}^\infty$ *and satisfies the Vlasov-type equation*

$$\begin{aligned}\frac{\partial}{\partial t}\rho_t(x) = &-\rho_t(x)\int_{\Gamma_0} e_\lambda(\rho_t,\xi) D_x^V(\xi)d\lambda(\xi) \\ &+\int_{\Gamma_0} e_\lambda(\rho_t,\xi) B_x^V(\xi)d\lambda(\xi).\end{aligned} \tag{4.25}$$

Proof. 1. The proof for the first statement is similar to the analogous one one in Proposition 4.1.

2. The same arguments as for the proof of Proposition 3.5 show that, for any $\alpha \in (0;1)$, $\mathcal{K}_{\alpha C} \subset \mathrm{Dom}(\hat{L}_V^*)$. Next, by (4.11), let us now take $\theta \in (a_1 + \frac{a_2}{C}; \frac{3}{2})$. Then we can set $\alpha_0 := \frac{a_2}{C(\theta - a_1)} \in (0;1)$. The second statement can be handled now in much the same way as in Theorem 3.8.

3. Since $\rho_0 \in \bar{B}_{\alpha C}^\infty$ implies $k_0 \in \overline{\mathcal{K}_{\alpha C}}$ then the Cauchy problem (4.24) has a unique solution in $\overline{\mathcal{K}_{\alpha C}}$. On the other hand, according to (4.13), for any

$\rho_t \in \bar{B}^\infty_{\alpha C}$

$$
\begin{aligned}
(\hat{L}^*_V e_\lambda(\rho_t))(\eta) = &- \sum_{x\in\eta} e_\lambda(\rho_t, \eta) \int_{\Gamma_0} e_\lambda(\rho_t, \xi) D^V_x(\xi) d\lambda(\xi) \\
&+ \sum_{x\in\eta} e_\lambda(\rho_t, \eta \setminus x) \int_{\Gamma_0} e_\lambda(\rho_t, \xi) B^V_x(\xi) d\lambda(\xi). \qquad (4.26)
\end{aligned}
$$

Combining (4.26) with the equality

$$
\frac{\partial}{\partial t} e_\lambda(\rho_t, \eta) = \sum_{x\in\eta} \rho_t(x) e_\lambda(\rho_t, \eta \setminus x),
$$

we can assert that $k_t = e_\lambda(\rho_t)$ is a solution to (4.24), with ρ_t given by (4.25). □

Remark 4.2. The question about existence and uniqueness of solutions to the Vlasov-type equation (4.25) in some ball $\bar{B}^\infty_{\alpha C}$ of $L^\infty(\mathbb{R}^d)$ shall be solved separately in each concrete model, see e.g. [8, 7].

Our next goal is to study the question about convergence of the semigroups $\hat{U}_\varepsilon(t)$ to $\hat{U}_V(t)$ in $\mathcal{L}_C$.

We begin by proving the following abstract statement.

Lemma 4.3. *Let X be a Banach space, and let $(A_\varepsilon, \mathfrak{D}_\varepsilon)$, $(B_\varepsilon, \mathfrak{D}_\varepsilon)$, $\varepsilon \geq 0$ be closed, densely defined operators on X. Suppose that there exists $\beta > 0$ and $z \in \mathbb{C}$ with $\operatorname{Re} z > \beta$ such that $z \in \rho(A_\varepsilon)$ for all $\varepsilon \geq 0$ and*

$$
\kappa := \sup_{\varepsilon>0} \left\| (A_\varepsilon - z1\!\!1)^{-1} \right\| < \infty, \qquad (4.27)
$$

$$
\sigma := \sup_{\varepsilon\geq 0} \left\| B_\varepsilon (A_\varepsilon - z1\!\!1)^{-1} \right\| < 1, \qquad (4.28)
$$

$$
(A_\varepsilon - z1\!\!1)^{-1} \xrightarrow{s} (A_0 - z1\!\!1)^{-1}, \quad \varepsilon \to 0, \qquad (4.29)
$$

$$
B_\varepsilon (A_\varepsilon - z1\!\!1)^{-1} \xrightarrow{s} B_0 (A_0 - z1\!\!1)^{-1}, \quad \varepsilon \to 0. \qquad (4.30)
$$

