UNIQUENESS RESULT FOR NONNEGATIVE SOLUTIONS OF A LARGE CLASS OF INEQUALITIES ON RIEMANNIAN MANIFOLDS

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Abstract. We consider a large class of differential inequalities on complete connected Riemannian manifolds, and provide a sufficient condition in terms of volume growth for the uniqueness of nonnegative solutions to the differential inequalities.

1. Introduction

The purpose of this paper is to give the sufficient condition for the uniqueness of nonnegative solutions to a large class of differential inequalities

$$Lu + V(x)u^{\sigma} \leq 0, \tag{1.1}$$

on a geodesic complete noncompact connected N-dimensional Riemannian manifold M^N with $N \geq 2$. Here

$$Lu = \sum_{i=1}^{N} \frac{d}{dx_i} A_i(x, u, \nabla u), \tag{1.2}$$

where $A_i(x, \eta, \xi)$ are Carathéodorian functions defined on $M^N \times [0, \infty) \times TM^N$, and TM^N is the tangent bundle of M^N. V is a positive measurable locally integralable function on M^N.

Let $m \geq 1$ be an arbitrary given number. We say that the operator L belongs to the class $A(m)$ if there exists a positive constant C such that for almost all $x \in M^N$, all $\eta \in [0, \infty)$, and all ξ, $\zeta \in T_x M^N$, the following inequalities holds

$$\begin{cases} (A(x, \eta, \xi), \xi) \geq 0, \\ |(A(x, \eta, \xi), \zeta)| \leq C|\zeta| \left(A(x, \eta, \xi), \xi\right)^{\frac{m-1}{m}} \end{cases} \tag{1.3}$$

where $(\cdot, \cdot)$ is inner product given by the Riemannian metric.

The definition of such class operator was firstly introduced by Miklyukov [10, 11]. The operators of such class are quite common, for example:

(1) m-Laplacian operator:

$$L_1 u = div(|\nabla u|^{m-2} \nabla u), \quad m > 1. \tag{1.4}$$

(2) Mean Curvature type operator:

$$L_2 u = div\left(\frac{|\nabla u|^{m-2} \nabla u}{\sqrt{1 + |\nabla u|^m}}\right), \quad m > 1. \tag{1.5}$$

and

$$L_3 u = div\left(\frac{|\nabla u|^{m-2} \nabla u}{\sqrt{1 + |\nabla u|^2}}\right), \quad m > 1. \tag{1.6}$$

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1 Supported by IGK of University Bielefeld.
Keywords and phrases. inequalities; Riemannian manifolds; volume growth; uniqueness.
2010 *Mathematics Subject Classification.* Primary: 35J99, Secondary: 58J05.

(3) Nonlinear operator:

$$L_4 u = div\left(a(x,u,\nabla u)|\nabla u|^{m-2}\nabla u\right), \quad m>1. \tag{1.7}$$

The definition of L in (1.3) is less restrictive than the following one defined by

$$|A(x,\eta,\xi)| \le C_1|\xi|^{m-1}, \quad ,|(A(x,\eta,\xi),\xi)| \ge C_2|\xi|^m, \tag{1.8}$$

for some positive constant C_1, C_2. For example, by choosing $a(x,\eta,\xi)$ appropriately, the operator L_4 belongs to $A(m)$ but not necessarily satisfies (1.8).

Generally speaking, the operator Lu defined by (1.3) may belong to several classes denoted by $A(m_1),\cdots,A(m_k)$, where $m_1 \le m_2 \le \cdots \le m_k$. For example, the operators L_2 and L_3 belong to $A(m-1)$ and $A(m)$ at the same time. Throughout the paper, when we say that L belongs to the class of $A(m)$, we mean m is the largest value m_k.

The purpose of this paper is to provide very simple geometric condition of volume growth on M^N to suffice that the only nonnegative solution u of (1.1) is identical zero. We emphasize here there is no curvature assumption throughout the paper.

First, let us give our assumption on manifolds. Let M^N be a geodesic complete noncompact connected manifold. Denote by μ the Riemannian measure, and by $B(x,r)$ the geodesic ball on M^N of radius r centered at $x \in M^N$. Given that $d(\cdot,\cdot)$ is geodesic distance, and x_0 is a reference point on M. Denote $B_r := B(x_0,r)$ for simplicity, where $r = d(x,x_0)$. Assume that $V(x) \in L^\infty_{loc}(M^N)$ throughout the paper.

The problem of investigating the uniqueness of nonnegative solutions has attracted a lot of attention, especially in the Euclidean space. For example, when $M^N = \mathbb{R}^N$ with $N \ge 2$, $V(x) \equiv 1$, the problem (1.1) was systematically investigated by Kurta [9]. By using the nonlinear capacity arguments, he obtained many nonexistence results. For specific operator L, let us recommend the papers of Mitidieri and Pokhozhaev [12, 13, 15] for a more comprehensive description. The related problems have also been studied in massive literatures, for example, [2, 3, 4, 5, 16, 17] and the references therein.

