Nearly hyperharmonic functions
and Jensen measures

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Abstract

Let \((X, \mathcal{H})\) be a \(\mathcal{P}\)-harmonic space and assume for simplicity that constants are harmonic. Given a numerical function \(\varphi\) on \(X\) which is locally lower bounded, let

\[ J_\varphi(x) := \sup \{ \int^* \varphi \, d\mu(x) : \mu \in J_x(X) \}, \quad x \in X, \]

where \(J_x(X)\) denotes the set of all Jensen measures \(\mu\) for \(x\), that is, \(\mu\) is a compactly supported measure on \(X\) satisfying \(\int u \, d\mu \leq u(x)\) for every hyperharmonic function on \(X\). The main purpose of the paper is to show that, assuming quasi-universal measurability of \(\varphi\), the function \(J_\varphi\) is the smallest nearly hyperharmonic function majorizing \(\varphi\) and that \(J_\varphi = \varphi \lor \hat{J}_\varphi\), where \(\hat{J}_\varphi\) is the lower semicontinuous regularization of \(J_\varphi\). So, in particular, \(J_\varphi\) turns out to be at least “as measurable as” \(\varphi\).

This improves recent results, where the axiom of polarity was assumed. The preparations about nearly hyperharmonic functions on balayage spaces are closely related to the study of strongly supermedian functions triggered by J.-F. Mertens more than forty years ago.

Keywords: Jensen measure; nearly hyperharmonic function; strongly supermedian function.

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1 Representing measures for positive hyperharmonic functions

Let \((X, \mathcal{W})\) be a balayage space \((X\) a locally compact space with countable base and \(\mathcal{W}\) the set of positive hyperharmonic functions on \(X\), see [4] or [10]). The fine topology on \(X\) (it is finer than the initial topology) is the coarsest topology such that all functions in \(\mathcal{W}\) are continuous. Let \(\mathcal{C}(X)\) denote the set of all continuous real functions on \(X\), and let us fix a strictly positive function \(u_0 \in \mathcal{W} \cap \mathcal{C}(X)\). Further, let \(\mathcal{P}(X)\) be the set of all continuous real potentials, that is, functions \(p \in \mathcal{W} \cap \mathcal{C}(X)\) such that \(p/v\) vanishes at infinity for some strictly positive \(v \in \mathcal{W} \cap \mathcal{C}(X)\). We shall say that a numerical function \(\varphi\) on \(X\) is \(\mathcal{P}\)-bounded, if \(|\varphi| \leq p\) for some \(p \in \mathcal{P}(X)\); the set of all \(\mathcal{P}\)-bounded functions in \(\mathcal{C}(X)\) will be denoted by \(\mathcal{C}_\mathcal{P}(X)\). For every
numerical function \( \varphi \) on \( X \), let \( \hat{\varphi} \) denote its lower semicontinuous regularization, that is, \( \hat{\varphi}(x) := \liminf_{y \to x} \varphi(y) \) for every \( x \in X \). If \( \mathcal{V} \) is a subset of \( \mathcal{W} \) and \( v := \inf \mathcal{V} \), then \( \hat{v} \in \mathcal{W} \) and \( \hat{v}(x) = \hat{v}^f(x) := f\liminf_{y \to x} v(x) \), \( x \in X \) (lower limit with respect to the fine topology).

We recall that, for every numerical function \( \varphi \geq 0 \), a reduced function \( R_{\varphi} \) and a swept function \( \hat{R}_{\varphi} \) are defined by

\[
(1.1) \quad R_{\varphi} := \inf \{ u \in \mathcal{W} : u \geq \varphi \} \quad \text{and} \quad \hat{R}_{\varphi} := \hat{R}_{\varphi}.
\]

In particular, we have \( R_{\varphi}^A := R_{\varphi|A} \leq v \) and \( \hat{R}_{\varphi}^A := \hat{R}_{\varphi|A} \leq R_{\varphi}^A \) for \( A \subset X \) and \( v \in \mathcal{W} \), which leads to reduced measures \( \varepsilon_x^A \) and swept measures \( \hat{\varepsilon}_x^A \), \( x \in X \), characterized by \( \int v d\varepsilon_x^A = R_{\varepsilon}^A(x) \) and \( \int v d\hat{\varepsilon}_x^A = \hat{R}_{\varepsilon}^A(x) \), \( v \in \mathcal{W} \). Let us observe that \( \varepsilon_x^A = \hat{\varepsilon}_x^A \) for every \( x \in A^c \), since \( \hat{R}_v^A = R_v^A \) on \( A^c \) (see [4, VI.2.4]). If \( x \in A \), then \( \varepsilon_x^A = \varepsilon_x \) and, by [4, VI.9.2], \( \varepsilon_x^A = \varepsilon_x^A \{ x \} \varepsilon_x + (1 - \varepsilon_x^A \{ x \}) \varepsilon_x^A \setminus \{ x \} \).

