THE NONRELATIVISTIC LIMIT IN RADIATION HYDRODYNAMICS: I. WEAK ENTROPY SOLUTIONS FOR A MODEL PROBLEM

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Abstract. This paper is concerned with a model system for radiation hydrodynamics in multiple space dimensions. The system depends singularly on the light speed c and consists of a scalar nonlinear balance law coupled via an integral-type source term to a family of radiation transport equations. We first show existence of entropy solutions to Cauchy problems of the model system in the framework of functions of bounded variation. This is done by using differences schemes and discrete ordinates. Then we establish strong convergence of the entropy solutions, indexed with c, as c goes to infinity. The limit function satisfies a scalar integro-differential equation.

1. Introduction

The dynamics of a radiating fluid is governed by the Euler equations of compressible hydrodynamics coupled to a radiation transport equation via an integral-type source term. See [10, 13] for the full system of equations. In [3, 4], the following model

$$\begin{aligned} u_t + \mathrm{div} f(u) &= \int_{\mathcal{S}^{d-1}} \big(I(x,t,\omega) - B(u) \big)\, d\omega \\ \frac{1}{c} I_t + \omega \cdot \nabla I &= B(u) - I. \end{aligned} \tag{1.1}$$

was derived from the full system. In this model, the unknowns are non-negative functions $u = u(x,t)$ and $I = I(x,t,\omega)$ for $(x,t,\omega) \in \mathbb{R}^d \times [0,\infty) \times \mathcal{S}^{d-1}$ with $\mathcal{S}^{d-1}$ being the unit sphere in $\mathbb{R}^d$. The parameter $c > 1$ stands for the speed of light. The subscript t denotes the partial derivative with respect to the time variable t, while div and ∇ are the usual divergence and gradient operators with respect to the spatial variable x. The dot "$\cdot$" between two vectors denotes the scalar product. The flux $f = (f_1, \ldots, f_d)^T : [0,\infty) \to \mathbb{R}^d$ and $B : [0,\infty) \to \mathbb{R}$ are given functions of u.

It was pointed out in [3, 4] that the relation of system (1.1) to the full system of radiation hydrodynamics is similar to the relation of scalar nonlinear conservation laws to the Euler equations of compressible hydrodynamics. In particular, u is a lumped variable for the original hydromechanical unknowns (density, velocity, and temperature) and I is the radiation intensity. For applications, $B(u)$ is the Planck function νu^4, with ν being a positive constant, which is increasing for $u \geq 0$. This monotonicity of $B(u)$, not the specific form, is crucial to our analysis.

The goal of this paper is to investigate what we call the *non-relativistic limit*, i.e., the limit as the light speed c in (1.1) tends to infinity. We shall prove under quite general assumptions that for each $c > 1$, there is an entropy solution (u^c, I^c) to (1.1) with some

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initial data. Moreover, we show that as c goes to infinity, u^c converges, almost everywhere, to an entropy solution of the *non-local* scalar equation

$$u_t + \text{div} f(u) = \phi * B(u) - B(u) \tag{1.2}$$

with corresponding initial data. Here $*$ denotes convolution in $\mathbb{R}^d$ and the kernel ϕ is defined through

$$\phi(x) = \frac{e^{-|x|}}{|x|^{d-1}}, \qquad x \in \mathbb{R}^d \setminus \{0\}.$$

Our analysis starts with discrete-ordinate models derived from (1.1) by replacing $\mathcal{S}^{d-1}$ with its finite subsets. As in [17], we prove existence of entropy solutions to Cauchy problems of the discrete-ordinate models by showing convergence of a difference scheme in the framework of functions of bounded variation. The estimates rely crucially on the *relaxation* structure [18] of equations (1.1). The result is summarized in Theorem 3.8. Next we let the number of ordinates tend to infinity and show existence of entropy solutions to Cauchy problems of (1.1). Here it is important that the equations are linear with respect to I, which enables us to use the weak-star convergence, since uniform BV-estimates for I are not available. The result is summarized in Theorem 4.1. Since the entropy solutions thus obtained obey some c-independent estimates, we analyze the nonrelativistic limit $c \to \infty$ in Sec. 4.2 and thus show existence of entropy solutions to (1.2) (Theorem 2.1). By formally solving a linear equation for I with the method of characteristics, we arrive at the convolution source term in the scalar model problem (1.2).

Note that stable explicit numerical integrations of (1.1) would require very small time steps due to the factor c^{-1} in the transport equation (see also Remark 3.1 below). Because of this fact, in numerics one usually uses the hydrodynamical equations coupled with the stationary transport equation which results formally from (1.1) by setting $c = \infty$ [10]. Our analysis shows the validity of the ad-hoc approach in the regime of weak entropy solutions.

We mention some related work for the equations of radiation hydrodynamics. In [7], Kawashima *et al.* proposed a scalar model problem (coupled with an elliptic equation). That is a one-dimensional problem and therefore has only two directions of radiation. For that model problem, Ito [6] established the existence of weak solutions in the framework of functions of bounded variation. See also [16] for early work on related problems. In addition, we refer to [8, 12] for threshold behaviours of equations of the form (1.2).

The paper is organized as follows. In Section 2 we present the main result and introduce discrete-ordinate models for (1.1). Section 3 is devoted to a difference scheme for the discrete-ordinate models. The main result is proved in Section 4.

2. The Main Result

The aim of this section is to present our main result for Cauchy problems of (1.1) with initial data

$$u(x,0) = u_0(x), \qquad I(x,0,\omega) = I_0(x,\omega). \tag{2.1}$$

We recall that the space $BV(\mathbb{R}^d)$ consists of all measurable functions $u = u(x)$ such that

$$|u|_{BV(\mathbb{R}^d)} := \limsup_{|z|\to 0,\, z\in\mathbb{R}^d\setminus\{0\}} \left\{ \frac{1}{|z|} \int_{\mathbb{R}^d} |u(x+z) - u(x)|\, dx \right\} < \infty. \tag{2.2}$$

Our main result reads as

Theorem 2.1. *Suppose the flux functions $f_j = f_j(u)$ are continuously differentiable, $B = B(u)$ is continuous and increasing with respect to u, $u_0 \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and $I_0 \in L^\infty(\mathbb{R}^d \times \mathcal{S}^{d-1})$ satisfies ess $\sup_{\omega\in\mathcal{S}^{d-1}} |I_0(.,\omega)|_{BV(\mathbb{R}^d)} < \infty$.*

Then, for each $c > 1$, the Cauchy problem (1.1) with (2.1) has an entropy solution (u^c, I^c). Moreover, as c tends to infinity, u^c converges to an entropy solution to (1.2) with initial data u_0.