Then z belongs to the resolvent set of $L_\varepsilon := A_\varepsilon + B_\varepsilon$, $\varepsilon \geq 0$ and

$$
(L_\varepsilon - z1\!\!1)^{-1} \xrightarrow{s} (L_0 - z1\!\!1)^{-1}, \quad \varepsilon \to 0.
$$

Proof. For any $\varepsilon \geq 0$ we set $C_\varepsilon := (A_\varepsilon - z1\!\!1)^{-1}$, then we have $\operatorname{Ran}(C_\varepsilon) = \operatorname{Dom}(A_\varepsilon) = \operatorname{Dom}(B_\varepsilon) = \operatorname{Dom}(L_\varepsilon) = \mathfrak{D}_\varepsilon$. Therefore, for any $z \in \rho(A_\varepsilon)$ one can write

$$
L_\varepsilon - z1\!\!1 = A_\varepsilon + B_\varepsilon - z1\!\!1 = \left(B_\varepsilon (A_\varepsilon - z1\!\!1)^{-1} + 1\!\!1 \right) (A_\varepsilon - z1\!\!1).
$$

By (4.28), the operator $B_\varepsilon (A_\varepsilon - z1\!\!1)^{-1} + 1\!\!1 = B_\varepsilon C_\varepsilon + 1\!\!1$ is invertible with bounded inverse D_ε. Moreover,

$$
\|D_\varepsilon\| \leq \frac{1}{1 - \|B_\varepsilon C_\varepsilon\|} \leq \frac{1}{1-\sigma}. \qquad (4.31)
$$

Therefore, we have that $z \in \rho(L_\varepsilon)$ and

$$(L_\varepsilon - z1\!\!1)^{-1} = (A_\varepsilon - z1\!\!1)^{-1} \left(B_\varepsilon C_\varepsilon + 1\!\!1\right)^{-1} = C_\varepsilon D_\varepsilon. \tag{4.32}$$

Next,

$$\begin{aligned} D_\varepsilon - D_0 &= (B_\varepsilon C_\varepsilon + 1\!\!1)^{-1} - (B_0 C_0 + 1\!\!1)^{-1} \\ &= (B_\varepsilon C_\varepsilon + 1\!\!1)^{-1} \left((B_0 C_0 + 1\!\!1) - (B_\varepsilon C_\varepsilon + 1\!\!1)\right) (B_0 C_0 + 1\!\!1)^{-1} \\ &= D_\varepsilon \left(B_0 C_0 - B_\varepsilon C_\varepsilon\right) D_0, \end{aligned}$$

thus, according to (4.31) and (4.30), for any $x \in X$

$$\begin{aligned} \|D_\varepsilon x - D_0 x\| &\le \|D_\varepsilon\| \cdot \|\left(B_0 C_0 - B_\varepsilon C_\varepsilon\right) D_0 x\| \\ &\le \frac{1}{1-\sigma} \|\left(B_0 C_0 - B_\varepsilon C_\varepsilon\right) D_0 x\| \to 0, \quad \varepsilon \to 0. \end{aligned}$$

Hence, $D_\varepsilon \overset{s}{\longrightarrow} D_0$. Then, using (4.32) and (4.29), we have for any $x \in X$

$$\begin{aligned} &\left\|(L_\varepsilon - z1\!\!1)^{-1} x - (L_0 - z1\!\!1)^{-1} x\right\| \\ &= \left\|C_\varepsilon D_\varepsilon x - C_0 D_0 x\right\| = \left\|C_\varepsilon (D_\varepsilon - D_0) x + (C_\varepsilon - C_0) D_0 x\right\| \\ &\le \left\|C_\varepsilon\right\| \cdot \left\|(D_\varepsilon - D_0) x\right\| + \left\|(C_\varepsilon - C_0) D_0 x\right\| \\ &\le \kappa \cdot \left\|(D_\varepsilon - D_0) x\right\| + \left\|(C_\varepsilon - C_0) D_0 x\right\| \to 0, \quad \varepsilon \to 0. \end{aligned}$$

The statement is proven. □

Now we are able to prove result about convergence in $\mathcal{L}_C$.