Let us turn to the results in Riemannian manifolds setting. The celebrated idea of studying uniqueness of nonnegative solutions in terms of volume of geodesic ball was due to Cheng and Yau [1]. They proved that if for all large enough r

$$\mu(B_r) \le Cr^2, \tag{1.9}$$

then any positive solution to $\Delta u \le 0$ is identical constant.

Very recently, this idea was used by Kontradtiev, Grigor'yan and the author [7, 8, 18] to investigate the inequality in the form

$$div(A(x)\nabla u) + V(x)u^\sigma \le 0,$$

where $\sigma > 1$. In [8], when $A(x) = Id, V(x) = 1$, Grigor'yan and the author proved that if

$$\mu(B_r) \le Cr^{\frac{2\sigma}{\sigma-1}} \ln^{\frac{1}{\sigma-1}} r$$

holds for all large enough r, then the only nonnegative solution of

$$\Delta u + u^\sigma \le 0$$

is identical zero. Moreover, the exponents of $\frac{2\sigma}{\sigma-1}$ and $\frac{1}{\sigma-1}$ are sharp, and cannot be relaxed.

Let us define the weak nonnegative solution of (1.1). For convenience, introduce some notation

$$A_u = (A(x,u,\nabla u),\nabla u). \tag{1.10}$$

and

$$W^{1,m}_{loc}(M^N) := \{f | f \in L^m_{loc}(M^N),\ \nabla f \in L^m_{loc}(M^N)\}, \tag{1.11}$$

and denote by $W_c^{1,m}(M^N)$ the subspace of $W_{loc}^{1,m}(M^N)$ of functions with compact support.

Here is our definition of the solution:

Definition 1.1. A function u on M^N is called a weak nonnegative solution of (1.1), if $u \in W_{loc}^{1,m}(M^N)$, and $A_u \in L_{loc}^1(M^N)$ and for any nonnegative function $\psi \in W_c^{1,m}(M^N)$, the following inequality holds

$$-\int_{M^N} (A(x,u,\nabla u), \nabla\psi) d\mu + \int_{M^N} u^\sigma \psi d\mu \leq 0, \tag{1.12}$$

where $(\cdot,\cdot)$ is the inner product in $T_x(M^N)$ given by Riemannian metric.

Remark 1.2. If u is a weak nonnegative solution of (1.1), and the operator L belongs to the class $A(m)$, we know

$$\begin{aligned}\int_{M^N} (A(x,u,\nabla u), \nabla\psi) d\mu &\leq C \int_{M^N} |\nabla\psi| A_u^{\frac{m-1}{m}} d\mu \\ &\leq C \left(\int_{M^N} |\nabla\psi|^m d\mu \right)^{\frac{1}{m}} \left(\int_{supp(\psi)} A_u d\mu \right)^{\frac{m-1}{m}} \\ &< \infty.\end{aligned}$$

Hence, by the definition of the solution, we know the second integral in (1.12) is bounded.

Define

$$p = \frac{m\sigma}{\sigma - m + 1}, \quad q = \frac{m-1}{\sigma - m + 1}. \tag{1.13}$$

and introduce a new measure ν defined by

$$d\nu = V^{-\frac{m-1}{\sigma - m + 1}} d\mu. \tag{1.14}$$

Assume that V satisfies the following condition : for some nonnegative constants δ_1, δ_2, the following inequality

$$cr^{-\delta_1} \leq V \leq Cr^{\delta_2}, \tag{\textbf{V}}$$

holds for all large enough r.

Here is our main result.

Theorem 1.3. Assume that operator L in (1.1) belongs to the class of $A(m)$ with $1 < m < \sigma - 1$. Assume also that **(V)** holds with $\delta_1, \delta_2 \geq 0$. If the following inequality

$$\nu(B_r \setminus B_1) \leq Cr^p \ln^q r, \tag{1.15}$$

holds for all large enough r, then the only nonnegative solution of (1.1) is identical zero.

Remark 1.4. We still do not know the sharpness of exponents p and q in (1.15) for the operators of this class. However, in many specific cases, the exponents are sharp, one can refer to [7, 18, 19].

Notation. The letters $C, C', C_0, C_1, ...$ denote positive constants whose values are unimportant and may vary at different occurrences.

2. First proof of Theorem 1.3

Let u be a nonnegative solution of (1.1). Fix some ball B_R, and $R > 0$ to be chosen later. Take a Lipschitz function φ on M^N with compact support, such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in a neighborhood of $\bar{B}_R$. Particularly, $\varphi \in W_c^{1,m}(M^N)$. We use the following test function for (1.12):

$$\psi_\rho(x) = \varphi(x)^s (u+\rho)^{-t}, \tag{2.1}$$

where $\rho > 0$ is a parameter near zero, and s will be chosen to be a large enough fixed constant, and t will take arbitrarily small positive values near zero.