For every \( x \in X \), let \( \mathcal{M}_x(\mathcal{W}) \) denote the convex set of all representing measures for \( x \) with respect to \( \mathcal{W} \), that is, (positive Radon) measures \( \mu \) on \( X \) such that, for every \( w \in \mathcal{W} \),

\[
(1.2) \quad \int w \, d\mu \leq w(x).
\]

Since every function in \( \mathcal{W} \) is an increasing limit of a sequence in \( \mathcal{P}(X) \), (1.2) holds for functions in \( \mathcal{W} \), if it holds for functions in \( \mathcal{P}(X) \). Let \( \mathcal{B}, \mathcal{B}^* \) respectively denote the \( \sigma \)-algebra of all Borel, \( (\mathcal{B}) \)-universally measurable sets in \( X \). By [4, VI.12.5, 2.2, 4.3, 4.4],

\[
(1.3) \quad E := \{ \varepsilon_x^A : A \subset X \} = \{ \varepsilon_x \} \cup \{ \varepsilon_x^A : A \in \mathcal{B}, \ A \ finely \ closed, \ x \notin A \}
\]

is the set of extreme points of \( \mathcal{M}_x(\mathcal{W}) \). The set \( \mathcal{M}_x(\mathcal{W}) \) is weak*-compact, that is, for every sequence \( (\mu_m) \) in \( \mathcal{M}_x(\mathcal{W}) \), there exists a subsequence \( (\mu_{m_k}) \) and \( \mu \in \mathcal{M}_x(\mathcal{W}) \) such that \( \lim_{k \to \infty} \int f \, d\mu_{m_k} = \int f \, d\mu \) for every \( f \in \mathcal{C}_p(X) \) (see [4, VI.10.1]). So we know by Choquet’s theorem that, for every \( \mu \in \mathcal{M}_x(\mathcal{W}) \), there exists a probability measure \( \rho \) on \( E \) such that, for every \( f \in \mathcal{C}_p(X) \),

\[
(1.4) \quad \int f \, d\mu = \int (\int f \, d\nu) \, d\rho(\nu),
\]

and then (1.4) holds for every Borel measurable function \( f \geq 0 \) on \( X \). (We might note that, for a given \( \mu \), the measure \( \rho \) does not have to be unique; see [11]).

By definition, a subset \( P \) of \( X \) is polar if \( \hat{R}_{u_0}^P = 0 \). Every polar set \( P \) is contained in a polar set in \( \mathcal{B} \) (see [4, VI.2.2]). Let \( \hat{\mathcal{B}}, \mathcal{B}^* \) denote the \( \sigma \)-algebra of all sets \( A \) in \( X \) for which there exists a set \( B \) in \( \mathcal{B}, \mathcal{B}^* \) respectively such that the symmetric difference \( A \triangle B \) is polar.

If \( \mu \in \mathcal{M}_x(\mathcal{W}) \), \( x \in X \), then \( \mu \) does not charge polar sets \( P \) in \( X \setminus \{ x \} \). Indeed, given \( \varepsilon > 0 \), there exists a function \( w \in \mathcal{W} \) such that \( w = u_0 \) on \( P \) and \( w(x) < \varepsilon \), and we have \( \int_P u_0 \, d\mu \leq \int w \, d\mu \leq w(x) < \varepsilon \) (cf. [5, Corollary 1.8]), whence \( \mu^*(P) = 0 \). So we know that \( \hat{\mathcal{B}}^* \) is contained in the completions of \( \mathcal{B} \) with respect to the measures \( \mu \in \mathcal{M}_x(\mathcal{W}) \), \( x \in X \).
2 Nearly hyperharmonic positive functions

Let $U_c$ denote the set of all relatively compact open sets in $X$ and let us say that a positive numerical function $u$ on $X$ is nearly hyperharmonic if

\[ (2.1) \quad \int^* u \, d\varepsilon^V_x \leq u(x) \quad \text{for all } V \in U_c \text{ and } x \in X. \]

This generalizes the definition given for harmonic spaces in [1, Section II.1] and [6, p. 119]). As for harmonic spaces we easily obtain the following.

**Proposition 2.1.** The set $N^+$ of all nearly hyperharmonic positive functions on $X$ has the following properties:

(i) $N^+$ is a convex cone containing $W$.

(ii) For every $u \in N^+$, $\hat{u} = \hat{u}^f \in W$.

(iii) If $(u_m)$ is a sequence in $N^+$ and $u_m \uparrow u$, then $u \in N^+$ and $\hat{u}_m \uparrow \hat{u}$.

(iv) For every subset $V$ of $N^+$, $\inf V \in N^+$.

Given $\varphi : X \to [0, \infty]$, let

\[ (2.2) \quad N_\varphi := \inf \{ u \in N^+ : u \geq \varphi \} \quad \text{and} \quad \hat{N}_\varphi := \hat{N}_\varphi. \]

Proposition 2.1 immediately yields the following.

**Proposition 2.2.**

(i) $N_\varphi \leq R_\varphi$, and $N_\varphi$ is the smallest majorant of $\varphi$ in $N^+$.

(ii) If $\varphi_1, \varphi_2, \ldots : X \to [0, \infty]$ and $\varphi_m \uparrow \varphi$, then $N_{\varphi_m} \uparrow N_\varphi$ and $\hat{N}_{\varphi_m} \uparrow \hat{N}_\varphi$.

(iii) If $\varphi$ is finely lower semicontinuous, then $N_\varphi = R_\varphi$.

For all functions $\varphi : X \to [0, \infty]$ and $x \in X$, let

\[ M_\varphi(x) := \sup \left\{ \int^* \varphi \, d\mu : \mu \in M_x(W) \right\}, \]

\[ M'_\varphi(x) := \sup \left\{ \int^* \varphi \, d\varepsilon^V_x : x \in V \in U_c \right\}, \]

\[ M''_\varphi(x) := \sup \left\{ \int^* \varphi \, d\varepsilon^K_x : K \text{ compact in } X \setminus \{x\} \right\}, \]

\[ M'''_\varphi(x) := \sup \left\{ \int^* \varphi \, d\varepsilon^A_x : A \in B, A \text{ finely closed, } x \notin A \right\}, \]

where we may replace the upper integrals by integrals, if $\varphi$ is $\tilde{\mathcal{B}}^*$-measurable.

**Proposition 2.3.** Let $\varphi$ be a positive numerical function on $X$. Then

\[ (2.3) \quad M'_\varphi = M''_\varphi = M'''_\varphi \leq M_\varphi. \]

If $\varphi$ is $\tilde{\mathcal{B}}^*$-measurable, then $\varphi \vee M'_\varphi = M_\varphi$. 