Theorem 4.4. *Let conditions* (4.8), (4.9)*, and* (4.11) *are satisfied. Suppose that convergences* (4.16), (4.17) *take place for all* $\eta \in \Gamma_0$ *as well as in the sense of* $\mathcal{L}_C$*. Assume also that there exists* $\sigma > 0$ *such that (cf.* (4.18)*) either*

$$d_\varepsilon(x, \xi) \le \sigma D_x^V(\emptyset) \qquad \text{or} \qquad d_\varepsilon(x, \xi) \ge \sigma D_x^V(\emptyset) \tag{4.33}$$

is satisfied for all $\xi \in \Gamma_0$ *and for a.a.* $x \in \mathbb{R}^d$*. Then* $\hat{U}_\varepsilon(t) \overset{s}{\longrightarrow} \hat{U}_V(t)$ *in* $\mathcal{L}_C$ *uniformly on finite time intervals.*

Proof. First of all note that $\mathcal{L}_C$-convergence in (4.16), (4.17) together with (4.18) yields (4.20), (4.21) provided (4.8), (4.9) hold. Then, by Propositions 4.1, 4.2, the semigroups $\hat{U}_\varepsilon(t)$, $\hat{U}_V(t)$ exist in $\mathcal{L}_C$. To prove their convergence it is enough to show the strong convergence of the resolvent corresponding to the generators of this semigroup, see e.g. [5, Theorem III.4.8]. To verify this, we apply Lemma 4.3 taking $A_\varepsilon = L_0^{(\varepsilon)}$, $B_\varepsilon = L_1^{(\varepsilon)}$, $L_\varepsilon = \hat{L}_{\varepsilon,\mathrm{ren}}$, $\mathfrak{D}_0 = \mathcal{D}^V$, $\mathfrak{D}_\varepsilon = \mathcal{D}^{(\varepsilon)}$, $\varepsilon > 0$. Below we check the conditions of this lemma.

Let us fix any $z > 0$. It is easily seen that (4.27) is satisfied since

$$\left\|\left(L_0^{(\varepsilon)} - z1\!\!1\right)^{-1}\right\| \le \frac{1}{z}$$

for all $\varepsilon \in (0;1]$. Clearly, (4.14) implies (4.28). Let $G \in \mathcal{L}_C$. Then

$$\begin{aligned}&\big\|\big(L_0^{(\varepsilon)} - z1\!\!1\big)^{-1}G - \big(L_0^V - z1\!\!1\big)^{-1}G\big\|_C\\ &\leq \int_{\Gamma_0} \frac{\big|D^{(\varepsilon)}(\eta) - D^V(\eta)\big|}{\big(z + D^V(\eta)\big)\big(z + D^{(\varepsilon)}(\eta)\big)}|G(\eta)|d\lambda(\eta).\end{aligned}$$

By (4.18), for all $\eta \in \Gamma_0$

$$D^{(\varepsilon)}(\eta) \to D^V(\eta), \quad \varepsilon \to 0. \tag{4.34}$$

Then the inequality

$$\frac{\big|D^{(\varepsilon)}(\eta) - D^V(\eta)\big|}{\big(z + D^V(\eta)\big)\big(z + D^{(\varepsilon)}(\eta)\big)} \leq \frac{1}{z + D^V(\eta)} + \frac{1}{z + D^{(\varepsilon)}(\eta)} \leq \frac{2}{z}$$

implies (4.29) by the dominated convergence theorem.

Let inequality $d_\varepsilon(x,\xi) \leq \sigma D_x^V(\emptyset)$ hold for all $\xi \in \Gamma_0$ and a.a. $x \in \mathbb{R}^d$. Then, by Lemma 2.1,