Since $\frac{1}{u+\rho}$ is bounded, hence, ψ_ρ has compact support and is bounded. The identity

$$\nabla \psi_\rho = -t\varphi^s (u+\rho)^{-t-1}\nabla u + s\varphi^{s-1}(u+\rho)^{-t}\nabla\varphi,$$

implies that $\nabla\psi_\rho \in L^m(M^N)$, hence, $\psi_\rho \in W_c^{1,m}(M^N)$. We obtain from (1.12) that

$$\begin{aligned} & t\int_{M^N} \varphi^s (u+\rho)^{-t-1} A_u d\mu + \int_{M^N} \varphi^s V u^\sigma (u+\rho)^{-t} d\mu \\ \le\ & s\int_{M^N} \varphi^{s-1}(u+\rho)^{-t}(A(x,u,\nabla u),\nabla\varphi) d\mu. \end{aligned} \tag{2.2}$$

Estimating the right-hand side of (2.2) by the following Young inequality

$$\int_{M^N} fg d\mu \le \epsilon \int_{M^N} |f|^{p_0} d\mu + C_\epsilon |g|^{p_0'} d\mu, \tag{2.3}$$

where $\frac{1}{p_0} + \frac{1}{p_0'} = 1$. Letting $p_0 = \frac{m}{m-1}$, and using (1.3), we obtain

$$\begin{aligned} & s\int_{M^N} \varphi^{s-1}(u+\rho)^{-t}(A(x,u,\nabla u),\nabla\varphi) d\mu \\ \le\ & Cs\int_{M^N} \varphi^{s-1}(u+\rho)^{-t} A_u^{\frac{m-1}{m}} |\nabla\varphi| d\mu \\ =\ & C\int_{M^N} [t^{\frac{1}{p_0}} \varphi^{\frac{s}{p_0}} (u+\rho)^{-\frac{t+1}{p_0}} A_u^{\frac{m-1}{m}}][\frac{s}{t^{p_0}} \varphi^{\frac{s}{p_0'}-1} (u+\rho)^{1-\frac{t+1}{p_0'}} |\nabla\varphi|] d\mu \\ \le\ & \frac{t}{2}\int_{M^N} \varphi^s (u+\rho)^{-t-1} A_u d\mu \\ & + C\frac{s^m}{t^{m-1}} \int_{M^N} \varphi^{s-m}(u+\rho)^{m-t-1}|\nabla\varphi|^m d\mu. \end{aligned}$$

Substituting the above into (2.2), and cancelling out the half of the first term in (2.2), we obtain

$$\begin{aligned} & \frac{t}{2}\int_{M^N} \varphi^s (u+\rho)^{-t-1} A_u d\mu + \int_{M^N} \varphi^s V u^\sigma (u+\rho)^{-t} d\mu \\ \le\ & C\frac{s^m}{t^{m-1}} \int_{M^N} \varphi^{s-m}(u+\rho)^{m-t-1}|\nabla\varphi|^m d\mu. \end{aligned} \tag{2.4}$$

Using Young inequality again to the right-hand side of (2.4) with

$$p_1 = \frac{\sigma - t}{m-t-1}, \quad p_1' = \frac{\sigma - t}{\sigma - m + 1},$$

we obtain

$$\begin{aligned}
&\frac{s^m}{t^{m-1}}\int_{M^N}\varphi^{s-m}(u+\rho)^{m-t-1}|\nabla\varphi|^m d\mu\\
=&\int_{M^N}[\varphi^{\frac{s}{p_1}}V^{\frac{1}{p_1}}(u+\rho)^{\frac{\sigma-t}{p_1}}][\frac{s^m}{t^{m-1}}\varphi^{\frac{s}{p_1'}-m}V^{-\frac{1}{p_1}}|\nabla\varphi|^m]d\mu\\
\le&\frac{1}{2}\int_{M^N}\varphi^s V(u+\rho)^{\sigma-t}d\mu\\
&+C\left(\frac{s^m}{t^{m-1}}\right)^{\frac{\sigma-t}{\sigma-m+1}}\int_{M^N}\varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}V^{-\frac{m-t-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu. \qquad (2.5)
\end{aligned}$$

Using in the right-hand side of (2.5) the simple inequality

$$\left(\frac{s^m}{t^{m-1}}\right)^{\frac{\sigma-t}{\sigma-m+1}}\le\left(\frac{s^m}{t^{m-1}}\right)^{\frac{\sigma}{\sigma-m+1}}.$$

and combining (2.5) with (2.4), we obtain that

$$\begin{aligned}
&\frac{t}{2}\int_{M^N}\varphi^s(u+\rho)^{-t-1}A_u d\mu+\int_{M^N}\varphi^s Vu^\sigma(u+\rho)^{-t}d\mu\\
\le&\frac{1}{2}\int_{M^N}\varphi^s V(u+\rho)^{\sigma-t}d\mu+Ct^{-\frac{\sigma(m-1)}{\sigma-m+1}}\int_{M^N}\varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}V^{-\frac{m-t-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu, \qquad (2.6)
\end{aligned}$$

where the value of s is absorbed into constant C.

