ON THE UNIQUENESS OF NONNEGATIVE SOLUTIONS OF DIFFERENTIAL INEQUALITIES WITH GRADIENT TERMS ON RIEMANNIAN MANIFOLDS

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Abstract. We investigate the uniqueness of nonnegative solutions to the following differential inequality

$$div(A(x)|\nabla u|^{m-2}\nabla u) + V(x)u^{\sigma_1}|\nabla u|^{\sigma_2} \le 0, \tag{1}$$

on a noncompact complete Riemannian manifold, where A, V are positive measurable functions, $m > 1$, and σ_1, $\sigma_2 \ge 0$ are parameters such that $\sigma_1 + \sigma_2 > m - 1$.

Our purpose is to establish the uniqueness of nonnegative solution to (1) via very natural geometric assumption on volume growth.

1. Introduction

These years the uniqueness of nonnegative solutions of various differential inequalities and systems has attracted much attention [2, 3, 6, 7, 9, 12, 13, 15, 16, 17], and these studies arise naturally in geometry and physics.

In this paper, our interest is to investigate the nonexistence of non-negative solution to the following differential inequality

$$div(A(x)|\nabla u|^{m-2}\nabla u) + V(x)u^{\sigma_1}|\nabla u|^{\sigma_2} \le 0, \quad \text{on } M, \tag{1.1}$$

on a geodesically complete noncompact connected Riemannian manifold M. Here div and ∇ are the Riemannian divergence and gradient respectively, A is non-negative measurable function, and V is a positive measurable function. $m > 1$, and σ_1, $\sigma_2 \ge 0$ are parameters such that $\sigma_1 + \sigma_2 > m - 1$. When $A = 1$, $\Delta_m u := div(|\nabla u|^{m-2}\nabla u)$ is aslo well known as m-Laplacian.

The purpose of this paper is to establish the uniqueness of nonnegative solution via very natural geometric assumption volume growth, without any curvature assumptions. The difficulty here is that how to decide the speed of the volume growth to guarantee the uniqueness of nonnegative solution.

Let us first give our setting on Riemannian manfolds: let M be a geodesically complete noncompact connected Riemannian manifold. Denote by μ the Riemannian measure on M and by $d(x, y)$ the geodesic distance between $x, y \in M$. Let $B(x, r)$ be the geodesic ball centered at $x \in M$ of radius r. Let x_0 be some reference point on M.

Let us introduce two paramters

$$p = \frac{m\sigma_1 + \sigma_2}{\sigma_1 + \sigma_2 - m + 1}, \quad q = \frac{m - 1}{\sigma_1 + \sigma_2 - m + 1}, \tag{1.2}$$

and another measure ν defined by

$$d\nu = A^{\frac{\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}} V^{-\frac{m-1}{\sigma_1+\sigma_2-m+1}} d\mu. \tag{1.3}$$

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Throughout the paper, we say that A, V satisfies **(VA)** condition: if there exist a nonnegative pair (δ_1, δ_2) and two positive constants c_0, C_0 such that

$$c_0(1+r)^{-\delta_1} \le \frac{V}{A} \le C_0(1+r)^{\delta_2}, \quad \text{for large enough } r. \tag{VA}$$

where $r := d(x, x_0)$.

Here are our main results.

Theorem 1.1. *Assume that* **(VA)** *holds with some* $\delta_1, \delta_2 \ge 0$. *If for some* $x_0 \in M$, *the following inequality*

$$\nu(B(x_0, r)) \le Cr^p \ln^q r, \tag{1.4}$$

holds for all large enough r, *then the only nonnegative solution of* (1.1) *is constant.*

When $A = V = 1$, we have the following corollary

Corollary 1.2. *If for some* $x_0 \in M$, *the following inequality*

$$\mu(B(x_0, r)) \le Cr^p \ln^q r, \tag{1.5}$$

holds for all large enough r, *then the only nonnegative solution of* (1.1) *is constant.*

For the first time the study of uniqueness of nonnegative solutions of semilinear equation began from Gidas and Spruck's work [5]. Later, the inequality

$$\Delta u + u^\sigma \le 0, \quad \text{in } \mathbb{R}^n \tag{1.6}$$

was considered by Ni and Serrin [18, 19], which is a particular case of (1.1. They proved that in $\mathbb{R}^2$, (1.6) has no positive solution, while in $\mathbb{R}^n$ with $n \ge 3$ (1.6) has no positive solution if and only if $\sigma \le \frac{n}{n-2}$.

The first uniqueness result in the Riemannian manifold setting in terms of volume growth is due to Cheng and Yau's pioneering work [1]. They proved that if the volume of geodesic balls of a complete Riemannian manifold grows at most a quadratic polynomial, namely, if

$$\mu(B(x_0, r)) \le Cr^2, \quad \text{for large enough } r, \tag{1.7}$$

then any positive superharmonic function on M is constant.

Grigor'yan relaxed the condition (1.7) to the integral form [8]: if

$$\int^\infty \frac{r}{\mu(B(x_0, r))} dr = \infty, \tag{1.8}$$

then any superharmonic function on M is constant, or equivalently to say, M is parabolic.

Holopainen generalized the notion of parabolicity to m-parabolicity [10, 11], and obtained that if

$$\int^\infty \left(\frac{r}{\mu(B(x_0, r))}\right)^{\frac{1}{m-1}} dr = \infty, \tag{1.9}$$

then M is m-parabolic, or equivalently, any positive m-superharmonic function u ($\Delta_m u \le 0$) is constant.

Very recently, in [9], Grigor'yan and the author reinvestigated (1.6) in the Riemannian manifold setting, they proved that under the following volume growth hypothesis

$$\mu(B(x_0, r)) \le cr^{\frac{2\sigma}{\sigma-1}} \ln^{\frac{1}{\sigma-1}} r, \quad \text{for all large enough } r.$$

then any nonnegative solution to (1.6) is identical zero. Moreover, the exponents $\frac{2\sigma}{\sigma-1}$ and $\frac{1}{\sigma-1}$ cannot be relaxed. Their methods are based on the carefully chosen families of test functions.

