A 6-DIMENSIONAL SIMPLY CONNECTED COMPLEX AND SYMPLECTIC MANIFOLD WITH NO KÄHLER METRIC

GIOVANNI BAZZONI, MARISA FERNÁNDEZ, AND VICENTE MUÑOZ

Abstract. We construct a simply connected compact manifold which has complex and symplectic structures but does not admit Kähler metric, in the lowest possible dimension where this can happen, that is, dimension 6. Such a manifold is automatically formal and has even odd-degree Betti numbers but it does not satisfy the Lefschetz property for any symplectic form.

1. Introduction

A Kähler manifold \((M, J, \omega)\) is a smooth manifold \(M\) of dimension \(2n\) endowed with an integrable almost complex structure \(J\) and a symplectic form \(\omega\) such that \(g(X, Y) = \omega(X, JY)\) defines a Riemannian metric, called Kähler metric. In order to check that a compact manifold does not carry any Kähler metric, one can use a collection of known topological obstructions to the existence of such a structure: theory of Kähler groups, evenness of odd-degree Betti numbers, Lefschetz property or the formality of the rational homotopy type (see [1, 7, 23]).

If \(M\) is a compact Kähler manifold, then it has a complex and a symplectic structure. However, the converse is not true. The first example of a compact manifold admitting complex and symplectic structures but no Kähler metric is the Kodaira-Thurston manifold [16, 21]. This 4-manifold is not simply connected (it is actually a nilmanifold) hence the fundamental group plays a key role in this property. The classification of complex and symplectic nilmanifolds of dimension 6 was given by Salamon in [20]. Generalizations to higher dimension \(2n \geq 6\) of the Kodaira-Thurston manifold are the generalized Iwasawa manifolds considered in [6]. Such manifolds have complex and symplectic structures but carry no Kähler metric. Note that, in dimension 2, every oriented surface admits a Kähler metric.

If one restricts attention to manifolds with trivial fundamental group, then every complex manifold of real dimension 4 admits a Kähler structure. Indeed, by the Enriques-Kodaira classification [16], if \(M\) is a complex surface whose first Betti number \(b_1\) is even (this holds in particular when \(b_1 = 0\)), then \(M\) is deformation equivalent to a Kähler surface (see also [2, Theorem 3.1, page 144] for a direct proof of this fact). We point out that Gompf [13] has constructed the first examples of simply connected compact symplectic but not complex 4-manifolds. Also Fintushel and Stern [12] have given a family of simply connected symplectic 4-manifolds not admitting complex structures (the latter was proved by Park [19]).

In dimensions higher than 4, we have the following results. The first examples of simply connected compact symplectic non-Kählerian manifolds were given in dimension 6 by Gompf in the aforementioned paper [13] and in dimension \(\geq 10\) by McDuff in [17] (these examples are not known to admit complex structures). Fine and Panov in [10] (see also [11]) have produced

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simply connected symplectic 6-manifolds with $c_1 = 0$ which do not have a compatible complex structure (but it is not known if they admit Kähler structures). Furthermore, Guan in [14] constructed the first family of simply connected, compact and holomorphic symplectic non-Kählerian manifolds of (real) dimension $4n \geq 8$. On the other hand, the first and third authors have proved [3] that the 8-dimensional manifold $X$ constructed in [9] is an example of a simply connected, symplectic and complex manifold which does not admit a Kähler structure (since it is not formal). For higher dimensions $2n = 8 + 2k$, $k \geq 1$, one can take $X \times \mathbb{CP}^k$. This is simply connected, complex and symplectic but not Kähler. Thus, a natural question arises:

Does there exist a 6-dimensional simply connected, compact, symplectic and complex manifold which does not admit Kähler metrics?

In this paper we answer this question in the affirmative by proving the following result:

**Theorem 1.1.** There exists a 6-dimensional, simply connected, compact, symplectic and complex manifold which carries no Kähler metric.

In order to construct such an example, we start with a 6-dimensional nilmanifold $M$ admitting both a complex structure $J$ and a symplectic structure $\omega$. Then we quotient it by a finite group preserving $J$ and $\omega$ to obtain a simply connected, 6-dimensional orbifold $\widetilde{M}$ with an orbifold complex structure $\tilde{J}$ and an orbifold symplectic form $\tilde{\omega}$. By Hironaka Theorem [15], there is a complex resolution $(\tilde{M}_c, \tilde{J})$ of $(\widetilde{M}, \tilde{J})$. As in [3], we resolve symplectically the singularities of $(\widetilde{M}, \tilde{\omega})$ to obtain a smooth symplectic 6-manifold $(\tilde{M}_s, \tilde{\omega})$. However, in our situation, the singular locus of the orbifold $\widetilde{M}$ does not consist only of a discrete set of points, in contrast with [3]. For a complex and symplectic orbifold, we provide conditions under which the complex and the symplectic resolution of singularities are diffeomorphic (Theorem 3.1). Using this we prove that the resolutions $\tilde{M}_c$ and $\tilde{M}_s$ are diffeomorphic. Thus, $\tilde{M} = \tilde{M}_c$ is not only a complex manifold but also a symplectic one.

Next, we show that $\tilde{M}$ is simply connected (Proposition 6.1), this resulting from the careful choice of the action of the finite group on $M$. Since $\tilde{M}$ is a 6-dimensional simply connected compact manifold, then $b_1(\tilde{M}) = 0$, and $b_3(\tilde{M})$ is even by Poincaré duality. Also $\tilde{M}$ is automatically formal by [3, Theorem 3.2]. Therefore, to ensure that $\tilde{M}$ does not carry any Kähler metric, we use the Lefschetz property; more precisely, we prove that the map $L[\Omega] : H^2(\tilde{M}) \to H^4(\tilde{M})$ given by the cup product with $[\Omega]$ is not an isomorphism for any possible symplectic form $\Omega$. Again the choice of nilmanifold $M$ and finite group action makes possible to have a non-zero $[\beta] \in H^2(\tilde{M})$ such that $[\beta] \wedge [\alpha_1] \wedge [\alpha_2] = 0$ for every $[\alpha_1], [\alpha_2] \in H^2(\tilde{M})$, which gives the result.

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2. Orbifolds

Definition 2.1. A (smooth) \( n \)-dimensional orbifold is a Hausdorff, paracompact topological space \( X \) endowed with an atlas \( \mathcal{A} = \{(U_p, \bar{U}_p, \Gamma_p, \varphi_p)\} \) of orbifold charts, that is \( U_p \subset X \) is a neighbourhood of \( p \in X \), \( \bar{U}_p \subset \mathbb{R}^n \) an open set, \( \Gamma_p \subset \text{GL}(n, \mathbb{R}) \) a finite group acting on \( \bar{U}_p \), and \( \varphi_p : \bar{U}_p \to U_p \) is a \( \Gamma_p \)-invariant map with \( \varphi_p(0) = p \), inducing a homeomorphism \( \bar{U}_p/\Gamma_p \cong U_p \).

