

On well-posedness of difference schemes for abstract elliptic problems in $L^p([0, T]; E)$ spaces

Allaberen Ashyralyev^{1*} and Sergey Piskarev^{2**}

¹ Department of Mathematics, Fatih University,
34500 Buyukcekmece, Istanbul, Turkey

² Scientific Research Computer Center, Moscow State University,
Vorobjevy Gory, Moscow 119899, Russia

ABSTRACT

This paper is devoted to the numerical analysis of abstract elliptic differential equations in $L^p([0, T]; E)$ spaces. The presentation uses general approximation scheme and is based on C_0 -semigroup theory and a functional analysis approach. For the solutions of difference scheme of the second order accuracy, the almost coercive inequality in $L^p_{\tau_n}([0, T]; E_n)$ spaces with the factor $\min\{|\ln \frac{1}{\tau_n}|, 1 + |\ln \|B_n\|_{B(E_n)}|\}$ is obtained. In the case of UMD space E_n we establish a coercive inequality for the same scheme in $L^p_{\tau_n}([0, T]; E_n)$ under the condition of R-boundedness.

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*Corresponding author. Department of Mathematics, Fatih University, Buyukcekmece, 34500 Istanbul, Turkey. Fax: +90-212-889-09-32. E-mail: aashyr@fatih.edu.tr

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1 Introduction

The importance of coercive (maximal regularity, well-posedness) inequalities is well-known [6], [7], [24]. The coercive inequality in the discrete case have at least two advantages. First, even if the continuous coercive inequality does not hold for a certain problem, it is possible to get discrete inequalities, but with the coercive constant depending on the discretization step size. Second, in the discrete case the coercive inequality can be formulated in different ways and validity of the inequality may depend on this formulation. In a recent paper [5], we have discussed the well-posedness in $L^p_n([0, T]; E_n)$ spaces of the difference schemes for abstract parabolic problems.

In present paper, we will investigate the well-posedness in $L^p_n([0, T]; E_n)$ spaces of the difference schemes for abstract elliptic problems. The paper is organized as follows. In Section 2 we give description of the general approximation scheme, where discretisation is considered, introduce the notion of positive operators and give an information on operator-function constructions using Dunford

integral. Section 3 gives us orientation in discrete coercive inequalities and draws for the reader almost complete picture of discrete coercive inequalities for abstract elliptic problems obtained during last 20 years. As we will see in Subsections 3.2 - 3.3 there are complete pictures of existence of coercive inequalities in $C([0, T]; E)$ and $C^{\beta, \gamma}([0, T]; E)$ spaces and it's discrete analogues. We prove also that if coercive inequalities in $C([0, T]; E)$ holds then either A is bounded or E contains a closed subspace which is isomorphic to c_0 . In the mean time the situation in Subsection 3.4 is not complete in the sense that in general one got only extrapolation Theorem 3.12 and one could get coercive inequality just for interpolation space $E_{n, \alpha, p}$ as in Theorem 3.13. Section 4 presents the main results of the paper, where we are completing the picture in $L^p_{\tau_n}([0, T]; E_n)$ spaces. First, we establish in $L^p_{\tau_n}([0, T]; E_n)$ for general Banach spaces E_n the coercive inequalities with the logarithm, as it is shown also for the case of $C_{\tau_n}([0, T]; E_n)$ in Section 3. Secondly, we concern to UMD space case. A Banach space E has the UMD-property, whenever Hilbert transform $Hf(t) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{1}{t-s} f(s) ds$ extends to a bounded operator on $L^p(\mathbb{R}; E)$ for some (all) $p \in (1, \infty)$. It is well known, that all subspaces and quotient spaces of $L^q(\Omega, \mu)$ with $1 < q < \infty$ have this property.

A set $\mathcal{T} \subset B(E)$ is called R -bounded, if there is a constant $C < \infty$, such that for all $Q_1, \dots, Q_k \in \mathcal{T}$ and $x_1, \dots, x_k \in E, k \in \mathbb{N}$,

$$(1.1) \quad \int_0^1 \left\| \sum_{j=0}^k r_j(\xi) Q_j(x_j) \right\| d\xi \leq C \int_0^1 \left\| \sum_{j=0}^k r_j(\xi) x_j \right\| d\xi,$$

where $\{r_j(\cdot)\}$ is a sequence of independent symmetric $\{-1, 1\}$ -valued random variables, e. g. the Rademacher functions $r_j(t) = \text{sign}(\sin(2^j \pi t))$ on $[0, 1]$.