In [20], the author investigated a more general semilinear inequalities, and obtained the uniqueness result in terms of volume growth. It is worth mentioning that two different proofs were given, relying on two different choices of families of test functions.

Let us recall some results of (1.1) in the Euclidean setting. Mitidieri and Pokhozhaev in [14] obtained that: when $A(x) = V(x) = 1$, in $\mathbb{R}^n$ with $n > m > 1$. If

$$\sigma_1 + \sigma_2 \frac{n-1}{n-m} \leq \frac{n(m-1)}{n-m}. \tag{1.10}$$

then (1.1) has no positive solutions except constants. By Corollary 1.2, we see that if for large enough r (1.5) holds, then the only nonnegative solution of (1.1) is constant. Note that in $\mathbb{R}^n$

$$\mu(B(0,r)) = cr^n,$$

so that the condition (1.5) is equivalent to

$$n \leq p = \frac{m\sigma_1 + \sigma_2}{\sigma_1 + \sigma_2 - m + 1},$$

which in turn is equivalent to (1.10). Therefore, the result of [14] is covered by Corollary 1.2.

For the case $V(x) = |x|^{-\gamma_2}$, $A(x) = |x|^{\gamma_1}$ for $|x| \geq 1$, the problem (1.1) in $\mathbb{R}^n$ with $n > m > 1$ was studied by Filippucci [4], who proved that if

$$\begin{cases} 0 \leq \sigma_2 < m-1, \quad m-n < \gamma_1 < m - \gamma_2 - \sigma_2, \\ m - 1 - \sigma_2 < \sigma_1 \leq \frac{(n-\gamma_2)(m-1)}{n-m+\gamma_1} - \sigma_2 \frac{n-1+\gamma_1}{n-m+\gamma_1}. \end{cases} \tag{1.11}$$

then (1.1) has no positive solutions except constants. Let us compare the result of [4] with our Theorem 1.1. Using the measure ν in (1.3), we obtain in $\mathbb{R}^n$ for large enough r

$$\begin{aligned} \nu(B(0,r)) &= \int_{B(0,r)} A^{\frac{\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}} V^{-\frac{m-1}{\sigma_1+\sigma_2-m+1}} d\mu \\ &= C\int_1^r r^{\frac{\gamma_1(\sigma_1+\sigma_2)}{\sigma_1+\sigma_2-m+1}} r^{\frac{(m-1)\gamma_2}{\sigma_1+\sigma_2-m+1}} r^{n-1} dr + C_1 \\ &\approx r^{\frac{\gamma_1(\sigma_1+\sigma_2)+(m-1)\gamma_2}{\sigma_1+\sigma_2-m+1}+n}, \end{aligned}$$

where μ is the Lebesgue measure. The condition (1.4) is then equivalent to

$$\frac{\gamma_1(\sigma_1+\sigma_2)+(m-1)\gamma_2}{\sigma_1+\sigma_2-m+1} + n \leq p = \frac{m\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}, \tag{1.12}$$

which in turn is equivalent to (1.11). Under (1.11), we obtain that (1.1) has no positive solutions except constants. Thus, our Theorem 1.1 covers the aforementioned results in $\mathbb{R}^n$.

Next we will explain in which sense the solutions of (1.1) are defined. Define

$$d\omega = Ad\mu.$$

Set

$$W^{1,m}_{loc}(M,\omega) = \{f : M \to \mathbb{R} | f \in L^m_{loc}(M,\omega), \nabla f \in L^m_{loc}(M,\omega)\}, \tag{1.13}$$

where ∇f is understood in distributional sense. Denote by $W^{1,m}_c(M,\omega)$ the subspace of $W^{1,m}_{loc}(M,\omega)$ of functions with compact support.

Definition 1.1. A function u on M is called a weak solution of the inequality (1.1), if $u \geq 0$, $u \in W^{1,m}_{loc}(M,\omega)$, $V|\nabla u|^{\sigma_2} \in L^1_{loc}(M,\mu)$, and for any nonnegative function $\psi \in W^{1,m}_c(M,\omega)$, the following inequality holds:

$$-\int_M A(x)|\nabla u|^{m-2}(\nabla u, \nabla\psi)d\mu + \int_M V(x)u^{\sigma_1}|\nabla u|^{\sigma_2}\psi d\mu \leq 0, \tag{1.14}$$

where $(\cdot,\cdot)$ is the inner product in T_xM given by the Riemannian metric.

Remark. Using the defintion of ψ, we have

$$\begin{aligned} & \left|\int_M A(x)|\nabla u|^{m-2}(\nabla u, \nabla\psi)d\mu\right| \\ \leq & \int_{supp(\psi)} A|\nabla u|^{m-1}|\nabla\psi|d\mu \\ \leq & \left(\int_{supp(\psi)} A|\nabla u|^m d\mu\right)^{\frac{m-1}{m}} \left(\int_{supp(\psi)} A|\nabla\psi|^m d\mu\right)^{\frac{1}{m}}, \end{aligned}$$

Since $u \in W^{1,m}_{loc}(M,\omega)$ and $\psi \in W^{1,m}_c(M,\omega)$. Therefore, the first term in (1.14) is finite, which implies the finiteness of the second term, that is

$$\int_M Vu^{\sigma_1}|\nabla u|^{\sigma_2}\psi d\mu < \infty.$$

The structure of this paper is as follows: in Section 2, we show the proof of Theorem 1.1. In Section 3, we show some interesting applications of Theorem 1.1. In Section 4, the sharpness of p, q in Theorem 1.1 in some cases are showed.

NOTATION. The letters $C, C_0, C_1, ...$ denote positive constants whose values are unimportant and may vary at different occurrences.