The charts are compatible in the following sense: if \( q \in U_q \cap U_p \) then there exist a connected neighbourhood \( V \subset U_q \cap U_p \) and a diffeomorphism \( f : \varphi^{-1}_p(V)_0 \to \varphi^{-1}_q(V) \), where \( \varphi^{-1}_p(V)_0 \) is the connected component of \( \varphi^{-1}_p(V) \) containing \( q \), such that \( f(\sigma(x)) = \rho(\sigma)(f(x)) \), for any \( x \), and \( \sigma \in \text{Stab}_{\Gamma_p}(q) \), where \( \rho : \text{Stab}_{\Gamma_p}(q) \to \Gamma_q \) is a group isomorphism.

For each \( p \in X \), let \( n_p = \# \Gamma_p \) be the order of the orbifold point (if \( n_p = 1 \) the point is smooth, also called non-orbifold point). The singular locus of the orbifold is the set \( \mathcal{S} = \{ p \in X \mid n_p > 1 \} \). Therefore \( \mathcal{S} - \mathcal{M} \) is a smooth \( n \)-dimensional manifold. The singular locus \( \mathcal{S} \) is stratified: if we write \( \mathcal{S}_k = \{ p \mid n_p = k \} \), and consider its closure \( \overline{\mathcal{S}_k} \), then \( \overline{\mathcal{S}_k} \) inherits the structure of an orbifold. In particular \( \mathcal{S}_k \) is a smooth manifold, and the closure consists of some points of \( \mathcal{S}_{k+1} \), \( k \geq 2 \).

We say that the orbifold is locally oriented if \( \Gamma_p \subset \text{GL}_+(n, \mathbb{R}) \) for any \( p \in X \). As \( \Gamma_p \) is finite, we can choose a metric on \( \bar{U}_p \) such that \( \Gamma_p \subset \text{SO}(n) \). An element \( \sigma \in \Gamma_p \) admits a basis in which it is written as \( \sigma = \text{diag} \left( \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix}, \ldots, \begin{pmatrix} \cos \theta_r & -\sin \theta_r \\ \sin \theta_r & \cos \theta_r \end{pmatrix} \right), \ldots, \begin{pmatrix} 1 & \ldots & 1 \\ \ldots & \ldots & \ldots \end{pmatrix} \), for \( \theta_1, \ldots, \theta_r \in (0, 2\pi) \). In particular, the set of points fixed by \( \sigma \) is of codimension \( 2r \). Therefore the set of singular points \( \mathcal{S} \cap U_p \) is of codimension \( \geq 2 \), and hence \( \mathcal{M} - \mathcal{S} \) is connected (if \( X \) is connected). Also we say that the orbifold \( X \) is oriented if it is locally oriented and \( X - \mathcal{S} \) is oriented.

A natural example of orbifold appears when we take a smooth manifold \( M \) and a finite group \( \Gamma \) acting on \( M \) effectively. Then \( \widetilde{M} = M/\Gamma \) is an orbifold. If \( M \) is oriented and the action of \( \Gamma \) preserves the orientation, then \( \widetilde{M} \) is an oriented orbifold. Note that for every \( \widetilde{p} \in \widetilde{M} \), the group \( \Gamma_{\widetilde{p}} \) is the stabilizer of \( p \in M \), with \( \widetilde{p} = \pi(p) \) under the natural projection \( \pi : M \to \widetilde{M} \).

Definition 2.2. A complex orbifold is a 2\( n \)-dimensional orbifold \( X \) whose orbifold charts have \( U_p \subset \mathbb{C}^n \), \( \Gamma_p \subset \text{GL}(n, \mathbb{C}) \), and in the compatibility of charts the maps \( f \) are biholomorphisms. Note that \( X \) is automatically oriented.

If \( M \) is a complex manifold and \( \Gamma \) is a finite group acting effectively on \( M \) by biholomorphisms, then \( \widetilde{M} = M/\Gamma \) is a complex orbifold.

The complex structure of a complex orbifold \( X \) can be given by the orbifold \((1, 1)\)-tensor \( J \) with \( J^2 = -\text{Id} \). This is given by tensors \( J_p \) on each \( U_p \) defining the complex structure, which are \( \Gamma_p \)-equivariant, for each \( p \in X \), and which agree under the functions \( f \) defining the compatibility of charts.

Definition 2.3. A complex resolution of a complex orbifold \((X, J)\) is a complex manifold \( \widetilde{X} \) together with a holomorphic map \( \pi : \widetilde{X} \to X \) which is a biholomorphism \( \widetilde{X} - E \to X - S \), where \( S \subset X \) is the singular locus and \( E = \pi^{-1}(S) \) is the exceptional locus.
Let $X$ be an orbifold. An orbifold $k$-form $\alpha$ consists of a collection of $k$-forms $\alpha_p$ on each $\tilde{U}_p$ which are $\Gamma_p$-equivariant and that match under the compatibility maps between different charts.

**Definition 2.4.** A symplectic orbifold $(X, \omega)$ consists of a $2n$-dimensional oriented orbifold $X$ and an orbifold 2-form $\omega$ such that $d\omega = 0$ and $\omega^n > 0$ everywhere.

If $M$ is a symplectic manifold and $\Gamma$ is a finite group acting effectively on $M$ by symplectomorphisms, then $\tilde{M} = M/\Gamma$ is a symplectic orbifold.

**Definition 2.5.** A symplectic resolution of a symplectic orbifold $(X, \omega)$ consists of a smooth symplectic manifold $(\tilde{X}, \tilde{\omega})$ and a map $\pi: \tilde{X} \to X$ such that:

- $\pi$ is a diffeomorphism $\tilde{X} - E \to X - S$, where $S \subset X$ is the singular locus and $E = \pi^{-1}(S)$ is the exceptional locus.
- $\tilde{\omega}$ and $\pi^*\omega$ agree in the complement of a small neighbourhood of $E$.

3. Desingularization of Orbifold Points

In this section we suppose that $X$ is an oriented orbifold whose singular locus $S$ consists of a discrete set of points. Assume that $X$ admits a complex structure $J$ and a symplectic structure $\omega$. Therefore we have a complex orbifold $(X, J)$ and a symplectic orbifold $(X, \omega)$.

It is well-known that $(X, J)$ admits a complex resolution $(\tilde{X}_c, \tilde{J})$ by Hironaka’s desingularization [15]. Also, the symplectic orbifold $(X, \omega)$ admits a symplectic resolution $(\tilde{X}_s, \tilde{\omega})$ by Theorem 3.3 in [5]. We want to compare the two resolutions.

