HIGHER GENERATION FOR PURE BRAID GROUPS

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Abstract. We exhibit some families of subgroups of the pure braid group that are highly generating, in the sense of Abels and Holz. In one class of examples, the relevant geometric object is a complex termed the restricted arc complex of a surface. Another arises by considering “dangling braiges,” introduced by Bux, Fluch, Schwandt, Witzel and the author.

Introduction

The notion of a family of subgroups of a group being highly generating was introduced by Abels and Holz [AH93]. It is a very natural condition, with many strong consequences, but to date few examples have been explicitly constructed of highly generating subgroups for “interesting” groups. One prominent existing example, given by Abels and Holz, is standard parabolic subgroups of Coxeter groups, or standard parabolic subgroups of groups with a BN-pair. The relevant geometry is given by Coxeter complexes and buildings. Higher generation is also used in [MMV98] as a tool to calculate the Bieri-Neumann-Strebel-Renz invariants of right-angled Artin groups.

As an addition to the collection of interesting examples, we produce two classes of families of subgroups of the n-string pure braid group \(PB_n\) that we show to be highly generating. In the first case the geometry is given by certain complexes of arcs on a surface. In the second case the geometry is given by complexes of what are called “dangling braiges.” In both cases, the definitions and techniques are heavily informed by the paper [BFS+12], in which arc complexes and braige complexes are used to prove that the braided Thompson’s groups \(V_{br}\) and \(F_{br}\) are of type \(F_\infty\).

In Section 1 we recall some definitions and results from [AH93], and establish a criterion for detecting coset complexes in Proposition 1.5. In Section 2.1 we define the restricted arc complex on a surface, and in Section 2.2 we define the complex of dangling pure braiges. The relevant families of subgroups of \(PB_n\) are defined in the paragraphs before Lemma 2.6 and Corollary 2.11, and in Definition 2.12. Finally in Section 3 we calculate the connectivity of these complexes and deduce that the families of subgroups are highly generating. See Propositions 3.6 and 3.13 for the exact bounds.

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1. Higher generation

Higher generation is defined using nerves of coverings of groups by cosets. The relevant definitions are as follows.

Definition 1.1 (Nerve). Let \(X\) be a set and \(\mathcal{U}\) a collection of subsets covering \(X\). The nerve of the cover \(\mathcal{U}\), denoted \(\mathcal{N}(\mathcal{U})\), is a simplicial complex with vertex set \(\mathcal{U}\), such that pairwise distinct vertices \(U_0, \ldots, U_k\) span a \(k\)-simplex if and only if \(U_0 \cap \cdots \cap U_k \neq \emptyset\).

The type of nerve we are interested in is the following coset complex.

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**Definition 1.2** (Coset complex and higher generation). Let $G$ be a group and $\mathcal{F}$ a family of subgroups. Let $\mathcal{U} := \prod_{H \in \mathcal{F}} G/H$ be the covering of $G$ by cosets of subgroups in $\mathcal{F}$. We call $\mathcal{N}(\mathcal{U})$ the coset complex of $G$ with respect to $\mathcal{F}$, and denote it $CC(G, \mathcal{F})$. We say that $\mathcal{F}$ $n$-generates $G$ if $CC(G, \mathcal{F})$ is $(n-1)$-connected, and $\infty$-generates $G$ if $CC(G, \mathcal{F})$ is contractible.

The following theorem indicates some ways higher generation can be used. The first part says that 1-generation equals generation, and the second part says that a 2-generating family yields a decomposition of $G$ as an amalgamated product.

**Theorem 1.3.** [AH93, Theorem 2.4] Let $\mathcal{F} = \{H_\alpha \mid \alpha \in \Lambda\}$ be a family of subgroups of $G$.

1. $\mathcal{F}$ is 1-generating if and only if $\bigcup \{H_\alpha \mid \alpha \in \Lambda\}$ generates $G$.
2. $\mathcal{F}$ is 2-generating if and only if the natural map $\prod_{\alpha \in \Lambda} H_\alpha \to G$ is an isomorphism.

Here by $\prod_{\alpha \in \Lambda} H_\alpha$ we mean the amalgamated product of the $H_\alpha$ over their intersections. We remark that another equivalent condition in part (1) is that the map $\prod_{\alpha \in \Lambda} H_\alpha \to G$ be surjective.

An important observation about coset complexes is that the action of the group on the complex has a very nice fundamental domain.

**Observation 1.4** (Fundamental domain). With the above notation, assume $\mathcal{F}$ is finite. Since $\bigcap_{H \in \mathcal{F}} H \neq \emptyset$, we see that $\mathcal{F}$ itself is the vertex set of a maximal simplex in $CC(G, \mathcal{F})$. This maximal simplex, which we call $C$, is a fundamental domain for the action of $G$ on $CC(G, \mathcal{F})$ by left multiplication.

**Proof.** For any simplex $\sigma$ in $CC(G, \mathcal{F})$, there exist $H_0, \ldots, H_k \in \mathcal{F}$ and $g \in G$ such that the vertices of $\sigma$ are the cosets $gH_i$ for $0 \leq i \leq k$. Then $g^{-1}\sigma$ is a face of $C$. This shows that every $G$-orbit intersects $C$, and indeed intersects $C$ uniquely since if $gH_i = H_j$ then $g \in H_i = H_j$. □

A sort of converse of this observation is the following proposition, which allows us to detect highly generating families of subgroups as stabilizers of “nice” actions.

**Proposition 1.5** (Detecting coset complexes). Let $G$ be a group acting by simplicial automorphisms on a simplicial complex $X$, with a single maximal simplex $C$ as fundamental domain. Let $\mathcal{F} := \{\text{Stab}_G(v) \mid v \text{ is a vertex of } C\}$. Then $CC(G, \mathcal{F})$ is isomorphic to $X$ as a simplicial $G$-complex.

**Proof.** Define a map $\phi: CC(G, \mathcal{F}) \to X$ by sending the coset $g\text{Stab}_G(v)$ to the vertex $gv$ of $X$. This is a $G$-invariant map between the 0-skeleta, and it induces a simplicial map since the vertices of a simplex in $CC(G, \mathcal{F})$ can be represented as cosets with a common left representative. Since $C$ is a fundamental domain, $\phi$ is bijective. □

A good first example is when $X$ is a tree, on which a group $G$ acts edge transitively and without inversion. Then Theorem 1.3 and Proposition 1.5 imply that $G$ decomposes as an amalgamated product. Namely, if $e$ is a fundamental domain with endpoints $v$ and $w$, then $G = G_v *_{G_e} G_w$ (this is standard Bass-Serre theory). Indeed, the vertex stabilizers are not just 2-generating, but $\infty$-generating.

This example is generalized by looking at groups acting on buildings.

**Example 1.6** (Buildings). Let $G$ be a group acting chamber transitively on a building $\Delta$, by type preserving automorphisms. See [AB08] for the relevant background. Let $C$ be the fundamental chamber, and let $\mathcal{F} := \{\text{Stab}_G(v) \mid v \text{ is a vertex of } C\}$. Then $CC(G, \mathcal{F}) \cong \Delta$, and so $\mathcal{F}$ is highly generating for $G$. More precisely, if $\Delta$ is spherical of dimension $n$ then $\mathcal{F}$ is $n$-generating, and if $\Delta$ is not spherical then $\mathcal{F}$ is $\infty$-generating. If the action is not just chamber transitive, but is even Weyl transitive, as in [AB08, Chapter 6], then the stabilizers Stab$_G(v)$ are precisely the maximal standard parabolic subgroups. An even stronger condition is that the action is strongly transitive, in which case $G$ has a $BN$-pair, and we recover the situation in [AH93, Section 3.2].