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Proof. Of course, $M'_\varphi \vee M''_\varphi \leq M'''_\varphi \leq M_\varphi$. Let us fix $x \in X$.

Let $A$ be a finely closed Borel set, $x \notin A$, and $b < \int^* \varphi \, d\varepsilon^A_x$. By [4, VI.4.6], $\varepsilon^A_x$ is supported by $A$. So there exists a compact $K$ in $A$ such that $\int^* 1_K \varphi \, d\varepsilon^A_x > b$. By [4, VI.9.4], $1_K \varepsilon^A_x \leq \varepsilon^K_x$, hence $b < \int^* 1_K \varphi \, d\varepsilon^A_x \leq \int^* \varphi \, d\varepsilon^K_x \leq M''_\varphi(x)$. Thus $M'''_\varphi(x) \leq M''_\varphi(x)$.

Next let $K$ be a compact set in $X \setminus \{x\}$, $b < c < \int^* \varphi \, d\varepsilon^K_x$ and $p \in P(X)$, $p > 0$. Then there exists $m \in \mathbb{N}$ such that $\varphi' := \varphi \wedge (mp)$ satisfies $\int^* \varphi' \, d\varepsilon^K_x > c$. Since $q := mp$ is a potential, there exists, by [4, II.5.2], a relatively compact open neighborhood $U$ of $\{x\} \cup K$ such that $\int q \, d\varepsilon^c_x < c - b$. Let us define $V := U \setminus K$ and $\nu := 1_{U^c} \varepsilon^V_x$. By [4, VI.9.4],

$$\nu \leq \varepsilon^U_x \quad \text{and} \quad \varepsilon^K_x = 1_K \varepsilon^V_x + \nu^K.$$

Therefore

$$\int q \, d\nu^K \leq \int q \, d\nu \leq \int q \, d\varepsilon^V_x < c - b \quad \text{and} \quad c < \int^* \varphi' \, d\varepsilon^K_x \leq \int^* \varphi \, d\varepsilon^K_x + \int q \, d\nu^K.$$

So $b < \int^* \varphi \, d\varepsilon^K_x$, and we conclude that $M'''_\varphi(x) \leq M''_\varphi(x)$ completing the proof of the equalities in (2.3).

Finally, we suppose that $\varphi$ is $\bar{B}^*$-measurable and fix $\mu \in M_x(W)$. Let us assume for the moment that $\int \varphi \, d\mu < \infty$. There exist positive Borel measurable functions $f, g$ on $X$ such that $f \leq \varphi \leq g$ and

$$\int f \, d\mu = \int \varphi \, d\mu = \int g \, d\mu. \tag{2.4}$$

Using the integral representation (1.4), we see that $\int f \, d\nu = \int g \, d\nu$ for $\rho$-a.e. $\nu \in E$, and hence

$$\int f \, d\nu = \int \varphi \, d\nu \leq \varphi(x) \vee M'''_\varphi(x) \quad \text{for} \quad \rho$-a.e. $\nu \in E$.

Thus $\int \varphi \, d\mu = \int f \, d\mu \leq \varphi(x) \vee M'''_\varphi(x)$, by (1.4) and (2.4). In the general case, we apply the previous considerations to the functions $\varphi \wedge (m\mu_0)$, $m \in \mathbb{N}$, and let $m \to \infty$. \hfill $\square$

**Corollary 2.4.** Let $u$ be a positive numerical function on $X$ and $x \in X$. Then the following properties are equivalent:

(i) The function $u$ is nearly hyperharmonic.

(ii) For every subset $A$ of $X \setminus \{x\}$, $\int^* u \, d\varepsilon^A_x \leq u(x)$.

(iii) For every compact $K$ in $X \setminus \{x\}$, $\int^* u \, d\varepsilon^K_x \leq u(x)$.

If $u$ is $\bar{B}^*$-measurable, then these properties hold if and only if $\int u \, d\mu \leq u(x)$ for every $\mu \in M_x(W)$.

So our (positive) nearly hyperharmonic functions are functions which in [2, 3, 7, 8, 15, 17] (mostly assuming additional measurability properties) are called strongly supermedian.
3 Identity of $M_\varphi$ and $N_\varphi$, Mertens’ formula

In this section, we shall give a fairly straightforward proof for the following result.

**Theorem 3.1.** For every $\tilde{\mathcal{B}}^*$-measurable numerical function $\varphi \geq 0$ on $X$,

$$M_\varphi = N_\varphi = \varphi \lor \hat{N}_\varphi.$$ 

In particular, $M_\varphi$ is the smallest nearly hyperharmonic majorant of $\varphi$, and $M_\varphi$, $N_\varphi$ are (at least) “as measurable as $\varphi$”, that is, if $\mathcal{A}$ is any $\sigma$-algebra on $X$ such that $\mathcal{B} \subset \mathcal{A} \subset \tilde{\mathcal{B}}^*$ and $\varphi$ is $\mathcal{A}$-measurable, then $M_\varphi$, $N_\varphi$ are $\mathcal{A}$-measurable.

**Remark 3.2.** In a more general setting, this has been shown by different methods for the smaller class of functions $\varphi \geq 0$ which are nearly Borel measurable or, slightly more general, nearly analytic (see [14, 8, 2, 3]).

The following simple possibility of replacing $\varphi$ by a smaller function when dealing with envelopes such as $R_\varphi$ and $N_\varphi$ will be useful.

