UNIQUENESS RESULTS FOR NON-NEGATIVE SOLUTIONS OF QUASI-LINEAR INEQUALITIES ON RIEMANNIAN MANIFOLDS

YUHUA SUN[‡]

ABSTRACT. We investigate the uniqueness of nonnegative solution to the following differential inequality

$$div(|\nabla u|^{m-2}\nabla u) + u^\sigma \leq 0, \tag{1}$$

on a noncompact complete Riemannian manifold, where $m > 1$ and $\sigma > m - 1$ are parameters. Our main result is as follows: If the volume of a geodesic ball of radius r with a fixed center x_0 is bounded by $Cr^{\frac{m\sigma}{\sigma-m+1}} \ln^{\frac{m-1}{\sigma-m+1}} r$ for large enough r, then the only non-negative solution to (1) is identical zero.

We also show the sharpness of exponents $\frac{m\sigma}{\sigma-m+1}$ and $\frac{m-1}{\sigma-m+1}$.

1. INTRODUCTION

In this paper, our purpose is to investigate the nonexistence of non-negative solution to the following differential inequality

$$div(|\nabla u|^{m-2}\nabla u) + u^\sigma \leq 0. \tag{1.1}$$

on a geodesically complete connected Riemannian manifold M. Here div and ∇ are the Riemannian divergence and gradient respectively. $m > 1$ and $\sigma > m-1$ are given parameters. The operator $\Delta_m u := div(|\nabla u|^{m-2}\nabla u)$ is well-known and is called the $m-$Laplacian.

Many studies about non-linear elliptic differential inequalities on Riemannian manifolds similar to (1.1) arise naturally in geometry and physics. In the Euclidean setting such problems have been investigated by many authors. For example, in a series of papers[14, 16, 17, 18] E. Mitidieri and S. I. Pohozaev investigated a priori estimates and the nonexistence of solutions. P. Pucci, J. Serrin and H. Zou [20, 22] established maximum principles. Liouville type theorems are obtained in [3, 4, 24], and [11] is devoted to the comparison theorem. In the context of Riemannian manifolds, S. Y. Cheng, S.-T. Yau [2] proved that if the volume grows at most quadratic , then there is no nontrivial positive superharmonic function on the manifold. See also [7, 8, 9, 21], for further references.

Consider in $M = \mathbb{R}^n$ the inequality

$$\Delta u + u^\sigma \leq 0, \tag{1.2}$$

which is a particular case of (1.1) with $m = 2$. It is well known, that in $\mathbb{R}^2$ the only non-negative solution to (1.2) is identical zero, while in $\mathbb{R}^n$ with $n \geq 3$, the same result holds if and only if $\sigma \leq \frac{n}{n-2}$(cf. [19]).

Mitidieri and Pohozaev [15] obtained a similar result for (1.1) in $\mathbb{R}^n$. In particular, they proved that if

$$0 < m - 1 < \sigma \leq \frac{n(m-1)}{n-m}, \quad \text{and} \quad n > m. \tag{1.3}$$

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then (1.1) has no positive solution from a certain natural class.

Let us turn to results on Riemannian manfolds. Let M be a geodesically complete non-compact connected Riemannian manifold. Denote by μ the Riemannian measure on M and by $d(x,y)$ the geodesic distance between $x,y\in M$. Let $B(x,r)$ be the geodesic ball centered at $x\in M$ of radius r. Fix some $x_0\in M$ and Set $V(r):=\mu(B(x_0,r))$.

In [10, Theorem 2.2], Holopainen proved the following result for solution to

$$\Delta_m u\leq 0, \tag{1.4}$$

Namely, if $m>1$ and

$$\int^{\infty}\left(\frac{r}{V(r)}\right)^{\frac{1}{m-1}}dr=\infty, \tag{1.5}$$

then any non-negative solution to (1.4) is identical constant. In particular, this implies that any non-negative solution to

$$\Delta_m u+u^{\sigma}\leq 0.$$

is identical zero. Note that (1.5) is satisfied if $V(r)\leq Cr^m$ or even if $V(r)\leq Cr^m\ln^{m-1}r$.

In [9], Grigor'yan and the author investigated (1.1) with $m=2$, that is

$$\Delta u+u^{\sigma}\leq 0,$$

and proved that under the following volume growth hypothesis

$$V(r)\leq cr^{\frac{2\sigma}{\sigma-1}}\ln^{\frac{1}{\sigma-1}}r,\quad \text{for all large enough } r.$$

then the only non-negative solution to (1.1) is identical zero.

In the present paper, we concentrate our attention on obtaining optimal volume growth conditions that ensure the the uniqueness of non-negative solution to (1.1) for all $m>1$ and $\sigma>m-1$.

Before we state the results, let us first give the definition of a solution to (1.1). Set

$$W^{1,m}_{loc}(M)=\left\{f:M\to\mathbb{R}|f\in L^m_{loc}(M),\nabla f\in L^m_{loc}(M)\right\}, \tag{1.6}$$

where ∇f is understood in distributional sense. Denote by $W^{1,m}_c(M)$ the subspace of $W^{1,m}_{loc}(M)$ of functions with compact support.

Definition 1.1. A function u on M is called a non-negative weak solution of the inequality (1.1) if $u\in W^{1,m}_{loc}(M)$, $u\geq 0$ and for any non-negative function $\psi\in W^{1,m}_c(M)$, the following inequality holds:

$$-\int_M(|\nabla u|^{m-2}\nabla u,\nabla\psi)d\mu+\int_M u^{\sigma}\psi d\mu\leq 0, \tag{1.7}$$

where $(\cdot,\cdot)$ is the inner product in T_xM given by the Riemannian metric.