We also have examples from the world of Artin groups.
Example 1.7 (Deligne complexes). Background for this example can be found in [CD95]. Let \((A,S)\) be an Artin system with associated Coxeter system \((W,S)\). For \(T \subseteq S\) let \(A_T\) (respectively \(W_T\)) be the subgroup generated by \(T\). Let \(\mathcal{F} := \{A_T \mid T \subseteq S\}\) and \(\mathcal{F} := \{A_T \mid T \subseteq S\text{ with }|W_T| < \infty\}\). The coset complexes \(CC(A,\mathcal{F})\) and \(CC(A,\mathcal{F})\) are, up to homotopy equivalence, the Deligne complex and modified Deligne complex of \(A\). The connectivity of these complexes, and hence the higher generation properties of these families of subgroups, is tied to the \(K(\pi,1)\) Conjecture described in [CD95]. Namely, \(\mathcal{F}\) is conjecturally \(\infty\)-generating; see [CD95, Conjecture 2]. This is known to hold for many Artin groups, including for braid groups.

2. Arc complexes and braige complexes

In this section we define and analyze some complexes on which the braid group and pure braid group act. After recalling some background on braid groups, in the first subsection we look at the restricted arc complex on a surface, and in the second subsection we look at the complex of dangling (pure) braigs. Arc complexes are well-studied, and the restricted arc complex here will provide a coset complex for \(PB_n\) using arc stabilizers as subgroups. Braige complexes are more esoteric (the name "braige" would only be familiar if the reader has seen [BFS12]), but these will provide a coset complex for \(PB_n\) using subgroups obtained via the "strand cloning maps." These subgroups are smaller than the arc stabilizers, and more visualizable when using strand pictures for braids. We will save the connectivity calculations for Section 3, after which we will conclude that these families of subgroups are highly generating.

Before introducing the complexes, we quickly recall some background on braid groups. Let \(B_n\) be the braid group on \(n\) strands. This is the Artin group for the symmetric group \(\Sigma_n\), so we have a projection \(B_n \to \Sigma_n\). The kernel of this projection is the pure braid group on \(n\) strands, \(PB_n\). Moreover, \(B_n\) is isomorphic to the mapping class group of the \(n\)-punctured disk \(D_n\) [Bir74], and \(PB_n\) is obtained by treating the punctures as distinguished points and only considering homeomorphisms taking each puncture to itself.

2.1. Arc complexes. Let \(S\) be a connected surface, possibly with boundary \(\partial S\). Let \(P \subseteq S \setminus \partial S\) be a set of \(n\) points in the interior of the surface. An arc is a simple path in \(S \setminus \partial S\) that intersects \(P\) precisely at its endpoints, and whose endpoints are distinct. Our reference for arc complexes is [Hat91]. This definition of arc is different from the definition in [Hat91], in that we do not allow the endpoints of a given arc to coincide. Also, in [Hat91], points in \(P\) may be contained in \(\partial S\), which we do not allow.

Let \(\{\alpha_0, \ldots, \alpha_k\}\) be a collection of arcs. If the \(\alpha_i\) are pairwise disjoint except possibly at endpoints, and if no distinct \(\alpha_i\) and \(\alpha_j\) are homotopic relative \(P\), we call \(\{\alpha_0, \ldots, \alpha_k\}\) an arc system. The homotopy classes, relative \(P\), of arc systems form the simplices of a simplicial complex, with the face relation given by passing to subcollections of arcs.

Definition 2.1 (Arc complex). Let \(\Gamma\) be a simplicial graph with \(|P|\) vertices, and identify \(P\) with the set of vertices of \(\Gamma\). Call an arc compatible with \(\Gamma\) if its endpoints are connected by an edge in \(\Gamma\). Let \(\mathcal{H}(\Gamma)\) be the arc complex on \((S,P)\) corresponding to \(\Gamma\), that is the simplicial complex with a \(k\)-simplex for each arc system \(\{\alpha_0, \ldots, \alpha_k\}\) such that all the \(\alpha_i\) are compatible with \(\Gamma\). When extra precision is required, we will write \(\mathcal{H}(S,P,\Gamma)\).

Given an arc system \(\sigma = \{\alpha_0, \ldots, \alpha_k\}\) in \(\mathcal{H}(\Gamma)\), denote by \(\Gamma_\sigma\) the following subgraph of \(\Gamma\). Every vertex of \(\Gamma\) is a vertex of \(\Gamma_\sigma\), and an edge \(e\) of \(\Gamma\) is in \(\Gamma_\sigma\) if and only if the endpoints of \(e\) are the endpoints of some \(\alpha_i\). Call \(\Gamma_\sigma\) faithful if it has precisely \((k+1)\) edges. Since we only consider simplicial graphs, i.e., there are no loops or multiple edges, this condition is equivalent to saying that no distinct \(\alpha_i, \alpha_j\) share both endpoints (they may share one).

Of particular interest is the complete graph \(K_n\), which is the graph with \(n\) vertices and a single edge between any two distinct vertices. We remark that \(\mathcal{H}(K_n)\) is a proper subcomplex of the complex \(\mathcal{A}(S,P)\) in [Hat91], since we only consider arcs with two distinct endpoints. We are also interested in the family of subgraphs of linear graphs \(L_n\). The linear graph \(L_n\) has \(n\) vertices labeled 1 through \(n\), and \(n-1\) edges, one connecting the vertex labeled \(i\) to the vertex labeled \(i+1\) for each \(1 \leq i < n\). We will be interested in subgraphs of \(L_n\) having the same vertex set as \(L_n\).
Terminological convention: From now on, a subgraph $\Gamma'$ of a graph $\Gamma$ always has the same vertex set as $\Gamma$.

An important subcomplex of $\mathcal{HA}(\Gamma)$ is the matching complex on the surface $S$ with distinguished points $P$, introduced in [BFS+12]. First we define the matching complex of a graph, and then the matching complex on a surface.

**Definition 2.2** (Matching complex of a graph). Let $\Gamma$ be a subgraph of $K_n$. The matching complex $M(\Gamma)$ of $\Gamma$ is the simplicial complex with a $k$-simplex for each collection of $k+1$ pairwise disjoint edges $\{e_0, \ldots, e_k\}$ of $\Gamma$, and face relation given by passing to subcollections. Observe that if $\Gamma'$ is a subgraph of $\Gamma$ then $M(\Gamma')$ is a subcomplex of $M(\Gamma)$.

**Definition 2.3** (Matching complex on a surface). Let $\mathcal{MA}(K_n)$ be the subcomplex of $\mathcal{HA}(K_n)$ whose simplices are given by arc systems whose arcs are pairwise disjoint including at their endpoints. For a subgraph $\Gamma$ of $K_n$ let $\mathcal{MA}(\Gamma)$ be the preimage of $M(\Gamma)$ under the map $\mathcal{MA}(K_n) \to M(K_n)$ that sends an arc with endpoints labeled $i$ and $j$ to the edge of $K_n$ with endpoints $i$ and $j$. We call $\mathcal{MA}(\Gamma)$ the matching complex on $(S, P)$ corresponding to $\Gamma$. We may also write $\mathcal{MA}(S, P, \Gamma)$.

The complex we are presently interested in is a complex $\mathcal{RA}(\Gamma)$, which we will call the restricted arc complex.

**Definition 2.4** (Restricted arc complex). The restricted arc complex $\mathcal{RA}(\Gamma)$ on $(S, P)$ corresponding to $\Gamma$ is the subcomplex of $\mathcal{HA}(\Gamma)$ consisting of arc systems $\sigma$ for which $\Gamma_\sigma$ is faithful. We may also write $\mathcal{RA}(S, P, \Gamma)$.