2. PROOFS OF THEOREM 1.1

Let u be a nonnegative weak solution to (1.1). x_0 is the reference point as before in Theorem 1.1. Denote $B_R := B(x_0, R)$, and fix a Lipschitz function φ on M with compact support, such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in a neighborhood of B_R. In particular, we have $\varphi \in W^{1,m}_c(M,\omega)$. Take the following test function for (1.14):

$$\psi_\rho(x) = \varphi(x)^s(u(x)+\rho)^{-t}, \tag{2.1}$$

where the value of $\rho > 0$ is a parameter near zero, s is some fixed bigger enough constant, and t is variable and can be chosen arbitrarily close to 0. By Definition 1.1, we know that ψ_ρ has compact support and is locally bounded. Since

$$\nabla\psi_\rho = -t(u+\rho)^{-t-1}\varphi^s\nabla u + s(u+\rho)^{-t}\varphi^{s-1}\nabla\varphi,$$

we see that, $\nabla\psi_\rho \in L^m(M,\omega)$. It follows that, $\psi_\rho \in W^{1,m}_c(M,\omega)$. We obtain from (1.14) that

$$\begin{aligned} & t\int_M \varphi^s A(x)(u+\rho)^{-t-1}|u|^m d\mu + \int_M \varphi^s V(x)u^{\sigma_1}(u+\rho)^{-t}|\nabla u|^{\sigma_2}d\mu \\ \leq & \ s\int_M \varphi^{s-1}A(x)(u+\rho)^{-t}|\nabla u|^{m-2}(\nabla u, \nabla\varphi)d\mu. \end{aligned} \tag{2.2}$$

thus

$$\begin{aligned} & t\int_M \varphi^s A(x)(u+\rho)^{-t-1}|\nabla u|^m d\mu + \int_M \varphi^s V(x)u^{\sigma_1}(u+\rho)^{-t}|\nabla u|^{\sigma_2} d\mu \\ \leq\ & s\int_M \varphi^{s-1}A(x)(u+\rho)^{-t}|\nabla u|^{m-1}|\nabla\varphi| d\mu. \end{aligned} \tag{2.3}$$

In what follows, we use the following Young's inequality

$$\int_M fg d\mu \leq \varepsilon\int_M |f|^{p_0} d\mu + C_\varepsilon\int_M |g|^{p_0'} d\mu, \tag{2.4}$$

where $\varepsilon > 0$ is arbitrary. When t is small enough, (p_0, p_0') is a Hölder conjugate couple such that

$$p_0 = \frac{m\sigma_1 - t(m-\sigma_2)+\sigma_2}{m\sigma_1 - \sigma_1 + t - t(m-\sigma_2)} > 1, \quad p_0' = \frac{m\sigma_1 - t(m-\sigma_2)+\sigma_2}{\sigma_1+\sigma_2-t} > 1.$$

Applying (2.4) to the right-hand side integral of (2.3), we obtain

$$\begin{aligned} & s\int_M \varphi^{s-1}A(x)(u+\rho)^{-t}|\nabla u|^{m-1}|\nabla\varphi| d\mu \\ =\ & \int_M \left(t^{\frac{1}{p_0}}\varphi^{\frac{s}{p_0}}A(x)^{\frac{1}{p_0}}(u+\rho)^{-\frac{t+1}{p_0}}|\nabla u|^{\frac{m}{p_0}}\right) \\ & \times\left(\frac{s}{t^{\frac{1}{p_0}}}\varphi^{s-1-\frac{s}{p_0}}A(x)^{\frac{1}{p_0'}}(u+\rho)^{-t+\frac{t+1}{p_0}}|\nabla u|^{m-1-\frac{m}{p_0}}|\nabla\varphi|\right) d\mu \\ \leq\ & \varepsilon t\int_M \varphi^s A(x)(u+\rho)^{-t-1}|\nabla u|^m d\mu \\ & + C_\varepsilon\frac{s^{p_0'}}{t^{\frac{p_0'}{p_0}}}\int_M \varphi^{p_0'(s-1)-\frac{sp_0'}{p_0}}A(x)(u+\rho)^{-p_0't+\frac{p_0'(t+1)}{p_0}}|\nabla u|^{(m-1)p_0'-\frac{mp_0'}{p_0}}|\nabla\varphi|^{p_0'} d\mu \\ \leq\ & \varepsilon t\int_M \varphi^s A(x)(u+\rho)^{-t-1}|\nabla u|^m d\mu \\ & + C_\varepsilon\frac{s^{p_0'}}{t^{p_0'-1}}\int_M \varphi^{s-p_0'}A(x)(u+\rho)^{p_0'-t-1}|\nabla u|^{m-p_0'}|\nabla\varphi|^{p_0'} d\mu. \end{aligned}$$

Letting $\varepsilon = \frac{1}{2}$, substituting the above into (2.3), and cancelling out the half of the first term in (2.3), we obtain

$$\begin{aligned} & \frac{t}{2}\int_M \varphi^s A(x)(u+\rho)^{-t-1}|\nabla u|^m d\mu + \int_M \varphi^s V(x)u^{\sigma_1}(u+\rho)^{-t}|\nabla u|^{\sigma_2} d\mu \\ \leq\ & C\frac{s^{p_0'}}{t^{p_0'-1}}\int_M \varphi^{s-p_0'}A(x)(u+\rho)^{p_0'-t-1}|\nabla u|^{m-p_0'}|\nabla\varphi|^{p_0'} d\mu. \end{aligned} \tag{2.5}$$

Using (2.4) once more to the right-hand side of (2.5) with another Hölder conjugate couple (p_1, p_1') satisfying

$$p_1 = \frac{\sigma_2}{m-p_0'} = \frac{\sigma_1+\sigma_2-t}{m-t-1}, \quad p_1' = \frac{\sigma_2}{\sigma_2-m+p_0'} = \frac{\sigma_1+\sigma_2-t}{\sigma_1+\sigma_2-m+1},$$