**Proposition 3.3.** Let $F$ be a convex cone of numerical functions on a set $Y$ and $f_0 \in F$, $0 < f_0 < \infty$. For every numerical function $\varphi \geq 0$ on $Y$, let

$$F_\varphi := \inf\{f \in F : f \geq \varphi\}.$$ 

Then $F_{\varphi 1_{X \setminus A}} = F_\varphi$ for every numerical function $\varphi \geq 0$ on $Y$ and every $A \subset X$ such that $\alpha \varphi \leq F_\varphi \lor (M f_0)$ on $A$ for some $\alpha, M \in (1, \infty)$.

**Proof.** Clearly, it suffices to consider the case, where $\varphi$ is not identically zero on $A$. Trivially, $u := F_{\varphi 1_{X \setminus A}} \leq F_\varphi$. For the reverse inequality let $\varepsilon > 0$, $v := u + \varepsilon f_0$,

$$\beta := \inf\{b \in (0, \infty) : \varphi \leq bv \text{ on } A\} \quad \text{and} \quad \gamma := 1 \lor \beta.$$ 

Then $0 < \beta \leq M/\varepsilon$ and $\varphi \leq \gamma v$ on $X$. Hence $(\beta/\alpha) v(x) < \varphi(x)$ for some $x \in A$ and $F_{\varphi} \leq \gamma v$, which leads to $\beta v(x) < \alpha \varphi(x) \leq F_{\varphi}(x) \leq \gamma v(x)$. Thus $\gamma = 1$, $v \geq F_{\varphi}$, and the proof is completed letting $\varepsilon \to 0$. 

Let $\varphi : X \to [0, \infty]$ be $\tilde{\mathcal{B}}^*$-measurable. Since $\varphi \leq N_\varphi$ and $N_\varphi \in \mathcal{N}^+$, we obtain, by Corollary 2.4, that

$$M_\varphi \leq M_{N_\varphi} \leq N_\varphi.$$ 

To prove the reverse inequality we start considering the case, where $\varphi$ is upper semicontinuous and $\mathcal{P}$-bounded. We first recall the following ([10, Corollary 1.2.2]).

**Proposition 3.4.** For all upper semicontinuous $\mathcal{P}$-bounded positive functions $\psi, \psi_1, \psi_2, \ldots$ on $X$ the following holds:

- The function $R_\psi$ is upper semicontinuous. It is harmonic on $X \setminus \text{supp}(\psi)$.
- If $\psi$ is continuous, then $R_\psi$ is continuous.
- If $\psi_m \downarrow \psi$, then $R_{\psi_m} \downarrow R_\psi$. 

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The following consequence of the theorem of Hahn-Banach is known in more general situations (see e.g. [16, p. 226]). For the convenience of the reader we include a complete proof.

**PROPOSITION 3.5.** Let $\psi \geq 0$ be upper semicontinuous and $\mathcal{P}$-bounded. Then, for every $x \in X$, there exists $\mu \in \mathcal{M}_x(X)$ such that $\int \psi \, d\mu = R_\psi(x)$.

**Proof.** (a) Let $x \in X$ and $\varphi \in \mathcal{C}_p(X)$, $\varphi \geq 0$. Since the mapping $f \mapsto R_{f^+}(x)$ is sublinear on $\mathcal{C}_p(X)$, there exists a linear form $\mu$ on $\mathcal{C}_p(X)$ such that

$$\mu(\varphi) = R_\varphi(x) \quad \text{and} \quad \mu(f) \leq R_{f^+}(x) \quad \text{for every} \quad f \in \mathcal{C}_p(X).$$

If $f \in \mathcal{C}_p(X)$ and $f \leq 0$, then $\mu(f) \leq R_0(x) = 0$. Therefore $\mu$ is a measure on $X$. Of course, $\mu(p) \leq p(x)$ for every $p \in \mathcal{P}(X)$, and hence $\mu \in \mathcal{M}_x(W)$.

(b) There exist $\varphi_m \in \mathcal{C}_p(X)$ such that $\varphi_m \downarrow \psi$. By (a), for every $m \in \mathbb{N}$, there exists a measure $\mu_m \in \mathcal{M}_x(W)$ such that $\mu_m(\varphi_m) = R_{\varphi_m}(x)$. We may (passing to a subsequence) assume without loss of generality that the sequence $(\mu_m)$ converges to a measure $\mu \in \mathcal{M}_x(W)$ (that is, $\lim_{m \to \infty} \mu_m(f) = \mu(f)$ for every $f \in \mathcal{C}_p(X)$). Then, for every $k \in \mathbb{N}$,

$$R_\psi(x) = \lim_{m \to \infty} R_{\varphi_m}(x) = \lim_{m \to \infty} \mu_m(\varphi_m) \leq \lim_{m \to \infty} \mu_m(\varphi_k) = \mu(\varphi_k).$$

Letting $k \to \infty$, we get $R_\psi(x) \leq \int \psi \, d\mu$. Trivially $\int \psi \, d\mu \leq \int R_\psi \, d\mu \leq R_\psi(x)$. \hfill $\square$

**COROLLARY 3.6.** Let $\psi : X \to [0, \infty]$ be upper semicontinuous and $\mathcal{P}$-bounded. Then

$$\mathcal{M}_\psi = N_\psi = R_\psi = \psi \vee \hat{R}_\psi.$$

**Proof.** By Proposition 3.5, $R_\psi \leq \mathcal{M}_\psi$. By (3.1) and Proposition 2.2, $\mathcal{M}_\psi \leq N_\psi \leq R_\psi$. Therefore

$$\mathcal{M}_\psi = N_\psi = R_\psi.$$

To complete the proof it suffices to show that $N_\psi = \psi \vee \hat{N}_\psi$.