It is easy to obtain from the definition of the solution the boundedness of the following term

$$\int_{M^N}\varphi^s Vu^\sigma(u+\rho)^{-t}d\mu,$$

Then, the boundedness of $\int_{M^N}\varphi^s V(u+\rho)^{\sigma-t}d\mu$ follows by the boundedness of

$$\int_{M^N}\varphi^s Vu^\sigma(u+\rho)^{-t}d\mu,$$

and $V\in L^1_{loc}(M)$.

By Dominated Convergence theorem, we know

$$\lim_{\rho\downarrow 0}\int_{M^N}\varphi^s V(u+\rho)^{\sigma-t}d\mu=\int_{M^N}\varphi^s Vu^{\sigma-t}d\mu,$$

Letting $\rho\downarrow 0$ in (2.6) , applying Monotone Convergence theorem, we have

$$\begin{aligned}
&\frac{t}{2}\int_{M^N}\varphi^s u^{-t-1}A_u d\mu+\int_{M^N}\varphi^s Vu^{\sigma-t}d\mu\\
\le&\frac{1}{2}\int_{M^N}\varphi^s Vu^{\sigma-t}d\mu+Ct^{-\frac{\sigma(m-1)}{\sigma-m+1}}\int_{M^N}\varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}V^{-\frac{m-t-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu,
\end{aligned}$$

which is

$$\begin{aligned}
&\frac{t}{2}\int_{M^N}\varphi^s u^{-t-1}A_u d\mu+\frac{1}{2}\int_{M^N}\varphi^s Vu^{\sigma-t}d\mu\\
\le&Ct^{-\frac{\sigma(m-1)}{\sigma-m+1}}\int_{M^N}\varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}V^{-\frac{m-t-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu, \qquad (2.7)
\end{aligned}$$

Applying (1.12) once more, using another test function $\psi = \varphi^s$, we obtain

$$\begin{aligned}
&\int_{M^N} \varphi^s V u^\sigma d\mu \\
\leq\ & s\int_{M^N} \varphi^{s-1}(A(x,u,\nabla u),\nabla\varphi)d\mu \\
\leq\ & Cs\int_{M^N} \varphi^{s-1}A_u^{\frac{m-1}{m}}|\nabla\varphi|d\mu \\
\leq\ & Cs\left(\int_{M^N} \varphi^s u^{-t-1}A_u d\mu\right)^{\frac{m-1}{m}}\left(\int_{M^N}\varphi^{s-m}u^{(t+1)(m-1)}|\nabla\varphi|^m d\mu\right)^{\frac{1}{m}}. \qquad (2.8)
\end{aligned}$$

From (2.7), we obtain

$$\int_{M^N} \varphi^s u^{-t-1}A_u d\mu \leq Ct^{-1-\frac{\sigma(m-1)}{\sigma-m+1}}\int_{M^N}\varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}V^{-\frac{m-t-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu.$$

Substituting into (2.8) yields

$$\begin{aligned}
\int_{M^N}\varphi^s V u^\sigma d\mu \ \leq\ & C\left[t^{-1-\frac{\sigma(m-1)}{\sigma-m+1}}\int_{M^N}\varphi^{s-\frac{m(\sigma-t)}{\sigma-m+1}}V^{-\frac{m-t-1}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu\right]^{\frac{m-1}{m}} \\
&\times\left[\int_{M^N}\varphi^{s-m}u^{(t+1)(m-1)}|\nabla\varphi|^m d\mu\right]^{\frac{1}{m}}. \qquad (2.9)
\end{aligned}$$

Recalling that $\nabla\varphi = 0$ on B_R, and applying Hölder inequality to the last term of (2.9) with the Hölder couple

$$p_2 = \frac{\sigma}{(t+1)(m-1)},\quad p_2' = \frac{\sigma}{\sigma-(t+1)(m-1)},$$

we obtain

$$\begin{aligned}
&\int_{M^N}\varphi^{s-m}u^{(t+1)(m-1)}|\nabla\varphi|^m d\mu \\
=\ & \int_{M^N\setminus B_R}\left(\varphi^{\frac{s}{p_2}}V^{\frac{1}{p_2}}u^{(t+1)(m-1)}\right)\left(\varphi^{\frac{s}{p_2'}-m}V^{-\frac{1}{p_2}}|\nabla\varphi|^m\right)d\mu \\
\leq\ & \left(\int_{M^N\setminus B_R}\varphi^s V u^\sigma d\mu\right)^{\frac{(t+1)(m-1)}{\sigma}} \\
&\times\left(\int_{M^N\setminus B_R}\varphi^{s-\frac{m\sigma}{\sigma-(t+1)(m-1)}}V^{-\frac{(t+1)(m-1)}{\sigma-(t+1)(m-1)}}|\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}}d\mu\right)^{\frac{\sigma-(t+1)(m-1)}{\sigma}} \qquad (2.10)
\end{aligned}$$