We obtain

$$
\begin{aligned}
& \frac{Cs^{p_0'}}{t^{p_0'-1}}\int_M \varphi^{s-p_0'}A(x)(u+\rho)^{p_0'-t-1}|\nabla u|^{m-p_0'}|\nabla\varphi|^{p_0'}d\mu \\
= & \int_M [\varphi^{\frac{s}{p_1}}V(x)^{\frac{1}{p_1}}(u+\rho)^{p_0'-t-1}|\nabla u|^{m-p_0'}]\cdot[\frac{Cs^{p_0'}}{t^{p_0'-1}}\varphi^{\frac{s}{p_1'}-p_0'}V(x)^{-\frac{1}{p_1}}A(x)|\nabla\varphi|^{p_0'}]d\mu \\
\leq & \frac{1}{2}\int_M \varphi^s V(x)(u+\rho)^{(p_0'-t-1)p_1}|\nabla u|^{(m-p_0')p_1}d\mu \\
& +C_1\left(\frac{s^{p_0'}}{t^{p_0'-1}}\right)^{p_1'}\int_M \varphi^{s-p_0'p_1'}V(x)^{-\frac{p_1'}{p_1}}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu \\
= & \frac{1}{2}\int_M \varphi^s V(x)(u+\rho)^{\sigma_1-t}|\nabla u|^{\sigma_2}d\mu \\
& +C_1\left(\frac{s^{p_0'}}{t^{p_0'-1}}\right)^{p_1'}\int_M \varphi^{s-p_0'p_1'}V(x)^{1-p_1'}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu. \qquad (2.6)
\end{aligned}
$$

Combining (2.6) with (2.5), we obtain

$$
\begin{aligned}
& \frac{t}{2}\int_M \varphi^s A(x)(u+\rho)^{-t-1}|\nabla u|^m d\mu+\int_M \varphi^s V(x)u^{\sigma_1}(u+\rho)^{-t}|\nabla u|^{\sigma_2}d\mu \\
\leq & \frac{1}{2}\int_M \varphi^s V(x)(u+\rho)^{\sigma_1-t}|\nabla u|^{\sigma_2}d\mu \\
& +C_1 t^{-(p_0'-1)p_1'}\int_M \varphi^{s-p_0'p_1'}V(x)^{1-p_1'}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu, \qquad (2.7)
\end{aligned}
$$

where the term contains s is absorbed into constant C_1.

We know that

$$
\begin{aligned}
\int_M \varphi^s V(x)(u+\rho)^{\sigma_1-t}|\nabla u|^{\sigma_2}d\mu \leq & \; C\int_M \varphi^s V(x)u^{\sigma_1}(u+\rho)^{-t}|\nabla u|^{\sigma_2}d\mu \\
& +C\rho^{\sigma_1-t}\int_M \varphi^s V(x)|\nabla u|^{\sigma_2}d\mu,
\end{aligned}
$$

From (2.3), and by definition of the solution, we know

$$\int_M \varphi^s V(x)u^{\sigma_1}(u+\rho)^{-t}|\nabla u|^{\sigma_2}d\mu,$$

is bounded, and noting that by definition of the solution $V|\nabla u|^{\sigma_2}\in L^1_{loc}(M,\mu)$, we obtain

$$\int_M \varphi^s V(x)(u+\rho)^{\sigma_1-t}|\nabla u|^{\sigma_2}d\mu$$

is bounded.

Taking $\rho\to 0$ in (2.7), applying Monotone Convergence theorem to the left-hand side integrals, and Dominated Convergence theorem to the right-hand side integrals, we obtain

$$
\begin{aligned}
& \frac{t}{2}\int_M \varphi^s A(x)u^{-t-1}|\nabla u|^m d\mu+\int_M \varphi^s V(x)u^{\sigma_1-t}|\nabla u|^{\sigma_2}d\mu \\
\leq & \frac{1}{2}\int_M \varphi^s V(x)u^{\sigma_1-t}|\nabla u|^{\sigma_2}d\mu \\
& +C_1\left(\frac{s^{p_0'}}{t^{p_0'-1}}\right)^{p_1'}\int_M \varphi^{s-p_0'p_1'}V(x)^{1-p_1'}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu,
\end{aligned}
$$

which is

$$\begin{aligned}&\frac{t}{2}\int_M \varphi^s A(x)u^{-t-1}|\nabla u|^m d\mu + \frac{1}{2}\int_M \varphi^s V(x)u^{\sigma_1-t}|\nabla u|^{\sigma_2}d\mu\\ \leq\ & C_1 t^{-(p_0'-1)p_1'}\int_M \varphi^{s-p_0'p_1'}V(x)^{1-p_1'}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu, \end{aligned}\tag{2.8}$$

Applying (1.14) once more with another test function $\psi = \varphi^s$, we obtain

$$\begin{aligned}&\int_M \varphi^s V(x)u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\\ \leq\ & s\int_M \varphi^{s-1}A(x)|\nabla u|^{m-2}(\nabla u,\nabla\varphi)d\mu\\ \leq\ & s\int_M \varphi^{s-1}A(x)|\nabla u|^{m-1}|\nabla\varphi|d\mu\\ =\ & s\int_M [\varphi^{\frac{s}{p_2}}A(x)^{\frac{1}{p_2}}u^{-\frac{t+1}{p_2}}|\nabla u|^{\frac{m}{p_2}}]\cdot[\varphi^{(s-1)-\frac{s}{p_2}}A(x)^{\frac{1}{p_2'}}u^{\frac{t+1}{p_2}}|\nabla u|^{m-1-\frac{m}{p_2}}|\nabla\varphi|]d\mu\\ \leq\ & s\left(\int_M \varphi^s A(x)u^{-t-1}|\nabla u|^m d\mu\right)^{\frac{1}{p_2}}\\ &\times\left(\int_M \varphi^{(s-1)p_2'-\frac{sp_2'}{p_2}}A(x)u^{\frac{(t+1)p_2'}{p_2}}|\nabla u|^{(m-1)p_2'-\frac{mp_2'}{p_2}}|\nabla\varphi|^{p_2'}d\mu\right)^{\frac{1}{p_2'}}\\ =\ & s\left(\int_M \varphi^s A(x)u^{-t-1}|\nabla u|^m d\mu\right)^{\frac{1}{p_2}}\\ &\times\left(\int_M \varphi^{s-p_2'}A(x)u^{(t+1)(p_2'-1)}|\nabla u|^{m-p_2'}|\nabla\varphi|^{p_2'}d\mu\right)^{\frac{1}{p_2'}}. \end{aligned}\tag{2.9}$$