We could equivalently require that the subspace of $S$ given by the union of the arcs is a simplicial graph, i.e., has no multiple edges. In this way we can view $\mathcal{RA}(\Gamma)$ as the complex of embeddings of subgraphs of $\Gamma$ into $S$ that send vertices in a prescribed way to the points of $P$. The $\Gamma = L_n$ case is especially nice, since all of $L_n$ can be embedded into any connected surface. In fact, every simplex of $\mathcal{RA}(L_n)$ is a face of a maximal simplex of dimension $n-2$. See Figure 1 for some examples of arc systems.

**Remark 2.5.** Embedding graphs into surfaces is an interesting enterprise in its own right, so the complex $\mathcal{RA}(\Gamma)$ may be of further general interest. For instance, the dimension of $\mathcal{RA}(S, P, \Gamma)$ is one less than the number of edges in a maximal subgraph of $\Gamma$ embeddable into $(S, P)$.

Being the mapping class group of $D_n$, $B_n$ acts on $\mathcal{HA}(K_n)$, and this action stabilizes $\mathcal{MA}(K_n)$ and $\mathcal{RA}(K_n)$. For general $\Gamma$, $B_n$ will not necessarily stabilize $\mathcal{HA}(\Gamma)$, since general braids may not stabilize $P$ pointwise. However, pure braids do stabilize $P$ pointwise, and so $PB_n$ stabilizes $\mathcal{HA}(\Gamma)$, $\mathcal{MA}(\Gamma)$ and $\mathcal{RA}(\Gamma)$ for any $\Gamma$.

Denote by $[m]$ the set $\{1, \ldots, m\}$ for $m \in \mathbb{N}$. Let $S$ be the unit disk, and fix an embedding $L_n \hookrightarrow S$ of the linear graph with $n$ vertices into $S$. Let $P$ be the image of the vertex set, so $P$ is a
set of \( n \) points in \( S \), labeled 1 through \( n \). Under this embedding, the edges of \( L_n \) yield a maximal simplex of \( \mathcal{R}A(L_n) \), which we will denote \( C \). For each \( \emptyset \neq J \subseteq [n-1] \) define \( \sigma_J \) to be the face of \( C \) consisting only of those arcs with endpoints \( j, j+1 \) for \( j \in J \). In particular, \( \sigma_J \) is a \((|J|-1)\)-simplex in \( \mathcal{R}A(L_n) \).

For each \( \emptyset \neq J \subseteq [n-1] \) define
\[
PB_n^J := \text{Stab}_{PB_n}(\sigma_J)
\]
and set \( \mathcal{A} \mathcal{F}_n := \{ PB_n^J \mid \emptyset \neq J \subseteq [n-1] \text{ with } |J| = 1 \} \).

**Lemma 2.6.** The coset complex \( CC(PB_n, \mathcal{A} \mathcal{F}_n) \) and the restricted arc complex \( \mathcal{R}A(L_n) \) are isomorphic as simplicial \( PB_n \)-complexes.

**Proof.** It suffices by Proposition \ref{prop:simplicial-independence} to show that \( C \) is a fundamental domain for the action of \( PB_n \) on \( \mathcal{R}A(L_n) \). A maximal simplex of \( \mathcal{R}A(L_n) \) is an embedding of \( L_n \) into \( S \), such that the vertex labeled \( i \) maps to the point in \( P \) labeled \( i \), for each \( 1 \leq i \leq n \). Any such simplex is in the \( PB_n \)-orbit of \( C \). Moreover, if \( p \sigma_J = \sigma_K \) for \( p \in PB_n \) and \( \sigma_J, \sigma_K \) are faces of \( C \), then since \( p \) is pure we know that \( J = K \). We conclude that \( C \) is a fundamental domain. \( \square \)

In Section \ref{sec:connectivity} we will calculate the connectivity of \( \mathcal{R}A(L_n) \), and deduce that \( \mathcal{A} \mathcal{F}_n \) is highly generating for \( PB_n \). Before doing that, we describe another complex with a nice action.

**2.2. Braige complexes.** The terminology “braige” is short for braid-merge. For our purposes, a braige consists of a braid whose strands may merge together at the bottom, in some sense. A more formal definition is as follows.

**Definition 2.7 (Braigas).** A braige on \( n \) strands is a pair \( (b, \Gamma) \), consisting of a braid \( b \in B_n \) and a subgraph \( \Gamma \) of \( L_n \). If the edges of \( \Gamma \) are disjoint, we call \( (b, \Gamma) \) elementary. If the braid is pure, then the braige is a pure braige. See Figure 2 for some examples.

![Figure 2. A braige on 6 strands and an elementary pure braigae on 6 strands.](image)

As a remark, this definition is not the same as the one in \[ BFS+12 \]. Here a “merge” amounts to choosing some pairs of adjacent strands that should be stuck together at the bottom with edges. In \[ BFS+12 \], the merging is more subtle; strands merge two at a time, not in a square shape but in more of a triangle, and a new strand continues down out of the merge. This new strand may merge further with other strands, but one must keep track of the order of merging. However, the notions of elementary braigas are the same here and in \[ BFS+12 \], since it does not matter in which order the merges occur.

Let \( \mathcal{B}_n(L_n) \) be the set of all braigas on \( n \) strands. There is a left action of \( B_n \) on \( \mathcal{B}_n(L_n) \), via \( b(c, \Gamma) := (bc, \Gamma) \). We can think of \( \mathcal{B}_n(L_n) \) as a simplicial complex, where \( (b, \Gamma) \) is a face of \( (b', \Gamma') \) if \( b = b' \) and \( \Gamma' \) is a subgraph of \( \Gamma \). Restricting to pure braigas, we get the set \( \mathcal{P}B_n(L_n) \) of pure braigas, with an action of \( PB_n \). A nice feature of this is that \( (\text{id}, L_n) \) is a fundamental domain for the action of \( B_n \) on \( \mathcal{B}_n(L_n) \), or \( PB_n \) on \( \mathcal{P}B_n(L_n) \). However, it is easy to see that \( \mathcal{B}_n(L_n) \) and \( \mathcal{P}B_n(L_n) \) stand little chance of being connected, since we can only “move” by changing the merges, and not the braid. To get a highly connected complex, we will consider an equivalence relation on these complexes via the notion of “dangling.” First we need to define what it means for a strand in a braid to be a clone.

**Definition 2.8 (Clones).** Let \( b \in B_n \). Number the strands of \( b \) from left to right at their tops by \( 1, \ldots, n \). Let \( \rho_b \) be the permutation induced by \( b \) under \( B_n \rightarrow \Sigma_n \). Think of \( b \) as living in 3-space \( \mathbb{R}^3 \), with the top of the \( i \)-th strand at the point \( (i, 1, 0) \) and the bottom at \( (\rho_b(i), 0, 0) \), for
each \( i \in [n] \). In particular all the tops and bottoms of the strands are in the \( xy \)-plane. Note that for any given strand, \( b \) has a representation wherein that strand is entirely contained in the \( xy \)-plane. Now suppose that for some \( i \in [n-1] \), \( b \) can be represented in such a way that the \( i^{th} \) and \((i+1)^{st}\) strands are simultaneously in the \( xy \)-plane, and moreover, no strands of the braid other than those two intersect the closed region of the \( xy \)-plane bounded by the two strands and the line segments from \((i,0,0)\) to \((i+1,0,0)\) and from \((\rho_b(i),0,0)\) to \((\rho_b(i+1),0,0)\). In this case we will refer to the \((i+1)^{st}\) strand as a clone, specifically a clone of the \( i^{th} \) strand. Note that necessarily \( \rho_b(i+1) = \rho_b(i) + 1 \).

Our convention is to always consider the strand on the right to be the clone of the strand on the left, as opposed to the other way around. See Figure 3 for an example.

