McKean-Vlasov diffusions: from the asynchronization to the synchronization

Julian Tugaut
Fakultät für Mathematik
Universität Bielefeld
D-33615 Bielefeld
Germany

Abstract
We make the asymptotic analysis of the unique symmetric stationary measure of a self-stabilizing process in the small-noise limit. It has been proved in previous works that this measure converges with a linear rate in the asynchronized case and in the strictly synchronized case but it is slower in the intermediate case. The aim of this paper is to zoom around this phase transition.

Résumé
Diffusions de McKean-Vlasov : de l’asynchronisation à la synchronisation. On procède à l’analyse asymptotique de l’unique mesure stationnaire symétrique pour un processus auto-stabilisant à petit bruit. Il a été prouvé dans des travaux antécédents que cette mesure converge avec un taux linéaire dans le cas asynchrone et dans le cas strictement synchrone mais la convergence est moins rapide dans le cas intermédiaire. Le but de ce papier est de zoomer autour de cette transition de phase.

Introduction
We investigate the asymptotic behavior in the small-noise limit of the stationary measures of the following so-called self-stabilizing process:

\[ \begin{align*}
X_t &= X_0 + \sqrt{\epsilon} B_t - \int_0^t V'(X_s) \, ds - \int_0^t F' \ast u_s (X_s) \, ds \\
u_s &= \mathcal{L}(X_s)
\end{align*} \]

(I)

* Supported by the DFG-funded CRC 701, Spectral Structures and Topological Methods in Mathematics, at the University of Bielefeld.

Email address: jtugaut@math.uni-bielefeld.de (Julian Tugaut).
URL: http://www.math.uni-bielefeld.de/~jtugaut/ (Julian Tugaut).

Preprint submitted to the Académie des sciences
Here, * denotes the convolution. The motion of the process is generated by three forces. The first one is the derivative of a potential $V$. The second influence is a Brownian motion $(B_t)_{t \in \mathbb{R}_+}$. The third term - the so-called self-stabilizing term - represents the attraction between all the trajectories. We will assume that $V$ and $F$ are polynomial functions.

There is a unique strong solution for (I) if $E[X_0^{8q^2}] < \infty$ with $2q = \max \{\deg(V) ; \deg(F)\}$, see [HIP08]. The existence and the non-uniqueness of the stationary measures have been studied in [HT10a,Tug11]. In particular, under simple assumptions, if $V$ and $F$ are even, there is at least one symmetric stationary measure $u_\epsilon$. The convergence in long-time towards one of the stationary measures has been proved in [Tug10]. The asymptotic analysis of the measure(s) $u_\epsilon$ have been made in [HT10b,HT09]. Especially, the convergence has been the subject of [HT10b] and the rate of convergence has been studied in [HT09]. Under the convexity of $V''$, $F$ and $F''$ and the non-convexity of $V$, if we consider $u_\epsilon$ (whose limit is $u_0$), according to Theorem 1.2, Theorem 1.3 and Theorem 1.4 in [HT09], we obtain: for any $f \in C^4(\mathbb{R},\mathbb{R})$ with polynomial growth,

$$
\lim_{\epsilon \to 0} \frac{\log \left| \left( f, u_\epsilon \right) - \left( f, u_0 \right) \right|}{\log \epsilon} = \Lambda(\alpha) := \begin{cases} 
1 & \text{if } \alpha \neq \vartheta \\
1/m_0 & \text{if } \alpha = \vartheta,
\end{cases} \quad (\text{II})
$$

where $m_0 := \min \left\{ k \geq 2 \mid V^{(2k)}(0) + F^{(2k)}(0) > 0 \text{ or } F^{(2k)}(0) > 0 \right\}$, $\alpha := V''(0)$ and $\vartheta := -V''(0)$. This points out a discontinuity around the phase transition $\alpha = \vartheta$ between the asynchronization ($V''(0) + F''(0) < 0$) and the synchronization ($V''(0) + F''(0) \geq 0$). The term “synchronization” means that the interaction is so strong that all the stationary measures are pure states (Dirac measures) in the small-noise limit.

The aim of this note is to prove that we can observe intermediate rates of convergence around the condition $\alpha = \vartheta$ provided that the potential function $F$ depends on the small parameter $\epsilon$. This is based on a particular example, nevertheless extensions to general situations can easily be proved.

In order to produce intermediate rates, we construct some particular model. First we choose the reference environment (resp. interaction) function $V(x) := \frac{4}{\epsilon^3} - \frac{2}{\epsilon^2}$ (resp. $F(x) := \frac{4}{\epsilon^3} + \frac{2}{\epsilon^2}$). Obviously $\vartheta = \alpha = 1$.