where we have used the following conjugate pair

$$p_2 = \frac{m\sigma_1+\sigma_2(t+1)}{m\sigma_1-\sigma_1},\quad p_2' = \frac{m\sigma_1+\sigma_2(t+1)}{\sigma_1+\sigma_2(t+1)}.\tag{2.10}$$

From (2.8), we have

$$\int_M \varphi^s A(x)u^{-t-1}|\nabla u|^m d\mu \leq Ct^{-1-(p_0'-1)p_1'}\int_M \varphi^{s-p_0'p_1'}V(x)^{1-p_1'}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu.$$

Substituting this into (2.9) yields

$$\begin{aligned}&\int_M \varphi^s V(x)u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\\ \leq\ & C\left[t^{-1-(p_0'-1)p_1'}\int_M \varphi^{s-p_0'p_1'}V(x)^{1-p_1'}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu\right]^{\frac{1}{p_2}}\\ &\times\left[\int_M \varphi^{s-p_2'}A(x)u^{(t+1)(p_2'-1)}|\nabla u|^{m-p_2'}|\nabla\varphi|^{p_2'}d\mu\right]^{\frac{1}{p_2'}}. \end{aligned}\tag{2.11}$$

Noting that $\nabla\varphi = 0$ on B_R, and applying Hölder inequality to the last term of (2.11) with the following couple (p_3, p_3')

$$p_3 = \frac{\sigma_1+\sigma_2(t+1)}{(t+1)(m-1)},\quad p_3' = \frac{\sigma_1+\sigma_2(t+1)}{\sigma_1+\sigma_2(t+1)-(t+1)(m-1)},$$

It is easy to check that

$$(t+1)(p_2'-1)=\frac{(t+1)(m-1)}{\sigma_1+\sigma_2(t+1)}\sigma_1=\frac{\sigma_1}{p_3},$$
$$m-p_2'=\frac{(t+1)(m-1)}{\sigma_1+\sigma_2(t+1)}\sigma_2=\frac{\sigma_2}{p_3}.$$

We obtain

$$\begin{aligned}
&\int_M \varphi^{s-p_2'}A(x)u^{(t+1)(p_2'-1)}|\nabla u|^{m-p_2'}|\nabla\varphi|^{p_2'}d\mu\\
=\ &\int_{M\setminus B_R}\left(\varphi^{\frac{s}{p_3}}V(x)^{\frac{1}{p_3}}u^{(t+1)(p_2'-1)}|\nabla u|^{m-p_2'}\right)\left(\varphi^{\frac{s}{p_3'}-p_2'}V(x)^{-\frac{1}{p_3}}A(x)|\nabla\varphi|^{p_2'}\right)d\mu\\
=\ &\int_{M\setminus B_R}\left(\varphi^{\frac{s}{p_3}}V(x)^{\frac{1}{p_3}}u^{\frac{\sigma_1}{p_3}}|\nabla u|^{\frac{\sigma_2}{p_3}}\right)\left(\varphi^{\frac{s}{p_3'}-p_2'}V(x)^{-\frac{1}{p_3}}A(x)|\nabla\varphi|^{p_2'}\right)d\mu\\
\le\ &\left(\int_{M\setminus B_R}\varphi^sV(x)u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\right)^{\frac{1}{p_3}}\\
&\times\left(\int_{M\setminus B_R}\varphi^{\frac{sp_3'}{p_3}-p_2'p_3'}V(x)^{1-p_3'}A(x)^{p_3'}|\nabla\varphi|^{p_2'p_3'}d\mu\right)^{\frac{1}{p_3'}}. \qquad (2.12)
\end{aligned}$$

Substituting (2.12) into (2.11), we obtain

$$\begin{aligned}
&\int_M\varphi^sV(x)u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\\
\le\ &C\left[t^{-1-(p_0'-1)p_1'}\int_{M\setminus B_R}\varphi^{s-p_0'p_1'}V(x)^{1-p_1'}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu\right]^{\frac{1}{p_2}}\\
&\times\left(\int_{M\setminus B_R}\varphi^{\frac{sp_3'}{p_3}-p_2'p_3'}V(x)^{1-p_3'}A(x)^{p_3'}|\nabla\varphi|^{p_2'p_3'}d\mu\right)^{\frac{1}{p_2'p_3'}}\\
&\times\left(\int_{M\setminus B_R}\varphi^sV(x)u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\right)^{\frac{1}{p_2'p_3}}. \qquad (2.13)
\end{aligned}$$

Choosing s large enough, and recalling that $0\le\varphi\le1$, from (2.13), we obtain

$$\begin{aligned}
&\int_M\varphi^sV(x)u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\\
\le\ &C\left[t^{-1-(p_0'-1)p_1'}\int_{M\setminus B_R}V(x)^{1-p_1'}A(x)^{p_1'}|\nabla\varphi|^{p_0'p_1'}d\mu\right]^{\frac{1}{p_2}}\\
&\times\left(\int_{M\setminus B_R}V(x)^{1-p_3'}A(x)^{p_3'}|\nabla\varphi|^{p_2'p_3'}d\mu\right)^{\frac{1}{p_2'p_3'}}\\
&\times\left(\int_{M\setminus B_R}\varphi^sV(x)u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\right)^{\frac{1}{p_2'p_3}}.
\end{aligned}$$