Remark. Using the defintion of ψ, we have

$$\begin{aligned}\left|\int_M(|\nabla u|^{m-2}\nabla u,\nabla\psi)d\mu\right| &\leq \int_{supp(\psi)}\|\nabla u\|^{m-1}\|\nabla\psi\|d\mu\\ &\leq \left(\int_{supp(\psi)}\|\nabla u\|^m d\mu\right)^{\frac{m-1}{m}}\left(\int_{supp(\psi)}\|\nabla\psi\|^m d\mu\right)^{\frac{1}{m}},\end{aligned}$$

Since $u\in W^{1,m}_{loc}(M)$ and $\psi\in W^{1,m}_c(M)$. Therefore, the first term in (1.7) is finite, which implies the finiteness of the second term, that is $\int_M u^{\sigma}\psi d\mu<\infty$.

Assuming always $\sigma > m-1$, let us introduce two paramters

$$p = \frac{m\sigma}{\sigma - m + 1}, \quad q = \frac{m-1}{\sigma - m + 1}. \tag{1.8}$$

Here are our main results.

Theorem 1.1. *If for some $x_0 \in M$, the following inequality*

$$V(r) \leq C r^p \ln^q r, \tag{1.9}$$

holds for all large enough r, then the only non-negative solution to (1.1) is identical zero.

Both values of p, q are sharp, that is, the statement of Theorem 1.1 is not true for large values of p and q.

In section 2, we present the proof for Theorem 1.1. In section 3, we present an example showing the sharpness of parameters p, q.

Notation. The letters $C, C_0, C_1, \ldots$ denote positive constants whose values are unimportant and may vary at different occurrences.

2. Proof of Theorem 1.1

Let u be a non-negative weak solution to (1.1). x_0 is the reference point as before in Theorem 1.1. Denote $B_R := B(x_0, R)$, and fix a Lipschitz function φ on M with compact support, such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in a neighborhood of B_R. In particular, we have $\varphi \in W_c^{1,m}(M)$. Take the following test function for (1.7):

$$\psi_\rho(x) = \varphi(x)^s (u+\rho)^{-t}, \tag{2.1}$$

where ρ is a positive constant near zero, and t, s satisfy two condtions

$$0 < t < \min\left\{1, \frac{\sigma - m + 1}{2}\right\}, \quad s > \frac{4m\sigma}{\sigma - m + 1}. \tag{2.2}$$

Actually, the value of s is a large enough fixed constant, while t is variable and can be chosen arbitrarily close to 0.

Note that $\frac{1}{u+\rho}$ is bounded, hence ψ_ρ has compact support and is bounded. Since

$$\nabla \psi_\rho = -t(u+\rho)^{-t-1}\varphi^s \nabla u + s(u+\rho)^{-t}\varphi^{s-1}\nabla\varphi,$$

we see that, $\nabla\psi_\rho \in L^m(M)$. It follows that, $\psi_\rho \in W_c^{1,m}(M)$. We obtain from (1.7) that

$$\begin{aligned} & t\int_M \varphi^s (u+\rho)^{-t-1} \|\nabla u\|^m d\mu + \int_M \varphi^s u^\sigma (u+\rho)^{-t} d\mu \\ \leq \ & s\int_M \varphi^{s-1}(u+\rho)^{-t} \|\nabla u\|^{m-2} (\nabla u, \nabla\varphi) d\mu. \end{aligned}$$

thus

$$\begin{aligned} & t\int_M \varphi^s (u+\rho)^{-t-1} \|\nabla u\|^m d\mu + \int_M \varphi^s u^\sigma (u+\rho)^{-t} d\mu \\ \leq \ & s\int_M \varphi^{s-1}(u+\rho)^{-t} \|\nabla u\|^{m-1} \|\nabla\varphi\| d\mu. \end{aligned} \tag{2.3}$$

In what follows, we use the following Young's inequality

$$\int_M fg\,d\mu \leq \varepsilon \int_M |f|^{p_0} d\mu + C_\varepsilon \int_M |g|^{p_0'} d\mu, \tag{2.4}$$

where $\varepsilon > 0$ is arbitrary, and (p_0, p_0') is a Hölder conjugate couple such that

$$p_0 = \frac{m}{m-1}, \quad p_0' = m.$$

Applying (2.4) to the right-hand-side integral of (2.3), we obtain

$$\begin{aligned}
& s\int_M \varphi^{s-1}(u+\rho)^{-t}\left\|\nabla u\right\|^{m-1}\left\|\nabla\varphi\right\|d\mu \\
= & \int_M \left(t^{\frac{m-1}{m}}\varphi^{\frac{s(m-1)}{m}}(u+\rho)^{-\frac{(t+1)(m-1)}{m}}\left\|\nabla u\right\|^{m-1}\right) \\
& \times\left(\frac{s}{t^{\frac{m-1}{m}}}\varphi^{s-1-\frac{s(m-1)}{m}}(u+\rho)^{-t+\frac{(t+1)(m-1)}{m}}\left\|\nabla\varphi\right\|\right)d\mu \\
\leq & \ \varepsilon t\int_M \varphi^s(u+\rho)^{-t-1}\left\|\nabla u\right\|^m d\mu + C_\varepsilon\frac{s^m}{t^{m-1}}\int_M \varphi^{s-m}(u+\rho)^{m-t-1}\left\|\nabla\varphi\right\|^m d\mu.
\end{aligned}$$