First, let us look at the complex resolution of $(X, J)$. Consider $p \in S$, and let $U_p = \tilde{U}_{\pi(p)}/\Gamma_p$ be an orbifold neighbourhood. Recall that we denote $\varphi_p: \tilde{U}_p \to U_p$ the quotient map. By definition of complex orbifold, $\tilde{U}_p \subset \mathbb{C}^n = \mathbb{R}^{2n}$ and $\Gamma_p \subset \text{GL}(n, \mathbb{C})$. As $\Gamma_p$ is a finite group, we can choose a Kähler metric invariant by $\Gamma_p$. With a linear change of variables, we can transform the Kähler metric into standard form. That is, we can suppose that there is an inclusion

$$i: \Gamma_p \hookrightarrow U(n).$$

Shrinking $\tilde{U}_p$, if necessary, we can assume that $\tilde{U}_p = B_\epsilon(0)$, for some $\epsilon > 0$.

Consider now an algebraic resolution of the singularity of $Y = \mathbb{C}^n/\Gamma_p$, provided by [15]. Denote it $\pi: \tilde{Y} \to Y$, and let $E = \pi^{-1}(p)$ be the exceptional locus. Write $B = B_\epsilon(0)/\Gamma_p$ and $\tilde{B} = \pi^{-1}(B)$. The complex resolution is defined as the smooth manifold

$$\tilde{X}_c = (X - \{p\}) \cup_{\pi} \tilde{B},$$

where we identify with the map $\pi: \tilde{B} - E \to B - \{p\} = U_p - \{p\}$. This has a natural complex structure since $\pi$ is a biholomorphism.

Now we move to the construction of the symplectic resolution of $(X, \omega)$, as done in [5]. For $p \in S$, take an orbifold neighbourhood $U'_p = \tilde{U}'_{\pi(p)}/\Gamma'_p$, with $\varphi'_p: \tilde{U}'_p \to U'_p$. By the equivariant Darboux theorem, there is a $\Gamma'_p$-equivariant symplectomorphism $(\tilde{U}'_p, \omega_p) \cong (V, \omega_0)$, where $V \subset \mathbb{R}^{2n}$ is an open set, and $\omega_0$ is the standard symplectic form (shrinking $\tilde{U}'_p$ if necessary). So without loss of generality, we can assume that $\tilde{U}'_p \subset (\mathbb{R}^{2n}, \omega_0)$, where $\omega_0$ is the standard
sympathetic form, and $\Gamma_p' \subset \text{Sp}(2n, \mathbb{R})$. As $\Gamma_p'$ is a finite group, and $U(n) \subset \text{Sp}(2n, \mathbb{R})$ is the maximal compact subgroup, we can choose a complex structure $J$ on $\mathbb{R}^{2n}$ such that the pair $(J, \omega_0)$ determines a Kähler metric, which is invariant by $\Gamma_p'$. We perform a linear change of variables, which transforms the complex structure into standard form (so $\widetilde{U}'_p$ has the standard Kähler structure). Equivalently, we can suppose that there is an inclusion

(2) \[ \iota': \Gamma_p' \hookrightarrow U(n). \]

Shrinking $\widetilde{U}'_p$ if necessary, we can assume that $\widetilde{U}'_p = B_{\epsilon'}(0)$, for some $\epsilon' > 0$.

Consider an algebraic resolution of singularities of $Y' = \mathbb{C}^n / \Gamma_p'$, say $\pi': \tilde{Y}' \to Y'$, and let $E' = (\pi')^{-1}(p)$ be the exceptional locus. Write $B' = B_{\epsilon'}(0)/\Gamma_p'$ and $\tilde{B}' = (\pi')^{-1}(B')$. The symplectic resolution is defined as the smooth manifold

\[ \tilde{X}_s = (X - \{p\}) / \pi', \]

where $\tilde{B}' - E'$ and $B' - \{p\} = U_p' - \{p\}$ are identified by $\pi'$. This has a symplectic structure that is constructed by gluing the symplectic structure of $X - \{p\}$ and the Kähler form of $\tilde{B}'$ by a cut-off process, as done in Theorem 3.3 of [5].

Now we are going to compare $\tilde{X}_c$ and $\tilde{X}_s$. First note that for $p \in S$, we have $\Gamma_p \cong \Gamma_p'$. This follows from $\Gamma_p \cong \pi_1(B - \{p\})$ and $\Gamma_p' \cong \pi_1(B' - \{p\})$, and the fact that $B, B'$ are homeomorphic. So we shall denote $\Gamma_p' = \Gamma_p$ henceforth. We have the following result.

**Theorem 3.1.** If one can arrange that the inclusions $\iota$ and $\iota'$, given by (1) and (2), respectively, are such that $\iota = \iota'$ for every singular point $p \in S$, then there is a diffeomorphism $\tilde{X}_c \cong \tilde{X}_s$, which is the identity outside a small neighbourhood of the exceptional loci. In particular, $\tilde{X}_c$ admits both complex and symplectic structures.

**Proof.** The key point is obviously that if $\iota = \iota'$, then $Y' = Y$, so we can take $\tilde{Y}' = \tilde{Y}$ and $\pi' = \pi$ in the constructions above.

We fix a point $p \in S$, and construct the required isomorphism in a neighbourhood of the exceptional locus over that point. Consider the map (reducing $\epsilon > 0$ if necessary)

\[ f = (\varphi_p')^{-1} \circ \varphi_p: B_\epsilon(0) \to \tilde{U}_p \to B_{\epsilon'}(0) = \widetilde{U}'_p; \]

$f$ is $\Gamma_p$-equivariant and an open embedding (it might fail to be surjective) with $f(0) = 0$. We shall construct a map $F: B_\epsilon(0) \to B_{\epsilon'}(0)$ such that

- $F = \text{Id}$ in a small ball $B_{0,2\epsilon}(0)$,
- $F = f$ outside a slightly bigger ball $B_{0,9\epsilon}(0)$,
- $F$ is a $\Gamma_p$-equivariant diffeomorphism onto its image.

This gives a diffeomorphism $F: \tilde{X}_c \to \tilde{X}_s$, defined by $F$ on $B_\epsilon(0)/\Gamma_p - \{p\}$, extended by the identity on $\pi^{-1}(B_{0,2\epsilon}(0)/\Gamma_p)$, and also by the identity on $X - \pi^{-1}(B_{0,9\epsilon}(0)/\Gamma_p)$.