Using the new measure $d\nu = A^{\frac{\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}} V^{-\frac{m-1}{\sigma_1+\sigma_2-m+1}} d\mu$, we have

$$\begin{aligned}
&\int_M V(x)\varphi^s u^{\sigma_1}|\nabla u|^{\sigma_2} d\mu \\
\leq\ & C\left[t^{-1-(p_0'-1)p_1'} \int_{M\setminus B_R} \left(\frac{V}{A}\right)^{\frac{t}{\sigma_1+\sigma_2-m+1}} |\nabla\varphi|^{p_0'p_1'} d\nu\right]^{\frac{1}{p_2}} \\
&\times \left(\int_{M\setminus B_R} \left(\frac{V}{A}\right)^{-\frac{t\sigma_1(m-1)}{(\sigma_1+\sigma_2-m+1)[\sigma_1+\sigma_2(t+1)-(t+1)(m-1)]}} |\nabla\varphi|^{p_2'p_3'} d\nu\right)^{\frac{1}{p_2'p_3'}} \\
&\times \left(\int_{M\setminus B_R} V(x)\varphi^s u^{\sigma_1}|\nabla u|^{\sigma_2} d\mu\right)^{\frac{1}{p_2'p_3}}. \qquad (2.14)
\end{aligned}$$

We know $\int_M \varphi^s V(x) u^{\sigma_1}|\nabla u|^{\sigma_2} d\mu$ is finite in Introduction, it follows from (2.14) that

$$\begin{aligned}
&\left(\int_M \varphi^s V(x) u^{\sigma_1}|\nabla u|^{\sigma_2} d\mu\right)^{1-\frac{1}{p_2'p_3}} \\
\leq\ & C\left[t^{-1-(p_0'-1)p_1'} \int_{M\setminus B_R} \left(\frac{V}{A}\right)^{\frac{t}{\sigma_1+\sigma_2-m+1}} |\nabla\varphi|^{p_0'p_1'} d\nu\right]^{\frac{1}{p_2}} \\
&\times \left(\int_{M\setminus B_R} \left(\frac{V}{A}\right)^{-\frac{t\sigma_1(m-1)}{(\sigma_1+\sigma_2-m+1)[\sigma_1+\sigma_2(t+1)-(t+1)(m-1)]}} |\nabla\varphi|^{p_2'p_3'} d\nu\right)^{\frac{1}{p_2'p_3'}}. \qquad (2.15)
\end{aligned}$$

We notice, that all integral terms in the right hand side of (2.15) have the form

$$\int_M |\nabla\varphi|^a \left(\frac{V}{A}\right)^b d\nu,$$

with the following two pairs of (a, b) such that

$$\begin{aligned}
a_1 &= p_0'p_1' = \frac{m\sigma_1 - t(m-\sigma_2) + \sigma_2}{\sigma_1+\sigma_2-m+1}, \\
b_1 &= \frac{t}{\sigma_1+\sigma_2-m+1}.
\end{aligned} \qquad (2.16)$$

and

$$\begin{aligned}
a_2 &= p_2'p_3' = \frac{m\sigma_1 + \sigma_2(t+1)}{\sigma_1+\sigma_2(t+1)-(t+1)(m-1)}, \\
b_2 &= -\frac{t\sigma_1(m-1)}{(\sigma_1+\sigma_2-m+1)[\sigma_1+\sigma_2(t+1)-(t+1)(m-1)]}.
\end{aligned} \qquad (2.17)$$

Besides, a could be written in the form

$$a = p + lt, \qquad (2.18)$$

with the following two respective values of l

$$l_1 = \frac{\sigma_2 - m}{\sigma_1+\sigma_2-m+1}, \quad l_2 = \frac{\sigma_1(m-\sigma_2)(m-1)}{[\sigma_1+\sigma_2(t+1)-(t+1)(m-1)](\sigma_1+\sigma_2-m+1)}. \qquad (2.19)$$

where $p = \frac{m\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}$ is defined as before in (1.2). Clearly, it is obvious that all the values of a and l are uniformly bounded, when t is small enough near zero.

Let $\{\tilde{\varphi}_k\}_{k\in\mathbb{N}}$ be a sequence satisfying the following condtions: each $\tilde{\varphi}_k$ is a Lipschitz function such that $supp(\tilde{\varphi}_k) \subset B_{2^k}$, $\tilde{\varphi}_k = 1$ in a neighborhood of $B_{2^{k-1}}$, and

$$|\nabla\tilde{\varphi}_k| \begin{cases} \leq \frac{C}{2^{k-1}}, & \text{for } x \in B_{2^k} \setminus B_{2^{k-1}}, \\ = 0, & \text{otherwise.} \end{cases} \tag{2.20}$$

where C does not depend on k.

Fix some $n \in \mathbb{N}$ and set

$$t = \frac{1}{n}, \tag{2.21}$$

and

$$\varphi_n = \frac{\sum_{k=n+1}^{2n} \tilde{\varphi}_k}{n}. \tag{2.22}$$

Note that $\varphi_n = 1$ on B_{2^n}, $\varphi_{2^{2n}} = 0$ outside $B_{2^{2n}}$, $0 \leq \varphi_n \leq 1$ on M. Note also that for any $a \geq 1$, using that $supp(\nabla\tilde{\varphi}_k)$ is disjoint with each other, we have

$$|\nabla\varphi_n|^a = \frac{\sum_{k=n+1}^{2n} |\nabla\tilde{\varphi}_k|^a}{n^a}, \tag{2.23}$$

Besides, $\varphi_n \in W^{1,m}_{loc}(M,\omega)$.

Consider the integral

$$J_n(a,b) = \int_M |\nabla\varphi_n|^a \left(\frac{V}{A}\right)^b d\nu, \tag{2.24}$$

where a, b take values from (2.16) and (2.17).