In [7] the coercive well-posedness in $L^p([0, T]; E)$ of the problem

$$(1.2) \quad u''(t) = Au(t) + f(t), \quad t \in [0, T], \quad u(0) = u^0, \quad u(T) = u^T,$$

has been considered under condition of positivity of operator A . In such case operator $-\sqrt{A}$ generates analytic C_0 -semigroup.

If $-\sqrt{A}$ generates an analytic exponentially decreasing C_0 -semigroup $\{\exp(-z\sqrt{A}) : |\arg(z)| \leq \delta\}$, on a Banach space E , then the following three sets are bounded in the operator norm

- (i) $\{\lambda(\lambda I \pm \sqrt{A})^{-1} : \lambda \in i\mathbb{R}, \lambda \neq 0\}$;
- (ii) $\{\exp(-t\sqrt{A}), t\sqrt{A} \exp(-t\sqrt{A}) : t > 0\}$;
- (iii) $\{\exp(-z\sqrt{A}) : |\arg z| \leq \delta\}$.

In the mean time if the space E is UMD space, then [20],[33],[34] the coercive well-posedness in $L^p([0, T]; E)$ of the problem

$$(1.3) \quad v'(t) = -\sqrt{A}v(t) + g(t), \quad v(0) = v^0, \quad t \geq 0,$$

holds iff one of the sets (i)-(iii) is R -bounded. Moreover, coercive well-posedness of (1.2) in $L^p([0, T]; E)$ with $u(0) = u(T) = 0$ implies [13] that (1.3) is coercive well-posed in $L^p([0, T]; E)$ and therefore the set $\{s(sI + A)^{-1} : s > 0\}$ is R -bounded.

So as our second aim to get a discrete coercive inequality for central difference scheme for (1.2) under condition of R -boundedness of the set $\{s(sI_n + A_n)^{-1} : s > 0\}$. In this paper we consider the difference scheme of second order accuracy only.

2 Discretization of positive operators and semigroups

This section contains the notions we use and operate in Sections 3 and 4.

2.1 General approximation scheme

Let $B(E)$ denote the Banach algebra of all linear bounded operators on a complex Banach space E . The set of all linear closed densely defined operators in E will be denoted by $\mathcal{C}(E)$. We denote by $\sigma(B)$ the spectrum of the operator B , by $\rho(B)$ the resolvent set of B . The general approximation scheme, due to [18], [30], [31], [32] can be described in the following way. Let E_n and E be Banach spaces and $\{p_n\}$ be a sequence of linear bounded operators $p_n : E \rightarrow E_n, p_n \in B(E, E_n), n \in \mathbb{N} = \{1, 2, \dots\}$, with the property:

$$\|p_n x\|_{E_n} \rightarrow \|x\|_E \text{ as } n \rightarrow \infty \text{ for any } x \in E.$$

Definition 2.1 *The sequence of elements $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, is said to be \mathcal{P} -convergent to $x \in E$ iff $\|x_n - p_n x\|_{E_n} \rightarrow 0$ as $n \rightarrow \infty$ and we write this $x_n \rightarrow x$.*

Definition 2.2 *The sequence of bounded linear operators $B_n \in B(E_n), n \in \mathbb{N}$, is said to be \mathcal{PP} -convergent to the bounded linear operator $B \in B(E)$ if for every $x \in E$ and for every sequence $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, such that $x_n \rightarrow x$ one has $B_n x_n \rightarrow Bx$. We write then $B_n \rightarrow B$.*

For general examples of notions of \mathcal{P} -convergence see [31].

Remark 2.1 *If we put $E_n = E$ and $p_n = I$ for each $n \in \mathbb{N}$, where I is the identity operator on E , then Definition 2.1 leads to the traditional pointwise convergent bounded linear operators which we denote by $B_n \rightarrow B$.*

In the case of unbounded operators, and we know in general infinitesimal generators are unbounded, we consider the notion of *compatibility*.