As usual, a pair of functions $(u^c, I^c) \in L^\infty_{loc}\big(\mathbb{R}^d \times [0,\infty)\big) \times L^\infty_{loc}\big(\mathbb{R}^d \times [0,\infty) \times \mathcal{S}^{d-1}\big)$ is called an entropy solution to (1.1) with (2.1) if

$$
\begin{aligned}
&\int_{\mathbb{R}^d\times[0,\infty)} [\eta(u^c)\psi_t + q(u^c)\cdot\nabla\psi]\,dxdt \\
&\quad\geq\ -\int_{\mathbb{R}^d} \eta(u_0)\psi(.,0)\,dx - \int_{\mathbb{R}^d\times[0,\infty)} \eta'(u^c)\psi \int_{\mathcal{S}^{d-1}} \big(I^c(.,\omega) - B(u^c)\big)\,d\omega, \\
&\int_{\mathbb{R}^d\times[0,\infty)} [c^{-1}I^c(.,\omega)\xi_t + I^c(.,\omega)\omega\cdot\nabla\xi]\,dxdt \\
&\quad=\ -\int_{\mathbb{R}^d} c^{-1}I_0(.,\omega)\xi(.,0)\,dx - \int_{\mathbb{R}^d\times[0,\infty)} \big(B(u^c) - I^c(.,\omega)\big)\xi\,dxdt
\end{aligned}
\tag{2.3}
$$

hold for all $\psi, \xi \in C_0^\infty(\mathbb{R}^d \times [0,\infty))$ with $\psi \geq 0$ and all entropy pairs (η, q). The equality in (2.3) means that two sides are equal as *measures* on $\mathcal{S}^{d-1}$. An entropy pair (η, q) consists of a convex function $\eta \in C^1$ and the entropy flux $q = (q_1, \dots, q_d)^T \in C^1$ satisfying the compatibility relation

$$\eta' q_j' = f_j', \qquad j = 1, \dots, d. \tag{2.4}$$

An entropy solution to (1.2) with initial data u_0 is a function $u^\infty \in L^\infty_{loc}(\mathbb{R}^d \times [0,\infty))$ such that

$$
\begin{aligned}
&\int_{\mathbb{R}^d\times[0,\infty)} [\eta(u^\infty)\psi_t + q(u^\infty)\cdot\nabla\psi]\,dxdt \\
&\quad\geq\ -\int_{\mathbb{R}^d} \eta(u_0)\psi(.,0)\,dx - \int_{\mathbb{R}^d\times[0,\infty)} \eta'(u^\infty)\psi\big(\phi * B(u^\infty) - B(u^\infty)\big)\,dxdt
\end{aligned}
\tag{2.5}
$$

holds for all nonnegative $\psi \in C_0^\infty(\mathbb{R}^d \times [0,T))$ and all entropy pairs (η, q).

The proof of Theorem 2.1 starts with discrete-ordinate approximations for (1.1), which we introduce here. We know from [11] that there is a set $\mathcal{N}$ of infinitely many positive integers such that the following construction can be made. For each $L \in \mathcal{N}$ there is a partition of the unit sphere $\mathcal{S}^{d-1}$ into L subsets $\Omega_1^L, \Omega_2^L, \dots, \Omega_L^L$ satisfying the following properties

$$
\begin{aligned}
\mathcal{S}^{d-1} &= \bigcup_{l\in\{1,\dots,L\}} \Omega_l^L, \\
\mathring{\Omega}_l^L \cap \mathring{\Omega}_k^L &= \emptyset \qquad \forall\ k \neq l, \\
\sigma_L := |\Omega_l^L| &= L^{-1}|\mathcal{S}^{d-1}|.
\end{aligned}
\tag{2.6}
$$

Here and henceforth $|A|$ denotes the $(d-1)$-dimensional Hausdorff-measure of a set A. The $\circ$-symbol in the second line of (2.6) denotes the interior of a subset of the sphere with respect to the $(d-1)$-dimensional topology. Moreover, we refer to [11] and suppose that there is a constant K_d, depending only on the dimension d, such that

$$\max_{l\in\{1,\dots,L\}} \operatorname{diam}(\Omega_l^L) \le K L^{-\frac{1}{d-1}}. \tag{2.7}$$

About such partitions, we remark as follows.

Remark 2.1. For our purpose, the equal-size property in (2.6) can be skipped, while it makes the presentation simple. A non-equal-size partition with property (2.7) is explicitly constructed in Appendix. On the other hand, it is shown in [11] that for any $d \le 9$ and any L, there is an equal-size partition with property (2.7). In fact, for $d = 2$ such a partition can be constructed in a straightforward way.

Once the partition is done, we choose an arbitrary but fixed vector $\omega_l^L \in \mathring{\Omega}_l^L$ for each l, this is the ordinate. For (1.1) with (2.1), the discrete-ordinate approximations are Cauchy problems of the form:

$$\begin{aligned} u_t + \operatorname{div} f(u) &= \sigma_L \sum_{l=1}^{L} \big(I_l - B(u)\big), \\ c^{-1} I_{l,t} + \omega_l^L \cdot \nabla I_l &= B(u) - I_l, \\ u(.,0) = u_0, &\qquad I_l(.,0) = \bar I_{l0}. \end{aligned} \tag{2.8}$$

Here

$$\bar I_{l0}(x) = \frac{1}{\sigma_L} \int_{\Omega_l^L} I_0(x,\omega)\, d\omega.$$

We will show the existence of entropy solutions for (2.8). As in (2.3) and (2.5), an entropy solution is a function $(u, I_1, \dots, I_L) \in L^\infty_{loc}(\mathbb{R}^d \times [0,\infty))^{L+1}$ such that

$$\begin{aligned} &\int_{\mathbb{R}^d\times[0,\infty)} [\eta(u)\psi_t + q(u)\cdot\nabla\psi]\, dxdt \\ &\quad\ge\ -\int_{\mathbb{R}^d} \eta(u_0)\psi(.,0)\, dx - \int_{\mathbb{R}^d\times[0,\infty)} \eta'(u)\psi\sigma_L \sum_{l=1}^{L} \big(I_l - B(u)\big)\, dxdt, \\ &\int_{\mathbb{R}^d\times[0,\infty)} [\eta(I_l)\psi_t + c\eta(I_l)\omega_l^L\cdot\nabla\psi]\, dxdt \\ &\quad\ge\ -\int_{\mathbb{R}^d} \eta(\bar I_{l0})\psi(.,0)\, dx - c\int_{\mathbb{R}^d\times[0,\infty)} \eta'(I_l)\big(B(u) - I_l\big)\psi\, dxdt \end{aligned} \tag{2.9}$$

holds for all nonnegative $\psi \in C_0^\infty(\mathbb{R}^d \times [0,\infty))$ and all entropy pairs (η, q).

3. Discrete-Ordinate Models

This section is devoted to the discrete-ordinate models in (2.8). In Sec. 3.1, we study a difference scheme for (2.8) and establish a number of *a priori* estimates of difference solutions. With the estimates, we show in Sect. 3.2 the convergence of the difference solutions to an entropy solution of the Cauchy problem (2.8).

3.1. **A Difference Scheme.** In this subsection, we analyse a difference scheme for the Cauchy problem (2.8). The parameter c is kept constant throughout the section.