Letting $\varepsilon=\frac{1}{2}$, substituting the above estimate into (2.3), and cancelling out the half of the first term in (2.3), we obtain

$$\begin{aligned}
& \frac{t}{2}\int_M \varphi^s(u+\rho)^{-t-1}\left\|\nabla u\right\|^m d\mu + \int_M \varphi^s u^\sigma(u+\rho)^{-t}d\mu \\
\leq & \ \frac{Cs^m}{t^{m-1}}\int_M \varphi^{s-m}(u+\rho)^{m-t-1}\left\|\nabla\varphi\right\|^m d\mu. \qquad (2.5)
\end{aligned}$$

Using (2.4) once more to the right hand side of (2.5) with another Hölder conjugate couple (p_1, p_1') satisfying

$$p_1=\frac{\sigma-t}{m-t-1},\quad p_1'=\frac{\sigma-t}{\sigma-m+1}$$

we obtain

$$\begin{aligned}
& \frac{Cs^m}{t^{m-1}}\int_M \varphi^{s-m}(u+\rho)^{m-t-1}\left\|\nabla\varphi\right\|^m d\mu \\
= & \int_M [\varphi^{\frac{s}{p_1}}(u+\rho)^{m-t-1}]\cdot[\tfrac{Cs^m}{t^{m-1}}\varphi^{\frac{s}{p_1'}-m}\left\|\nabla\varphi\right\|^m]d\mu \\
\leq & \ \frac{1}{2}\int_M \varphi^s(u+\rho)^{\sigma-t}d\mu \\
& +C_1\left(\frac{s^m}{t^{m-1}}\right)^{\frac{\sigma-t}{\sigma-m+1}}\int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}\left\|\nabla\varphi\right\|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu. \qquad (2.6)
\end{aligned}$$

Using in the right hand side of (2.6) the obvious inequality

$$\left(\frac{s^m}{t^{m-1}}\right)^{\frac{\sigma-t}{\sigma-m+1}}\leq\left(\frac{s^m}{t^{m-1}}\right)^{\frac{\sigma}{\sigma-m+1}},$$

and combining (2.6) with (2.5), we obtain

$$\begin{aligned}
& \frac{t}{2}\int_M \varphi^s(u+\rho)^{-t-1}\left\|\nabla u\right\|^m d\mu + \int_M \varphi^s u^\sigma(u+\rho)^{-t}d\mu \\
\leq & \ \frac{1}{2}\int_M \varphi^s(u+\rho)^{\sigma-t}d\mu \\
& +C_1 t^{-\frac{\sigma(m-1)}{\sigma-m+1}}\int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}\left\|\nabla\varphi\right\|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu, \qquad (2.7)
\end{aligned}$$

where the term contains s is absorbed into constant C_1.

Since

$$\begin{aligned}
\int_M \varphi^s(u+\rho)^{\sigma-t}d\mu &= \int_M \varphi^s(u+\rho)^\sigma(u+\rho)^{-t}d\mu \\
&\leq 2^{\sigma-1}\rho^{-t}\left(\int_M \varphi^s u^\sigma(u+\rho)^{-t}d\mu+\rho^{-t}\int_M \varphi^s d\mu\right), \qquad (2.8)
\end{aligned}$$

By the definition of solution, we know $\int_M \varphi^s u^\sigma (u+\rho)^{-t} d\mu$ is bounded, hence, the term

$$\int_M \varphi^s (u+\rho)^{\sigma-t} d\mu$$

is bounded. By Dominated Convergence theorem, letting $\rho \to 0$, we have

$$\lim_{\rho\to 0} \int_M \varphi^s (u+\rho)^{\sigma-t} d\mu = \int_M \varphi^s u^{\sigma-t} d\mu, \tag{2.9}$$

Letting $\rho \to 0$ in (2.7), applying Monotone Convergence theorem to the right-hand side integrals in 2.7, we obtain

$$\begin{aligned} & \lim_{\rho\to 0} \frac{t}{2} \int_M \varphi^s (u+\rho)^{-t-1} \|\nabla u\|^m d\mu + \lim_{\rho\to 0} \int_M \varphi^s u^\sigma (u+\rho)^{-t} d\mu \\ \leq \ & \lim_{\rho\to 0} \frac{1}{2} \int_M \varphi^s (u+\rho)^{\sigma-t} d\mu \\ & + C_1 t^{-\frac{\sigma(m-1)}{\sigma-m+1}} \int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}} \|\nabla\varphi\|^{\frac{m(\sigma-t)}{\sigma-m+1}} d\mu, \end{aligned}$$

combining with (2.9), which is

$$\begin{aligned} & \frac{t}{2} \int_M \varphi^s u^{-t-1} \|\nabla u\|^m d\mu + \frac{1}{2} \int_M \varphi^s u^\sigma u^{-t} d\mu \\ \leq \ & C_1 t^{-\frac{\sigma(m-1)}{\sigma-m+1}} \int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}} \|\nabla\varphi\|^{\frac{m(\sigma-t)}{\sigma-m+1}} d\mu, \end{aligned} \tag{2.10}$$

Applying (1.7) once more with another test function $\psi = \varphi^s$, we obtain

$$\begin{aligned} & \int_M \varphi^s u^\sigma d\mu \\ \leq \ & s \int_M \varphi^{s-1} \|\nabla u\|^{m-2} (\nabla u, \nabla\varphi) d\mu \\ \leq \ & s \int_M \varphi^{s-1} \|\nabla u\|^{m-1} \|\nabla\varphi\| d\mu \\ \leq \ & s \left(\int_M \varphi^s u^{-t-1} \|\nabla u\|^m d\mu \right)^{\frac{m-1}{m}} \left(\int_M \varphi^{s-m} u^{(t+1)(m-1)} \|\nabla\varphi\|^m d\mu \right)^{\frac{1}{m}}. \end{aligned} \tag{2.11}$$