**Figure 3.** The sixth strand is a clone of the fifth.

For each \( i \in [n-1] \) there is a cloning map \( \kappa_i : B_{n-1} \to B_n \) given by cloning the \( i^{th} \) strand. This is not a homomorphism, but becomes one when restricted to \( \kappa_i : PB_{n-1} \to PB_n \). For \( I = \{i_1, \ldots, i_r\} \subseteq [n-r] \), with \( i_1 < \cdots < i_r \), define the cloning map \( \kappa_I := \kappa_{i_1} \circ \cdots \circ \kappa_{i_r} : B_{n-r} \to B_n \). The restriction \( \kappa_I : PB_{n-r} \to PB_n \) is again a homomorphism. Now for \( J = \{j_1, \ldots, j_r\} \subseteq [n-1] \), with \( j_1 < \cdots < j_r \) let \( I_j \subseteq [n-r] \) be the set \( \{j_i - (i - 1) \mid 1 \leq i \leq r\} \). The point is that a braid \( b \in B_n \) is in the image of \( \kappa_I \) if and only if for each \( j \in J \), the \((j+1)^{st}\) strand is a clone of the \( j^{th} \) strand. Denote the subset of such braids by \( B^{(j)}_n \), and the subgroup of such pure braids by \( PB^{(j)}_n \). (The parentheses distinguish \( PB^{(j)}_n \) from the arc system stabilizer \( PB^{(j)}_n \) from the previous section.)

We can now define the equivalence relation between braids, given by “dangling.”

**Definition 2.9** (Dangling). Let \((b, \Gamma)\) be a braige on \( n \) strands, and number the vertices of \( \Gamma \) by \( 1, \ldots, n \) from left to right. Let \( J_\Gamma \subseteq [n-1] \) be the set of left endpoints of edges of \( \Gamma \). Now consider any braid \( c \) from the set \( B^{(J_\Gamma)}_n \). For each \( j \in J_\Gamma \), we know that \( \rho_c(j + 1) = \rho_c(j) + 1 \), so there is a subgraph of \( L_n \) whose edges are precisely those connecting \( \rho_c(j) \) and \( \rho_c(j) + 1 \) for \( j \in J_\Gamma \). Call this graph \( \Gamma^c \). The point is that, if we draw \( c \) below the braige, and “pull” the merges through \( c \), we get the braige \((bc, \Gamma^c)\). Now declare that \((b, \Gamma)\) is equivalent to \((bc, \Gamma^c)\) for each \( c \in B^{(J_\Gamma)}_n \). One checks that this is an equivalence relation, called equivalence under dangling. Denote the equivalence class of \((b, \Gamma)\) by \([[(b, \Gamma)]\] \), and call it a dangling braige. The idea is that the top of a braige is static, but the strands at the bottom are free to “dangle,” modulo the restriction that the merges remain rigid (and oriented) during the dangling. We analogously get the notion of a dangling pure braige, where we only consider \( c \) as above coming from \( PB^{(J_\Gamma)}_n \), so in particular \( \Gamma^c \) always equals \( \Gamma \) in the pure case. An example of dangling can be seen in Figure 4.

**Figure 4.** The first and second braiges are equivalent under pure dangling, but are not equivalent to the third one.
Let $\mathcal{B}_n(L_n)$ be the set of equivalence classes under dangling of braiges in $\mathcal{B}_n(L_n)$. The simplicial structure of the latter induces a simplicial structure on the former, for example the faces of $[(b, \Gamma)]$ are precisely of the form $[(bc, \Gamma')]$ for $c \in B_n(L_n)$ and $\Gamma'$ a subgraph of $\Gamma^c$. Also let $\mathcal{PB}_n(L_n)$ be the set of dangling pure braiges. The faces of a dangling pure braige $[(p, \Gamma)]$ are the dangling pure braiges of the form $[(pc, \Gamma')]$ for $c \in PB_n(L_n)$ and $\Gamma'$ a subgraph of $\Gamma$. Heuristically, in $\mathcal{B}_n(L_n)$ we can move around not only by changing the mergers, but now also by changing the braid in certain controlled ways, so $\mathcal{B}_n(L_n)$ and $\mathcal{PB}_n(L_n)$ stand a chance of being connected (for large enough $n$), and even highly connected. In the pure case we can also define $\mathcal{PB}_n(\Gamma)$ for any subgraph $\Gamma$ of $L_n$, by only considering braiges from $\mathcal{PB}_n(\Gamma)$. We also have the subcomplexes of dangling elementary braiges or dangling elementary pure braiges, denoted $\mathcal{EB}_n(L_n)$ and $\mathcal{EPB}_n(L_n)$ respectively. In the pure case we can use any $\Gamma$ instead of $L_n$, and get the complex $\mathcal{EPB}_n(\Gamma)$. This will be an important subcomplex for proving that $\mathcal{PB}_n(L_n)$ is highly connected.

The left action of $B_n$ on $\mathcal{B}_n(L_n)$ induces an action of $B_n$ on $\mathcal{B}_n(L_n)$; for $c \in B_n$ we have $c[(b, \Gamma)] := [(cb, \Gamma)]$. Similarly, $\mathcal{PB}_n$ acts from the left on $\mathcal{PB}_n(L_n)$, and indeed stabilizes $\mathcal{PB}_n(\Gamma)$ for any subgraph $\Gamma$ of $L_n$. The action of $\mathcal{PB}_n$ on $\mathcal{PB}_n(L_n)$ is of particular interest, since there is a fundamental domain consisting of a single maximal simplex, namely $[(id, L_n)]$. This tells us that $\mathcal{PB}_n(L_n)$ is a coset complex, using the family of stabilizers of faces of $[(id, L_n)]$.

**Lemma 2.10** (Stabilizers of dangling braiges). Let $\Gamma$ be a subgraph of $L_n$. Then the stabilizer $\text{Stab}_{\mathcal{PB}_n}([(id, \Gamma)])$ is precisely the subgroup $\mathcal{PB}_n(J_{\Gamma})$.

**Proof.** First let $p \in PB_n(J_{\Gamma})$. Then $p[(id, \Gamma)] = [(p, \Gamma)] = [(id, \Gamma)]$. Now suppose $p[(id, \Gamma)] = [(id, \Gamma)]$, so $[(p, \Gamma)] = [(id, \Gamma)]$. Then there exists $c \in PB_n(J_{\Gamma})$ such that $(p, \Gamma) = (c, \Gamma)$. But this implies that $p = c$, so we are done. \qed

Let $\mathcal{F}_n := \{PB_n(J_{\Gamma}) \mid \Gamma \text{ is a subgraph of } L_n\}$ be a subgraph of $L_n$ with one edge.

**Corollary 2.11.** $\text{CC}(PB_n, \mathcal{F}_n)$ is isomorphic to $\mathcal{PB}_n(L_n)$ as a simplicial $PB_n$-complex.

**Proof.** This is immediate from Proposition 2.15 since $[(id, L_n)]$ is a fundamental domain. \qed

In the next section we will calculate the connectivity of $\mathcal{RA}(L_n)$ and $\mathcal{PB}_n(L_n)$, and hence of $\text{CC}(PB_n, \mathcal{F}_n)$ and $\text{CC}(PB_n, \mathcal{F}_n)$, from which we deduce higher generation.

We close this section by setting up a generalization of the complexes we have constructed. Note that in the definition of $\mathcal{F}_n$ we require $|J| = 1$, and in the definition of $\mathcal{F}_n$ we require $\Gamma$ to have only one edge (this is the same as saying $|J_{\Gamma}| = 1$). The subgroups in these families consist of braids that, respectively, stabilize some arc, or feature at least one cloned strand. Of course, as $n$ grows, it becomes increasingly “easy” for a braid to be very complicated while still featuring a cloned strand, or stabilizing an arc. Hence, higher generation becomes an even more interesting question if we consider requirements like, e.g., all but 5 strands are clones. (Observe that any of the standard generators of $PB_n$ satisfy this very requirement.)