First of all, for $h > 0$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}^d$ we define

$$R_\alpha := \prod_{j=1}^{d} \left[\Big(\alpha_j - \frac{1}{2}\Big)h, \Big(\alpha_j + \frac{1}{2}\Big)h\right). \tag{3.1}$$

Let $\Delta t > 0$ and denote by e_j the j-th column of the unit matrix of order d. Our difference scheme reads as

$$\begin{aligned}
\frac{u_\alpha^{k+1} - u_\alpha^k}{\Delta t} + \frac{1}{h}\sum_{j=1}^{d} \Big(g_j(u_\alpha^k, u_{\alpha+e_j}^k) - g_j(u_{\alpha-e_j}^k, u_\alpha^k)\Big) &= \sigma_L \sum_{l=1}^{L} \Big(I_{l\alpha}^{k+1} - B(u_\alpha^{k+1})\Big), \\
\frac{I_{l\alpha}^{k+1} - I_{l\alpha}^k}{\Delta t} + \frac{1}{h}\sum_{j=1}^{d} \Big(h_{lj}(I_{l\alpha}^k, I_{l,\alpha+e_j}^k) - h_{lj}(I_{l,\alpha-e_j}^k, I_{l\alpha}^k)\Big) &= c\big(B(u_\alpha^{k+1}) - I_{l\alpha}^{k+1}\big), \\
(u_\alpha^0,\ I_{l\alpha}^0) &= \frac{1}{h^d}\int_{R_\alpha} \big(u_0(x),\ \bar{I}_{l0}(x)\big)\,dx.
\end{aligned} \tag{3.2}$$

for $k = 0, 1, 2, \cdots$. This is the usual semi-implicit upwind scheme for first-order equations of balance laws. For more details on numerical schemes, we refer to the text books [5, 9]. In (3.2), the numerical flux function

$$g_j(u, v) = f_j^+(u) + f_j^-(v)$$

for u is defined according to the well-known splitting

$$f_j^\pm(u) = \int^u \frac{|f_j'(w)| \pm f_j'(w)}{2} dw.$$

For the radiation intensity I_l, it is

$$h_{lj}^L(u, v) = c\big(\min\{\omega_{lj}^L, 0\}v + \max\{\omega_{lj}^L, 0\}u\big)$$

Set

$$\begin{aligned}
G_\alpha &= G(u_\alpha^k, u_{\alpha\pm e_1}^k, \ldots, u_{\alpha\pm e_d}^k) \\
&= u_\alpha^k - \frac{\Delta t}{h}\sum_{j=1}^{d} \Big(g_j(u_\alpha^k, u_{\alpha+e_j}^k) - g_j(u_{\alpha-e_j}^k, u_\alpha^k)\Big), \\
G_{l\alpha} &= G_l(I_{l\alpha}^k, I_{l\alpha\pm e_1}^k, \ldots, I_{l\alpha\pm e_d}^k) \\
&= I_{l\alpha}^k - \frac{\Delta t}{h}\sum_{j=1}^{d} \Big(h_{lj}^L(I_{l\alpha}^k, I_{l,\alpha+e_j}^k) - h_{lj}^L(I_{l,\alpha-e_j}^k, I_{l\alpha}^k)\Big).
\end{aligned} \tag{3.3}$$

Then the scheme in (3.2) can be rewritten as

$$\begin{aligned} u_\alpha^{k+1} &= G(u_\alpha^k, u_{\alpha\pm e_1}^k, \dots, u_{\alpha\pm e_d}^k) + \sigma_L \Delta t \sum_{l=1}^L \Big(I_{l\alpha}^{k+1} - B(u_\alpha^{k+1}) \Big), \\ I_{l\alpha}^{k+1} &= G_l(I_{l\alpha}^k, I_{l\alpha\pm e_1}^k, \dots, I_{l\alpha\pm e_d}^k) + c\Delta t \big(B(u_\alpha^{k+1}) - I_{l\alpha}^{k+1} \big). \end{aligned} \tag{3.4}$$

By definition, it is obvious that

$$G(u, \dots, u) = u, \qquad G_l(I, \dots, I) = I. \tag{3.5}$$

Moreover, it is straightforward to verify

Proposition 3.1. *Under the conditions of Theorem 2.1, let $a < b$ be two real numbers such that*

$$\big(u_0(x),\ I_0(x,\omega)\big) \in [a,b] \times [B(a), B(b)]$$

for almost every (x,ω). The functions $G : [a,b]^{2d+1} \to \mathbb{R}$ and $G_l : [B(a), B(b)]^{2d+1} \to \mathbb{R}$, as defined in (3.3), are increasing with respect to their arguments, provided that Δt satisfies the following CFL-like condition

$$\max\Big\{ \max_{u\in[a,b]} \{|f_1'(u)|, \dots, |f_d'(u)|\}, c \Big\} \frac{\Delta t}{h} \le 1. \tag{3.6}$$

Remark 3.1. The CFL condition (3.6) depends singularly on the light speed c which is large. Therefore, the scheme (3.2) is not useful for practice. We use it only for analysis. Moreover, we won't let c go to infinity before Δt and h tend to zero.

Since the scheme (3.2) is not completely explicit, it is not clear whether (3.2) can be solved in terms of the given quantities at the previous time $t = k\Delta t$. The following remark clarifies this point.

Remark 3.2. Applying the operation $c^{-1}\sigma_L \sum_{l=1}^L$ to the second equation in (3.4) and then adding it to the first equation, we obtain

$$u_\alpha^{k+1} + c^{-1}\sigma_L \sum_{l=1}^L I_{l\alpha}^{k+1} = G_\alpha + c^{-1}\sigma_L \sum_{l=1}^L G_{l\alpha}.$$

Moreover, from the first equation in (3.4) we deduce that

$$(1 + c\Delta t) u_\alpha^{k+1} + \sigma_L \Delta t L B(u_\alpha^{k+1}) = (1 + c\Delta t) G_\alpha + \sigma_L \Delta t \sum_{l=1}^L G_{l\alpha}. \tag{3.7}$$

Since the left-hand side is strictly increasing with respect to u_α^{k+1} and the right-hand side is known, u_α^{k+1} is uniquely determined. Substituting this u_α^{k+1} into the second equation in (3.4), we obtain $(1 + c\Delta t) I_{l\alpha}^{k+1}$.

The next lemma indicates an *a priori* L^∞-bound for the difference solutions from the scheme (3.2).

Lemma 3.2. *Under the conditions of Theorem 2.1, assume the CFL-like condition (3.6) is satisfied. Then it holds that*

$$(u_\alpha^k, I_{l\alpha}^k) \in [a,b] \times [B(a), B(b)]$$

for any $\alpha \in \mathbb{Z}^d$, any $l \in \{1, \dots, L\}$ and any $k \ge 0$.

Proof. We use induction on k. By definition, it is clear that

$$(u_\alpha^0, I_{l\alpha}^0) \in [a,b] \times [B(a), B(b)]$$

for any $\alpha \in \mathbb{Z}^d$ and any $l \in \{1, \dots, L\}$. Assume $(u_\alpha^k, I_{l\alpha}^k) \in [a,b] \times [B(a), B(b)]$. It follows from Proposition 3.1 and (3.5) that the right-hand side of the equation (3.7) takes values in the interval

$$[(1 + c\Delta t)a + \sigma_L \Delta t L B(a), (1 + c\Delta t)b + \sigma_L \Delta t L B(b)].$$

Since $B = B(u)$ is increasing, the left-hand side is strictly increasing with respect to u_α^{k+1}. Thus, from the structure of the left-hand side we see that u_α^{k+1} must take values in $[a,b]$. On the other hand, it follows from the second equation in (3.4) that

$$(1 + c\Delta t) I_{l\alpha}^{k+1} = G_{l\alpha} + c\Delta t B(u_\alpha^{k+1}).$$

Thanks to the monotonicity, the right-hand side obviously takes values in the interval

$$[B(a) + c\Delta t B(a), B(b) + c\Delta t B(b)] = (1 + c\Delta t)[B(a), B(b)].$$

Hence $I_{l\alpha}^{k+1} \in [B(a), B(b)]$ and the proof is complete. ☐

Having Lemma 3.2, we use (3.2) simply to obtain the following time-Lipschitz estimate.