The example is based on a small perturbation of this reference: we consider the association between $\vartheta$ and one of the following interaction potentials

$$
F^+_{\rho}(x) := \frac{x^4}{4} + \frac{(1 + \rho \epsilon^q)}{2} x^2 \quad \text{or} \quad F^-_{\rho}(x) := \frac{x^4}{4} + \frac{(1 - \rho \epsilon^q)}{2} x^2,
$$

with $\rho > 0$. In other words, Equation (I) in [HT09] leads to the study of symmetric invariant measures of the following self-stabilizing process:

$$
dX_t = \sqrt{\epsilon} dB_t - \left\{ 2X_t^3 + (3E[X_t^2] \pm \rho \epsilon^q) X_t \right\} dt. \quad (\text{III})
$$

We remove the odd moments appearing in the equation since we focus our attention to symmetric laws. We will see that, in this new timescale, the function $\Lambda$ is continuous in the right part of the dioptré.

The note is organized as follows. After presenting and justifying succinctly the general results (existence of a solution, existence of a unique symmetric stationary measure $u_\epsilon$ and the convergence towards $\delta_0$ with $\epsilon \to 0$), we will give the value of $\lim_{\epsilon \to 0} \log m_2(\epsilon) / \log \epsilon$ - where $m_2(\epsilon)$ is the second moment of $u_\epsilon$ - by using some integrals which already appeared in [BG10]. Then, the prefactor will be provided.

**Preliminaries**

There exists a unique strong solution for (III) (see [HIP08]) and a unique symmetric invariant measure (see Section 4.2 in [HT10a]) denoted by $u^\pm_\epsilon$ because we do not use $\epsilon$ in these proofs. We define $m^\pm_2(\epsilon) := \int_{\mathbb{R}} x^4 u^\pm_\epsilon(x) dx$. By Section 4.2 in [HT10a], the second moment $m^\pm_2(\epsilon)$ satisfies
\[ m^±_2(\epsilon) = \frac{\int_{\mathbb{R}^+} x^2 \exp \left[ -\frac{1}{\epsilon} (3m^+_2(\epsilon) \pm \rho \epsilon^\eta)x^2 + x^4 \right] \, dx}{\int_{\mathbb{R}^+} \exp \left[ -\frac{1}{\epsilon} (3m^+_2(\epsilon) \pm \rho \epsilon^\eta)x^2 + x^4 \right] \, dx}. \]  

(IV)

**Proposition 1** For all \( \eta > 0 \) and \( \rho > 0 \), the sequence of symmetric invariant measures \( (u^±_\epsilon, \epsilon > 0) \) converges weakly towards \( \delta_0 \).

**Proof:** Let us assume the existence of a positive constant \( C \) and a decreasing sequence \( (\epsilon_k)_{k \in \mathbb{N}} \) converging towards 0 such that \( m^±_2(\epsilon_k) > C \) for all \( k \in \mathbb{N} \) (for notational simplicity, we shall drop the index \( k \)). We apply the following change of variable \( x := \sqrt{t} y \) to (IV) and obtain \( m^±_2(\epsilon) = \epsilon \xi(m^+_2(\epsilon) \pm \rho \epsilon^\eta, \epsilon) \) where

\[ \xi(u, v) := \int_{\mathbb{R}^+} y^2 \nu_{u,v}(y) \, dy \quad \text{and} \quad \nu_{u,v}(y) := \frac{\exp \left[ -3uy^2 - vy^4 \right]}{\int_{\mathbb{R}^+} \exp \left[ -3uy^2 - vy^4 \right] \, dz}. \]  

(V)

Let \( u \in \mathbb{R}, v > 0 \). By Jensen’s inequality, \( \frac{\partial \xi}{\partial u} (u, v) < 0 \) and \( \frac{\partial \xi}{\partial v} (u, v) < 0 \). Since \( 3m_2(\epsilon) \pm \rho \epsilon^\eta > C \), for \( \epsilon \) small enough, we deduce that \( m^±_2(\epsilon) \leq \epsilon \xi(C/3, 0) \). The r.h.s tends towards 0 when \( \epsilon \) goes to 0 which is a nonsense because \( m^+_2(\epsilon) > C \).

\[ \square \]

**Main results**

We shall just estimate the convergence of \( m^+_2(\epsilon) \) as \( \epsilon \to 0 \) since \( \langle f, u^+_\epsilon \rangle - \langle f, u^+_0 \rangle \) is directly linked to \( m^+_2(\epsilon) \). Indeed: \( \langle f, u^+_\epsilon \rangle - \langle f, u^+_0 \rangle = f''(0)m^+_2(\epsilon) + O(m^+_2(\epsilon)) \). And, \( m^+_2(\epsilon) = o(m^+_2(\epsilon)) \) since \( u^+_\epsilon \) converges towards \( \delta_0 \).

**Proposition 2** Let \( \eta > 0 \) and \( \rho > 0 \). Then

\[ \lim_{\epsilon \to 0} \frac{\log m^+_2(\epsilon)}{\log \epsilon} = \Lambda_+(\eta) := 1 - \min \left\{ \eta; \frac{1}{2} \right\}. \]

**Proof:** Let us recall that \( m^+_2(\epsilon) = \epsilon \xi(m^+_2(\epsilon) + \rho \epsilon^\eta/3, \epsilon) \) where \( \xi \) is defined by (V). According to the proof of Proposition 1, the function \( \xi \) is decreasing with respect to both variables. Therefore we can compute some upper-bound of \( m^+_2(\epsilon) \) just by noting that \( m^+_2(\epsilon) + \rho \epsilon^\eta/3 \geq \max \{ m^+_2(\epsilon); \rho \epsilon^\eta/3 \} \).