Substituting (2.10) into (2.9), choosing s large enough, noting that $\varphi \leq 1$, we obtain

$$\begin{aligned}
&\int_{M^N}\varphi^s V u^\sigma d\mu \\
\leq\ & Ct^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}}\left(\int_{M^N}V^{-\frac{m-1-t}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu\right)^{\frac{m-1}{m}} \\
&\times\left(\int_{M^N}V^{-\frac{(t+1)(m-1)}{\sigma-(t+1)(m-1)}}|\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}}d\mu\right)^{\frac{\sigma-(t+1)(m-1)}{m\sigma}} \\
&\times\left(\int_{M^N\setminus B_R}\varphi^s V u^\sigma d\mu\right)^{\frac{(t+1)(m-1)}{m\sigma}}. \qquad (2.11)
\end{aligned}$$

From the definition of the solution, we know $\int_{M^N}\varphi^s V u^\sigma d\mu$ is finite. It follows from (2.11) that

$$\begin{aligned}&\left(\int_{M^N}\varphi^s V u^\sigma d\mu\right)^{1-\frac{(t+1)(m-1)}{m\sigma}}\\ \leq\ & Ct^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}}\left(\int_{M^N}V^{-\frac{m-1-t}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu\right)^{\frac{m-1}{m}}\\ &\times\left(\int_{M^N}V^{-\frac{(t+1)(m-1)}{\sigma-(t+1)(m-1)}}|\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}}d\mu\right)^{\frac{\sigma-(t+1)(m-1)}{m\sigma}}.\end{aligned}\tag{2.12}$$

Note that the first integral in the right-hand side of (2.12) has the following estimate

$$\int_{M^N}V^{-\frac{m-1-t}{\sigma-m+1}}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}d\mu\leq\int_{M^N}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}V^{\frac{t}{\sigma-m+1}}d\nu,\tag{2.13}$$

where we have used that $d\nu=V^{-\frac{m-1}{\sigma-m+1}}d\mu$. Similarly, the second integral in the right-hand side of (2.12) can be estimated as follows

$$\begin{aligned}&\int_{M^N}V^{-\frac{(t+1)(m-1)}{\sigma-(t+1)(m-1)}}|\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}}d\mu\\ \leq\ &\int_{M^N}|\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}}V^{-\frac{t\sigma(m-1)}{[\sigma-(t+1)(m-1)](\sigma-m+1)}}d\nu.\end{aligned}\tag{2.14}$$

Substituting that (2.13) and (2.14) into (2.11), we have

$$\begin{aligned}&\int_{M^N}\varphi^s V u^\sigma d\mu\\ \leq\ & Ct^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}}\left(\int_{M^N}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}V^{\frac{t}{\sigma-m+1}}d\nu\right)^{\frac{m-1}{m}}\\ &\times\left(\int_{M^N}|\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}}V^{-\frac{t\sigma(m-1)}{[\sigma-(t+1)(m-1)](\sigma-m+1)}}d\nu\right)^{\frac{\sigma-(t+1)(m-1)}{m\sigma}}\\ &\times\left(\int_{M^N\setminus B_R}\varphi^s V u^\sigma d\mu\right)^{\frac{(t+1)(m-1)}{m\sigma}}.\end{aligned}\tag{2.15}$$

Substituting that (2.13) and (2.14) into (2.12), we obtain

$$\begin{aligned}&\left(\int_{M^N}\varphi^s V u^\sigma d\mu\right)^{1-\frac{(t+1)(m-1)}{m\sigma}}\\ \leq\ & Ct^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}}\left(\int_{M^N}|\nabla\varphi|^{\frac{m(\sigma-t)}{\sigma-m+1}}V^{\frac{t}{\sigma-m+1}}d\nu\right)^{\frac{m-1}{m}}\\ &\times\left(\int_{M^N}|\nabla\varphi|^{\frac{m\sigma}{\sigma-(t+1)(m-1)}}V^{-\frac{t\sigma(m-1)}{[\sigma-(t+1)(m-1)](\sigma-m+1)}}d\nu\right)^{\frac{\sigma-(t+1)(m-1)}{m\sigma}}.\end{aligned}\tag{2.16}$$

Let $\{\tilde{\varphi}_k\}_{k\in\mathbb{N}}$ be a sequence satisfying that each $\tilde{\varphi}_k$ is a Lipschitz function such that $supp(\tilde{\varphi}_k)\subset B_{2^k}$, $\tilde{\varphi}_k=1$ in a neighborhood of $B_{2^{k-1}}$, and

$$|\nabla\tilde{\varphi}_k|\begin{cases}\leq\frac{C}{2^{k-1}} & \text{for } x\in B_{2^k}\setminus B_{2^{k-1}},\\ =0, & \text{otherwise.}\end{cases}\tag{2.17}$$

where C does not depend on k.