Write $f(x) = L(x) + R(x)$, where $L$ is the linear part and $|R(x)| \leq C|x|^2$, for some constant $C > 0$. Both these maps are $\Gamma_p$-equivariant. Take a smooth, non-decreasing function $\rho: [0, \epsilon] \to [0, 1]$ such that $\rho_1(t) = 0$ for $t \in [0, 0.8\epsilon]$ and $\rho_1(t) = 1$ for $t \in [0.9\epsilon, 1]$. Consider $g(x) = L(x) + \rho_1(|x|)R(x)$. Then, $g(x) = L(x)$ for $|x| \leq 0.8\epsilon$, $g(x) = f(x)$ for $|x| \geq 0.9\epsilon$, and $g(x)$ is $\Gamma_p$-equivariant because $\Gamma_p \subset \text{SO}(2n)$. Also

\[ dg(x) - L = \rho'_1(|x|)R(x)dx + \rho_1(|x|)dR(x). \]
Using that \(|\rho'_1(t)| \leq C/\epsilon\) and \(|dR(x)| \leq C|x|\) (we denote by \(C > 0\) uniform constants, that can vary from line to line) we have that \(|dg(x) - L| \leq C|x|\). For \(\epsilon > 0\) small enough, we have that \(g\) is a diffeomorphism onto its image.

For the next step, take the linear map \(L : \mathbb{R}^2n \to \mathbb{R}^2n\). We can choose orthonormal (oriented) basis in both origin and target so that \(L = \text{diag}(\lambda_1, \ldots, \lambda_{2n})\), where \(\lambda_i > 0\) are real numbers (the first vector of the basis is a unitary vector \(e_1\) such that \(|L(e_1)|\) is maximized; then \(L\) maps \(\langle e_1 \rangle^\perp\) to \(\langle L(e_1) \rangle^\perp\), and we proceed inductively). Consider the map

\[
h(x) = \begin{cases} x, & |x| \leq 0.4\epsilon, \\
x + \rho_2 \left( \left( \frac{|x| - 0.4\epsilon}{0.3\epsilon} \right)^\alpha \right) (L(x) - x), & 0.4\epsilon \leq |x| \leq 0.7\epsilon, \\
g(x), & |x| \geq 0.7\epsilon,
\end{cases}
\]

where \(\rho_2 : [0, 1] \to [0, 1]\) is smooth non-decreasing with \(\rho_2(t) = 0\) for \(t \in [0, \frac{1}{3}]\), and \(\rho_2(t) = 1\) for \(t \in [\frac{2}{3}, 1]\). Here \(\alpha > 0\) is a constant to be fixed soon.

Clearly \(h\) is \(\Gamma_\rho\)-equivariant, \(h(x) = f(x)\) off \(B_{0, \epsilon_0}(0)\), and \(h(x) = x\) in \(B_{0, 4\epsilon}(0)\) (but beware, we have chosen different coordinates on the origin \(\mathbb{R}^2n\) and the target \(\mathbb{R}^2n\), so \(h\) is not the identity in the ball). The map \(h\) is \(\mathcal{C}^\infty\) because for \(0.4\epsilon \leq |x| \leq 0.5\epsilon\) we have also \(h(x) = x\). Let us see that \(h\) is a diffeomorphism onto its image. It only remains to see this for \(0.5\epsilon \leq |x| \leq 0.7\epsilon\). Write \(y = h(x)\), so in our coordinates \(y_i = x_i + \rho_2(u)(\lambda_i - 1)x_i\), with \(u = \left( \frac{|x| - 0.4\epsilon}{0.3\epsilon} \right)^\alpha\). Then,

\[
dy_i = (1 + (\lambda_i - 1)\rho_2(u)) \, dx_i + (\lambda_i - 1)\rho_2(u) \frac{\alpha}{0.3\epsilon} \left( \frac{|x| - 0.4\epsilon}{0.3\epsilon} \right)^{\alpha - 1} x_i \gamma
\]

with \(\gamma = d|x| = \frac{1}{|x|} \sum x_j \, dx_j\). Write \(\delta_i = (1 + (\lambda_i - 1)\rho_2(u))\), so \(\delta_i\) takes values between 1 and \(\lambda_i\). We compute

\[
dy_1 \wedge \ldots \wedge dy_n = \delta_1 \ldots \delta_n \, dx_1 \wedge \ldots \wedge dx_n + \\
+ \sum \delta_1 \ldots \delta_i \ldots \delta_n \frac{(\lambda_i - 1)\rho_2(u)\alpha x_i}{0.3\epsilon} \left( \frac{|x| - 0.4\epsilon}{0.3\epsilon} \right)^{\alpha - 1} \, dx_1 \wedge \ldots \wedge (i) \wedge \ldots \wedge dx_n
\]

\[
= \delta_1 \ldots \delta_n \left( 1 + \alpha \sum \frac{(\lambda_i - 1)\rho_2(u)(|x| - 0.4\epsilon)^{\alpha - 1} x_i^2}{|x|\delta_i(0.3\epsilon)^\alpha} \right) \, dx_1 \wedge \ldots \wedge dx_n.
\]

In the sum, the numerator is bounded above by \(C(0.3\epsilon)^{\alpha + 1}\) and the denominator is bounded below by \(C^{-1}(0.3\epsilon)^{\alpha + 1}\), for some uniform (independent of \(\alpha\)) constant \(C > 0\). Hence choosing \(\alpha > 0\) small enough, we get that the above quantity does not vanish, hence \(h\) is a diffeomorphism onto its image.

After this step is done, recall that we have taken coordinates given by an orthonormal basis \(\{e_i\}\) on the origin \(\mathbb{R}^2n\), and by the orthonormal basis \(\{L(e_i)/\lambda_i\}\) on the target \(\mathbb{R}^2n\). Written with respect to the same coordinates, we have an orthogonal transformation \(M : \mathbb{R}^2n \to \mathbb{R}^2n\) so that \(h(x) = M\) on \(B_{0, 4\epsilon}(0)\). The final step is to change the isometry \(M \in \text{SO}(2n)\) by the identity. Take a smooth path \(M_t\) of matrices joining \(M_0 = \text{Id}\) with \(M_1 = M\). Take a smooth non-decreasing \(\rho_3 : [0, \epsilon] \to [0, 1]\) with \(\rho_3(t) = 0\) for \(t \in [0, 0.2\epsilon]\), and \(\rho_3(t) = 1\) for \(t \in [0.3\epsilon, \epsilon]\). The map \(F(x) = M_{\rho_3(|x|)}(x), |x| \leq 0.4\epsilon, \) and \(F(x) = h(x)\) for \(|x| \geq 0.4\epsilon, \) is the required map.