$$\begin{aligned}&\Big\|L_1^{(\varepsilon)}\big(L_0^{(\varepsilon)} - z1\!\!1\big)^{-1}G - L_1^V\big(L_0^V - z1\!\!1\big)^{-1}G\Big\|_C\\ &\leq \Big\|\big(L_1^{(\varepsilon)} - L_1^V\big)\big(L_0^V - z1\!\!1\big)^{-1}G\Big\|_C + \Big\|L_1^{(\varepsilon)}\Big(\big(L_0^{(\varepsilon)} - z1\!\!1\big)^{-1} - \big(L_0^V - z1\!\!1\big)^{-1}\Big)G\Big\|_C\\ &\leq \int_{\Gamma_0} \frac{|G(\xi)|}{z + D^V(\xi)} \sum_{x\in\xi} \int_{\Gamma_0} \Big|\varepsilon^{-|\eta|}K_0^{-1}d_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta) - D_x^V(\eta)\Big| C^{|\eta|}d\lambda(\eta)C^{|\xi|}d\lambda(\xi)\\ &\quad + \frac{1}{C}\int_{\Gamma_0} \frac{|G(\xi)|}{z + D^V(\xi)} \sum_{x\in\xi} \int_{\Gamma_0} \Big|\varepsilon^{-|\eta|}K_0^{-1}b_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta) - B_x^V(\eta)\Big| C^{|\eta|}d\lambda(\eta)C^{|\xi|}d\lambda(\xi)\\ &\quad + \int_{\Gamma_0} |G(\xi)| \frac{\big|D^{(\varepsilon)}(\xi) - D^V(\xi)\big|}{\big(z + D^V(\xi)\big)\big(z + D^{(\varepsilon)}(\xi)\big)} \sum_{x\in\xi} \int_{\Gamma_0} \varepsilon^{-|\eta|}\Big(\big|K_0^{-1}d_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta)\big|\\ &\qquad\qquad + \frac{1}{C}\big|K_0^{-1}b_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta)\big|\Big)C^{|\eta|}d\lambda(\eta)C^{|\xi|}d\lambda(\xi).\end{aligned} \tag{4.35}$$

Convergence in $\mathcal{L}_C$ for (4.16), (4.17) together with (4.34) implies that all three integrand functions of ξ appearing in (4.35) converge to 0 λ-a.s., as $\varepsilon \to 0$. To use dominated convergence theorem we will show that the following functions are uniformly bounded. Using (4.8), (4.20), and (4.33), we get

$$\begin{aligned}&\frac{1}{z + D^V(\xi)} \sum_{x\in\xi} \int_{\Gamma_0} \Big|\varepsilon^{-|\eta|}K_0^{-1}d_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta) - D_x^V(\eta)\Big| C^{|\eta|}d\lambda(\eta)\\ &\leq \frac{a_1}{z + D^V(\xi)} \sum_{x\in\xi}\big(d_\varepsilon(x,\xi) + D_x^V(\emptyset)\big) \leq \frac{a_1(1+\sigma)}{z + D^V(\xi)} \sum_{x\in\xi} D_x^V(\emptyset) \leq a_1(1+\sigma).\end{aligned}$$

Analogously, by (4.9), (4.21), and (4.33),

$$\frac{1}{z + D^V(\xi)} \sum_{x\in\xi} \int_{\Gamma_0} \Big|\varepsilon^{-|\eta|}K_0^{-1}b_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta) - B_x^V(\eta)\Big| C^{|\eta|}d\lambda(\eta) \leq a_1(1+\sigma).$$

According to (4.8), (4.9), and (4.33),

$$\begin{aligned}
&\frac{\big|D^{(\varepsilon)}(\xi)-D^V(\xi)\big|}{\big(z+D^V(\xi)\big)\big(z+D^{(\varepsilon)}(\xi)\big)}\sum_{x\in\xi}\int_{\Gamma_0}\varepsilon^{-|\eta|}\Big(\big|K_0^{-1}d_\varepsilon(x,\cdot\cup\xi\setminus x)(\eta)\big| \\
&\qquad+\frac{1}{C}\big|K_0^{-1}b_\varepsilon(x,\cdot\cup\xi\setminus x)(\eta)\big|\Big)C^{|\eta|}d\lambda(\eta) \\
\le&\frac{D^{(\varepsilon)}(\xi)+D^V(\xi)}{\big(z+D^V(\xi)\big)\big(z+D^{(\varepsilon)}(\xi)\big)}\sum_{x\in\xi}\Big(a_1 d_\varepsilon(x,\xi)+\frac{a_2}{C}d_\varepsilon(x,\xi)\Big) \\
\le&\frac{D^{(\varepsilon)}(\xi)}{\big(z+D^V(\xi)\big)\big(z+D^{(\varepsilon)}(\xi)\big)}\Big(a_1+\frac{a_2}{C}\Big)\sigma D^V(\xi) \\
&+\frac{D^V(\xi)}{\big(z+D^V(\xi)\big)\big(z+D^{(\varepsilon)}(\xi)\big)}\Big(a_1+\frac{a_2}{C}\Big)D^{(\varepsilon)}(\xi)\le\Big(a_1+\frac{a_2}{C}\Big)(1+\sigma).
\end{aligned}$$

Hence, (4.30) is proved.