To that end let us consider $x \in X$ such that $N_\psi(x) > \psi(x)$. Let $V_m$ be open neighborhoods of $x$ such that $V_m \downarrow \{x\}$. Then the functions $\psi_m := \psi 1_{X \setminus V_m}$ are upper semicontinuous and $\psi_m \uparrow \varphi := \psi 1_{X \setminus \{x\}}$. Hence $N_{\psi_m} \uparrow N_\varphi = N_\psi$ and $\hat{N}_{\psi_m} \uparrow \hat{N}_\psi$, by Propositions 2.1 and 3.3.

For every $m \in \mathbb{N}$, the function $R_{\psi_m}$ is harmonic on $V_m$, by Proposition 3.4, and hence $\hat{R}_{\psi_m}(x) = R_{\psi_m}(x)$. So, by (3.2) (applied to $\psi_m$),

$$\hat{N}_\psi(x) = \lim_{m \to \infty} \hat{N}_{\psi_m}(x) = \lim_{m \to \infty} N_{\psi_m}(x) = N_\psi(x).$$

\hfill $\square$

For every $\varphi : X \to [0, \infty]$, let $\Psi_\varphi$ denote the set of all bounded upper semicontinuous functions $0 \leq \psi \leq \varphi$ with compact support in $\{\varphi > 0\}$. We are now able to prove even more than announced in Theorem 3.1.
**THEOREM 3.7.** Let \( \varphi : X \to [0, \infty] \) be \( \tilde{B}^* \)-measurable. Then
\[
M_\varphi = N_\varphi = \varphi \vee \hat{N}_\varphi = \sup \{ N_\psi : \psi \in \Psi_\varphi \},
\]
and there is an increasing sequence \( (\psi_m) \) in \( \Psi_\varphi \) such that
\[
N_\varphi = \varphi \vee \sup_{m \in \mathbb{N}} R_{\psi_m} = \varphi \vee \sup_{m \in \mathbb{N}} \hat{R}_{\psi_m}.
\]
If \( \varphi \leq s \) for some \( s \in \mathcal{W} \cap \mathcal{C}(X) \), then \( M_\varphi \) is harmonic on any open set, where \( M_\varphi \geq \alpha \varphi \) for some \( \alpha > 1 \).

**Proof.** Clearly, \( M_\varphi = \sup \{ M_\psi : \psi \in \Psi_\varphi \} \), where \( M_\psi = N_\psi = R_\psi = \psi \vee \hat{R}_\psi \) for every \( \psi \in \Psi_\varphi \), by Corollary 3.6. Since \( (\varphi \wedge n)_1(x) \in \Psi_\varphi \), \( x \in X \), \( n \in \mathbb{N} \), we obtain that
\[
M_\varphi = \sup \{ N_\psi : \psi \in \Psi_\varphi \} = \varphi \vee \sup \{ \tilde{N}_\psi : \psi \in \Psi_\varphi \} \leq \varphi \vee \hat{N}_\varphi \leq N_\varphi.
\]
In particular, \( M_\varphi \) is \( \tilde{B}^* \)-measurable. By [4, I.1.7], there is an increasing sequence \( (\psi_m) \) in \( \Psi_\varphi \) such that \( \sup_{m \in \mathbb{N}} R_{\psi_m} = \sup \{ \hat{R}_\psi : \psi \in \Psi_\varphi \} \). Then \( \varphi \vee \hat{R}_{\psi_m} \uparrow M_\varphi \) as \( m \to \infty \).

We now claim that
\[
M_\varphi \in \mathcal{N}^+,
\]
and therefore \( N_\varphi \leq M_\varphi \). Having (3.5) this implies that (3.3) and (3.4) hold.

So let \( x \in V \subseteq \mathcal{U}_c \), \( \mu := \varepsilon \chi_x^\varphi \). To show that \( \int M_\varphi \, d\mu \leq M_\varphi(x) \) we may assume that \( \int \varphi \, d\mu < \infty \), since otherwise \( M_\varphi(x) = \infty \). Let \( a < b < \int M_\varphi \, d\mu \). Then there exist \( m \in \mathbb{N} \) and \( \psi \in \Psi_\varphi \) such that
\[
b < \int \varphi \vee \hat{R}_{\psi_m} \, d\mu \quad \text{and} \quad \int (\varphi - \psi) \, d\mu < b - a.
\]
Of course, we may assume that \( \psi_m \leq \psi \). Since trivially \( \varphi \vee \hat{R}_\psi - \psi \vee \hat{R}_\psi \leq \varphi - \psi \), we obtain that
\[
b < \int \varphi \vee \hat{R}_\psi \, d\mu \leq \int \psi \vee \hat{R}_\psi \, d\mu + (b - a),
\]
and hence
\[
a < \int \psi \vee \hat{R}_\psi \, d\mu = \int R_\psi \, d\mu \leq R_\psi(x) \leq M_\varphi(x).
\]
Thus \( \int M_\varphi \, d\mu \leq M_\varphi(x) \) proving (3.6).

Finally, suppose that \( \varphi \leq w \) for some \( w \in \mathcal{W} \cap \mathcal{C}(X) \) and let \( U \) be an open set, where \( \varphi \) vanishes. Then all functions \( g_m := R_{\psi_m} \rceil_U \) are harmonic on \( U \), by Proposition 3.4, and hence \( M_\varphi \rceil_U = \sup g_m \) is harmonic on \( U \). An application of Proposition 3.3 completes the proof. \( \square \)

**REMARK 3.8.** If the semipolar sets \( \{ \inf \mathcal{V} < \inf \mathcal{V} \} \), \( \mathcal{V} \subset \mathcal{W} \), are polar (axiom of polarity, Hunt’s hypothesis (H)) and \( \varphi : X \to [0, \infty] \) is \( \tilde{B} \)-measurable, then by [13, Theorem 2.2] and (3.4) there exists an increasing sequence \( (\psi_n) \) in \( \Psi_\varphi \) such that
\[
R_\varphi = \varphi \vee \sup_{n \in \mathbb{N}} R_{\psi_n} = N_\varphi.
\]

By [2, Theorem 6.3], (3.7) holds even without assuming the axiom of polarity.
4 Application to Jensen measures

In this section, let us suppose that \((X, \mathcal{W})\) is a harmonic space, that is, the harmonic measures \(\mu_x^V = \mathcal{Z}_x^V\), \(V\) relatively compact open in \(X\), \(x \in X\), are supported by the boundary \(\partial V\) of \(V\).