**Remark 3.2.** Let \(F : (\tilde{X}, \tilde{J}) \to (\tilde{X}, \tilde{\omega})\) be the diffeomorphism provided by Theorem 3.1. Then if we denote \(\tilde{\omega}' = F^*\tilde{\omega}\), we have that \(\tilde{X}\) admits a symplectic structure \(\tilde{\omega}'\) and a complex
structure $\tilde{J}$. These are not compatible in general, but they are compatible on a neighbourhood of the exceptional locus, and give a Kähler structure there.

4. A COMPLEX AND SYMPLECTIC 6-ORBIFOLD

Consider the complex Heisenberg group $G$, that is, the complex nilpotent Lie group of (complex) dimension 3 consisting of matrices of the form

$$\begin{pmatrix}
1 & u_2 & u_3 \\
0 & 1 & u_1 \\
0 & 0 & 1
\end{pmatrix}.$$ 

In terms of the natural (complex) coordinate functions $(u_1, u_2, u_3)$ on $G$, we have that the complex 1-forms $\mu = du_1$, $\nu = du_2$ and $\theta = du_3 - u_2 du_1$ are left invariant, and $d\mu = d\nu = 0$, $d\theta = \mu \wedge \nu$.

Let $\Lambda \subset \mathbb{C}$ be the lattice generated by 1 and $\zeta = e^{2\pi i/6}$, and consider the discrete subgroup $\Gamma \subset G$ formed by the matrices in which $u_1, u_2, u_3 \in \Lambda$. We define the compact (parallelizable) nilmanifold $M = \Gamma \backslash G$.

We can describe $M$ as a principal torus bundle $T^2 = \mathbb{C}/\Lambda \hookrightarrow M \rightarrow T^4 = (\mathbb{C}/\Lambda)^2$ by the projection $(u_1, u_2, u_3) \mapsto (u_1, u_2)$.

Consider the action of the finite group $\mathbb{Z}_6$ on $G$ given by the generator $\rho: G \rightarrow G$

$$(u_1, u_2, u_3) \mapsto (\zeta^4 u_1, \zeta u_2, \zeta^5 u_3).$$

This action satisfies that $\rho(p \cdot q) = \rho(p) \cdot \rho(q)$, for $p, q \in G$, where $\cdot$ denotes the natural group structure of $G$. Moreover, $\rho(\Gamma) = \Gamma$. Thus, $\rho$ induces an action on the quotient $M = \Gamma \backslash G$.

Denote by $\rho: M \rightarrow M$ the $\mathbb{Z}_6$-action. The action on 1-forms is given by

$$\rho^* \mu = \zeta^4 \mu, \quad \rho^* \nu = \zeta \nu, \quad \rho^* \theta = \zeta^5 \theta.$$ 

**Proposition 4.1.** $\tilde{M} = M/\mathbb{Z}_6$ is a 6-orbifold admitting complex and symplectic structures.

**Proof.** The nilmanifold $M$ is a complex manifold whose complex structure $J$ is the multiplication by $i$ at each tangent space $T_pM$, $p \in M$. Then one can check that $J$ commutes with the $\mathbb{Z}_6$-action $\rho$ on $M$, that is, $(\rho_*)_p \circ J_p = J_{\rho(p)} \circ (\rho_*)_p$, for any point $p \in M$. Hence, $J$ induces a complex structure on the quotient $\tilde{M} = M/\mathbb{Z}_6$.

Now we define the complex 2-form $\omega$ on $M$ given by

$$\omega = -i \mu \wedge \bar{\mu} + \nu \wedge \theta + \bar{\nu} \wedge \bar{\theta}.$$ 

Clearly, $\omega$ is a real closed 2-form on $M$ which satisfies $\omega^3 > 0$, that is, $\omega$ is a symplectic form on $M$. Moreover, $\omega$ is $\mathbb{Z}_6$-invariant. Indeed, $\rho^* \omega = -i \mu \wedge \bar{\mu} + \zeta^6 \nu \wedge \theta + \zeta^{-6} \bar{\nu} \wedge \bar{\theta} = \omega$.

Therefore $\tilde{M}$ is a symplectic 6-orbifold, with the symplectic form $\tilde{\omega}$ induced by $\omega$. $\square$

We denote by

$$\tilde{\pi}: M \rightarrow \tilde{M}$$

the natural projection. The orbifold points of $\tilde{M}$ are the following:
The standard Kähler form. The components of the singular locus determined before independently. Recall that

5.1. Resolution of the isolated orbifold points. This transforms (4) into

is obtained with

We have to do a change of variables to transform $K \subset \text{Sp}(6, \mathbb{R})$ into a subgroup of $\text{U}(3)$. This is obtained with

This transforms (1) into

the standard Kähler form. The $K$-action is given by $(v_1, v_2, v_3) \mapsto (\zeta^2 v_1, \zeta^4 v_2, \zeta v_3)$, so $\iota'((\zeta^2)) = \text{diag}(\zeta^2, \zeta^2, \zeta^4)$, and $\iota = \iota'$.
5.2. Resolution of the singular sets \( \hat{\pi}(S_{(p,q)}) \). Now we consider a connected component of the singular set which is homeomorphic to a 2-torus. There are 5 such components in \( \hat{M} \), all of them are images by \( \hat{\pi} \) of the sets \( S_{(p,q)} = \{(u_1, pu_1 + q) \mid u_1 \in \mathbb{C}/\Lambda \} \), where \( (p, q) \in I = \left( \left\{ (0, \frac{-1}{2}, \frac{1}{2}, \frac{1+i}{8}) \right\} \right)^2 - \{(0, 0)\} \).

Let us focus on one such component \( \hat{T} = \hat{\pi}(T) \equiv \mathbb{C}/\Lambda \). Then \( H \) fixes \( S_{(p,q)} \), and its orbit under \( K \) is given by \( S_{(p,q)} \), for three elements \( (p_1, q_1) = (p, q), (p_2, q_2), (p_3, q_3) \in I \). Consider a neighbourhood \( U \) of \( T \subset M \) via

\[
T \times B_\varepsilon(0) \to U \\
(u_1, u_2, u_3) \mapsto (u_1, u_2 + p, u_3 + pu_1 + q),
\]

where \( B_\varepsilon(0) \subset \mathbb{C}^2 \). The image is

\[
\hat{U} = \hat{\pi}(U) \cong U/H \cong T \times (B_\varepsilon(0)/H),
\]

where \( H \cong \mathbb{Z}_2 \) acts as \( (u_2, u_3) \mapsto (-u_2, -u_3) \).