Fix some $n \in \mathbb{N}$ and set

$$t = \frac{1}{n}, \tag{2.18}$$

and

$$\varphi_n = \frac{\sum_{k=n+1}^{2n} \tilde{\varphi}_k}{n}, \tag{2.19}$$

Note that $\varphi_n = 1$ on B_{2^n}, $\varphi_n = 0$ outside $B_{2^{2n}}$, $0 \leq \varphi_n \leq 1$ on M. Note that for any $a \geq 1$, using that $supp(\nabla\tilde{\varphi}_k)$ are disjoint, we have

$$|\nabla\varphi_n|^a = \frac{\sum_{k=n+1}^{2n} |\nabla\tilde{\varphi}_k|^a}{n^a}. \tag{2.20}$$

It is easy to see that

$$\varphi_n \in W^{1,m}_{loc}(M^N).$$

Consider the integral

$$J_n(a,b) = \int_{M^N} |\nabla\varphi_n|^a V^b d\mu, \tag{2.21}$$

where a, b are taking values from

$$(a,b) = \begin{cases} \left(\frac{m(\sigma-t)}{\sigma-m+1}, \frac{t}{\sigma-m+1}\right), \\ \left(\frac{m\sigma}{\sigma-(t+1)(m-1)}, -\frac{t\sigma(m-1)}{[\sigma-(t+1)(m-1)](\sigma-m+1)}\right), \end{cases} \tag{2.22}$$

We write in the form

$$a = p + lt, \tag{2.23}$$

with the corresponding two values of l

$$l_1 = -\frac{m}{\sigma-m+1}, \quad l_2 = \frac{m\sigma(m-1)}{[\sigma-(t+1)(m-1)](\sigma-m+1)}, \tag{2.24}$$

where $p = \frac{m\sigma}{\sigma-m+1}$.

When $b \geq 0$, we know

$$\begin{aligned} J_n(a,b) &= \int_{M^N} |\nabla\varphi_n|^a V^b d\nu \\ &= \int_{M^N} \frac{\sum_{k=n+1}^{2n} |\nabla\tilde{\varphi}_k|^a}{n^a} V^b d\nu \\ &\leq \sum_{k=n+1}^{2n} \int_{B_{2^k}\setminus B_{2^{k-1}}} \frac{|\nabla\tilde{\varphi}_k|^a}{n^a} V^b d\nu \\ &\leq C \sum_{k=n+1}^{2n} \int_{B_{2^k}\setminus B_{2^{k-1}}} \left(\frac{2^{1-k}}{n}\right)^a r^{\delta_2 b} d\nu \\ &\leq C \sum_{k=n+1}^{2n} \left(\frac{2^{1-k}}{n}\right)^a (2^k)^{\delta_2 b} \nu(B_{2^k} \setminus B_1) \\ &\leq C \sum_{k=n+1}^{2n} \left(\frac{2^{1-k}}{n}\right)^a (2^k)^{\delta_2 b} \nu(B_{2^k} \setminus B_1). \end{aligned} \tag{2.25}$$

Note that $a=p+lt$, $n+1\le k\le 2n$,

$$\begin{aligned}\left(\frac{2^{1-k}}{n}\right)^a (2^k)^{\delta_2 b} &= \left(\frac{2^{-k}}{n}\right)^p \left(\frac{2^{-k}}{n}\right)^{lt} (2^k)^{\delta_2 b}\\ &\le \left(\frac{2^{-k}}{n}\right)^p (2^k)^{\delta_2 b} \sup_{n+1\le k\le 2n} \left(\frac{2^{-k}}{n}\right)^{lt}\\ &\le C\left(\frac{2^{-k}}{n}\right)^p (2^k)^{\delta_2 b}. \end{aligned}\tag{2.26}$$

Substituting (2.26) into (2.25), and using the volume growth (1.15), we obtain

$$\begin{aligned} J_n(a,b) &\le C\sum_{k=n+1}^{2n}\left(\frac{2^{-k}}{n}\right)^p (2^k)^{\delta_2 b}\nu(B_{2^k}\setminus B_1)\\ &\le C\sum_{k=n+1}^{2n}\left(\frac{2^{-k}}{n}\right)^p (2^k)^{\delta_2 b}(2^k)^p\ln^q(2^k)\\ &\le C\frac{1}{n^p}\sum_{k=n+1}^{2n} k^q 2^{k\delta_2 b}\\ &\le Cn^{q+1-p}2^{2n\delta_2 b}\\ &\le Cn^{-\frac{\sigma(m-1)}{\sigma-m+1}}2^{2n\delta_2 b}. \end{aligned}\tag{2.27}$$