**Definition 2.12** (More restrictive families). Let $s \in \mathbb{N}$. Define $\mathcal{F}_n^s := \{PB_n \mid J \subseteq [n-1] \text{ with } |J| = s\}$. Also define $\mathcal{F}_n^{n-1} := \{PB_n(J_{\Gamma}) \mid \Gamma \text{ is a subgraph of } L_n \text{ with } s \text{ edges}\}$. Hence $\mathcal{F}_n^1 = \mathcal{F}_n$ and $\mathcal{F}_n^{n-1} = \{Z(PB_n)\}$, and also $\mathcal{F}_n^1 = \mathcal{F}_n$ and $\mathcal{F}_n^{n-1} = \{\{1\}\}$.

### 3. Connectivity of the complexes

For $\ell \in \mathbb{Z}$ define $\eta(\ell) := \lfloor \frac{\ell - 2}{4} \rfloor$. The main goal of this section is to prove that $\mathcal{RA}(L_n)$ and $\mathcal{PB}_n(L_n)$ are $(\eta(n) - 1)$-connected.

The only blackbox we will use in this section is the following result from $\text{BFS}^{12}$. The proof there is informed by techniques from the proof of Proposition 5.2 in $\text{Put12}$.

**Theorem 3.1.** [BFS$^{12}$ Theorem 3.9] Let $\Gamma_m$ be a subgraph of $L_n$ with $m$ edges. Then $\mathcal{MA}(\Gamma_m)$ is $(\eta(m + 1) - 1)$-connected.

The function $\eta$ is defined slightly differently in $\text{BFS}^{12}$, hence the unusual-looking connectivity bound here. The point it that $L_n$ has $n - 1$ edges, so in particular $\mathcal{MA}(L_n)$ is $(\eta(n) - 1)$-connected.
3.1. Connectivity of arc complexes. Our first goal is to deduce the connectivity of $\mathcal{RA}(L_n)$ from Theorem 3.1. Let $\Gamma_m$ be a subgraph of $L_n$ with $m$ edges. For a $k$-simplex $\sigma = \{\alpha_0, \ldots, \alpha_k\}$ in $\mathcal{RA}(\Gamma_m)$, define $r(\sigma)$ to be the number of points in $P$ that are used as endpoints of arcs in $\sigma$. Then define the defect $d(\sigma)$ to be $2(k+1) - r(\sigma)$. Let $h$ be the function on the barycentric subdivision $\mathcal{RA}(\Gamma_m)$ of $\mathcal{RA}(\Gamma_m)$ given by $h(\sigma) = (d(\sigma), -\dim(\sigma))$, ordered lexicographically. Note that $d(\sigma) = 0$ if and only if the arcs are all disjoint, even at their endpoints. Hence, thinking of $h$ as a height function on the vertices of $\mathcal{RA}(\Gamma_m)$, in the sense of [BB97], we observe that the sublevel set $(\mathcal{RA}(\Gamma_m))^{d=0}$ is precisely $\mathcal{MA}(\Gamma_m)$. Hence we can compare the homotopy types of the two complexes using discrete Morse theory, with [BB97] Corollary 2.6 as the guide. The key is to inspect the descending links with respect to $h$. This is very similar to the procedure used in [BFS12] to deduce connectivity of $\mathcal{MA}(K_n)$ from connectivity of $\mathcal{HA}(K_n)$.

**Corollary 3.3.** $\mathcal{RA}(\Gamma_m)$ is $(\eta(m+1) - 1)$-connected.

**Proof.** We know that $\mathcal{MA}(\Gamma_m)$ is $(\eta(m+1) - 1)$-connected by Theorem 3.1. We claim that the inclusion $\mathcal{MA}(\Gamma_m) \to \mathcal{RA}(\Gamma_m)$ induces a surjection in homotopy $\pi_k$ for $k \leq \eta(m+1) - 1$, from which the proposition follows. To prove the claim, it suffices by [BB97] Corollary 2.6 to prove that for $\sigma \in \mathcal{RA}(\Gamma_m) \setminus \mathcal{MA}(\Gamma_m)$, i.e., $h(\sigma) > 0$, the descending link $lk_\downarrow(\sigma)$ is $(\eta(m+1) - 2)$-connected. We suppose that $\sigma$ is a $k$-simplex, with $\sigma = \{\alpha_0, \ldots, \alpha_k\}$.

There are two types of arc systems in $lk_\downarrow(\sigma)$. First, we could have $\sigma' < \sigma$ and $h(\sigma') < h(\sigma)$. Then $\sigma'$ is obtained from $\sigma$ by removing arcs and strictly decreasing the defect. Call the full subcomplex $\mathcal{RA}(\Gamma_m)$ of $lk_\downarrow(\sigma)$ spanned by these $\sigma'$ the down-link. Second, we could have $\sigma > \sigma$ and $h(\sigma) > h(\sigma)$. Here $\sigma$ is obtained by adding new arcs to $\sigma$, so that the new arcs are all disjoint from each other and from any existing arcs, even at endpoints. Call the full subcomplex of $lk_\downarrow(\sigma)$ spanned by such $\sigma$ the up-link. Any simplex in the down-link is a face of every simplex in the up-link, so $lk_\downarrow(\sigma)$ is the join of the down-link and up-link.

First consider the down-link. A face $\sigma'$ of $\sigma$ fails to be in the down-link if and only if each arc in $\sigma \setminus \sigma'$ is disjoint from every other arc of $\sigma$, since then and only then do $\sigma$ and $\sigma'$ have the same defect. Let $\sigma_0$ be the face of $\sigma$ consisting precisely of all such arcs, if any exist. Since $d(\sigma) > 0$, we know $\sigma_0 \neq \sigma$. The boundary of $\sigma$ is a $(k-1)$-sphere, and the complement in the boundary of the down-link is either empty, or is a cone with cone point $\sigma_0$. Hence the down-link is either a $(k-1)$-sphere or is contractible, so in particular is $(k-2)$-connected. At this point we may assume without loss of generality that the down-link is a $(k-1)$-sphere, and so every arc in $\sigma$ shares an endpoint with some other arc in $\sigma$. This means that every edge of $\Gamma_\sigma$ shares an endpoint with some other edge of $\Gamma_\sigma$. In particular $k \geq 1$.

Now consider the up-link. The simplices in the up-link are given by adding arcs to $\sigma$ that are all disjoint from each other and from the arcs in $\sigma$. Consider the connected surface $S' := S \setminus \{\alpha_0, \ldots, \alpha_k\}$, obtained by cutting out the arcs $\alpha_i$. If $P' := S' \cap P$, then $|P'| = n - r(\sigma)$. Also let $\Gamma_m' \setminus 2k-2$ be the subgraph of $\Gamma_m$ obtained by removing the edges of $\Gamma_\sigma$, and all edges sharing a vertex with any of these, so $\Gamma_m' \setminus 2k-2$ has at most $m - 2k - 2$ edges (here we use the fact that every edge of $\Gamma_\sigma$ shares an endpoint with some other edge of $\Gamma_\sigma$). The up-link of $\sigma$ is isomorphic to the matching complex $\mathcal{MA}(S', P', \Gamma_m' \setminus 2k-2)$, which is $(\eta(m - 2k - 1) - 1)$-connected. Since $lk_\downarrow(\sigma)$ is the join of the down- and up-links, we conclude that $lk_\downarrow(\sigma)$ is $(\eta(m - 2k - 1) - 1)$-connected. We have $\eta(m - 2k - 1) + k - 1 \geq 4m - 2k - 3 + k - 2 \geq \eta(m + 1) + k \geq 2 \geq 2 \geq \eta(m + 1) - 2$ since $k \geq 1$, and so we are done.