Lemma 3.3. *Under the conditions of Lemma 3.2, let M be a positive integer. Then we have*

$$\sum_{|\alpha| \le M} |u_\alpha^{k+1} - u_\alpha^k| \le C\Delta t \left(M^d + h^{-1} \sum_{\alpha, j} |u_{\alpha+e_j}^k - u_\alpha^k| \right),$$

$$c^{-1} \sum_{|\alpha| \le M} |I_{l\alpha}^{k+1} - I_{l\alpha}^k| \le C\Delta t \left(M^d + h^{-1} \sum_{\alpha, j} |I_{l,\alpha+e_j}^k - I_{l\alpha}^k| \right)$$

for all $k \in \mathbb{N}$. Here C is a generic constant depending only on $d, a, b, |\mathcal{S}^{d-1}|$ and the functions B, f'.

Now we turn to the L^1-stability of the difference scheme (3.2).

Lemma 3.4. *Let $(u_\alpha^k, I_{l\alpha}^k)$ and $(\tilde{u}_\alpha^k, \tilde{I}_{l\alpha}^k)$ be two solutions to the difference scheme (3.2) that satisfies the conditions of Lemma 3.2. Then the L^1-contraction estimate*

$$\sum_\alpha |u_\alpha^{k+1} - \tilde{u}_\alpha^{k+1}| + c^{-1}\sigma_L \sum_{l,\alpha} |I_{l\alpha}^{k+1} - \tilde{I}_{l\alpha}^{k+1}| \le \sum_\alpha |u_\alpha^k - \tilde{u}_\alpha^k| + c^{-1}\sigma_L \sum_{l,\alpha} |I_{l\alpha}^k - \tilde{I}_{l\alpha}^k|$$

holds for all $k \in \mathbb{N}$.

Proof. ¿From (3.4) we deduce that

$$\begin{aligned} u_\alpha^{k+1} - \tilde{u}_\alpha^{k+1} &= G_\alpha - \tilde{G}_\alpha + \sigma_L \Delta t \sum_{l=1}^L \big((I_{l\alpha}^{k+1} - \tilde{I}_{l\alpha}^{k+1}) - B(u_\alpha^{k+1}) + B(\tilde{u}_\alpha^{k+1}) \big), \\ I_{l\alpha}^{k+1} - \tilde{I}_{l\alpha}^{k+1} &= G_{l\alpha} - \tilde{G}_{l\alpha} - c\Delta t \big(I_{l\alpha}^{k+1} - \tilde{I}_{l\alpha}^{k+1} - B(u_\alpha^{k+1}) + B(\tilde{u}_\alpha^{k+1}) \big). \end{aligned}$$

For $\alpha \in \mathbb{Z}^d$, define

$$s_\alpha = \operatorname{sign}(u_\alpha^{k+1} - \tilde{u}_\alpha^{k+1}), \qquad s_{l\alpha} = \operatorname{sign}(I_{l\alpha}^{k+1} - \tilde{I}_{l\alpha}^{k+1}).$$

We have

$$
\begin{aligned}
|u_\alpha^{k+1}-\tilde u_\alpha^{k+1}| &= s_\alpha(G_\alpha-\tilde G_\alpha)+s_\alpha\sigma_L\Delta t\sum_{l=1}^{L}\big((I_{l\alpha}^{k+1}-\tilde I_{l\alpha}^{k+1})-B(u_\alpha^{k+1})+B(\tilde u_\alpha^{k+1})\big),\\
|I_{l\alpha}^{k+1}-\tilde I_{l\alpha}^{k+1}| &= s_{l\alpha}(G_{l\alpha}-\tilde G_{l\alpha})-s_{l\alpha}c\Delta t\big(I_{l\alpha}^{k+1}-\tilde I_{l\alpha}^{k+1}-B(u_\alpha^{k+1})+B(\tilde u_\alpha^{k+1})\big).
\end{aligned}
$$

Since $B=B(u)$ is increasing with respect to u, we see that

$$
\begin{aligned}
&(s_{l\alpha}-s_\alpha)(B(u_\alpha^{k+1})-B(\tilde u_\alpha^{k+1}))\\
&= s_{l\alpha}(B(u_\alpha^{k+1})-B(\tilde u_\alpha^{k+1}))-|B(u_\alpha^{k+1})-B(\tilde u_\alpha^{k+1})|\\
&\le 0.
\end{aligned}
$$

Similarly, we have

$$(s_\alpha-s_{l\alpha})(I_{l\alpha}^{k+1}-\tilde I_{l\alpha}^{k+1})\le 0.$$

Consequently, we arrive at

$$
\begin{aligned}
&|u_\alpha^{k+1}-\tilde u_\alpha^{k+1}|+c^{-1}\sigma_L\sum_{l=1}^{L}|I_{l\alpha}^{k+1}-\tilde I_{l\alpha}^{k+1}|\\
&\le |G_\alpha-\tilde G_\alpha|+c^{-1}\sigma_L\sum_{l=1}^{L}|G_{l\alpha}-\tilde G_{l\alpha}|\\
&\quad+\sigma_L\Delta t\sum_{l=1}^{L}\big[(s_\alpha-s_{l\alpha})(I_{l\alpha}^{k+1}-\tilde I_{l\alpha}^{k+1})+(s_{l\alpha}-s_\alpha)(B(u_\alpha^{k+1})-B(\tilde u_\alpha^{k+1}))\big]\\
&\le |G_\alpha-\tilde G_\alpha|+c^{-1}\sigma_L\sum_{l=1}^{L}|G_{l\alpha}-\tilde G_{l\alpha}|.
\end{aligned}
\tag{3.8}
$$

On the other hand, we may as well assume that

$$\sum_\alpha|u_\alpha^k-\tilde u_\alpha^k|+c^{-1}\sigma_L\sum_{l,\alpha}|I_{l\alpha}^k-\tilde I_{l\alpha}^k|<\infty.$$

(Otherwise, the lemma is trivially true). This implies that

$$|u_\alpha^k-\tilde u_\alpha^k|,\quad |I_{l\alpha}^k-\tilde I_{l\alpha}^k|\longrightarrow 0 \qquad \text{as} \qquad |\alpha|\to\infty. \tag{3.9}$$

Now we follow [1] and set

$$\hat u_\alpha^k=\max\{u_\alpha^k,\tilde u_\alpha^k\},\qquad \hat G_\alpha=G(\hat u_\alpha^k,\hat u_{\alpha\pm e_1}^k,\dots,\hat u_{\alpha\pm e_d}^k).$$

Thanks to the monotonicity of G (Proposition 3.1), we have $\hat G_\alpha\ge\max\{G_\alpha,\tilde G_\alpha\}$ and thereby

$$
\begin{aligned}
&|G_\alpha-\tilde G_\alpha|\le|G_\alpha-\hat G_\alpha|+|\hat G_\alpha-\tilde G_\alpha|\\
&= (\hat G_\alpha-G_\alpha)+(\hat G_\alpha-\tilde G_\alpha)\\
&= (\hat u_\alpha^k-u_\alpha^k)+(\hat u_\alpha^k-\tilde u_\alpha^k)+\\
&\quad+[(\hat G_\alpha-\hat u_\alpha^k)-(G_\alpha-u_\alpha^k)]+[(\hat G_\alpha-\hat u_\alpha^k)-(\tilde G_\alpha-\tilde u_\alpha^k)]\\
&= |u_\alpha^k-\tilde u_\alpha^k|+[(\hat G_\alpha-\hat u_\alpha^k)-(G_\alpha-u_\alpha^k)]+[(\hat G_\alpha-\hat u_\alpha^k)-(\tilde G_\alpha-\tilde u_\alpha^k)].
\end{aligned}
$$