On the other hand, we obtain from (2.10) that

$$\int_M \varphi^s u^{-t-1} \|\nabla u\|^m d\mu \leq C t^{-1-\frac{\sigma(m-1)}{\sigma-m+1}} \int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}} \|\nabla\varphi\|^{\frac{m(\sigma-t)}{\sigma-m+1}} d\mu.$$

Substituting this into (2.11) yields

$$\begin{aligned} \int_M \varphi^s u^\sigma d\mu \ \leq \ & C \left[t^{-1-\frac{\sigma(m-1)}{\sigma-m+1}} \int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}} \|\nabla\varphi\|^{\frac{m(\sigma-t)}{\sigma-m+1}} d\mu \right]^{\frac{m-1}{m}} \\ & \times \left[\int_M \varphi^{s-m} u^{(t+1)(m-1)} \|\nabla\varphi\|^m d\mu \right]^{\frac{1}{m}}. \end{aligned} \tag{2.12}$$

Recalling that $\nabla\varphi = 0$ on B_R, and applying Hölder inequality to the last term of (2.12) with the following Hölder couple (p_2, p_2')

$$p_2 = \frac{\sigma}{(t+1)(m-1)}, \quad p_2' = \frac{\sigma}{\sigma-(t+1)(m-1)},$$

we obtain

$$\begin{aligned}
& \int_M \varphi^{s-m} u^{(t+1)(m-1)} \|\nabla\varphi\|^m \, d\mu \\
= \; & \int_{M\setminus B_R} \left(\varphi^{\frac{s}{p_2}} u^{(t+1)(m-1)}\right) \left(\varphi^{\frac{s}{p_2'}-m} \|\nabla\varphi\|^m\right) d\mu \\
\leq \; & \left(\int_{M\setminus B_R} \varphi^s u^\sigma d\mu\right)^{\frac{(t+1)(m-1)}{\sigma}} \\
& \times \left(\int_{M\setminus B_R} \varphi^{s-\frac{m\sigma}{\sigma-(t+1)(m-1)}} \|\nabla\varphi\|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}} \, d\mu\right)^{1-\frac{(t+1)(m-1)}{\sigma}}. \qquad (2.13)
\end{aligned}$$

Substituting (2.13) into (2.12), we obtain

$$\begin{aligned}
& \int_M \varphi^s u^\sigma d\mu \\
\leq \; & C t^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}} \left(\int_M \varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}} \|\nabla\varphi\|^{\frac{m(\sigma-t)}{\sigma-m+1}} \, d\mu\right)^{\frac{m-1}{m}} \\
& \times \left(\int_M \varphi^{s-\frac{m\sigma}{\sigma-(t+1)(m-1)}} \|\nabla\varphi\|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}} \, d\mu\right)^{\frac{1}{m}-\frac{(t+1)(m-1)}{m\sigma}} \\
& \times \left(\int_{M\setminus B_R} \varphi^s u^\sigma d\mu\right)^{\frac{(t+1)(m-1)}{m\sigma}}. \qquad (2.14)
\end{aligned}$$

Noting $s > \frac{4m\sigma}{\sigma-m+1}$, and $t < \frac{\sigma-m+1}{2}$ in (2.2), and recalling that $0 \leq \varphi \leq 1$, from (2.14), we obtain

$$\begin{aligned}
& \int_M \varphi^s u^\sigma d\mu \\
\leq \; & C t^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}} \left(\int_M \|\nabla\varphi\|^{\frac{m(\sigma-t)}{\sigma-m+1}} \, d\mu\right)^{\frac{m-1}{m}} \\
& \times \left(\int_M \|\nabla\varphi\|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}} \, d\mu\right)^{\frac{1}{m}-\frac{(t+1)(m-1)}{m\sigma}} \\
& \times \left(\int_{M\setminus B_R} \varphi^s u^\sigma d\mu\right)^{\frac{(t+1)(m-1)}{m\sigma}}. \qquad (2.15)
\end{aligned}$$

Since $\int_M \varphi^s u^\sigma d\mu$ is finite due to Remark in Introduction, it follows from (2.15) that

$$\begin{aligned}
& \left(\int_M \varphi^s u^\sigma d\mu\right)^{1-\frac{(t+1)(m-1)}{m\sigma}} \\
\leq \; & C t^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}} \left(\int_M \|\nabla\varphi\|^{\frac{m(\sigma-t)}{\sigma-m+1}} \, d\mu\right)^{\frac{m-1}{m}} \\
& \times \left(\int_M \|\nabla\varphi\|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}} \, d\mu\right)^{\frac{1}{m}-\frac{(t+1)(m-1)}{m\sigma}}. \qquad (2.16)
\end{aligned}$$

We see, that all integral terms in the right hand side of (2.16) have the form

$$\int_M \|\nabla\varphi\|^a \, d\mu,$$

with the following two values of a such that

$$a = \frac{m(\sigma - t)}{\sigma - m + 1} \quad \text{and} \quad a = \frac{m\sigma}{\sigma - (t+1)(m-1)}. \tag{2.17}$$

Consequently, a could be written in the form

$$a = p + bt, \tag{2.18}$$

with the following two respective values of b

$$b = -\frac{m}{\sigma - m + 1} \quad \text{and} \quad b = \frac{m\sigma(m-1)}{[\sigma - (t+1)(m-1)](\sigma - m + 1)}. \tag{2.19}$$

where $p = \frac{m\sigma}{\sigma - m+1}$ is defined as before in (1.8). Clearly, the both values of a and b are uniformly bounded, when t is near zero.