The next corollary is immediate, keeping in mind that with our notation $L_n$ has $n-1$ edges.

**Corollary 3.3.** $\mathcal{RA}(L_n)$ is $(\eta(n) - 1)$-connected.

**Corollary 3.4.** $\mathcal{CC}(PB_n, \mathcal{AF}_n)$ is $(\eta(n) - 1)$-connected, and hence $\mathcal{AF}_n$ is $(\eta(n)$-generating for $PB_n$.

**Proof.** This is immediate from Lemma 2.6 and Corollary 3.3.

We also want to show that the families $\mathcal{AF}_n$ from Definition 2.12 are highly generating. For $s > 1$, the coset complex $\mathcal{CC}(PB_n, \mathcal{AF}^s_n)$ is obtained up to homotopy equivalence from $\mathcal{CC}(PB_n, \mathcal{AF}^{s-1}_n)$ by removing the open stars of the vertices, i.e., the cosets $pPB_n$ for $|J| = s-1$. Hence the problem
amounts to showing high connectivity of links. This is more or less the procedure done in the proof of Theorem 3.3 in [AH98], in the context of buildings. It is a bit harder here though; links in buildings are themselves buildings, but links in restricted arc complexes are not themselves restricted arc complexes. Nonetheless, we can get the right connectivity without too much extra work.

Lemma 3.5 (Links in \( \mathcal{RA}(\Gamma_m) \)). Let \( \sigma = \{\alpha_0, \ldots, \alpha_k\} \) be a \( k \)-simplex in \( \mathcal{RA}(\Gamma_m) \) for \( \Gamma_m \) as above (with \( m \) edges). Then the link \( \text{lk}_{\mathcal{RA}(\Gamma_m)}(\sigma) \) is \((\eta(m-k) - 1)\)-connected.

To make precise the terminology, here by “link” we mean the subcomplex of simplices \( \tau \) disjoint from \( \sigma \) for which there exists a simplex with \( \tau \) and \( \sigma \) as faces.

Proof. Set \( L := \text{lk}_{\mathcal{RA}(\Gamma_m)}(\sigma) \). An arc system \( \tau \) is in \( L \) if and only if each arc of \( \tau \) is distinct from, but compatible with, every \( \alpha_i \). For such a \( \tau \), by retracting each arc \( \alpha_i \) to a point, \( \tau \) maps to an arc system in \( \mathcal{RA}(\Gamma_{m-(k+1)}) \). Here \( \Gamma_{m-(k+1)} \) is a subgraph of \( \Gamma_m \) with \( m-(k+1) \) edges. More formally, for \( 0 \leq d \leq k \) consider the homotopy equivalence of surfaces \( r_d: S \to S_d \), obtained by collapsing \( \alpha_i \) to a point, for each \( 0 \leq i \leq d \). Recall \( S = D_n \), and here \( S_d \) is just our name for the copy of \( D_{n-(d+1)} \) obtained by collapsing these arcs. Here we do not think of \( D_n \) as a punctured disk, but rather as a disk with \( n \) distinguished points; hence \( r_d \) is really a homotopy equivalence. Also let \( P_d \) be the image of \( P \) under \( r_d \). We have induced maps of complexes \( R_d: L \to \mathcal{RA}(\Gamma_{m-(d+1)}) \). Note that these maps are surjective, but not injective; see Figure 5 for an example of the non-injectivity. Note however that the connectivity of \( \mathcal{RA}(\Gamma_{m-(k+1)}) \) is precisely the connectivity we are trying to verify for \( L \).

The \( r_d \) also induce epimorphisms \( \phi_d: PB_n \to PB_{n-(d+1)} \), with kernels \( K_d := \ker(\phi_d) \). Also declare \( K_{-1} \) to be the trivial subgroup. Note that \( K_{-1} \leq K_0 \leq \cdots \leq K_k \). Colloquially, the pure braids \( p \) in \( K_d \setminus K_{d-1} \) are precisely those that do “twist” \( \alpha_d \) but don’t twist any \( \alpha_i \) for \( i > d \). For \( p \in K_k \), define \( D(p) := \min\{d+1 \mid p \in K_d\} \). We will call \( D(p) \) the deviation of \( p \); note that \( D(p) = 0 \) if and only if \( p = \text{id} \). Now fix a map \( s_{id}: S_k \to S \) with \( s_{id} \circ r_k \) homotopic to the identity. This essentially amounts to fixing a choice of how to “blow up” each arc \( \alpha_i \) to get from \( S_k \) back to \( S \). We get an induced map \( \iota_{id}: \mathcal{RA}(\Gamma_{m-(k+1)}) \to L \), with \( R_k \circ \iota_{id} \) equal to the identity on \( \mathcal{RA}(\Gamma_{m-(k+1)}) \). For each \( p \in K_k \), set \( \iota_p := \iota_{id} \circ \iota_{id} \). These maps are all injective simplicial maps that can be thought of as different choices of how to blow up each \( \alpha_i \), and we see that \( R_k \circ \iota_p \) is the identity for all \( p \). Every arc system in \( L \) is the image of an arc system in \( \mathcal{RA}(\Gamma_{m-(k+1)}) \) under some \( \iota_p \), so \( L = \bigcup_{p \in PB_n} \text{Im}(\iota_p) \). Also, each \( \text{Im}(\iota_p) \) is isomorphic to \( \mathcal{RA}(\Gamma_{m-(k+1)}) \), and hence is an \((\eta(m-k) - 1)\)-connected subcomplex of \( L \). We now need to glue these \( \text{Im}(\iota_p) \) together in a clever order, always along \((\eta(m-k) - 2)\)-connected relative links, from which we will deduce that \( L \) is \((\eta(m-k) - 1)\)-connected.

The measurement \( D(p) \) provides such an order. For \( 0 \leq d \leq k \) let \( L^d := \bigcup_{D(p) \leq d} \text{Im}(\iota_p) \). We claim that \( L^d \) is \((\eta(m-k) - 1)\)-connected for all \( d \). The base case \( d = 0 \) is clear. For a given \( d \), the intersection \( \text{Im}(\iota_p) \cap \text{Im}(\iota_q) \) with \( p \neq q \) and \( D(p) = d + 1 \) is contained in \( L^d \). This is because \( p \) and \( q \) must twist the arc \( \alpha_d \) differently, and so if \( \beta \) is an arc in \( \text{Im}(\iota_p) \cap \text{Im}(\iota_q) \) then \( \beta \) cannot share endpoints with \( \alpha_d \). For this reason, we can build up from \( L^0 \) to \( L^{d+1} \) by attaching the \( \text{Im}(\iota_p) \) with deviation \( d + 1 \), in any order, and the relative links will always be in \( L^d \). Now, for \( p \) with \( D(p) = d + 1 \), we attach \( \text{Im}(\iota_p) \) to \( L^d \) along the intersection \( \text{Im}(\iota_p) \cap L^d \). This intersection consists precisely of those arc systems in \( \text{Im}(\iota_p) \) that do not use arcs sharing endpoints with \( \alpha_d \). Applying \( R_k \) (so retracting each \( \alpha_i \) to a point), this gives us the subcomplex of \( \mathcal{RA}(\Gamma_{m-(k+1)}) \) whose arcs are disjoint from the endpoint obtained by collapsing \( \alpha_d \). But this is just \( \mathcal{RA}(\Gamma') \) for \( \Gamma' \) a subgraph of \( \Gamma_{m-(k+1)} \) with at most two fewer edges. This is \((\eta(m-k) - 2)\)-connected, and so we are done. □

![Figure 5. Distinct arcs in the link of \( \sigma \) that map to the same arc under \( R_d \).](image-url)
Proposition 3.6. For \( s \in \mathbb{N} \), \( CC(PB_n, \mathcal{F}^s_n) \) is \((\eta(n - (s - 1)) - 1)\)-connected, and hence \( \mathcal{F}^s_n \) is \((\eta(n - (s - 1)))\)-generating for \( PB_n \).