Since the scheme is conservative, it follows from (3.9) that

$$\sum_\alpha [(\hat{G}_\alpha - \hat{u}^k_\alpha) - (G_\alpha - u^k_\alpha)] = 0, \quad \sum_\alpha [(\hat{G}_\alpha - \hat{u}^k_\alpha) - (\tilde{G}_\alpha - \tilde{u}^k_\alpha)] = 0.$$

Therefore, we get

$$\sum_\alpha |G_\alpha - \tilde{G}_\alpha| \le \sum_\alpha |u^k_\alpha - \tilde{u}^k_\alpha|.$$

Similarly, we have

$$\sum_{l,\alpha} |G_{l\alpha} - \tilde{G}_{l\alpha}| \le \sum_{l,\alpha} |I^k_{l\alpha} - \tilde{I}^k_{l\alpha}|.$$

By substituting the last two inequalities into (3.8), we complete the proof. □

By taking

$$(\tilde{u}^k_\alpha, \tilde{I}^k_{l\alpha}) = (u^k_{\alpha+e_j}, I^k_{l,\alpha+e_j})$$

in Lemma 3.4, we get the following corollary on BV-estimates of the difference solutions.

Corollary 3.5. *Let $(u^k_\alpha, I^k_{l\alpha})$ be a solution to the difference scheme (3.2) and satisfy the conditions of Lemma 3.2. Then the BV estimate*

$$\sum_\alpha |u^k_\alpha - u^k_{\alpha+e_j}| + c^{-1}\sigma_L \sum_{l,\alpha} |I^k_{l\alpha} - I^k_{l,\alpha+e_j}| \le \sum_\alpha |u^0_\alpha - u^0_{\alpha+e_j}| + c^{-1}\sigma_L \sum_{l,\alpha} |I^0_{l\alpha} - I^0_{l,\alpha+e_j}|$$

holds for $k \ge 0$ and $j = 1, 2, \ldots, d$.

We conclude this subsection with an entropy property of the difference solutions. Notice that the coupling in system (2.8) is only due to the source terms. For such weakly coupled systems, a proof of the next lemma can be found in [14, Lemma 4.3].

Lemma 3.6. *Let $(u^k_\alpha, I^k_{l\alpha})$ be a solution to the difference scheme (3.2) that satisfies the conditions of Lemma 3.2. Then, for any smooth convex function $\eta = \eta(u)$, there exist Lipschitz continuous functions r_j and s^L_{lj} $(j = 1, 2, \ldots, d; l = 1, 2, \ldots, L)$ of two variables such that for all $\alpha \in \mathbb{Z}^d$ and $k \ge 0$, the following cell entropy inequalities hold:*

$$\begin{aligned}\eta(u^{k+1}_\alpha) \le & \eta(u^k_\alpha) - \frac{\Delta t}{h} \sum_{j=1}^d \big(r_j(u^k_\alpha, u^k_{\alpha+e_j}) - r_j(u^k_{\alpha-e_j}, u^k_\alpha)\big) \\ & + \eta'(u^{k+1}_\alpha)\sigma_L \sum_{l=1}^L \big(I^{k+1}_{l\alpha} - B(u^{k+1}_\alpha)\big),\end{aligned}$$

$$\begin{aligned}\eta(I^{k+1}_{l\alpha}) \le & \eta(I^k_{l\alpha}) - \frac{\Delta t}{h} \sum_{j=1}^d \big(s^L_{lj}(I^k_{l\alpha}, I^k_{l,\alpha+e_j}) - s^L_{lj}(I^k_{l,\alpha-e_j}, I^k_{l\alpha})\big) \\ & + c\eta'(I^{k+1}_{l\alpha})\big(B(u^{k+1}_\alpha) - I^{k+1}_{l\alpha}\big).\end{aligned}$$

Moreover, the Lipschitz continuous functions (numerical entropy fluxes) satisfy the following consistency relations

$$r_j(u,u) = \int_0^u \eta'(w) f'_j(w)\, dw \quad , \quad s^L_{lj}(I,I) = c\omega^L_{lj}\eta(I).$$

3.2. **Existence of Entropy Solutions.** In this subsection, we show the convergence of the difference scheme (3.2) to the Cauchy problem (2.8). To this end, we define

$$\big(u^h(x,t), I_l^h(x,t)\big) := \big(u_\alpha^k, I_{l\alpha}^k\big) \quad \text{for} \quad (x,t) \in R_\alpha \times [k\Delta t, (k+1)\Delta t). \tag{3.10}$$

For u^h and I_l^h defined thus, we have

Lemma 3.7. *Under the conditions of Lemma 3.2, the piecewise constant functions u^h, I_l^h defined in (3.10) satisfy the following estimates*

$$\big(u^h(x,t), I_l^h(x,t)\big) \in [a,b] \times [B(a), B(b)] \quad \textit{for all} \quad (x,t), \tag{3.11}$$

$$|u^h(.,t)|_{BV(\mathbb{R}^d)} + \frac{\sigma_L}{c}\sum_{l=1}^{L} |I_l^h(.,t)|_{BV(\mathbb{R}^d)} \le |u_0|_{BV(\mathbb{R}^d)} + \frac{\sigma_L}{c}\sum_{l=1}^{L} |\bar{I}_{l0}|_{BV(\mathbb{R}^d)}, \tag{3.12}$$

$$\|u^h(\cdot,t)-u^h(\cdot,t_1)\|_{L^1(|x|\le R)},\ c^{-2}\sigma_L\sum_{l=1}^{L}\|I_l^h(\cdot,t)-I_l^h(\cdot,t_1)\|_{L^1(|x|\le R)} \le C_R(|t-t_1|+\Delta t) \tag{3.13}$$

for all $t, t_1 \ge 0$ and all $R > 0$. In (3.13), the generic constant C_R depends on R.

Proof. The inclusion in (3.11) follows directly from Lemma 3.2. For (3.12), we observe from the definition of BV-seminorm (2.2) that

$$\big(|u^h(.,t)|_{BV(\mathbb{R}^d)},\ |I_l^h(.,t)|_{BV(\mathbb{R}^d)}\big) = h^{d-1}\sum_{j=1}^{d}\sum_{\alpha\in\mathbb{Z}^d}\big(|u_{\alpha+e_j}^k - u_\alpha^k|,\ |I_{l\alpha+e_j}^k - I_{l\alpha}^k|\big), \tag{3.14}$$

where k is an integer such that $t \in [k\Delta t, (k+1)\Delta t)$. Thus, the inequality (3.12) simply follows from Corollary 3.5.