In the case $d_\varepsilon(x,\xi)\ge\sigma D_x^V(\emptyset)$, $\xi\in\Gamma_0$, we rewrite l.h.s. of (4.30) in a another manner. Namely,

$$\begin{aligned}
&\Big\|L_1^{(\varepsilon)}\big(L_0^{(\varepsilon)}-z1\!\!1\big)^{-1}G-L_1^V\big(L_0^V-z1\!\!1\big)^{-1}G\Big\|_C \\
\le&\Big\|\big(L_1^{(\varepsilon)}-L_1^V\big)\big(L_0^\varepsilon-z1\!\!1\big)^{-1}G\Big\|_C+\Big\|L_1^V\Big(\big(L_0^{(\varepsilon)}-z1\!\!1\big)^{-1}-\big(L_0^V-z1\!\!1\big)^{-1}\Big)G\Big\|_C \\
\le&\int_{\Gamma_0}\frac{|G(\xi)|}{z+D^{(\varepsilon)}(\xi)}\sum_{x\in\xi}\int_{\Gamma_0}\Big|\varepsilon^{-|\eta|}K_0^{-1}d_\varepsilon(x,\cdot\cup\xi\setminus x)(\eta)-D_x^V(\eta)\Big|C^{|\eta|}d\lambda(\eta)C^{|\xi|}d\lambda(\xi) \\
&+\frac{1}{C}\int_{\Gamma_0}\frac{|G(\xi)|}{z+D^{(\varepsilon)}(\xi)}\sum_{x\in\xi}\int_{\Gamma_0}\Big|\varepsilon^{-|\eta|}K_0^{-1}b_\varepsilon(x,\cdot\cup\xi\setminus x)(\eta)-B_x^V(\eta)\Big|C^{|\eta|}d\lambda(\eta)C^{|\xi|}d\lambda(\xi) \\
&+\int_{\Gamma_0}|G(\xi)|\frac{\big|D^{(\varepsilon)}(\xi)-D^V(\xi)\big|}{\big(z+D^V(\xi)\big)\big(z+D^{(\varepsilon)}(\xi)\big)}\sum_{x\in\xi}\int_{\Gamma_0}\Big(D_x^V(\eta)+\frac{1}{C}B_x^V(\eta)\Big)C^{|\eta|}d\lambda(\eta)C^{|\xi|}d\lambda(\xi).
\end{aligned}$$

Repeating all estimates done for the first alternative of (4.33) we get the desired result. □

Remark 4.3. Note that in all examples considered in [10] the function $d_\varepsilon(x,\xi)$ is monotone in ε. Taking into account (4.18), condition (4.33) becomes natural.

Example 1 (revisited)**.** Let us consider for $\varepsilon\in[0;1]$, $s\in[0;1]$

$$d_\varepsilon(x,\gamma)=\exp\Big\{\varepsilon s\sum_{y\in\gamma\setminus x}\phi(x-y)\Big\},\quad b_\varepsilon(x,\gamma)=z\exp\Big\{\varepsilon(s-1)\sum_{y\in\gamma}\phi(x-y)\Big\},$$

Analogously to the previous computations,

$$\int_{\Gamma_0} \left|K_0^{-1} d_\varepsilon\left(x, \cdot \cup \xi\right)\right|(\eta)\, \varepsilon^{-|\eta|} C^{|\eta|} d\lambda\left(\eta\right) = d_\varepsilon(x,\xi) e^{C\varepsilon^{-1}\beta_{\varepsilon s}}$$

$$\int_{\Gamma_0} \left|K_0^{-1} b_\varepsilon\left(x, \cdot \cup \xi\right)\right|(\eta)\, \varepsilon^{-|\eta|} C^{|\eta|} d\lambda\left(\eta\right) = b_\varepsilon(x,\xi) e^{C\varepsilon^{-1}\beta_{\varepsilon(s-1)}} \le z d_\varepsilon(x,\xi) e^{C\varepsilon^{-1}\beta_{\varepsilon(s-1)}},$$

since $\phi \ge 0$. Let $s \in (0;1]$. Suppose that $\tilde{\beta} := \int_{\mathbb{R}^d} \phi(x) e^{\phi(x)} dx < \infty$. Then for $\tau \in [-1;1]$, $\varepsilon \in [0,1]$

$$\varepsilon^{-1}\beta_{\varepsilon\tau} \le \varepsilon^{-1} \int_{\mathbb{R}^d} \varepsilon|\tau|\phi(x) \sup_{\tau\in[-1,1]} e^{\varepsilon\tau\phi(x)} dx \le \tilde{\beta}.$$