Let $\{\tilde{\varphi}_k\}_{k\in\mathbb{N}}$ be a sequence satisfying the following condtions: each $\tilde{\varphi}_k$ is a Lipschitz function such that $supp(\tilde{\varphi}_k) \subset B_{2^k}$, $\tilde{\varphi}_k = 1$ in a neighborhood of $B_{2^{k-1}}$, and

$$\|\nabla\tilde{\varphi}_k\| \begin{cases} \leq \frac{C}{2^{k-1}}, & \text{for } x \in B_{2^k} \setminus B_{2^{k-1}}, \\ = 0, & \text{otherwise.} \end{cases} \tag{2.20}$$

where C does not depend on k.

Fix some $n \in \mathbb{N}$ and set

$$t = \frac{1}{n}, \tag{2.21}$$

and

$$\varphi_n = \frac{\sum_{k=n+1}^{2n} \tilde{\varphi}_k}{n}. \tag{2.22}$$

Note that $\varphi_n = 1$ on B_{2^n}, $\varphi_{2^{2n}} = 0$ outside $B_{2^{2n}}$, $0 \leq \varphi_n \leq 1$ on M. Note also that for any $a \geq 1$, using that $supp(\nabla\tilde{\varphi}_k)$ is disjoint with each other, we have

$$\|\nabla\varphi_n\|^a = \frac{\sum_{k=n+1}^{2n} \|\nabla\tilde{\varphi}_k\|^a}{n^a}, \tag{2.23}$$

It is easy to see that

$$\varphi_n \in W^{1,m}_{loc}(M).$$

Consider the integral

$$J_n(a) = \int_M \|\nabla\varphi_n\|^a \, d\mu, \tag{2.24}$$

where a is as above.

Substituting (2.22) into (2.24), applying (2.23) and (2.20), we obtain

$$\begin{aligned} J_n(a) &= \int_M \|\nabla\varphi_n\|^a \, d\nu \\ &= \int_M \frac{\sum_{k=n+1}^{2n} \|\nabla\tilde{\varphi}_k\|^a}{n^a} d\mu \\ &= \sum_{k=n+1}^{2n} \int_{B_{2^k}\setminus B_{2^{k-1}}} \frac{\|\nabla\tilde{\varphi}_k\|^a}{n^a} d\mu \\ &\leq C \sum_{k=n+1}^{2n} \left(\frac{2^{1-k}}{n}\right)^a \mu(B_{2^k}) \\ &\leq C \sum_{k=n+1}^{2n} \left(\frac{2^{-k}}{n}\right)^a \mu(B_{2^k}), \end{aligned} \tag{2.25}$$

where we have used that a is uniformly bounded. Noting that $a = p + bt$, $n+1 \leq k \leq 2n$, and substituting $t = \frac{1}{n}$, if $b > 0$, we obtain

$$\begin{aligned} \left(\frac{2^{-k}}{n}\right)^a &= \left(\frac{2^{-k}}{n}\right)^p \left(\frac{2^{-k}}{n}\right)^{bt} \\ &\leq \left(\frac{2^{-k}}{n}\right)^p. \end{aligned}$$

If $b < 0$, since $|b|$ is uniformly bounded, then

$$\begin{aligned} \left(\frac{2^{-k}}{n}\right)^a &= \left(\frac{2^{-k}}{n}\right)^p \left(\frac{2^{-k}}{n}\right)^{bt} \\ &\leq \left(\frac{2^{-k}}{n}\right)^p \left(n2^{2n}\right)^{\frac{|b|}{n}} \\ &\leq \left(\frac{2^{-k}}{n}\right)^p \sup_n \left(n2^{2n}\right)^{\frac{|b|}{n}} \\ &< C\left(\frac{2^{-k}}{n}\right)^p. \end{aligned}$$

Thus, in both cases, we obtain

$$\left(\frac{2^{-k}}{n}\right)^a \leq C\left(\frac{2^{-k}}{n}\right)^p. \tag{2.26}$$

Using (2.26) and (1.9), recalling that by (1.8) $p = \frac{m\sigma}{\sigma-m+1}$, $q = \frac{m-1}{\sigma-m+1}$, we obtain

$$\begin{aligned} J_n(a) &\le C \sum_{k=n+1}^{2n} \left(\frac{2^{-k}}{n}\right)^p \mu(B_{2^k}) \\ &\le C \sum_{k=n+1}^{2n} \left(\frac{2^{-k}}{n}\right)^p (2^k)^p \ln^q(2^k) \\ &\le C\frac{1}{n^p} \sum_{k=n+1}^{2n} k^q \\ &\le Cn^{q+1-p} \\ &\le Cn^{-\frac{(m-1)\sigma}{\sigma-m+1}}. \end{aligned} \tag{2.27}$$

Setting $\varphi = \varphi_n$ in (2.16), we obtain

$$\begin{aligned} &\left(\int_M \varphi_n^s u^\sigma d\mu\right)^{1-\frac{(t+1)(m-1)}{m\sigma}} \\ \le\ & ct^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}} \left(J_n(\frac{m(\sigma-t)}{\sigma-m+1})\right)^{\frac{m-1}{m}} \\ &\times \left(J_n(\frac{m\sigma}{\sigma-(t+1)(m-1)})\right)^{\frac{1}{m}-\frac{(t+1)(m-1)}{m\sigma}}. \end{aligned} \tag{2.28}$$