We see that the complex structure on \( \hat{\pi}(\tilde{S}) \) is the product complex structure. Also, the symplectic structure \( \omega = i du_1 \wedge d\bar{u}_1 + du_2 \wedge d\bar{u}_2 + d\bar{u}_3 \wedge \bar{d}u_3 \) is the product of the natural symplectic structure of \( \mathbb{C}/\Lambda \) with an orbifold symplectic structure on \( B_\varepsilon(0)/H \). Using the construction of Section \( \ref{section:resolution} \) we have a desingularization

\[
\tilde{Y} \to B_\varepsilon(0)/H
\]

which is a smooth manifold endowed with both a complex structure and a symplectic structure coinciding with the given ones outside a small neighbourhood of the exceptional locus \( E \). The condition \( t = t' \) of Theorem \( \ref{thm:resolution} \) is trivially satisfied, since \( s(\rho^3) = \rho^3 = -1 \). Multiplying by \( T = \mathbb{C}/\Lambda \), we have that

\[
\tilde{U} = T \times \tilde{Y}
\]

is a smooth manifold endowed with a complex structure \( \tilde{J} \), and a symplectic structure \( \tilde{\omega} \), which coincide with those of \( \tilde{U} \) outside a small neighbourhood of the exceptional locus \( T \times E \subset \tilde{U} \).

The complex and the symplectic resolutions of \( \hat{M} \) in a neighbourhood of \( \hat{T} \) are obtained by replacing \( \hat{U} \subset \hat{M} \) with \( \tilde{U} \). The two resolutions are diffeomorphic by the considerations above.

5.3. Resolution of the singular set \( \hat{\pi}(S_0) \). Finally we consider the connected component of the singular set which is homeomorphic to a 2-sphere. This is \( \hat{S}_0 = \hat{\pi}(S_0) \), where \( S_0 = \{(u_1, 0, 0) \mid u_1 \in \mathbb{C}/\Lambda \} \). As before, a neighbourhood of \( S_0 \) in \( M \) is of the form

\[
U_0 = (\mathbb{C}/\Lambda) \times B_\varepsilon(0),
\]

where \( B_\varepsilon(0) \subset \mathbb{C}^2 \). The action of \( H = \mathbb{Z}_2 \) is trivial on \( \mathbb{C}/\Lambda \) and as \( \pm 1 \) on \( \mathbb{C}^2 \). The action of \( K = \mathbb{Z}_3 \) is of the form \( \rho^2(u_1, u_2, u_3) = (\zeta^2 u_1, \zeta^2 u_2, \zeta^4 u_3) \).

Let us focus on \( B_\varepsilon(0)/H \). By the construction of Section \( \ref{section:resolution} \) we have a complex desingularization \( (\tilde{Y}_c, \tilde{J}) \to B_\varepsilon(0)/H \). The holomorphic action of \( K \) on \( B_\varepsilon(0) \) induces an action on \( (\tilde{Y}_c, \tilde{J}) \). Also, there is a symplectic desingularization \( (\tilde{Y}_s, \tilde{\omega}) \to B_\varepsilon(0)/H \). The action of \( K \) on \( B_\varepsilon(0) \) induces an action on \( (\tilde{Y}_s, \tilde{\omega}) \). This follows by taking an orbifold chart of the singular point that is \((H \times K)\)-equivariant, using the equivariant Darboux theorem.
By Theorem 3.1 there is a diffeomorphism $F: (\tilde{Y}_c, \tilde{J}) \to (\tilde{Y}_s, \tilde{\omega})$. Let us see that $F$ can be taken to be $K$-equivariant. This follows by the arguments in the proof of Theorem 3.1 by using that $\iota: H \times K \to U(2)$ and $\iota': H \to K \to U(2)$ are equal. For the complex case, $\iota$ is given by the representation $(u_2, u_3) \mapsto (\zeta u_2, \zeta^5 u_3)$, so $\iota(\zeta) = \text{diag}(\zeta, \zeta^5)$. For the symplectic case, we have to do a change of variables to transform $H \times K \subset \text{Sp}(4, \mathbb{R})$ into a subgroup of $U(2)$. This is given by

$$v_2 = \frac{1}{\sqrt{2}}(u_2 - iu_3), \quad v_3 = \frac{1}{\sqrt{2}}(u_2 - iu_3),$$

which transforms $\omega = du_2 \wedge du_3 + d\bar{u}_2 \wedge d\bar{u}_3$ into the standard Kähler form $-i dv_2 \wedge d\bar{v}_2 - i dv_3 \wedge d\bar{v}_3$. As $(v_2, v_3) \mapsto (\zeta v_2, \zeta^5 v_3)$, we have that $\iota'(\zeta) = \text{diag}(\zeta, \zeta^5)$. Hence $\iota = \iota'$.

This produces a desingularization $\tilde{Y} \to B_4(0)/H$ with a symplectic and a complex structure, which match the given ones outside a small neighbourhood of the exceptional set $E \subset \tilde{Y}$, which are compatible (they give a Kähler structure) in a smaller neighbourhood of $E$, by Remark 3.2, and which have an action of $K$ preserving both the complex and symplectic structures. A desingularization of

$$U_0/H = (\mathbb{C}/\Lambda) \times (B_4(0)/H)$$

is given by substituting a neighbourhood of $\tilde{S}_0 = (\mathbb{C}/\Lambda) \times \{0\}$ by $(\mathbb{C}/\Lambda) \times \tilde{Y}$. The fixed points of action of $K$ in $U_0/H$ lie on $\tilde{S}_0$, hence the fixed points of the action of $K$ on the desingularization of $U_0/H$ lie in the exceptional divisor. In this part of the manifold, we have a Kähler structure, so the symplectic and complex desingularization are the same.

This means that $(U_0/H)/K \cong U_0/(H \times K)$ admits a desingularization $\tilde{V}$ with a complex and a symplectic structure. The resolution of $\tilde{M}$ in a neighbourhood of $\tilde{S}_0$ is obtained by substituting $\tilde{T}(U_0) = U_0/(H \times K) \subset \tilde{M}$ with $\tilde{V}$.

All together, we get a smooth 6-manifold $\tilde{M}$ with a complex structure and a symplectic structure, and with a map

$$\pi: \tilde{M} \to \hat{\tilde{M}},$$

which is simultaneously a complex and a symplectic resolution.

6. Topological properties of $\hat{\tilde{M}}$

In this section, we are going to complete the proof of Theorem 3.1 by proving that $\hat{\tilde{M}}$ is simply-connected and that it does not admit a Kähler structure.