Substituting (2.22) into (2.24), applying (2.23) and (2.20), when $b > 0$, we obtain

$$\begin{aligned} J_n(a,b) &= \int_M |\nabla\varphi_n|^a \left(\frac{V}{A}\right)^b d\nu \\ &= C\int_M \frac{\sum_{k=n+1}^{2n} |\nabla\tilde{\varphi}_k|^a}{n^a} \left(\frac{V}{A}\right)^b d\nu \\ &\leq \sum_{k=n+1}^{2n} \int_{B_{2^k}\setminus B_{2^{k-1}}} \frac{|\nabla\tilde{\varphi}_k|^a}{n^a}(1+r)^{b\delta_2} d\nu \\ &\leq C\sum_{k=n+1}^{2n} \left(\frac{2^{1-k}}{n}\right)^a (2^k)^{b\delta_2}\nu(B_{2^k}) \\ &\leq C\sum_{k=n+1}^{2n} \left(\frac{2^{-k}}{n}\right)^a (2^k)^{b\delta_2}\nu(B_{2^k}), \end{aligned} \tag{2.25}$$

where we have used that a is uniformly bounded. Noting that $a = p + bt$, $n+1 \leq k \leq 2n$, and substituting $t = \frac{1}{n}$, if $b > 0$, we obtain

$$\begin{aligned} \left(\frac{2^{-k}}{n}\right)^a (2^k)^{b\delta_2} &= \left(\frac{2^{-k}}{n}\right)^p \left(\frac{2^{-k}}{n}\right)^{lt} (2^k)^{b\delta_2} \\ &\leq \left(\frac{2^{-k}}{n}\right)^p \sup_{n\leq k\leq 2n, t=\frac{1}{n}} \left(\frac{2^{-k}}{n}\right)^{\frac{l}{n}} (2^k)^{b\delta_2} \\ &\leq C\left(\frac{2^{-k}}{n}\right)^p. \end{aligned}$$

Thus, using (1.4), recalling that by (1.2) $p=\frac{m\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}$, $q=\frac{m-1}{\sigma_1+\sigma_2-m+1}$, we obtain

$$\begin{aligned} J_n(a,b) &\leq C\sum_{k=n+1}^{2n}\left(\frac{2^{-k}}{n}\right)^p \nu(B_{2^k}) \\ &\leq C\sum_{k=n+1}^{2n}\left(\frac{2^{-k}}{n}\right)^p (2^k)^p\ln^q(2^k) \\ &\leq C\frac{1}{n^p}\sum_{k=n+1}^{2n} k^q \\ &\leq Cn^{q+1-p} \\ &\leq Cn^{-\frac{(m-1)\sigma_1}{\sigma_1+\sigma_2-m+1}}. \end{aligned} \tag{2.26}$$

Similarly, if $b<0$, we also have

$$J_n(a,b)\leq Cn^{-\frac{(m-1)\sigma_1}{\sigma_1+\sigma_2-m+1}}. \tag{2.27}$$

Setting $\varphi=\varphi_n$ in (2.15), we obtain

$$\begin{aligned} &\left(\int_M \varphi_n^s V u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\right)^{1-\frac{1}{p_2'p_3}} \\ \leq\ & ct^{-\frac{1}{p_2}-\frac{(p_0'-1)p_1'}{p_2}}\left(J_n(a_1,b_1)\right)^{\frac{1}{p_2}}\left(J_n(a_2,b_2)\right)^{\frac{1}{p_2'p_3'}}. \end{aligned} \tag{2.28}$$

where $(a_i,b_i)_{i=1,2}$ are defined in (2.16) and (2.17).

Substituting (2.26), (2.27) into (2.28), using $t=\frac{1}{n}$ as before, we obtain

$$\begin{aligned} &\left(\int_M \varphi_n^s V u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\right)^{1-\frac{1}{p_2'p_3}} \\ \leq\ & Cn^{\frac{1}{p_2}+\frac{(p_0'-1)p_1'}{p_2}}\left(n^{-\frac{(m-1)\sigma_1}{\sigma_1+\sigma_2-m+1}}\right)^{\frac{1}{p_2}} \\ &\times\left(n^{-\frac{(m-1)\sigma_1}{\sigma_1+\sigma_2-m+1}}\right)^{\frac{1}{p_2'p_3'}}, \end{aligned} \tag{2.29}$$

The exponents in the power of n in the right hand side of (2.29) is equal to

$$\frac{1}{p_2}+\frac{(p_0'-1)p_1'}{p_2}-\frac{(m-1)\sigma_1}{p_2(\sigma_1+\sigma_2-m+1)}-\frac{(m-1)\sigma_1}{p_2'p_3'(\sigma_1+\sigma_2-m+1)}=0$$

Therefore, we obtain

$$\left(\int_M \varphi_n V u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\right)^{1-\frac{1}{p_2'p_3}}\leq C<\infty. \tag{2.30}$$

Recalling that $\varphi_n=1$ on B_{2^n}, and taking the lim sup of both sides in (2.30) as $n\to\infty$, we obtain

$$\int_M V u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu \quad\leq\quad C<\infty. \tag{2.31}$$

The same argument can be used in (2.14), which implies

$$\int_M \varphi_n^s V u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\leq C\left(\int_{M\setminus B_{2^n}}\varphi_n^s V u^{\sigma_1}|\nabla u|^{\sigma_2}d\mu\right)^{\frac{1}{p_2'p_3}}, \tag{2.32}$$

Using that $0 \leq \varphi_n \leq 1$ and $\varphi_n|_{B_{2^n}} = 1$ once more, we obtain

$$\int_{B_{2^n}} V u^{\sigma_1} |\nabla u|^{\sigma_2} d\mu \leq C \left(\int_{M \setminus B_{2^n}} V u^{\sigma_1} |\nabla u|^{\sigma_2} d\mu \right)^{\frac{1}{p_2' p_3}}, \tag{2.33}$$

Letting $n \to \infty$, and using (2.31), we obtain

$$\int_M V u^{\sigma_1} |\nabla u|^{\sigma_2} d\mu = 0,$$

which implies that $u \equiv const$.