Substituting (2.27) into (2.28), using $t = \frac{1}{n}$ as before, we obtain

$$\begin{aligned} &\left(\int_M \varphi_n^s u^\sigma d\mu\right)^{1-\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}} \\ \le\ & Cn^{\frac{m-1}{m}+\frac{\sigma(m-1)^2}{m(\sigma-m+1)}} \left(n^{-\frac{(m-1)\sigma}{\sigma-m+1}}\right)^{\frac{m-1}{m}} \\ &\times \left(n^{-\frac{(m-1)\sigma}{\sigma-m+1}}\right)^{\frac{1}{m}-\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}}, \end{aligned} \tag{2.29}$$

the exponent in the power of n in the right hand side of (2.29) is then equal to

$$\frac{m-1}{m} + \frac{\sigma(m-1)^2}{m(\sigma-m+1)} - \frac{(m-1)\sigma}{\sigma-m+1} \cdot \frac{m-1}{m} - \frac{(m-1)\sigma}{\sigma-m+1} \cdot [\frac{1}{m} - \frac{m-1}{m\sigma} - \frac{m-1}{nm\sigma}]$$

a (careful) computation shows that all the terms here that do not contain n miraculously cancel out, so that the above expression reduces to

$$\frac{(m-1)^2}{nm(\sigma-m+1)}$$

Therefore, we obtain

$$\left(\int_M \varphi_n^s u^\sigma d\mu\right)^{1-\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}} \le Cn^{\frac{(m-1)^2}{nm(\sigma-m+1)}}. \tag{2.30}$$

Recalling that $\varphi_n = 1$ on B_{2^n}, and taking the lim sup of both sides in (2.30) as $n \to \infty$, we obtain

$$\begin{aligned} \int_M u^\sigma d\mu &\le C \limsup_{n\to\infty} n^{\frac{(m-1)^2}{nm(\sigma-m+1)}/[1-\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}]} \\ &< \infty. \end{aligned} \tag{2.31}$$

The same computation can be used in (2.15), which implies

$$\int_M \varphi_n^s u^\sigma d\mu \le C \left(\int_{M\setminus B_{2^n}} \varphi_n^s u^\sigma d\mu \right)^{\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}}, \tag{2.32}$$

Using that $0 \le \varphi_n \le 1$ and $\varphi_n|_{B_{2^n}} = 1$ once more, we obtain

$$\int_{B_{2^n}} u^\sigma d\mu \le C \left(\int_{M\setminus B_{2^n}} u^\sigma d\mu \right)^{\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}}, \tag{2.33}$$

Letting $n \to \infty$, and using (2.31), we obtain

$$\int_M u^\sigma d\mu = 0,$$

which implies that $u \equiv 0$.

3. The sharpness of p, q

In this section, we show the sharpness of parameters p and q in Theorem 1.1. Obviously, it suffices to verify the sharpness of q.

Fix $p = \frac{m\sigma}{\sigma-m+1}$, and choose any $q > \frac{m-1}{\sigma-m+1}$. We will construct an example of a manifold M satisfying the volume growth condition (1.9) with these values p,q and admitting a positive solution u of (1.1).

We will use the following proposition, which is a simplified version of [25, Theorem 2.2], where a more general ordinary differential equation has been investigated.

Proposition 3.1. Let $m > 1$, $\sigma > 0$ be a constant. Let $\beta(r)$ be a positive C^1-function on $[r_0, \infty)$ satisfying

$$\int_{r_0}^\infty (\beta(s))^{-\frac{1}{m-1}} ds < \infty. \tag{3.1}$$

Define the function $\gamma(r)$ on $[r_0, \infty)$ by

$$\gamma(r) = \int_r^\infty (\beta(s))^{-\frac{1}{m-1}} ds. \tag{3.2}$$

If

$$\int_{r_0}^\infty \beta(s)\gamma(s)^\sigma ds < \infty, \tag{3.3}$$

then the following equation

$$(\beta(r)|y'|^{m-2}y')' + \beta(r)y^\sigma = 0, \tag{3.4}$$

has at least one positive solution on $[r_0, \infty)$, which satisfies

$$y(r) = O(\gamma(r)), \quad \text{as } r \to \infty. \tag{3.5}$$

By definition in [25], a solution of (3.4) is a C^1-function y, such that $|y'|^{m-2}y'$ is also C^1, and (3.4) is satisfied.

Let M be $(\mathbb{R}^n, g)$ with the following Riemannian metric

$$g = dr^2 + \psi(r)^2 d\theta^2, \tag{3.6}$$

where (r, θ) are the polar coordinates in $\mathbb{R}^n$ and $\psi(r)$ is a smooth, positive function on $(0, \infty)$ such that

$$\psi(r) = \begin{cases} r, & \text{for small enough } r, \\ \left(r^{p-1}\ln^q r\right)^{\frac{1}{n-1}}, & \text{for large enough } r. \end{cases} \tag{3.7}$$

thus, in a neighborhood of 0, the metric g is exactly Euclidean, which can be extended smoothly to the origin. Hence, $M = (\mathbb{R}^n, g)$ is a complete Riemannian manifold.

By (3.6), the geodesic ball $B_r = B(0, r)$ on M coincides with the Euclidean ball $\{|x| < r\}$. Denote by $S(r)$ the surface area of B_r in M. It follows from (3.6) that $S(r) = \omega_n \psi^{n-1}(r)$, that is

$$S(r) = \omega_n \begin{cases} r^{n-1}, & \text{for small enough } r, \\ r^{p-1}\ln^q r, & \text{for large enough } r, \end{cases} \tag{3.8}$$

where ω_n is the surface area of the unit ball in $\mathbb{R}^n$. The Riemannian volume of the ball B_r can be determined by

$$V(r) = \mu(B_r) = \int_0^r S(\tau)\, d\tau, \tag{3.9}$$

whence it follows that, for large enough r,

$$V(r) \le C r^p \ln^q r. \tag{3.10}$$

Hence, the manifold M satisfies the volume growth condition of Theorem 1.1.