Given an open set \(U\) in \(X\), let \(*\mathcal{H}(U)\) denote the set of all hyperharmonic functions on \(U\), that is, lower semicontinuous numerical functions \(w > -\infty\) on \(U\) such that \(\int w \, d\mathcal{Z}_x^V \leq w(x)\) for all open \(V\), which are relatively compact in \(U\), and \(x \in V\).

Given \(x \in U\), let \(\mathcal{J}_x(U)\) denote the set of all Jensen measures for \(x\) with respect to \(U\), that is, measures \(\mu\) with compact support in \(U\) satisfying

\[
\int w \, d\mu \leq w(x) \quad \text{for every } w \in *\mathcal{H}(U).
\]

In fact, it suffices to know (4.1) for all \(w \in *\mathcal{H}(U) \cap \mathcal{C}(U)\), since every \(w \in *\mathcal{H}(U)\) is an increasing limit of functions in \(*\mathcal{H}(U) \cap \mathcal{C}(U)\).

Since \(\mathcal{W} = \{ w \in *\mathcal{H}(X): w \geq 0 \}\) and \(*\mathcal{H}(X)|_U \subset *\mathcal{H}(U)\), we have

\[
\mathcal{J}_x(U) \subset \mathcal{J}_x(X) \subset \mathcal{M}_x(\mathcal{W}), \quad x \in U
\]

(where we consider measures in \(\mathcal{J}_x(U)\) as measures on \(X\)). It will be convenient to introduce also the union \(\mathcal{J}_x'(X)\) of all \(\mathcal{J}_x(U)\), \(U\) open relatively compact in \(X\), \(x \in U\) (see [12] for properties implying that \(\mathcal{J}_x'(X) = \mathcal{J}_x(X)\)).

Finally, for every locally lower bounded function \(\varphi\) on \(X\) which is \(\mathcal{B}^*\)-measurable, we define functions \(J_\varphi\) and \(J_\varphi'\) on \(X\) by

\[
J_\varphi(x) := \sup\{ \int \varphi \, d\mu: \mu \in \mathcal{J}_x(X) \} \quad \text{and} \quad J_\varphi'(x) := \sup\{ \int \varphi \, d\mu: \mu \in \mathcal{J}_x'(X) \}.
\]

If \(\varphi \geq 0\), then obviously \(M_\varphi' \leq J_\varphi' \leq J_\varphi \leq M_\varphi\). Therefore Proposition 2.3 and Theorem 3.7 immediately yield the following.

**THEOREM 4.1.** Let \(\varphi\) be a positive \(\mathcal{B}^*\)-measurable numerical function on \(X\). Then

\[
J_\varphi = J_\varphi' = \varphi \lor M_\varphi' = M_\varphi = \varphi \lor \hat{N}_\varphi.
\]

In particular, \(J_\varphi\) is Borel measurable if \(\varphi\) is Borel measurable.

Similarly as in [13] we may now extend this result to functions \(\varphi\) which are not necessarily positive. To that end let \(\mathcal{N}\) denote the set of all nearly hyperharmonic functions on \(X\), that is, locally lower bounded functions \(w: X \to ]-\infty, \infty]\) such that \(\int w \, d\mathcal{Z}_x^V \leq w(x)\) for all \(x \in X\) and relatively compact open neighborhoods \(V\) of \(x\). We immediately get the following generalization of Proposition 2.1.

**PROPOSITION 4.2.** The set \(\mathcal{N}\) of all nearly hyperharmonic functions on \(X\) has the following properties:

(i) \(\mathcal{N}\) is a convex cone containing \(*\mathcal{H}(X)\).

(ii) For every \(u \in \mathcal{N}\), \(\hat{u} = \hat{u}' \in *\mathcal{H}(X)\).

(iii) If \((u_m)\) is a sequence in \(\mathcal{N}\) and \(u_m \uparrow u\), then \(u \in \mathcal{N}\) and \(\hat{u}_m \uparrow \hat{u}\).
(iv) For every subset $V$ of $\mathcal{N}$ which is locally lower bounded, $\inf V \in \mathcal{N}$.

Extending the definitions of $J_\varphi$, $J'_{\varphi}$, and $M'_{\varphi}$ in an obvious way, we get the following.

**COROLLARY 4.3.** Let $\varphi$ be a locally lower bounded $\tilde{\mathcal{B}}^*$-measurable numerical function on $X$ such that $\varphi + h \geq 0$ for some harmonic function $h$ on $X$. Then

$$J_\varphi = J'_{\varphi} = \varphi \lor M'_{\varphi} = N_\varphi = \varphi \lor N_\varphi.$$  

In particular, $J_\varphi$ is Borel measurable if $\varphi$ is Borel measurable.

**Proof.** It suffices to observe that $\varphi + h$ is $\tilde{\mathcal{B}}^*$-measurable and obviously $J_\varphi = J_{\varphi + h} - h$, $J'_{\varphi} = J'_{\varphi + h} - h$, $M'_{\varphi} = M'_{\varphi + h} - h$, and $N_\varphi = N_{\varphi + h} - h$. □

Localizing this result we may deal with functions $\varphi$ which are locally lower bounded.