**Proposition 6.1.** $\hat{\tilde{M}}$ is simply connected.

*Proof.* We fix base points $p_0 = (0, 0, 0) \in M$ and $\hat{p}_0 = \hat{\pi}(p_0) \in \hat{\tilde{M}}$. There is an epimorphism of fundamental groups

$$\Gamma = \pi_1(M, p_0) \to \pi_1(\hat{\tilde{M}}, \hat{p}_0),$$

since the $\mathbb{Z}_6$-action has a fixed point [4, Chapter II, Corollary 6.3]. Now the nilmanifold $M$ is a principal 2-torus bundle over the 4-torus $T^4$, so we have an exact sequence

$$\mathbb{Z}^2 \to \Gamma \to \mathbb{Z}^4.$$

The group $\Gamma = \pi_1(M, p_0)$ is thus generated by the images of the fundamental groups of the surfaces $\Sigma_1 = \{(u_1, 0, 0)\}$, $\Sigma_2 = \{(0, u_2, 0)\}$ and $\Sigma_3 = \{(0, 0, u_3)\}$ in $M$. The image $\hat{\pi}(\Sigma_1)$ is a 2-sphere, since $\hat{\pi}: \Sigma_1 \to \hat{\pi}(\Sigma_1)$ is a degree 3 map with three ramification points
of order 3 (namely $(\frac{1}{2}a(1 + \zeta), 0, 0)$, with $a = 0, 1, 2$). The image of $\Sigma_2$ is also a 2-sphere, since $\pi: \Sigma_2 \to \tilde{\pi}(\Sigma_2)$ is a degree 6 map with one point of order 6, $(0, 0, 0)$, two of order 3, $(0, \frac{1}{2}b(1 + \zeta), 0), b = 1, 2$, and three of order 2 (namely $(0, p, 0), p = \frac{1}{2}, \frac{\zeta}{2}, \frac{1+\zeta}{2}$). Analogously, $\tilde{\pi}(\Sigma_3)$ is a 2-sphere. This proves that $\pi_1(\tilde{M}, \tilde{p}_0) = \{1\}$.

Now, we look at the resolution process. Let $S \subset \tilde{M}$ be the singular locus and suppose $p \in S$ is an isolated orbifold point. The resolution replaces a neighbourhood $B = B_s(0)/\Gamma_p$ of $p$ with a smooth manifold $\tilde{B}$, such that $\pi: \tilde{B} \to B$ is a complex resolution of singularities. The manifold $\tilde{B}$ is simply connected by [22] Theorem 4.1. A Seifert-Van Kampen argument gives that $\pi_1(\tilde{M})$ is the amalgamated sum of $\pi_1(\tilde{M} - \{p\})$ and $\pi_1(\tilde{B})$ along $\pi_1(\partial B)$. Also $\pi_1(\tilde{M})$ is the amalgamated sum of $\pi_1(\tilde{M} - E)$ and $\pi_1(\tilde{B})$ along $\pi_1(\partial B)$. As $\pi_1(\tilde{B}) = \pi_1(B) = \{1\}$, we have that $\pi_1(\tilde{M}) = \pi_1(\tilde{M})$.

Suppose now that we have a connected component $S'$ of the singular locus $S$ of positive dimension. Let $E' = \pi^{-1}(S')$ be the corresponding exceptional locus. The invariance of the fundamental group under resolution is proved along the same lines as before if we know that the map $\pi: E' \to S'$ induces an isomorphism $\pi_1(E') \to \pi_1(S')$. In our case, we have two possibilities: if $S' = \tilde{\pi}(S_{(p,q)}) \cong T^2$, then $E' = T^2 \times E$, where $E$ is the exceptional divisor of the resolution $\tilde{Y} \to B_s(0)/H$, which is clearly simply connected, and the result follows.

The second possibility is $S' = \tilde{\pi}(S_0)$. In this case, the exceptional divisor over $S'$ is the exceptional divisor of the resolution of

$$((\mathbb{C}/\Lambda) \times (\mathbb{C}^2/H))/K.$$

The resolution of $\mathbb{C}^2/H$ is done by blowing-up $\mathbb{C}^2$ at the origin,

$$\mathbb{C}^2 = \{(a, b, [u : v]) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid av = bu\},$$

and then quotienting by $H = \{\pm \text{Id}\}$. Clearly, the fundamental groups of $(\mathbb{C}/\Lambda) \times (\mathbb{C}^2/H)$ and $(\mathbb{C}/\Lambda) \times (\mathbb{C}^2/H)$ coincide. The action of $K$ is given by $(a, b, [u : v]) \mapsto ((\zeta^2a, \zeta^4b), [u : \zeta^2v])$, with fixed points $(0, 0, [1 : 0])$ and $(0, 0, [0 : 1])$. The fixed points of $K$ on $((\mathbb{C}/\Lambda) \times (\mathbb{C}^2/H)$ occur when $K$ fixes both factors. Therefore, all fixed points are isolated, and the second resolution does not alter the fundamental group.

In order to prove that $\tilde{M}$ does not admit a Kähler structure, we are going to check that it does not satisfy the Lefschetz condition for any symplectic form. For this, it is necessary to understand the cohomology $H^*(\tilde{M})$.

We start by computing the cohomology of $\tilde{M}$. By Nomizu theorem [13], the cohomology of the nilmanifold $\tilde{M}$ is:

$$H^0(M, \mathbb{C}) = \{1\},$$

$$H^1(M, \mathbb{C}) = \{[\mu], [\bar{\mu}], [\nu], [\bar{\nu}]\},$$

$$H^2(M, \mathbb{C}) = \{[\mu \wedge \bar{\mu}], [\mu \wedge \nu], [\bar{\mu} \wedge \nu], [\nu \wedge \bar{\nu}], [\mu \wedge \theta], [\bar{\mu} \wedge \theta], [\nu \wedge \theta], [\bar{\nu} \wedge \theta]\},$$

$$H^3(M, \mathbb{C}) = \{[\mu \wedge \bar{\mu} \wedge \theta], [\mu \wedge \bar{\mu} \wedge \bar{\theta}], [\nu \wedge \bar{\nu} \wedge \theta], [\nu \wedge \bar{\nu} \wedge \bar{\theta}], [\mu \wedge \nu \wedge \theta], [\bar{\mu} \wedge \nu \wedge \bar{\theta}], [\mu \wedge \bar{\nu} \wedge \theta], [\bar{\mu} \wedge \bar{\nu} \wedge \bar{\theta}]\},$$

$$H^4(M, \mathbb{C}) = \{[\mu \wedge \bar{\mu} \wedge \nu \wedge \theta], [\mu \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\theta}], [\mu \wedge \nu \wedge \bar{\nu} \wedge \theta], [\mu \wedge \nu \wedge \bar{\nu} \wedge \bar{\theta}], [\mu \wedge \bar{\mu} \wedge \nu \wedge \theta], [\mu \wedge \bar{\mu} \wedge \nu \wedge \bar{\theta}], [\mu \wedge \nu \wedge \theta \wedge \bar{\theta}], [\mu \wedge \nu \wedge \bar{\theta} \wedge \theta], [\mu \wedge \bar{\mu} \wedge \theta \wedge \bar{\theta}], [\mu \wedge \bar{\mu} \wedge \bar{\theta} \wedge \theta]\},$$

$$\cdots$$
\[ H^5(M, \mathbb{C}) = \langle [\mu \wedge \bar{\mu} \wedge \nu \wedge \theta \wedge \bar{\theta}], [\mu \wedge \bar{\mu} \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}], [\mu \wedge \nu \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}] \rangle, \]
\[ H^6(M, \mathbb{C}) = \langle [\mu \wedge \bar{\mu} \wedge \nu \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}] \rangle. \]