To show (3.13), we let k_1 be such an integer that $t_1 \in [k_1\Delta t, (k_1+1)\Delta t)$. Without loss of generality, we assume $k_1 \le k$. Then we deduce from Definition (3.10) and Lemma 3.3 that

$$\begin{aligned}
\|u^h(\cdot,t)-u^h(\cdot,t_1)\|_{L^1(|x|\le R)} &= \sum_{|\alpha|\le Rh^{-1}} |u_\alpha^k - u_\alpha^{k_1}|h^d \\
&\le \sum_{n=k_1}^{k-1}\sum_{|\alpha|\le Rh^{-1}} |u_\alpha^{n+1} - u_\alpha^n|h^d \\
&\le C(k-k_1)\Delta t\Big(R^d + h^{d-1}\sum_{j,\alpha}|u_{\alpha+e_j}^n - u_\alpha^n|\Big) \\
&\le C_R(|t-t_1|+\Delta t),
\end{aligned}$$

where the last step uses (3.14) and (3.12) together with the uniform boundedness of $|u_0|_{BV(\mathbb{R}^d)}$ and $|I_0(.,\omega)|_{BV(\mathbb{R}^d)}$ assumed in Theorem 2.1. Likewise, we have

$$c^{-2}\sigma_L\sum_{l=1}^{L}\|I_l^h(\cdot,t) - I_l^h(\cdot,t_1)\|_{L^1(|x|\le R)} \le C_R(|t-t_1|+\Delta t).$$

Thus, the inequality (3.13) is verified. This completes the proof. ☐

Having Lemma 3.7, we are in a position to prove the existence of entropy solutions to the Cauchy problem (2.8).

Theorem 3.8. *Assume the conditions of Theorem 2.1. Then, as h and Δt go to zero but always satisfy the CFL-like condition (3.6), $\{(u^h, I_1^h, \dots, I_L^h)\}_{h>0}$ defined in (3.10) has a subsequence converging almost everywhere to an entropy solution $(u, I_1, \dots, I_L)$ to the Cauchy problem (2.8). Moreover, the solution fulfills the following estimates*

$$(u(x,t), I_1(x,t), \dots, I_L(x,t)) \in [a,b] \times [B(a), B(b)]^L \quad \text{for almost all } (x,t), \tag{3.15}$$

$$|u(.,t)|_{BV(\mathbb{R}^d)} + \frac{\sigma_L}{c}\sum_{l=1}^{L} |I_l(.,t)|_{BV(\mathbb{R}^d)} \le |u_0|_{BV(\mathbb{R}^d)} + \frac{\sigma_L}{c}\sum_{l=1}^{L} |\bar{I}_{l0}|_{BV(\mathbb{R}^d)}, \tag{3.16}$$

$$\|u(.,t) - u(.,t_1)\|_{L^1(|x|\le R)}, \quad \frac{\sigma_L}{c^2}\sum_{l=1}^{L} \|I_l(.,t) - I_l(.,t_1)\|_{L^1(|x|\le R)} \le C_R|t-t_1| \tag{3.17}$$

for all $t, t_1 > 0$ and all $R > 0$. In (3.17), the generic constant C_R depends on R.

Proof. Thanks to the estimates in (3.11) and (3.12), we deduce from the Fréchet-Kolmogorov theorem [19] that for each $t \ge 0$, set

$$S := \left\{ \left(u^h(.,t), I_1^h(.,t), \dots, I_L^h(.,t)\right) \right\}_{h>0}$$

is precompact in $L^1_{loc}(\mathbb{R}^d)^{L+1}$. Then for each $t \ge 0$, the set has a subsequence (denoted in the same way) converging to a certain $(u(.,t), I_1(.,t), \dots, I_L(.,t))$ in $L^1_{loc}(\mathbb{R}^d)^{L+1}$. Choose a countable and dense subset of the time interval $[0,\infty)$. By a standard diagonalization argument, we see that the set S has a subsequence converging to $(u(.,t), I_1(.,t), \dots, I_L(.,t))$ in $L^1_{loc}(\mathbb{R}^d)^{L+1}$ for all t in the dense subset. Moreover, we exploit the estimates in (3.13) and deduce that the subsequence converges to $(u(.,t), I_1(.,t), \dots, I_L(.,t))$ in $L^1_{loc}(\mathbb{R}^d)^{L+1}$ and thereby almost everywhere for all $t \ge 0$. Having this convergence, we follows the proof of the Lax-Wendroff theorem (see, e.g. [5, Theorem 1.1, Chapter III]) and deduce from Lemma 3.6 that $(u, I_1, \dots, I_L)$ is an entropy solution to the Cauchy problem (2.8). The estimates in (3.15)–(3.17) follow simply from the above convergence result and the estimates in (3.11)–(3.13). This completes the proof. □

Remark 3.3. In [15], it was proven that entropy solutions of weakly coupled systems like (2.8) are unique if they exist. Thus, the last proof shows the convergence of the difference scheme (3.2) to the unique entropy solution of the Cauchy problem (2.8).

4. A Proof of the Main Result

This section is devoted to proving our main result Theorem 2.1. We separate the proof into two parts: existence and nonrelativistic limit. In the first subsection, we show the existence of weak entropy solutions to the Cauchy problem (1.1) subject to (2.1). The definition of entropy solutions is given in (2.3).

4.1. **Existence of Entropy Solutions.**

To begin with, we recall that entropy solutions to the discrete-ordinate model (2.8) depend on the parameters $\sigma_L = |\mathcal{S}^{d-1}|/L$ and c. To make visible the dependency on σ_L, we denote by $(u^L, I_1^L, \dots, I_L^L)$ the unique entropy solution to (2.8) constructed in Theorem 3.8. Moreover, we define

$$I^L = I^L(x,t,\omega) = I_l^L(x,t), \qquad \text{for } \omega \in \Omega_l^L,$$

for each $(x,t) \in \mathbb{R}^d \times [0,\infty)$. This definition implies

$$\int_{\mathcal{S}^{d-1}} I^L(x,t,\omega)\,d\omega = \sigma_L \sum_l I_l^L(x,t) \qquad \text{for each} \quad (x,t), \tag{4.1}$$

and, together with the estimate (3.15),

$$\begin{array}{lll} u^L(x,t) & \in [a,b] & a.e. \quad (x,t) \in \mathbb{R}^d \times [0,\infty), \\ I^L(x,t,\omega) & \in [B(a),B(b)] & a.e. \quad (x,t,\omega)) \in \mathbb{R}^d \times [0,\infty) \times \mathcal{S}^{d-1}. \end{array} \tag{4.2}$$

Note that the functions u^L and I^L depend on the parameter c.

Theorem 4.1. *Under the conditions of Theorem 2.1, for each $c > 1$ the Cauchy problem (1.1) subject to (2.1) has an entropy solution $(u^c, I^c) = \big(u^c(x,t), I^c(x,t,\omega)\big)$ satisfying following estimates*

$$\begin{array}{lll} u^c(x,t) & \in [a,b] & a.e. \quad (x,t) \in \mathbb{R}^d \times [0,\infty), \\ I^c(x,t,\omega) & \in [B(a),B(b)] & a.e. \quad (x,t,\omega) \in \mathbb{R}^d \times [0,\infty) \times \mathcal{S}^{d-1}, \end{array} \tag{4.3}$$

$$|u^c(.,t)|_{BV(\mathbb{R}^d)} \le |u_0|_{BV(\mathbb{R}^d)} + \frac{|\mathcal{S}^{d-1}|}{c} \text{ess sup}_{\omega\in\mathcal{S}^{d-1}} |I_0(.,\omega)|_{BV(\mathbb{R}^d)}, \tag{4.4}$$

$$|u^c(.,t) - u^c(.,t_1)|_{L^1(|x|\le R)} \le C_R|t-t_1| \tag{4.5}$$

for all $t, t_1 \ge 0$ and all $R > 0$. In (4.5), the generic constant C_R depends on R.