Proof. It suffices to show that for \(|J| = s - 1\), the link of \( PB_n^J \) in \( CC(PB_n, \mathcal{F}^{s-1}_n) \) is \((\eta(n - (s - 1)) - 1)\)-connected. Equivalently, we need the link of \( \sigma_j \) in \( RA(L_n) \) to be \((\eta(n - (s - 1)))\)-connected. Since \( \sigma_j \) is a \((|J| - 1)\)-simplex, its link is \((\eta(n - (|J| - 1)))\)-connected by Lemma 3.5 (since \( L_n \) has \( n - 1 \) edges), and since \(|J| = s - 1\), we conclude that indeed the link is \((\eta(n - (s - 1)))\)-connected. \( \square \)

3.2. Connectivity of braige complexes. Now we inspect \( CC(PB_n, \mathcal{F}_n) \), or more accurately \( PB_n(L_n) \). To pass from the world of arcs to the world of braiges, we will project the braiges onto arcs in the following way. For each \( J \subseteq [n - 1] \), let \( \sigma_J \) be the simplex of \( MA(L_n) \) defined before Lemma 2.6. Consider the action of \( PB_n \) on \( RA(L_n) \) as a right action, and define a map

\[
\pi: PB_n(L_n) \to RA(L_n)
\]

where \( J_\ell \) is as in Definition 2.9. We will use \( \pi \) to also denote the restrictions \( \mathcal{F}PB_n(L_n) \to MA(L_n) \), \( PB_n(\Gamma) \to RA(\Gamma) \) and \( \mathcal{F}PB_n(\Gamma) \to MA(\Gamma) \) for \( \Gamma \) a subgraph of \( L_n \). As in [BFS +12], think of \( \pi \) as the procedure of combing the braid straight and watching where the arcs get moved.

Proposition 3.7 (Braige connectivity from arc connectivity). For \( \Gamma_m \) be a subgraph of \( L_n \) with \( m \) edges, \( \mathcal{F}PB_n(\Gamma_m) \) is \((\eta(m + 1) - 1)\)-connected.

Proof. By Theorem 3.1, \( MA(\Gamma_m) \) is \((\eta(m + 1) - 1)\)-connected. Let \( \sigma = \{a_0, \ldots, a_k\} \) be a \( k \)-simplex in \( MA(\Gamma_m) \). The link \( lk(\sigma) \) of \( \sigma \) in \( MA(\Gamma_m) \) for \( \Gamma' \) a subgraph of \( \Gamma_m \) with at least \( m - 3(k + 1) \) edges, so \( lk(\sigma) \) is \((\eta(m - 3(k + 1) - 1)\)-connected, and hence \((\eta(m + 1) - 2)\)-connected. It now suffices by [Qui78] Theorem 9.1 to prove that the fiber \( \pi^{-1}(\sigma) \) is \((\eta(m + 1) - 2)\)-connected (here we treat a simplex as a closed cell). Indeed, we will prove that \( \pi^{-1}(\sigma) \) is the join of the fibers \( \pi^{-1}(\alpha_i) \) of the vertices \( \alpha_i \) of \( \sigma \). See also Proposition 4.3 in [BFS +12].

Let \( \mathcal{F} \mathcal{F} := \bigvee_{i=0}^k \pi^{-1}(\alpha_i) \) be the join of the vertex fibers. Clearly \( \pi^{-1}(\sigma) \subseteq \mathcal{F} \mathcal{F} \). Also, the \( r \)-skeleton of \( \mathcal{F} \mathcal{F} \) is contained in \( \pi^{-1}(\sigma) \). Now suppose that the same is true of the \( r \)-skeleton for \( r > 0 \). An \((r + 1)\)-simplex in \( \mathcal{F} \mathcal{F} \) is the join of a \( 0 \)-simplex and an \( r \)-simplex, both of which are contained in \( \pi^{-1}(\sigma) \). It now suffices to prove the following claim.

Claim: Let \( [(p, E)] \) be a vertex in \( \mathcal{F}PB_n(\Gamma_m) \), so \( p \in PB_n \) and \( E \) is a one-edge subgraph of \( \Gamma_m \). Let \( [(q, \Gamma)] \) be a simplex in \( \mathcal{F}PB_n(\Gamma_m) \) such that \( \pi([(q, \Gamma)]) \) does not contain \( \pi([(p, E)]) \) but does share a simplex with \( \pi([(p, E)]) \) in \( MA(\Gamma_m) \). Then \( [(q, \Gamma)] \) shares a simplex with \( [(p, E)] \) in \( \mathcal{F}PB_n(\Gamma_m) \).

This hypothesis is rephrased in terms of arcs as: \((\Gamma) q^{-1} \) shares a simplex with \((E)p^{-1} \). By acting from the left with \( PB_n \), we can assume without loss of generality that \( p = \text{id} \), so we have \( \pi([(p, E)]) = E \). Let \( \beta_i \) be \( \pi^{-1}(\alpha_i) \), chosen so that \( E \) is disjoint from the \( \beta_i \), even at endpoints (remember we are in \( MA(\Gamma_m) \), not just \( RA(\Gamma_m) \)). This is possible by the hypothesis, and implies that the dangling equivalence class \([(q, \Gamma)] \) contains a representative in which the \((j + 1)\)st strand is a loop of the \( j \)th strand, for \( j \) and \( j + 1 \) are the endpoints of the edge of \( E \). We can assume \( (q, \Gamma) \) itself is such a representative, in which case the dangling braige \([(q, \Gamma \cup E)] \) is a simplex of \( \mathcal{F}PB_n(\Gamma_m) \) containing \([(q, \Gamma)] \) and \([(p, E)], \) proving the claim. \( \square \)

It might be possible to mimic this proof using \( \pi: PB_n(\Gamma) \to RA(\Gamma) \) instead, and get the connectivity of \( PB_n(L_n) \) right away, but the downside is that the fibers are not joins of vertex fibers. Hence one would have to do extra work to show that fibers have the right connectivity.

To calculate the connectivity of \( PB_n(L_n) \), we will use a similar procedure as for \( RA(\Gamma_m) \). Namely, we will build up from \( \mathcal{F}PB_n(\Gamma_m) \) to \( PB_n(\Gamma_m) \) using discrete Morse theory. A \( k \)-simplex in \( PB_n(\Gamma_m) \) is a dangling equivalence class of a pair \((p, \Gamma)\), for \( p \in PB_n \) and \( \Gamma \) a subgraph of \( \Gamma_m \) with \( k + 1 \) edges. Let \( r(\Gamma) \) be the number of vertices that are endpoints of an edge in \( \Gamma \). Then define the defect \( d(p, \Gamma) \) to be \( 2(k + 1) - r(\Gamma) \). Extend these definitions to the dangling equivalence classes, and observe that \( \mathcal{F}PB_n(\Gamma_m) \) is the \( d = 0 \) sublevel set of \( PB_n(\Gamma_m) \). We now apply Morse theory, as before.

Proposition 3.8. \( PB_n(\Gamma_m) \) is \((\eta(m + 1) - 1)\)-connected.