In what follows we prove the existence of a weak positive solution of

$$\Delta_m u + u^\sigma \le 0,$$

on M. This solution u will depend only on the polar radius r, so that we write $u = u(r)$. The construction of u will be done in two steps.

Step 1. For a function $u = u(r)$, the inequality (1.1) becomes

$$[S|u'|^{m-2}u']' + Su^\sigma \le 0. \tag{3.11}$$

Note that for large enough r_0

$$\begin{aligned} \int_{r_0}^\infty S(r)^{-\frac{1}{m-1}} dr &= \int_{r_0}^\infty \frac{1}{(\omega_n r^{p-1}\ln^q r)^{\frac{1}{m-1}}} dr \\ &= \int_{r_0}^\infty \frac{1}{\omega_n^{\frac{1}{m-1}} r^{\frac{p-1}{m-1}} \ln^{\frac{q}{m-1}} r} dr \\ &< \infty, \end{aligned} \tag{3.12}$$

this is because $p = \frac{m\sigma}{\sigma - m + 1} > m$. For all $r \ge r_0$, we have

$$\gamma(r) := \int_r^\infty (S(\tau))^{-\frac{1}{m-1}} d\tau = \int_r^\infty \frac{d\tau}{(\omega_n \tau^{p-1}\ln^q \tau)^{\frac{1}{m-1}}} \le \frac{C}{r^{\frac{p-1}{m-1}-1} \ln^{\frac{q}{m-1}} r}.$$

It follows that

$$\begin{aligned}\int_{r_0}^{\infty} S(\tau)\gamma^{\sigma}(\tau)d\tau &\leq C\int_{r_0}^{\infty}\frac{\tau^p\ln^q\tau}{\tau^{\frac{\sigma(p-1)}{m-1}-\sigma}\ln^{\frac{\sigma q}{m-1}}\tau}\frac{d\tau}{\tau}\\ &\leq C\int_{r_0}^{\infty}\frac{1}{\tau^{\frac{\sigma(p-1)}{m-1}-\sigma-p}\ln^{\frac{\sigma q}{m-1}-q}\tau}\frac{d\tau}{\tau}\\ &\leq C\int_{r_0}^{\infty}\frac{1}{\ln^{\frac{q(\sigma-m+1)}{m-1}}\tau}\frac{d\tau}{\tau}\\ &< \infty, \end{aligned}\tag{3.13}$$

where we have used that $p=\frac{m\sigma}{\sigma-m+1}$ and $q>\frac{m-1}{\sigma-m+1}$.

Applying Proposition 3.1 with $\beta(r)=S(r)$, we obtain that there exists some C^1 solution u of (3.11) on $[r_0,\infty)$, such that

$$u(r)=O(\gamma(r))=O(r^{-\frac{m}{\sigma-m+1}}\ln^{-\frac{q}{m-1}}r),\quad \text{as } r\to\infty.$$

In particular, $u(r)\to 0$ as $r\to\infty$. By increasing r_0 if necessary, we can assume that $u'(r_0)<0$.

Step 2. Consider the following eigenvalue problem in a ball B_ρ of M

$$\begin{cases} div(|\nabla v|^{m-2}\nabla v)+\lambda_\rho|v|^{m-2}v=0, & \text{in } B_\rho,\\ v|_{\partial B_\rho}=0. \end{cases}\tag{3.14}$$

We denote by λ_ρ the principal (smallest) eigenvalue of this problem. It is known from [13] that $\lambda_\rho>0$ and the corresponding eigenfunction $v_\rho>0$ in B_ρ. Hence, we rewrite (3.14) in the following form

$$\begin{cases} div(|\nabla v|^{m-2}\nabla v)+\lambda_\rho v^{m-1}=0, & \text{in } B_\rho,\\ v|_{\partial B_\rho}=0. \end{cases}\tag{3.15}$$

Moreover, by [13, Theorem 1.3] and [26], we know the principal eigenvalue λ_ρ is simple, and v_ρ depends only on the polar radius, we have $v_\rho=v_\rho(r)$. From [23] and [27], we know $v_\rho(r)$ is $C^{1,\beta}$ for some $\beta\in(0,1)$. Normalizing v_ρ, we can assume that $v_\rho(0)=1$, while $v_\rho|_{\partial B_\rho}=0$.

Therefore, for a radial function v_ρ, the equation (3.15) becomes

$$[S|v_\rho'|^{m-2}v_\rho']'+\lambda_\rho S v_\rho^{m-1}=0,\tag{3.16}$$

where also $v_\rho(\rho)=0$, $v_\rho(0)=1$, $v_\rho'(0)=0$, and $v_\rho>0$ in $(0,\rho)$.