The bound (4.11) will be proved once we show $e^{C\tilde{\beta}}\bigl(1+\frac{z}{C}\bigr) < \frac{3}{2}$. If $s=0$ then, similarly, we need $\beta := \int_{\mathbb{R}^d} \phi(x)dx < \infty$ and $\frac{z}{C}e^{C\beta} < \frac{1}{2}$. Note also that the conditions $\beta < \infty$ and $\bar{\phi} = \sup_{\mathbb{R}^d} \phi(x) < \infty$ yield $\tilde{\beta} \le e^{\bar{\phi}}\beta < \infty$. For the case $s = 0$ condition (4.10) holds automatically. If $s \in (0;1]$ one should assume $\bar{\phi} < \infty$ then $\nu = e^{s\bar{\phi}}$ (uniformly by $\varepsilon \in (0;1]$). Then to guarantee (4.12) we need $e^{C\tilde{\beta}}\bigl(1+\frac{z}{C}e^{s\bar{\phi}}\bigr) < \frac{3}{2}$. Therefore, under such conditions we obtain statement of Proposition 4.1. Next,

$$\begin{aligned}\lim_{\varepsilon\to 0} \varepsilon^{-|\eta|}\bigl(K_0^{-1} d_\varepsilon\left(x,\cdot\cup\xi\right)\bigr)(\eta) &= \lim_{\varepsilon\to 0} \exp\Bigl\{\varepsilon s \sum_{y\in\xi}\phi(x-y)\Bigr\} e_\lambda\left(\frac{e^{\varepsilon s\phi(x-\cdot)}-1}{\varepsilon}, \eta\right)\\ &= e_\lambda\bigl(s\phi(x-\cdot),\eta\bigr) =: D_x^V(\eta); \qquad (4.36)\end{aligned}$$

and, analogously,

$$\lim_{\varepsilon\to 0} \varepsilon^{-|\eta|}\bigl(K_0^{-1} b_\varepsilon\left(x,\cdot\cup\xi\right)\bigr)(\eta) = z e_\lambda\bigl((s-1)\phi(x-\cdot),\eta\bigr) =: B_x^V(\eta). \qquad (4.37)$$

Since $D_x^V(\emptyset) = 1 \le d_\varepsilon(x,\eta)$, the second alternative of (4.33) is satisfied. In order to use Proposition 4.2 and Theorem 4.4 we need to verify the convergences (4.36) and (4.37) in $\mathcal{L}_C$ (recall that this implies (4.20) and (4.21), see the proof of Theorem 4.4). To do this let us note that for any $\tau \in [-1;1]$

$$\begin{aligned}&\left|\exp\Bigl\{\varepsilon\tau\sum_{y\in\xi}\phi(x-y)\Bigr\} e_\lambda\left(\frac{e^{\varepsilon\tau\phi(x-\cdot)}-1}{\varepsilon},\eta\right) - e_\lambda\bigl(\tau\phi(x-\cdot),\eta\bigr)\right|\\ &\le \max\Bigl\{\exp\Bigl\{\tau\sum_{y\in\xi}\phi(x-y)\Bigr\},1\Bigr\} e_\lambda\left(\frac{\bigl|e^{\varepsilon\tau\phi(x-\cdot)}-1\bigr|}{\varepsilon},\eta\right) + e_\lambda\bigl(|\tau|\phi(x-\cdot),\eta\bigr)\\ &\le \left(\max\Bigl\{\exp\Bigl\{\tau\sum_{y\in\xi}\phi(x-y)\Bigr\},1\Bigr\}+1\right) e_\lambda\bigl(\phi(x-\cdot),\eta\bigr),\end{aligned}$$

and the last function of η belongs to $\mathcal{L}_C$ for all $\xi \in \Gamma_0$ and a.a. $x \in \mathbb{R}^d$ provided $\phi \in L^1(\mathbb{R}^d)$. By (2.6), the Vlasov equation (4.25) now has the following form

$$\frac{\partial}{\partial t}\rho_t(x) = -\rho_t(x)\exp\bigl\{s(\rho_t * \phi)(x)\bigr\} + z\exp\bigl\{(s-1)(\rho_t * \phi)(x)\bigr\}.$$

Here and below $*$ means usual convolution of functions in $\mathbb{R}^d$.