Note that we have not claimed the uniqueness of the entropy solution (u^c, I^c).

Proof. In this proof, the light speed c is fixed. Thus, u and I are used to stand for u^c and I^c, respectively. Thanks to (4.2), the set $\{(u^L, I^L) : L = 1, 2, \ldots\}$ is bounded in $L^\infty\big(\mathbb{R}^d \times [0,\infty)\big) \times L^\infty\big(\mathbb{R}^d \times [0,\infty) \times \mathcal{S}^{d-1}\big)$. Thus, the set has a subsequence (denoted in the same way) such that as L goes to infinity,

$$\begin{array}{lll} u^L & \stackrel{*}{\rightharpoonup} & u \quad \text{in } L^\infty\big(\mathbb{R}^d \times [0,\infty)\big), \\ I^L & \stackrel{*}{\rightharpoonup} & I \quad \text{in } L^\infty\big(\mathbb{R}^d \times [0,\infty) \times \mathcal{S}^{d-1}\big). \end{array} \tag{4.6}$$

On the other hand, based on estimates (3.16) and (3.17) of Theorem 3.8, we follow the proof of Theorem 3.8 to deduce that there is a further subsequence of $\{u^L\}_{L\in\mathcal{N}}$ (again denoted in the same way) such that

$$u^L \to u \qquad \text{in} \quad L^1_{loc}\big(\mathbb{R}^d \times [0,\infty)\big). \tag{4.7}$$

Thanks to this strong convergence, the estimates (4.4) and (4.5) follow from those of u^L in (3.16) and (3.17). Moreover, the estimate (4.3) follows from the weak-$*$ convergence (4.6) and the estimate (4.2).

Next we show that (u, I) obtained above is an entropy solution to the Cauchy problem (1.1) subject to (2.1). In fact, the strong convergence (4.7) implies that

$$\int_{\mathbb{R}^d\times[0,\infty)} \big[\eta(u^L)\psi_t + q(u^L)\cdot\nabla\psi\big]\,dxdt \to \int_{\mathbb{R}^d\times[0,\infty)} \big[\eta(u)\psi_t + q(u)\cdot\nabla\psi\big]\,dxdt.$$

Moreover, it follows from (4.1), (4.7) and (4.6) that

$$
\begin{aligned}
&\int_{\mathbb{R}^d\times[0,\infty)} \eta'(u^L)\psi\sigma_L \sum_{l=1}^{L} \big(I_l^L - B(u^L)\big)\,dxdt \\
&= \int_{\mathbb{R}^d\times[0,\infty)} \eta'(u^L)\psi \left(\int_{\mathcal{S}^{d-1}} \big(I^L - B(u^L)\big)\,d\omega\right) dxdt \\
&= \int_{\mathbb{R}^d\times[0,\infty)\times\mathcal{S}^{d-1}} \eta'(u^L)\psi\big(I^L - B(u^L)\big)\,dxdtd\omega \\
&\to \int_{\mathbb{R}^d\times[0,\infty)\times\mathcal{S}^{d-1}} \eta'(u)\psi\big(I - B(u)\big)\,dxdtd\omega.
\end{aligned}
$$

Here we have used the fact that the weak-$*$ convergence (4.6) and the strong convergence (4.7) together allow us to pass the limit for the product $\eta'(u^L)I^L$. Thus, we see from Theorem 3.8 that (u, I) satisfies the first inequality in (2.3).

Furthermore, for each $\omega \in \mathcal{S}^{d-1}$ and each $L \in \mathcal{N}$, there is an $l \in \{1, \ldots, L\}$ such that $\omega \in \Omega_l^L$. Then by definition we have $I^L(x,t,\omega) = I_l^L(x,t)$ for all (x,t). Because of Theorem 3.8, I_l^L is a weak solution of the second equation in (2.8):

$$
\begin{aligned}
&\int_{\mathbb{R}^d\times[0,\infty)} \big[c^{-1}I^L(.,\omega)\xi_t + I^L(.,\omega)\omega_l^L\cdot\nabla\xi\big]\,dxdt \\
&= -\int_{\mathbb{R}^d}\left(\frac{1}{c\sigma_L}\int_{\Omega_l^L} I_0(.,\omega)\,d\omega\right)\xi(.,0)\,dx - \int_{\mathbb{R}^d\times[0,\infty)} \big(B(u^L) - I^L(.,\omega)\big)\xi\,dxdt
\end{aligned}
\tag{4.8}
$$

for any $\xi = \xi(x,t) \in C_0^\infty\big(\mathbb{R}^d\times[0,\infty)\big)$. Note that due to (2.7) we have $|\omega_l^L - \omega| \le \operatorname{diam}(\Omega_l^L) \to 0$ as L goes to infinity. Thanks to the weak-$*$ and strong convergence (4.6) and (4.7), we deduce from (4.8) that

$$
\begin{aligned}
&\int_{\mathcal{S}^{d-1}}\int_{\mathbb{R}^d\times[0,\infty)} g(\omega)\big[c^{-1}I(.,\omega)\xi_t + I(.,\omega)\omega\cdot\nabla\xi\big]\,dxdtd\omega \\
&= -\int_{\mathcal{S}^{d-1}}\int_{\mathbb{R}^d} g(\omega)c^{-1}I_0(.,\omega)\xi(.,0)\,dxd\omega - \int_{\mathcal{S}^{d-1}}\int_{\mathbb{R}^d\times[0,\infty)} g(\omega)\big(B(u) - I(.,\omega)\big)\xi\,dxdtd\omega.
\end{aligned}
$$

This holds for any $g \in C^0(\mathcal{S}^{d-1})$ and $\xi \in C_0^\infty\big(\mathbb{R}^d \times [0,\infty)\big)$. Hence (u, I) also satisfies the second equation in (2.3) and the proof is complete. □

4.2. **Nonrelativistic Limit.** Finally, we analyse the nonrelativistic limit of (u^c, I^c) obtained in Theorem 4.1 as $c \to \infty$, that is, the second part of Theorem 2.1. More precisely, we will prove in this subsection the following result.

Theorem 4.2. *Under the conditions of Theorem 2.1, there is a bounded measurable function u^∞ and a subsequence (denoted in the same way) of the set $\{(u^c, I^c)\}_{c>1}$ such that as $c \to \infty$,*

$$\begin{aligned} u^c &\to u^\infty && \textit{in} \quad L^1_{loc}\big(\mathbb{R}^d \times [0,\infty)\big), \\ I^c &\overset{*}{\rightharpoonup} \int_{-\infty}^0 e^s B(u^\infty(x+s\omega,t))\,ds, && \textit{in} \quad L^\infty\big(\mathbb{R}^d \times [0,\infty) \times \mathcal{S}^{d-1}\big). \end{aligned}$$

Moreover, the function u^∞ is an entropy solution to the Cauchy problem of (1.2) with initial data u_0 and satisfies

$$\begin{aligned} u^\infty(x,t) \in [a,b] \quad & a.e.\, (x,t) \in \mathbb{R}^d \times [0,\infty), \\ |u^\infty(.,t)|_{BV(\mathbb{R}^d)} &\le |u_0|_{BV(\mathbb{R}^d)}, \\ |u^\infty(.,t) - u^\infty(.,t_1)|_{L^1(|x|\le R)} &\le C_R|t-t_1| \end{aligned} \tag{4.9}$$

for all $t, t_1 \ge 0$ and all $R > 0$. In (4.9), the generic constant C_R depends on R.