From (3.16), we obtain $[S|v_\rho'|^{m-2}v_\rho']'\leq 0$, so that the function $S|v_\rho'|^{m-2}v_\rho'$ is decreasing. Since $S|v_\rho'|^{m-2}v_\rho'$ vanishes at $r=0$, it follows that $S|v_\rho'|^{m-2}v_\rho'(r)\leq 0$ and, hence $v_\rho'(r)\leq 0$ for all $r\in(0,\rho)$. Hence, the function $v_\rho(r)$ is decreasing for $r<\rho$ which together with the boundary conditions implies that $0\leq v_\rho\leq 1$. Since $\sigma>m-1$, it follows that v_ρ is a positive solution in B_ρ of the inequality

$$div(|\nabla v_\rho|^{m-2}\nabla v_\rho)+\lambda_\rho v_\rho^{\sigma}\leq 0.\tag{3.17}$$

Let us show that $\lambda_\rho\to 0$ as $\rho\to\infty$. Indeed, it is known that

$$\lim_{\rho\to\infty}\lambda_\rho=\lambda_{\min}(M)$$

where $\lambda_{\min}(M)$ is the essential m-first eigenvalue of $-\Delta_m$ in $W^{1,m}(M)$(cf. [12]).

We know from [12, Theorem 1.4](when $m=2$, one also could see [1])

$$\lambda_{\min}(M)\leq\left(\limsup_{\rho\to\infty}\frac{\ln V(\rho)}{m\rho}\right)^m,\tag{3.18}$$

It follows from (3.10) that $\lim_{\rho\to\infty}\lambda_\rho=\lambda_{\min}(M)=0$.

In what follows we consider only integer values of ρ, and consider sequence $\{v_k\}_{k=1}^{\infty}$. Let us show that the sequence $\{v_k\}$ satisfy that $v_k \to 1$ and $v_k' \to 0$ locally uniformly as $k \to \infty$, By the above analysis, we know v_k is decreasing. It follows that $v_k' \leq 0$. Integrating (3.16), noting that $v_k'(0) = 0$, we obtain

$$|v_k'|^{m-1}(r) = \frac{\lambda_k \int_0^r S(t) v_k^{m-1}(t) dt}{S(r)}. \tag{3.19}$$

Note that $0 \leq v_k \leq 1$ and (3.9), it follows that

$$|v_k'|^{m-1}(r) \leq \lambda_k \frac{V(r)}{S(r)},$$

since $\lambda_k \to 0$ as $k \to \infty$, we obtain that

$$v_k' \to 0, \tag{3.20}$$

uniformly on any bounded range of r as $k \to \infty$. The identity

$$v_k(r) = 1 + \int_0^r v_k'(t) dt, \tag{3.21}$$

implies that

$$v_k \to 1, \tag{3.22}$$

uniformly on any bounded range of r as $k \to \infty$.

Choose ρ large enough so that $\rho > r_0$ and

$$\frac{v_\rho'}{v_\rho}(r_0) > \frac{u'}{u}(r_0), \tag{3.23}$$

where u is the function constructed in the first step. Indeed, it is possible to achieve (3.23) by choosing $\rho = k$ with large enough i because by (3.20) and (3.22)

$$\frac{v_k'}{v_k}(r_0) \to 0, \quad \text{as } k \to \infty,$$

whereas $\frac{u'}{u}(r_0) < 0$ by construction in Step 1.

Let us fix $\rho > r_0$ for which (3.23) is satisfied, and compare the functions $u(r)$ and $v_\rho(r)$ in the interval $[r_0, \rho)$. Set

$$\theta = \inf_{r \in [r_0, \rho)} \frac{u(r)}{v_\rho(r)}.$$

Since v_ρ vanishes at ρ and, hence,

$$\frac{u(r)}{v_\rho(r)} \to +\infty, \quad \text{as } r \to \rho_-,$$

and, at $r = r_0$, by (3.23)

$$\left(\frac{u}{v_\rho}\right)'(r_0) = \frac{u' v_\rho - u v_\rho'}{v^2}(r_0) < 0,$$

so that u/v_ρ is strictly decreasing at r_0 and cannot have minimum at r_0. Hence, $\frac{u}{v_\rho}$ attains its minimum at an interior point $\xi \in (r_0, \rho)$, and at this point we have

$$\left(\frac{u}{v_\rho}\right)'(\xi) = 0.$$

It follows that

$$u(\xi) = \theta v_\rho(\xi) \quad \text{and} \quad u'(\xi) = \theta v_\rho'(\xi) \tag{3.24}$$

The function $u(r)$ has been defined for $r \geq r_0$, in particular, for $r \geq \xi$, whereas $v_\rho(r)$ has been defined for $r \leq \rho$, in particular, for $r \leq \xi$. Now we merge the two definitions by redefining/extending the function $u(r)$ for all $0 < r < \xi$ by setting $u(r) = \theta v_\rho(r)$.

It follows from (3.24) that $u \in C^1(M)$, in particular, $u \in W^{1,m}_{loc}(M)$. By (3.17), u satisfies the following inequality in B_ξ:

$$div(|\nabla u|^{m-2}\nabla u) + \frac{\lambda_\rho}{\theta^{\sigma-m+1}} u^\sigma \leq 0. \tag{3.25}$$

By (1.1), u satisfies the following inequality in $M \setminus B_{r_0}$:

$$div(|\nabla u|^{m-2}\nabla u) + u^\sigma \leq 0. \tag{3.26}$$

Combining (3.25) and (3.26), we obtain that u satisfies on M the following inequality

$$div(|\nabla u|^{m-2}\nabla u) + \delta u^\sigma \leq 0, \tag{3.27}$$

where $\delta = \min\{\lambda_\rho/\theta^{\sigma-m+1}, 1\}$. Finally, changing $u \mapsto cu$ where $c = \delta^{-\frac{1}{\sigma-m+1}}$ we obtain a positive solution to (1.1) on M, which concludes this example.

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Department of Mathematics, University Bielefeld, 33501 Bielefeld, Germany
E-mail address: `ysun@math.uni-bielefeld.de`