Proof. By Proposition 3.7, \( \mathcal{F}PB_n(\Gamma_m) \) is \((\eta(m + 1) - 1)\)-connected. Mimicking the proof of Proposition 3.2 it suffices to prove that for \( \sigma \in PB_n(\Gamma_m) \setminus \mathcal{F}PB_n(\Gamma_m) \), the descending link \( \text{lk}_{\mathcal{F}}(\sigma) \) is \((\eta(m + 1) - 2)\)-connected. Let \( \sigma \) be such a \( k \)-simplex, say \( \sigma = [(p, \Gamma)] \). The down-link is either \( S^{k-1} \),
or contractible if $\Gamma$ has an isolated edge. Suppose there is no such isolated edge, so the down-link is $S^{k-1}$. Now, the up-link is obtained by dangling and then adding extra edges to the graph, such that the new edges are disjoint from $\Gamma$ and from each other. Since $\Gamma$ has no isolated edges, there are at most $2(k+1)$ edges of $\Gamma_m$ that share an endpoint with an edge of $\Gamma$. Hence the up-link of $\sigma$ is isomorphic to $E\mathcal{PB}_\ell(\Gamma_{m-2k-2})$ for some $\ell$, which is $(\eta(m-2k-1)-1)$-connected. The calculation from the proof of Proposition 3.2 now tells us that $\text{lk}_k(\sigma)$ is $(\eta(m+1)-2)$-connected.

**Corollary 3.9.** $\mathcal{PB}_n(L_n)$ is $(\eta(n)-1)$-connected.

**Corollary 3.10.** CC$(PB_n, \mathcal{RF}_n)$ is $(\eta(n)-1)$-connected, and hence $\mathcal{RF}_n$ is $\eta(n)$-generating for $PB_n$.

**Example 3.11.** For $n \geq 6$, CC$(PB_n, \mathcal{RF}_n)$ is connected, so $PB_n$ has a generating set in which each generator features at least one cloned strand. Indeed, the standard generating set from Section 1.3.1 of [KT08] satisfies this property for $n \geq 6$, and fails for $n < 6$. For $n \geq 10$, CC$(PB_n, \mathcal{RF}_n)$ is simply connected, so $PB_n$ is 2-generated by $\mathcal{RF}_n$. Hence there exists a presentation for $PB_n$ in which every generator features a cloned strand, and the relations all arise from relations in the subgroups of braids with a cloned strand. Again we note that the standard presentation works precisely in this range.

We conclude by showing that the families $\mathcal{RF}_n^s$ for $s \in \mathbb{N}$, defined in Definition 2.12 are highly generating as well. For $s > 1$, the coset complex CC$(PB_n, \mathcal{RF}_n^s)$ is obtained up to homotopy equivalence from CC$(PB_n, \mathcal{RF}_n^{s-1})$ by removing the open stars of vertices, i.e, cosets $pPB_n^{(J)}$ for $|J| = s-1$, just like in the arc case.

**Lemma 3.12** (Links in $\mathcal{PB}_n(\Gamma_m)$). Let $\sigma$ be a $k$-simplex in $\mathcal{PB}_n(\Gamma_m)$ for $\Gamma_m$ as above (with $m$ edges). Then the link $\text{lk}_{\mathcal{PB}_n(\Gamma_m)}(\sigma)$ is $(\eta(m-k)-1)$-connected.

**Proof.** Links in the braid case are nicer than links in the arc case, since they are actually isomorphic to smaller braide complexes. In the arc case, namely in the proof of Lemma 3.3, we related a given link to a smaller arc complex, via a map that was not an isomorphism. In the present case, we claim that $\text{lk}_{\mathcal{PB}_n(\Gamma_m)}(\sigma)$ is just isomorphic to $\mathcal{PB}_{n-(k+1)}(\Gamma_{m-(k+1)})$, for $\Gamma_{m-(k+1)}$ a graph with $m-(k+1)$ edges, and then the connectivity result is immediate. Say $\sigma = [(p, \Gamma_{k+1})]$ for $\Gamma_{k+1}$ a subgraph of $\Gamma_m$ with $k+1$ edges. Let $L := \text{lk}_{\mathcal{PB}_n(\Gamma_m)}(\sigma)$. The simplices in $L$ are dangling braides of the form $\tau = [(pq, \Gamma)]$, where $q \in PB_n^{(J_{k+1})}$ and $\Gamma$ is a subgraph of $\Gamma_m$ having no edges in common with $\Gamma_{k+1}$. The first condition ensures that $\tau$ and $\sigma$ share a simplex, namely $[(pq, \Gamma \cup \Gamma_{k+1})]$, and the second condition ensures that $\tau$ and $\sigma$ are disjoint. Acting from the left with $PB_n$, we can assume $p = \text{id}$. We have a map $\phi: L \to \mathcal{PB}_{n-(k+1)}(\Gamma_{m-(k+1)})$, where $\Gamma_{m-(k+1)}$ is the graph with $n-(k+1)$ vertices that is obtained from $\Gamma_m$ by retracting each edge of $\Gamma_{k+1}$ to a point. The map $\phi$ sends $\tau = [(q, \Gamma)]$ to $[(q', \Gamma')]$, where $\Gamma'$ is the image of $\Gamma$ under the retraction $\Gamma_m \to \Gamma_{m-(k+1)}$, and $q'$ is the preimage of $q$ under the cloning map $\kappa_{J_{k+1}}$. See Figure 6 for an example. Since $q'$ is uniquely determined by $q$, we have an inverse $\phi^{-1}$, induced by the cloning map. (This is the essential difference from the arc case, that there is only one way to “blow up” a braide via cloning.) Since $\phi$ and $\phi^{-1}$ are of course simplicial maps, we conclude that $\phi$ is a simplicial isomorphism, and the result follows.

**Figure 6.** The map $\phi$ takes an element of $\text{lk}_{\mathcal{PB}_4(L_4)}(\sigma)$ to an element of $\mathcal{PB}_4(L_4)$. Here $\sigma$ is $[(\text{id}, E_4)]$, for $E_4$ the subgraph with a single edge indicated by the dashed line.
Proposition 3.13. For \( s \in \mathbb{N} \), \( \text{CC}(PB_n, \mathcal{B}^s_n) \) is \( (\eta(n - (s - 1)) - 1) \)-connected, and hence \( \mathcal{B}^s_n \) is \( \eta(n - (s - 1)) - 1 \)-generating for \( PB_n \).

Proof. As in the proof of Proposition 3.6 it suffices to prove that for \( \Gamma \) with \( s - 1 \) edges, the link of the \((s - 2)\)-simplex \([\text{id}, \Gamma]\) in \( PB_n(L_n) \) is \( (\eta(n - (s - 1)) - 1) \)-connected. Since \( L_n \) has \( n - 1 \) edges, this follows from Lemma 3.12.

Example 3.14. To generalize the previous example, we have that for any \( n \geq 6 \), \( \mathcal{B}^n_n \) is 1-generating for \( PB_n \). This means that \( PB_n \) has a set of generators such that in each generator, all but 5 strands are clones (indeed the standard generators have this property). Similarly for \( n \geq 10 \), \( \mathcal{B}^{n-9}_n \) is 2-generating for \( PB_n \), so \( PB_n \) has a presentation in which each relation can be realized by using only 9 non-clone strands. Again, the standard presentation fits the bill.

Example 3.15. In the situation of arcs, the swing presentation for \( PB_n \), described in Section 4 of [MM09], provides an explicit example of \( \mathcal{A}^n_n \) being 1-generating for \( n \geq 6 \) and \( \mathcal{A}^{n-9}_n \) being 2-generating for \( n \geq 10 \). In this presentation the generators are Dehn twists, each of which must stabilize at least one arc of the form \( \sigma_j \), as soon as \( n \geq 6 \). Each relation in [MM09, Theorem 4.10] (specifically the second presentation) is a product of Dehn twists, and for \( n \geq 10 \) this product stabilizes at least one arc of the form \( \sigma_j \). See Figure 7 for an example.

Figure 7. With 6 points, each generator must stabilize an arc. With 10 points, each relation must stabilize an arc. The dashed lines indicate the arcs stabilized in the examples. The relation pictured here is a lantern relation, as in Figure 12 of [MM09].

References


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