**COROLLARY 4.4.** Let $\varphi$ be a locally lower bounded $\tilde{\mathcal{B}}^*$-measurable numerical function on $X$ such that, for every relatively compact open set $U$ in $X$, there exists a harmonic function $h$ on $X$ with $\varphi + h \geq 0$ on $U$. Then

$$J'_{\varphi} = \varphi \lor M'_{\varphi} = N_\varphi = \varphi \lor N_\varphi.$$  

In particular, $J'_{\varphi}$ is Borel measurable if $\varphi$ is Borel measurable.

**Proof.** Let $U_n$ be relatively compact open sets in $X$ such that $U_n \uparrow X$ as $n \to \infty$. For every $n \in \mathbb{N}$, we apply Corollary 4.3 to the harmonic space $(U_n, \ast \mathcal{H}^+(U_n))$ and obtain that, for $x \in U_n$,

$$\sup \{ \int \varphi \, d\mu : \mu \in J_x(U) \} = \varphi(x) \lor \sup \{ \int \varphi \, d\nu_x^V : x \in V \in U_c, \, V \subset U \} = \inf \{ w(x) : w \text{ nearly hyperharmonic on } U_n, \, w \geq \varphi \text{ on } U_n \} =: v_n(x),$$

where $v_n(x) = \varphi(x) \lor \hat{v}_n(x)$. Defining $v_n(x) := \varphi(x), \, x \in X \setminus U_n$, we easily see that the sequence $(v_n)$ is increasing to a nearly hyperharmonic function $v$ on $X$, where $v = N_\varphi = \varphi \lor N_\varphi$, by Proposition 4.2. The proof is completed letting $n \to \infty$. □

**REMARK 4.5.** By Remark 3.8, the results in this Section imply the results in [13, Section 3].

## 5 Some improvement of the measurability

Let us now return to the general situation of an arbitrary balayage space $(X, \mathcal{W})$. Sometimes we can say a bit more about the measurability of $N_\varphi$ (and hence on the measurability of $J_{\varphi}$ in Section 4).

We recall that a set $A$ in $X$ is called *thin at* $x \in X$ if $\varepsilon_{\varphi}^A \neq \varepsilon_x$. It is *totally thin* if it is thin at every $x \in X$. Every totally thin set is finely closed and contained in a totally thin Borel set. A *semipolar* set is a countable union of totally thin sets.

So the $\sigma$-algebra $\mathcal{B}^f$ of all finely Borel subsets of $X$ (that is, the smallest $\sigma$-algebra on $X$ containing all finely open sets) contains all semipolar sets. By [4, VI.5.16]), for every $B \in \mathcal{B}^f$, there are $B_1, B_2 \in \mathcal{B}$ such that $B_1 \subset B \subset B_2$ and $B_2 \setminus B_1$ is semipolar. Thus $\mathcal{B}^f$ is the smallest $\sigma$-algebra containing $\mathcal{B}$ and all semipolar sets. In particular, $\mathcal{B} \subset \mathcal{B}^f$. 

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EXAMPLE 5.1. Suppose for the moment that \( X = \mathbb{R}^d \times \mathbb{R} \), \( d \geq 1 \), and \( \mathcal{W} \) is the set of all positive hyperharmonic functions associated with the heat equation on \( \mathbb{R}^{d+1} \). Let \( S \) be any subset of \( \mathbb{R}^d \times \{0\} \). Then \( S \) is semipolar; it is polar if and only if \( S \) has outer \( d \)-dimensional Lebesgue measure zero. The function \( u := 1_{S} + 1_{\mathbb{R}^d \times (0, \infty)} \) is nearly hyperharmonic, \( \mathcal{B}^f \)-measurable, and \( \{ \hat{u} < u \} = S \).

PROPOSITION 5.2. Let \( S \in \tilde{\mathcal{B}}^* \) be semipolar. Then there exists a sequence \( (K_n) \) of compacts in \( S \) such that the set \( S \setminus \bigcup_{n=1}^{\infty} K_n \) is polar. In particular, \( S \in \tilde{\mathcal{B}} \).

Proof. By [9, Theorem 1.5] (which holds as well for balayage spaces), there exists a measure \( \mu \) on \( X \) such that \( \mu^*(B) > 0 \) for every subset \( B \) of \( S \) which is not polar. There exists a subset \( S_0 \in \mathcal{B}^* \) of \( S \) such that the set \( P_0 := S \setminus S_0 \) is polar. Moreover, there exists a sequence \( (K_n) \) of compacts in \( S_0 \) such that \( P_1 := S_0 \setminus \bigcup_{n=1}^{\infty} K_n \) is a \( \mu \)-null set, and hence polar. Since \( P_0 \cup P_1 \) is polar, the proof is finished. \( \square \)

The equivalence in the following proposition is of no interest, if we know that \( u = \inf \mathcal{V}, \mathcal{V} \subset \mathcal{W} \), since then \( u \) is obviously finely upper semicontinuous and the set \( \{ \hat{u} < u \} \) is semipolar by [4, VI.5.11].

PROPOSITION 5.3. For every \( u \in \mathcal{N}^+ \) the following statements are equivalent:

(i) The set \( \{ \hat{u} < u \} \) is semipolar.

(ii) The function \( u \) is finely Borel measurable.

Proof. (i) \( \Rightarrow \) (ii): For every \( t > 0 \), the set \( \{ u \geq t \} \) is the union of \( \{ \hat{u} \geq t \} \in \mathcal{B} \) and the semipolar set \( \{ u \geq t \} \setminus \{ \hat{u} \geq t \} \).