Similarly, for the case of $b\le 0$, we have

$$J_n(a,b)\le Cn^{-\frac{\sigma(m-1)}{\sigma-m+1}}2^{-2n\delta_1 b}. \tag{2.28}$$

Taking the sequence of $\{\varphi_n\}$ in (2.16), we obtain

$$\begin{aligned} &\left(\int_{M^N}\varphi_n^s V u^\sigma d\mu\right)^{1-\frac{(t+1)(m-1)}{m\sigma}}\\ \le\ & Ct^{-\frac{m-1}{m}-\frac{\sigma(m-1)^2}{m(\sigma-m+1)}}\left(J_n\left(\frac{m(\sigma-t)}{\sigma-m+1},\frac{t}{\sigma-m+1}\right)\right)^{\frac{m-1}{m}}\\ &\times\left(J_n\left(\frac{m\sigma}{\sigma-(t+1)(m-1)},-\frac{t\sigma(m-1)}{[\sigma-(t+1)(m-1)](\sigma-m+1)}\right)\right)^{\frac{\sigma-(t+1)(m-1)}{m\sigma}} \end{aligned}\tag{2.29}$$

Substituting (2.27) and (2.28), noting that $t=\frac{1}{n}$, we obtain

$$\begin{aligned} &\left(\int_{M^N}\varphi_n^s V u^\sigma d\mu\right)^{1-\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}}\\ \le\ & Cn^{\frac{m-1}{m}+\frac{\sigma(m-1)^2}{m(\sigma-m+1)}}\left(n^{-\frac{\sigma(m-1)}{\sigma-m+1}}2^{2n\delta_2\frac{\frac{1}{n}}{\sigma-m+1}}\right)^{\frac{m-1}{m}}\\ &\times\left(n^{-\frac{\sigma(m-1)}{\sigma-m+1}}2^{2n\delta_1\frac{\frac{1}{n}\sigma(m-1)}{[\sigma-(\frac{1}{n}+1)(m-1)](\sigma-m+1)}}\right)^{\frac{\sigma-(\frac{1}{n}+1)(m-1)}{m\sigma}}\\ \le\ & Cn^{\frac{(m-1)^2}{n(\sigma-m+1)}}2^{\frac{2(\delta_1+\delta_2)(m-1)}{m(\sigma-m+1)}}, \end{aligned}\tag{2.30}$$

Noting that $\varphi_n = 1$ on B_{2^n}, and taking the lim sup of both sides in (2.30) as $n \to \infty$, we obtain

$$\int_{M^N} V u^\sigma d\mu \le C < \infty. \tag{2.31}$$

Applying similar arguments to (2.15), we obtain that

$$\int_{M^N} \varphi_n^s V u^\sigma d\mu \le C \left(\int_{M^N \setminus B_{2^n}} \varphi_n^s V u^\sigma d\mu \right)^{\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}}, \tag{2.32}$$

since $\varphi_n = 1$ on B_{2^n}, we have

$$\int_{B_{2n}} V u^\sigma d\mu \le C \left(\int_{M^N \setminus B_{2^n}} \varphi_n^s V u^\sigma d\mu \right)^{\frac{(\frac{1}{n}+1)(m-1)}{m\sigma}}, \tag{2.33}$$

Combining with (2.31), letting $n \to \infty$, we obtain that

$$\int_{M^N} V u^\sigma d\mu = 0,$$

since $V > 0$ for almost all $x \in M^N$, thus $u \equiv 0$.

3. Examples

Our result could cover many known results in R^N, let us mention two of these examples.

Example 1. Let us investigate the following inequality

$$div(|\nabla u|^{m-2} \nabla u) + V(x) u^\sigma \le 0, \quad \text{in } \mathbb{R}^N, \tag{3.1}$$

where $V(x) = \frac{1}{|x|^\gamma}$ for $|x| \ge 1$; $N > m > \max\{1, \gamma\}$, and $\sigma > m - 1$.

By Filippucci's [6, Corollary 1.5], we know if

$$\sigma \le \frac{(N-\gamma)(m-1)}{N-m}. \tag{3.2}$$

then (3.1) has no positive solutions in some natural class. Compared to our result of Theorem 1.3, we know for large r

$$\begin{aligned} \nu(B_r \setminus B_1) &= \int_{B_r \setminus B_1} V^{-\frac{m-1}{\sigma-m+1}} d\mu \\ &= \omega_N \int_1^r s^{\frac{\gamma(m-1)}{\sigma-m+1}} s^{N-1} ds \\ &\approx C r^{N + \frac{\gamma(m-1)}{\sigma-m+1}}, \end{aligned} \tag{3.3}$$

where ω_N is the surface area of unite ball in $\mathbb{R}^N$, and μ is the Lebesgue measure.

Now, the condition (1.15) is equivalent to

$$N + \frac{\gamma(m-1)}{\sigma-m+1} \le p = \frac{m\sigma}{\sigma-m+1}, \tag{3.4}$$

which in turn is equivalent to (3.2).