Example 2 (revisited)**.** Let $d_\varepsilon(x,\gamma\setminus x) = m+\varepsilon\varkappa^- \sum_{y\in\gamma\setminus x} a^-(x-y)$, $b_\varepsilon(x,\gamma) = \varepsilon\varkappa^+ \sum_{y\in\gamma} a^+(x-y)$. Comparing with the previous notations we have changed $\varkappa^\pm$ onto $\varepsilon\varkappa^\pm$. Clearly, conditions (3.32), (3.33) implies the same inequalities for $\varepsilon\varkappa^\pm$. Note also that d_ε is decreasing in $\varepsilon \to 0$. Therefore, to apply all results of this section to BDLP-model we should prove the convergence (4.16), (4.17) in $\mathcal{L}_C$. Note, that

$$\begin{aligned}\varepsilon^{-|\eta|}K_0^{-1}d_\varepsilon\left(x,\cdot\cup\xi\right)(\eta) &= d_\varepsilon(x,\xi)\varepsilon^{-|\eta|}0^{|\eta|} + \varepsilon\varepsilon^{-|\eta|}\varkappa^- 1\!\!1_{\Gamma^{(1)}}(\eta)\sum_{y\in\eta}a^-(x-y)\\ &= d_\varepsilon(x,\xi)0^{|\eta|} + 1\!\!1_{\Gamma^{(1)}}(\eta)\sum_{y\in\eta}a^-(x-y)\\ &\to m0^{|\eta|} + 1\!\!1_{\Gamma^{(1)}}(\eta)\sum_{y\in\eta}a^-(x-y) =: D_x^V(\eta)\end{aligned}$$

and, analogously,

$$\begin{aligned}\varepsilon^{-|\eta|}K_0^{-1}b_\varepsilon\left(x,\cdot\cup\xi\right)(\eta) &= b_\varepsilon(x,\xi)0^{|\eta|} + 1\!\!1_{\Gamma^{(1)}}(\eta)\sum_{y\in\eta}a^+(x-y)\\ &\to 1\!\!1_{\Gamma^{(1)}}(\eta)\sum_{y\in\eta}a^+(x-y) =: B_x^V(\eta).\end{aligned}$$

The convergence in $\mathcal{L}_C$ is obvious now. The Vlasov equation has the following form

$$\frac{\partial}{\partial t}\rho_t(x) = \varkappa^+(a^+ * \rho_t)(x) - \varkappa^-\rho_t(x)(a^- * \rho_t)(x) - m\rho_t(x).$$

The existence and uniqueness of the solution to this equation was studied in [7].

Remark 4.4. By duality (3.35), Theorem 4.4 yields weak*-convergence of the semigroups $\hat{U}_\varepsilon^{\odot\alpha}(t)$ to $\hat{U}_V^{\odot\alpha}(t)$ in $\overline{\mathcal{K}_{\alpha C}}$. To prove such convergence in the strong sense we need additional analysis of their generators. The problem concerns the fact that we have explicit expression for the generator $\hat{L}_V^{\odot\alpha} = \hat{L}_V^*$ only on the core $\bigl\{k\in\mathcal{K}_{\alpha C} \bigm| \hat{L}_V^* k \in \overline{\mathcal{K}_{\alpha C}}\bigr\}$. However, we are able to show such convergence for the Glauber dynamics described in Example 1 for $s=0$ using modified technique (see [8]).

Acknowledgements

The financial support of DFG through the SFB 701 (Bielefeld University) and German-Ukrainian Project 436 UKR 113/97 is gratefully acknowledged.

References

[1] S. Albeverio, Y. Kondratiev, M. Röckner, Analysis and geometry on configuration spaces, J. Funct. Anal. 154 (1998) 444–500.

[2] B. Bolker, S. W. Pacala, Using moment equations to understand stochastically driven spatial pattern formation in ecological systems, Theor. Popul. Biol. 52 (1997) 179–197.