(ii) \( \Rightarrow \) (i): There is a semipolar Borel set \( S_0 \) such that the function \( u_0 := u1_{X \setminus S_0} \) is \( \mathcal{B} \)-measurable. Suppose that the set \( \{ \hat{u} < u \} \) is not semipolar. Then the Borel set \( A := \{ \hat{u} < u \} \setminus S_0 \) is not semipolar. So, by [4, VI.8.9], there is a measure \( \mu \neq 0 \) on \( X \) such that \( \mu(X \setminus A) = 0 \) and \( \mu \) does not charge semipolar sets. There exist functions \( \psi_m \in \Psi_{u_0} \) such that \( \psi_m \uparrow u_0 \) outside a \( \mu \)-null set \( B \in \mathcal{B} \). By Corollary 3.6,

\[
\psi_m \leq R_{\psi_m} = N_{\psi_m} \leq N_u = u, \quad m \in \mathbb{N}.
\]

Hence \( \{ \sup_{m \in \mathbb{N}} R_{\psi_m} < u \} \subset B \cup S_0 \). Further, the union \( S_1 \) of all sets \( \{ \hat{R}_{\psi_m} < R_{\psi_m} \}, \quad m \in \mathbb{N} \), is semipolar, and we obtain that

\[
\hat{u} \geq \sup_{m \in \mathbb{N}} R_{\psi_m} = \sup_{m \in \mathbb{N}} R_{\psi_m} = u \quad \text{on } X \setminus (B \cup S_0 \cup S_1).
\]

Thus \( A \subset B \cup S_1 \) and \( \mu(X) = \mu(A) = 0 \), a contradiction. \( \square \)

COROLLARY 5.4. If \( \varphi: X \to [0, \infty] \) is \( \mathcal{B}^f \cap \tilde{\mathcal{B}}^* \)-measurable, then the function \( N_{\varphi} \) is \( \tilde{\mathcal{B}} \)-measurable.

Proof. By Theorem 3.7, the function \( u := N_{\varphi} \) is \( \mathcal{B}^f \cap \tilde{\mathcal{B}}^* \)-measurable. Let \( t \in \mathbb{R} \) and \( S := \{ u \geq t \} \). Then \( S \in \tilde{\mathcal{B}}^* \) and \( S \) is semipolar, by Proposition 5.3. So \( S \in \tilde{\mathcal{B}} \), by Proposition 5.2, and \( \{ u \geq t \} = \{ \hat{u} \geq t \} \cup S \in \tilde{\mathcal{B}} \). \( \square \)

Further, let \( \mathcal{A}(X) \) denote the set of all numerical functions \( \varphi \) on \( X \) having the following property: For every \( t \in \mathbb{R} \), there exists an analytic set \( A \) in \( X \) such that the set \( \{ \varphi \geq t \} \triangle A \) is semipolar. By the discussion preceding Proposition 5.3, \( \varphi \in \mathcal{A}(X) \) for every finely Borel measurable function \( \varphi \).
PROPOSITION 5.5. Let \( \varphi \) be a positive function in \( A(X) \) which is \( \overline{\mathcal{B}}^* \)-measurable. Then \( N_{\varphi} \) is finely upper semicontinuous and \( \overline{\mathcal{B}}^* \)-measurable.

Proof. By Theorem 3.7, we know that \( u := N_{\varphi} \in A(X) \) and \( u \) is \( \overline{\mathcal{B}}^* \)-measurable. Let \( x \in X \) and \( u(x) < a \). We claim that \( \{ u < a \} \) is a fine neighborhood of \( x \). Indeed, suppose the contrary. Then the set \( A := \{ u \geq a \} \) is not thin at \( x \). Let \( A' \) be an analytic set such that \( A' \subset A \) and \( A \setminus A' \) is semipolar. We fix \( \eta \in (0,1) \) such that \( u(x) < a\eta^2 \). Since \( V := \{ u_0(x) > \eta u_0 \} \) is a neighborhood of \( x \), we know, by [4, VI.4.2], that either the analytic set \( A' \cap V \) or the semipolar set \( S := (A \setminus A') \cap V \) is not thin at \( x \).

If \( A' \cap V \) is not thin at \( x \), then, by [4, VI.10 and 1.3.5], there is a compact \( K \) in \( A' \cap V \) such that \( R^K_{u_0}(x) > \eta u_0(x) \). By definition of semipolar sets, \( S \) is the union of totally thin sets \( T_m, m \in \mathbb{N} \). By [4, VI.5.7], the union of finitely many totally thin sets is totally thin. Hence we may assume without loss of generality that \( T_m \uparrow S \) as \( m \to \infty \). If \( S \) is not thin at \( x \), we then obtain, by [4, VI.1.7], that \( R^K_{u_0}(x) > \eta u_0(x) \) for some \( m \in \mathbb{N} \).

Thus, in any case, there exists a finely closed set \( F \) such that

\[
F \subset A \cap V \in \overline{\mathcal{B}}^* \quad \text{and} \quad R^K_{u_0}(x) > \eta u_0(x).
\]

Since \( u \geq a > a\eta u_0(x)^{-1}u_0 \) on \( A \cap V \) and \( \varepsilon_x^F(X \setminus (A \cap V)) = 0 \), by [4, VI.4.6], we conclude that

\[
u(x) \geq \int u \, d\varepsilon_x^F \geq a\eta u_0(x)^{-1}\int u_0 \, d\varepsilon_x^F = a\eta u_0(x)^{-1}R^K_{u_0}(x) \geq a\eta^2 > u(x),
\]

a contradiction. By Corollary 5.4, the proof is finished.

COROLLARY 5.6. Let \( \varphi : X \to [0,\infty] \) be such that, for every \( t > 0 \), there exists an analytic set \( A \) in \( X \) such that the set \( \{ \varphi \geq t \} \triangle A \) is polar. Then \( N_{\varphi} \) is \( \overline{\mathcal{B}}^* \)-measurable.

References


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