Example 2. Consider the following differential inequality

$$div \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) + u^\sigma \le 0, \quad \text{in } \mathbb{R}^N, \tag{3.5}$$

where $N > 2$, $\sigma > 1$. This problem was considered by Mitidieri and Pokhozhaev in [14]. They obtained that if

$$\sigma \leq \frac{N}{N-2}, \tag{3.6}$$

then (3.5) has no positive solutions. Note that the operator belongs to the class $A(2)$, and $\nu(B_r \setminus B_1) = \mu(B_r \setminus B_1) \approx Cr^N$, using our result, we know if

$$N \leq \frac{2\sigma}{\sigma-1}, \tag{3.7}$$

then (3.5) has no positive solution. It is easy to check that (3.6) and (3.7) are equivalent.

References

[1] S.Y. Cheng, S.-T Yau, Differential equations on Riemannian manifolds and their geometric applications, Comm. Pure Appl. Math., **28** (1975), 333-354.

[2] G. Caristi, L. D'Ambrosio, and E. Mitidieri, Liouville Theorems for some nonlinear inequalities, Proc. Steklov Inst. Math. **260**, 90-111 (2008).

[3] G. Caristi, E. Mitidieri, Nonexistence of positive solutions of quasilinear equations. Adv. Differential Equ. **2** (1997), no. 3, 319-359.

[4] L. D'Ambrosio. Liouville type Theorems for Anisotropic Quasilinear Inequalities. Nonlinear Analysis TMA. **70** (2009), 2855-2869.

[5] L. D'Ambrosio, E. Mitidieri. A priori estimates, positivity results, and nonexistence theorems for quasilinear degenerate elliptic inequalities Adv. Math. 224 (2010), 967-1020.

[6] R. Filippucci, Nonexistence of positive weak solutions of elliptic inequalities, Nonlinear Anal. **70**(2009), 2903-2916.

[7] A. Grigor'yan, V. A. Kondratiev, On the existence of positive solutions of semi-linear elliptic inequalities on Riemannian manifolds, International Mathematical Series **12**, (2010), 203-218.

[8] A. Grigor'yan, Y. Sun, On nonnegative of the inequality $\Delta u + u^\sigma \leq 0$ on Riemannian manifolds, Comm. Pure Appl. Math., Vol.**67**(2014), No. 8, 1336-1352.

[9] V. V. Kurta, the nonexistence of positive solutions to some elliptic equations, (in Russian), Tr. Mat. Inst. Steklova, **65**(1999), 552-561. Engl. transl.: Proc. Steklov Inst. Math., **65** (1999) no.4, 462-469.

[10] V. M. Miklyukov, A new approach to the Bernstein theorem and related problems for equations of minimal surface type, Math. Sb., **108(150)**(1979), No.2, 268-289.

[11] V. M. Miklyukov, Capacitance and the generalized maximum principle for elliptic type quasilinear equations, Dokl. Akad. Nauk SSSR, **250**(1980), No. 6, 1318-1320.

[12] E. Mitidieri, S. I. Pokhozhaev, Absence of global positive solutions of quasilinear elliptic inequalities. (Russian) Dokl. Akad. Nauk. **359** (1998), no. 4, 456-460.

[13] E. Mitidieri, S. I. Pokhozhaev, Absence of positive solutions for quasilinear elliptic problems in $\mathbb{R}^N$. (Russian) Tran. Math. Inst. Steklova. **1999**, 192-222; translation in Proc. Steklov Inst. Math. **227**, no. 4 (1999), 186-216.

[14] E. Mitidieri, S. I. Pokhozhaev, Nonexistence of positive solutions for quasilinear elliptic problems on $\mathbb{R}^N$, Proc. Stek. Inst. Math, **227**(1999), 1-32.

[15] E. Mitidieri, S. I. Pokhozhaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, Tr. Math. Inst. Steklova, (in Russian) **234** (2001), 1-384. Engl. transl.: Proc. Steklov Inst. Math., **234** (2001) no.3, 1-362.

[16] W.-M, Ni, J. Serrin, Nonexistence theorems for quasilinear partial differential equations. Proceedings of the conference commemorating the 1st centennial of the Circolo Matematico di Palermo (Italian) (Palermo, 1984). Rend. Circ. Mat. Palermo (2) Suppl. No. **8** (1985), 171-185.

[17] W.-M, Ni, J. Serrin, Existence and nonexistence theorems for ground states of quasilinear partial differential equations: the anomalous case, Accad. Naz. Lincei, **77**(1986), 231-257.

[18] Y. Sun, Uniqueness result for non-negative solutions of semi-linear inequalities on Riemannian manifolds, J. Math. Anal. Appl. **419**(2014), no. 1, 643-661.

[19] Y. Sun, On nonexistence of nonnegative solutions of quasilinear inequalities on Riemannian manifolds, preprint.

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