- The first inequality leads to \( m^+_2(\epsilon) \leq \epsilon \xi(m^+_2(\epsilon), 0) \). The r.h.s can be computed by some change of variable: there exists some constant \( c_0 > 0 \) such that \( m^+_2(\epsilon) \leq c_0 \sqrt{\epsilon} \).

- The second inequality implies \( m^+_2(\epsilon) \leq \epsilon \xi(\rho \epsilon^\eta/3, 0) \). The same argument permits to obtain the existence of \( c_1 > 0 \) such that \( m^+_2(\epsilon) \leq c_1 \epsilon^{1-n} \).

Hence, for \( \epsilon \) small enough, we deduce the higher-bound \( m^+_2(\epsilon) \leq \max \{ c_1, c_2 \} \epsilon^{\Lambda_+((\eta))} \).

Let us now prove the lower-bound. By using the higher-bound, for \( \epsilon \) small enough, there exists \( c_3 > 0 \) such that \( m^+_2(\epsilon) + \rho \epsilon^\eta/3 \leq c_3 \epsilon^\min\{n, 1/2\} \). Since \( \xi \) is decreasing, we obtain: \( m^+_2(\epsilon) \geq \epsilon \xi(c_3 \epsilon^\min\{n, 1/2\}, \epsilon^{\min\{2n, 1\}}) \).

By a classical change of variable, it yields immediately \( m^+_2(\epsilon) \geq c_4 \epsilon^{\Lambda_+(\eta)} \) with \( c_4 > 0 \).

\[ \square \]

The same kind of convergence rate can be analyzed for \( m^-_2(\epsilon) \). The proof is quite similar to the one of Proposition 2 so we do not write the details.

**Proposition 3** Let \( \eta > 0 \) and \( \rho > 0 \). Then

\[ \lim_{\epsilon \to 0} \frac{\log m^-_2(\epsilon)}{\log \epsilon} = \Lambda_-(\eta) = \min \left\{ \eta; \frac{1}{2} \right\}. \]

We apply the change of variable \( y := \epsilon^{1/4}x \) in (IV) if \( \eta \geq \frac{1}{2} \). If \( \eta \leq \frac{1}{2} \), we proceed a reductio ad absurdum and we apply the change of variable \( y := \epsilon^{2/3}x \) in (IV).

Proposition 2 and Proposition 3 point out that the behavior of \( m^+_2(\epsilon) \) is not symmetric with respect to the critical threshold \( \alpha = \vartheta \). Some heuristic argument which could explain this difference is based on the limit measure. We know that the measure considered when \( \alpha > \vartheta \) is the trivial measure \( \delta_0 \) whereas the support of the limit measure for \( \alpha < \vartheta \) contains two points: \( u^-_\vartheta \) is then farther than \( \delta_0 \).

We can precise the asymptotic behavior of \( m^+_2(\epsilon) \). The proof is left to the attention of the reader since the method is the same than the ones used in Proposition 2 and in Proposition 3.
Corollary 4 Let $\eta > 0$ and $\rho > 0$. Then

$$
\lim_{\epsilon \to 0} \frac{m^+_{\epsilon}(\epsilon)}{\epsilon \Lambda^+_{\epsilon}(\eta)} = \lambda^+_{\epsilon}(\eta) := \begin{cases} 
(2\rho)^{-1} & \text{if } 2\eta < 1 \\
x_{\rho} & \text{if } 2\eta = 1 \\
x_{0} & \text{if } 2\eta > 1 
\end{cases}
$$

and

$$
\lim_{\epsilon \to 0} \frac{m^-_{\epsilon}(\epsilon)}{\epsilon \Lambda^-_{\epsilon}(\eta)} = \lambda^-_{\epsilon}(\eta) := \begin{cases} 
\rho/5 & \text{if } 2\eta < 1 \\
x_{-\rho} & \text{if } 2\eta = 1 \\
x_{0} & \text{if } 2\eta > 1 
\end{cases}
$$

Here, for all $r \in \mathbb{R}$, $x_r$ is the unique solution of $x_r = \xi(x_r + r/3, 1)$, see (V) for the definition of $\xi$.

By taking $\epsilon := 10^{-10}$, let us simulate $\xi^\pm(\eta) := \{\log(m^\pm_{\epsilon}(\epsilon)) - \log(\lambda^\pm_{\epsilon}(\eta))\}/\log \epsilon$ with respect to $\eta$. The continuous lines represent $\Lambda^\pm_{\epsilon}(\eta)$. The discontinuity appearing in the simulation is due to the prefactors in the asymptotic estimates.

Acknowledgements

It is a great pleasure to thank Samuel Herrmann for having helped me to simplify the proofs. Un très grand merci aussi à Manue et à Sandra pour tout.

References