Remark. One also could use the method developed in [20, 9].

3. Applications

Theorem 1.1 could be applied to get the uniqueness of bounded nonnegative m-superharmonic function, namely, the uniqueness of the following problem

$$\Delta_m v \leq 0, \quad \text{on } M, \tag{3.1}$$

where M is the same as before, i.e. a geodesically complete noncompact connected Riemannian manifold.

Let $u = \ln(v + 1)$. Since v is nonnegative, hence, u is also nonnegative. Moreover, an easy calculation shows that u satisfies the following inequality

$$e^{(m-1)u} \left(\Delta_m u + (m-1)|\nabla u|^m \right) \leq 0, \tag{3.2}$$

which simplifies to

$$\Delta_m u + (m-1)|\nabla u|^m \leq 0, \tag{3.3}$$

By changing $u \to cu$, we can get rid of the factor $m - 1$ in (3.3). By Theorem 1.1, we obtain that if

$$\mu(B(x_0, r)) \leq C r^m \ln^{m-1} r, \tag{3.4}$$

then the only bounded nonnegative solution of (3.3) is constant, and hence the only bounded nonnegative solution of (3.1) is constant.

Compared to the result obtained by Holopainen in [10, 11]. Obviously, (3.4) implies (1.9). However, the function $r \mapsto r^m \ln^{m-1} r$ is right on the borderline of divergence of the integral in (1.9), so that the condition cannot be significantly improved.

Another application of Theorem 1.1 is to investigate the following inequality

$$\Delta_m u + |\nabla u|^{m-2} \nabla B \cdot \nabla u + u^{\sigma_1} |\nabla u|^{\sigma_2} \leq 0, \quad \text{on } M, \tag{3.5}$$

where B is a given measurable function on M, and ∇B does not have any singular point, σ_1, σ_2 are defined as in 1. One can rewrite (3.5) as the following

$$e^{-B} div(e^B |\nabla u|^{m-2} \nabla u) + u^{\sigma_1} |\nabla u|^{\sigma_2} \leq 0,$$

which is equivalent to

$$div(e^B |\nabla u|^{m-2} \nabla u) + e^B u^{\sigma_1} |\nabla u|^{\sigma_2} \leq 0.$$

Thus, applying Theorem 1.1, we obtain the following result.

Corollary 3.1. If for some $x_0 \in M$, the following inequality

$$\nu(B(x_0, r)) \leq C r^p \ln^q r, \tag{3.6}$$

holds for all large enough r, where ν is defined by $d\nu = e^B d\mu$, p and q are defined by (1.2), then the only nonnegative solution of (3.5) is constant.

4. Sharpness of p, q

In this section, we show the sharpness of parameters p and q in Theorem 1.1.

The sharpness of p is already known in $\mathbb{R}^n$ with $n > m > 1$. The following example was given by Mitidieri and Pokhozhaev in [17]: If

$$\begin{cases} \sigma_1 > \frac{(n-\gamma_2)(m-1)}{n-m+\gamma_1} - \sigma_2\frac{n-1+\gamma_1}{n-m+\gamma_1}, \\ m-n < \gamma_1 < m-\gamma_2-\sigma_2, \\ \gamma_1(\sigma_1+\sigma_2)+\gamma_2(m-1)+n(\sigma_1+\sigma_2-m+1) > 0, \\ 0 \le \sigma_2 < m-1. \end{cases} \tag{4.1}$$

then the function

$$u(x) := \epsilon[1+|x|^{\frac{m-\sigma_2-\gamma_1-\gamma_2}{m-1-\sigma_2}}]^{-\frac{m-1-\sigma_2}{\sigma_1+\sigma_2-m+1}}$$

is a solution to (1.1) with $A(x) = |x|^{\gamma_1}$, $V(x) = |x|^{-\gamma_2}$, where ϵ is a suitable small positive constant. Actually, using the measure ν of (1.4)

$$\begin{aligned} \nu(B(o,r)) &= \int_{B(0,r)} A^{\frac{\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}} V^{-\frac{m-1}{\sigma_1+\sigma_2-m+1}} d\mu \\ &= \int_{B(0,r)} |x|^{\frac{\gamma_1(\sigma_1+\sigma_2)}{\sigma_1+\sigma_2-m+1}} |x|^{\frac{\gamma_2(m-1)}{\sigma_1+\sigma_2-m+1}} d\mu \\ &\approx r^p, \end{aligned} \tag{4.2}$$

where $p = \frac{\gamma_1(\sigma_1+\sigma_2)+\gamma_2(m-1)}{\sigma_1+\sigma_2-m+1} + n$. From the assumption (4.1), we know it is equivalent to

$$p > \frac{m\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}, \tag{4.3}$$

One could let p be close to $\frac{m\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1}$ from above, by carefully choosing γ_1, γ_2.

In what follows we show the sharpness of q in case of $A = 1$, $V = m-1$, $\sigma_1 = 0$ and $\sigma_2 = m$. Fix here

$$p = \frac{m\sigma_1+\sigma_2}{\sigma_1+\sigma_2-m+1} = m,$$

and for arbitrary $\epsilon > 0$, choose

$$q = \frac{m-1}{\sigma_1+\sigma_2-m+1} + \epsilon = m-1+\epsilon.$$

Recall that M is called m-parabolic, if any positive m-superharmonic function v on M is constant, namely $\Delta_m v \le 0$. Holopainen proved in [10] that M is m-parabolic if and only if (1.9) holds, provided that the volume doubling condition and the Poincaré inequality hold. Thus, if

$$\mu(B(x_0,r)) \le Cr^m \ln^{m-1+\epsilon} r, \quad \text{for large enough } r, \tag{4.4}$$

we know (1.9) does not hold any more. Thus, there exists a positive function v such that $\Delta_m v \le 0$. Letting $u = \ln(v+1)$, we know u is a positive solution of (3.3). Hence, the exponent $m-1$ is sharp here.

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