So the cohomology \( H^*(\hat{M}) = H^*(M)^{\mathbb{Z}_6} \) of \( \hat{M} \) is:
\[ H^0(\hat{M}, \mathbb{C}) = \langle 1 \rangle, \]
\[ H^1(\hat{M}, \mathbb{C}) = 0, \]
\[ H^2(\hat{M}, \mathbb{C}) = \langle [\mu \wedge \bar{\mu}], [\nu \wedge \bar{\nu}], [\nu \wedge \theta] \rangle, \]
\[ H^3(\hat{M}, \mathbb{C}) = 0, \]
\[ H^4(\hat{M}, \mathbb{C}) = \langle [\mu \wedge \bar{\mu} \wedge \nu \wedge \theta], [\mu \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\theta}], [\mu \wedge \nu \wedge \theta \wedge \bar{\theta}] \rangle, \]
\[ H^5(\hat{M}, \mathbb{C}) = 0, \]
\[ H^6(\hat{M}, \mathbb{C}) = \langle [\mu \wedge \bar{\mu} \wedge \nu \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}] \rangle. \]

**Proposition 6.2.** \( \hat{M} \) does not admit a Kähler structure since it does not satisfy the Lefschetz property for any symplectic form on \( \hat{M} \).

**Proof.** Let \( \Omega \) be a symplectic form on \( \hat{M} \). The Lefschetz map \( L_{[\Omega]} : H^2(\hat{M}) \to H^4(\hat{M}) \) is given by the cup product with \([\Omega]\). We show that there is a class \([\beta] \in H^2(\hat{M})\) which is in the kernel of \( L_{[\Omega]} \). We prove this by checking that \([\Omega] \wedge [\beta] \wedge [\alpha] = 0\), for any 2-form \([\alpha] \in H^2(\hat{M})\).

We need to determine the cohomology \( H^2(\hat{M}) \). For this, the first step is to construct a map \( H^2(\hat{M}) \to H^2(\hat{M}) \). Let \( h : M \to M \) be a map which:

- is the identity outside small neighbourhoods of each point with non-trivial isotropy,
- contracts a neighbourhood of each of the isolated 24 points with isotropy \( K \) onto the corresponding point,
- contracts a neighbourhood of each \( S_{(p,q)} \) onto \( S_{(p,q)} \) (fixing \( S_{(p,q)} \) pointwise),
- in a neighbourhood of \( S_0 \), is the composition of a contraction onto \( S_0 \) with a map that contracts neighbourhoods (in \( S_0 \)) of the 3 fixed points to the points, and
- is \( \mathbb{Z}_6 \)-equivariant.

\( h \) induces a map \( \hat{h} : \hat{M} \to \hat{M} \). Note that for any closed form \( \alpha \in \Omega^*(\hat{M}) \), \( \hat{h}^*(\alpha) \in \Omega^*(\hat{M}) \) is cohomologous to \( \alpha \) and can be lifted to a form \( \pi^* \hat{h}^*(\alpha) \in \Omega^*(\hat{M}) \), where \( \pi : \hat{M} \to \hat{M} \) is the resolution map. This induces a well-defined map
\[ \Psi = \pi^* \circ \hat{h}^* : H^*(\hat{M}) \to H^*(\hat{M}). \]

Now consider \( U = \hat{M} - S \), where \( S \subset \hat{M} \) is the singular locus and \( V \subset \hat{M} \) is a small neighbourhood of \( S \). Let also \( \hat{U} = \pi^{-1}(U) \) and \( \hat{V} = \pi^{-1}(V) \subset \hat{M} \). Using compactly supported de Rham cohomology, we have a diagram
\[
\begin{array}{ccccccc}
H^2_c(U) \oplus H^2_c(V) & \to & H^2_c(\hat{M}) & \to & H^2_c(U \cap V) & \to & H^3_c(U) \oplus H^3_c(V) \\
\downarrow & & \downarrow & & \downarrow \cong & & \downarrow \\
H^2_c(\hat{U}) \oplus H^2_c(\hat{V}) & \to & H^2_c(\hat{M}) & \to & H^3_c(U \cap \hat{V}) & \to & H^3_c(\hat{U}) \oplus H^3_c(\hat{V})
\end{array}
\]

Since \( V \) retracts onto a set of dimension 2, \( H^3(V) = 0 \). By Poincaré duality, \( H^3(V) = 0 \) as well. Now a simple diagram chasing proves that \( H^2(\hat{M}) = H^2_c(\hat{M}) \) is generated by \( H^2(\hat{M}) = H^2(\hat{M}) \) and \( H^2(\hat{V}) \).
Consider the closed form $\nu \wedge \bar{\nu} \in \Omega^2(\hat{M})$. Since $\nu \wedge \bar{\nu}|_{S_{(p,q)}} = 0$ for any surface $S_{(p,q)}$ and $\nu \wedge \bar{\nu}|_{S_0} = 0$ as well, the 2-cohomology class

$$[\beta] = \Psi([\nu \wedge \bar{\nu}])$$

vanishes on $\tilde{V}$. Clearly $[\beta] \wedge [\alpha_1] \wedge [\alpha_2] = 0$ if either $[\alpha_1], [\alpha_2] \in H^2_{\mathrm{c}}(\tilde{V})$. Moreover, one can check that $[\beta] \wedge [\alpha_1] \wedge [\alpha_2] = 0$, for $[\alpha_1], [\alpha_2] \in H^2(\hat{M})$, which completes the proof. □

References

Fakultät für Mathematik, Universität Bielefeld, Postfach 100301, D-33501 Bielefeld
E-mail address: gbazzoni@math.uni-bielefeld.de

Universidad del País Vasco, Facultad de Ciencia y Tecnología, Departamento de Matemáticas, Apartado 644, 48080 Bilbao, Spain
E-mail address: marisa.fernandez@ehu.es

Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain

Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), C/ Nicolás Cabrera 15, 28049 Madrid, Spain
E-mail address: vicente.munoz@mat.ucm.es