Proof. On the basis of the estimates (4.3)-(4.5) on (u^c, I^c) constructed in Theorem 4.1, we follow the proof of Theorem 4.1 to get analogues of (4.6) and (4.7). Namely, there exist bounded measurable functions u^∞, I^∞ and a subsequence (denoted in the same way) of the set $\{(u^c, I^c)\}_{c>1}$ such that as $c \to \infty$,

$$\begin{aligned} u^c &\to u^\infty \quad \text{in} \quad L^1_{loc}\big(\mathbb{R}^d \times [0,\infty)\big), \\ I^c &\overset{*}{\rightharpoonup} I^\infty \quad \text{in} \quad L^\infty\big(\mathbb{R}^d \times [0,\infty) \times \mathcal{S}^{d-1}\big). \end{aligned}$$

The strong convergence and the estimates on u^c in Theorem 4.1 ensures the estimates in (4.9). Moreover, it follows from the definition of entropy solutions (2.3) that (u^∞, I^∞) satisfies

$$\begin{aligned} &\int_{\mathbb{R}^d\times[0,\infty)} \big[\eta(u^\infty)\psi_t + q(u^\infty)\cdot\nabla\psi\big]\,dxdt \\ &\ge -\int_{\mathbb{R}^d} \eta(u_0)\psi(.,0)\,dx - \int_{\mathbb{R}^d\times[0,\infty)} \eta'(u^\infty)\psi\left(\int_{\mathcal{S}^{d-1}} \big(I^\infty(.,\omega) - B(u^\infty)\big)\,d\omega\right)dxdt, \\ &\int_{\mathbb{R}^d\times[0,\infty)} I^\infty(.,\omega)\omega\cdot\nabla\xi\,dxdt = -\int_{\mathbb{R}^d\times[0,\infty)} \Big(B(u^\infty) - I^\infty(.,\omega)\Big)\psi\,dxdt \end{aligned} \tag{4.10}$$

for all nonnegative $\psi \in C_0^\infty\big(\mathbb{R}^d\times[0,\infty)\big)$, all entropy pairs (η, q), and all $\xi \in C_0^\infty\big(\mathbb{R}^d\times(0,\infty)\big)$.

The equality in (4.10) indicates that I^∞ is a bounded weak solution of linear equation $\omega\cdot\nabla I = B(u^\infty) - I$, which is unique. Define $\tilde{I}^\infty = \tilde{I}^\infty(x,t,\omega)$ as

$$\tilde{I}^\infty(x,t,\omega) = \int_{-\infty}^0 e^s B(u^\infty(x+s\omega,t))\,ds.$$

It is easy to check that $\tilde{I}^\infty$ is also a bounded weak solution of the linear equation $\omega \cdot \nabla I = B(u^\infty) - I$. Therefore, we have $I^\infty = \tilde{I}^\infty$ almost everywhere and

$$\begin{aligned}\int_{\mathcal{S}^{d-1}} I^\infty(x,t,\omega)\,d\omega &= \int_{\mathcal{S}^{d-1}} \tilde{I}^\infty(x,t,\omega)\,d\omega \\ &= \int_{\mathcal{S}^{d-1}} \int_{-\infty}^0 e^s B(u^\infty(x+s\omega,t))\,dsd\omega \\ &= \int_{\mathbb{R}^d} \frac{e^{-|x-y|}}{|x-y|^{d-1}} B(u^\infty(y,t))\,dy.\end{aligned}$$

Substituting this into the inequality in (4.10), we show that u^∞ is an entropy solution to the Cauchy problem of (1.2) with initial data u_0. This completes the proof. □

5. Appendix

Here we present a possibly non-equal size partition of the unit sphere. To this end, we recall that the unit sphere $\mathcal{S}^{d-1}$ is the image of the map

$$F(\phi_1,\phi_2,\ldots,\phi_{d-1}) := \begin{pmatrix} \sin\pi\phi_{d-1}\sin\pi\phi_{d-2}\cdots\sin\pi\phi_2\sin 2\pi\phi_1 \\ \sin\pi\phi_{d-1}\sin\pi\phi_{d-2}\cdots\sin\pi\phi_2\cos 2\pi\phi_1 \\ \sin\pi\phi_{d-1}\sin\pi\phi_{d-2}\cdots\cos\pi\phi_2 \\ \cdots\cdots\cdots \\ \sin\pi\phi_{d-1}\cos\pi\phi_{d-2} \\ \cos\pi\phi_{d-1} \end{pmatrix}$$

from $[0,1]^{d-1} \subset \mathbb{R}^{d-1}$ into $\mathbb{R}^d$. This map $F = F(\phi_1,\phi_2,\ldots,\phi_{d-1})$ is well-defined in the whole $\mathbb{R}^{d-1}$, is of C^∞, and is one-to-one in $(0,1)^{d-1}$.

For any positive integer N and any multi-index $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_{d-1})$ with $0 \le \alpha_j \le N-1$ for all j, define

$$\Omega_\alpha = F\left(\prod_{j=1}^{d-1}\left[\frac{\alpha_j}{N},\frac{\alpha_j+1}{N}\right]\right).$$

In this way, the unit sphere is partitioned into $L = N^{d-1}$ pieces. Because F is one-to-one in $(0,1)^{d-1}$, the partition has the following property

$$F\left(\prod_{j=1}^{d-1}\left(\frac{\alpha_j}{N},\frac{\alpha_j+1}{N}\right)\right)\bigcap F\left(\prod_{j=1}^{d-1}\left(\frac{\beta_j}{N},\frac{\beta_j+1}{N}\right)\right) = \emptyset \qquad \text{if} \quad \alpha \neq \beta.$$

Furthermore, we take

$$\omega_\alpha = F\left(\frac{2\alpha_1+1}{2N},\frac{2\alpha_2+1}{2N},\ldots,\frac{2\alpha_{d-1}+1}{2N}\right) \in \Omega_\alpha.$$

To estimate the size and diameter of Ω_α, we first notice that F and its first-order derivatives are all bounded by 2π. Then the determinant of the $(d\times d)$-matrix

$$D := [F, F_{\phi_1}, F_{\phi_2}, \ldots, F_{\phi_{d-1}}]$$

is bounded by a constant K_d depending only on the dimension d. Thus, we easily see that

$$|\Omega_\alpha| = \int_{\alpha_1 N^{-1}}^{(\alpha_1+1)N^{-1}} \int_{\alpha_2 N^{-1}}^{(\alpha_2+1)N^{-1}} \dots \int_{\alpha_{d-1} N^{-1}}^{(\alpha_{d-1}+1)N^{-1}} |\det D| d\phi_1 d\phi_2 \dots d\phi_{d-1} \leq K_d N^{-(d-1)},$$

$$\mathrm{diam}(\Omega_\alpha) \leq \pi d \sqrt{d} N^{-1}.$$

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