[3] B. Bolker, S. W. Pacala, Spatial moment equations for plant competitions: Understanding spatial strategies and the advantages of short dispersal, American Naturalist 153 (1999) 575–602.

[4] U. Dieckmann, R. Law, Relaxation projections and the method of moments, in: The Geometry of Ecological Interactions, Cambridge University Press, Cambridge, UK, 2000, pp. 412–455.

[5] K.-J. Engel, R. Nagel, One-parameter semigroups for linear evolution equations, volume 194 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2000.

[6] D. Finkelshtein, Y. Kondratiev, O. Kutoviy, Correlation functions evolution for the Glauber dynamics in continuum, ArXiv: 1107.3487.

[7] D. Finkelshtein, Y. Kondratiev, O. Kutoviy, Operator approach to Vlasov scaling for some models of spatial ecology, ArXiv: 1104.2418.

[8] D. Finkelshtein, Y. Kondratiev, O. Kutoviy, Vlasov scaling for the Glauber dynamics in continuum, ArXiv: 1002.4762, *to appear in:* Infin. Dimens. Anal. Quantum Probab. Relat. Top..

[9] D. Finkelshtein, Y. Kondratiev, O. Kutoviy, Individual based model with competition in spatial ecology, SIAM J. Math. Anal. 41 (2009) 297–317.

[10] D. Finkelshtein, Y. Kondratiev, O. Kutoviy, Vlasov scaling for stochastic dynamics of continuous systems, J. Statistical Phys. 141 (2010) 158–178.

[11] D. Finkelshtein, Y. Kondratiev, O. Kutoviy, E. Zhizhina, An approximative approach for construction of the Glauber dynamics in continuum, ArXiv: 0910.4241, *to appear in:* Math. Nachr.

[12] D. Finkelshtein, Y. Kondratiev, M. J. Oliveira, Markov evolutions and hierarchical equations in the continuum I. One-component systems, J. Evol. Equ. 9 (2009) 197–233.

[13] N. L. Garcia, Birth and death processes as projections of higher dimensional poisson processes, Adv. in Appl. Probab. 27 (1995) 911–930.

[14] N. L. Garcia, T. G. Kurtz, Spatial birth and death processes as solutions of stochastic equations, ALEA Lat. Am. J. Probab. Math. Stat. 1 (2006) 281–303 (electronic).

[15] R. A. Holley, D. W. Stroock, Nearest neighbor birth and death processes on the real line, Acta Math. 140 (1978) 103–154.

[16] Y. Kondratiev, T. Kuna, Harmonic analysis on configuration space. I. General theory, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 5 (2002) 201–233.

[17] Y. Kondratiev, T. Kuna, Correlation functionals for Gibbs measures and Ruelle bounds, Methods Funct. Anal. Topology 9 (2003) 9–58.

[18] Y. Kondratiev, O. Kutoviy, On the metrical properties of the configuration space, Math. Nachr. 279 (2006) 774–783.

[19] Y. Kondratiev, O. Kutoviy, E. Zhizhina, Nonequilibrium Glauber-type dynamics in continuum, J. Math. Phys. 47 2006 113501, 17.

[20] Y. Kondratiev, E. Lytvynov, M. Röckner, Equilibrium Kawasaki dynamics of continuous particle systems, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 10 (2007) 185–209.

[21] Y. Kondratiev, R. Minlos, E. Zhizhina, One-particle subspace of the Glauber dynamics generator for continuous particle systems, Rev. Math. Phys. 16 (2004) 1073–1114.

[22] Y. Kondratiev, A. Skorokhod, On contact processes in continuum, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 9 (2006) 187–198.

[23] A. Lenard, States of classical statistical mechanical systems of infinitely many particles. I, Arch. Rational Mech. Anal. 59 (1975) 219–239.

[24] A. Lenard, States of classical statistical mechanical systems of infinitely many particles. II. Characterization of correlation measures, Arch. Rational Mech. Anal. 59 (1975) 241–256.

[25] D. J. Murrell, U. Dieckmann, R. Law, On moment closures for population dynamics in continuous space, Journal of Theoretical Biology 229 (2004) 421–432.

[26] C. Preston, Spatial birth-and-death processes, Bull. Inst. Internat. Statist. 46 (1975) 371–391, 405–408.

[27] D. Ruelle, Statistical mechanics: Rigorous results, W. A. Benjamin, Inc., New York-Amsterdam, 1969.