Definition 2.3 *The sequence of closed linear operators $\{A_n\}, A_n \in \mathcal{C}(E_n), n \in \mathbb{N}$, are said to be compatible with a closed linear operator $A \in \mathcal{C}(E)$ iff for each $x \in D(A)$ there is a sequence $\{x_n\}, x_n \in D(A_n) \subseteq E_n, n \in \mathbb{N}$, such that $x_n \rightarrow x$ and $A_n x_n \rightarrow Ax$. We write (A_n, A) are compatible.*

Usually in practice Banach spaces E_n are finite dimensional, although, in general, say for the case of a closed operator A , we have $\dim E_n \rightarrow \infty$ and $\|A_n\|_{B(E_n)} \rightarrow \infty$ as $n \rightarrow \infty$.

For analytic C_0 -semigroups the following ABC Theorem holds

Theorem 2.1 [19], [23] *Let operators A and A_n generate analytic C_0 -semigroups. The following conditions (A) and (B_1) are equivalent to condition (C_1) .*

(A) *Compatibility. There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ such that the resolvents converge*

$$(\lambda I_n - A_n)^{-1} \rightarrow (\lambda I - A)^{-1};$$

(B_1) *Stability. There are some constants $M \geq 1$ and ω such that*

$$\|(\lambda I_n - A_n)^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \text{ Re } \lambda > \omega, n \in \mathbb{N};$$

(C_1) *Convergence. For any finite $\mu > 0$ and some $0 < \theta < \frac{\pi}{2}$ we have*

$$\max_{\eta \in \Sigma(\theta, \mu)} \|\exp(\eta A_n) u_n^0 - p_n \exp(\eta A) u^0\| \rightarrow 0$$

as $n \rightarrow \infty$ whenever $u_n^0 \rightarrow u^0$. Here $\Sigma(\theta, \mu) = \{z \in \Sigma(\theta) : |z| \leq \mu\}$, and $\Sigma(\theta) = \{z \in \mathbb{C} : |\arg z| \leq \theta\}$.

2.2 Positive operators

An operator A acting in a Banach space E and having a dense domain $D(A)$ is called positive if the operator $\lambda I + A$ has bounded in E inverse and the estimate

$$(2.1) \quad \|(\lambda I + A)^{-1}\| \leq \frac{M_1}{1 + \lambda} \text{ for any } \lambda \geq 0$$

holds for some $1 \leq M_1 < \infty$. Note that from this estimate, evidently, it follows that the operator $\lambda I + A$ has bounded in E inverse for all complex numbers

$$\lambda = \sigma + i\tau \in G_\varepsilon^+ = G_\varepsilon^+(M_1), \quad 0 < \varepsilon < 1,$$

such that

$$|\tau| \leq \frac{1 - \varepsilon}{M_1}(1 + \sigma) \text{ for } \sigma \geq 0$$

and the estimate

$$\|(\lambda I + A)^{-1}\| \leq M_2 \varepsilon^{-1}(1 + |\lambda|)^{-1}$$

holds for some $M_2 \in [1, \infty)$ and $\varepsilon \in (0, 1)$. It means [7] that the spectrum $\sigma(A)$ of A is the subset of set $G_\varepsilon^- = -G_\varepsilon^+$ and inside of G_ε^- and on its boundary ∂G_ε^- the resolvent $(\lambda I + A)^{-1}$ is subject to the bound

$$(2.2) \quad \|(\lambda I + A)^{-1}\| \leq M_2 \varepsilon^{-1}(1 + |\lambda|)^{-1}, \quad \lambda \in \partial G_\varepsilon^-.$$

Let $\psi(z)$ be an analytic function on the neighborhood of $\sigma(A)$, and suppose that $\psi(\cdot)$ satisfies the estimate

$$(1 + |z|)^\alpha |\psi(z)| \leq M_3$$

for some $0 < \alpha < +\infty$, $1 \leq M_3 < +\infty$. Then the operator Cauchy-Riesz integral

$$\psi(A) = \frac{1}{2\pi i} \int_{\partial G_\varepsilon^-} \psi(z)(zI + A)^{-1} dz$$

converges in the operator norm and defines [21] a bounded linear operator $\psi(A)$, a function of A .

In particular, the function $\psi(A) = z^{-\alpha}$ defines a bounded operator $A^{-\alpha}$ whenever $\alpha > 0$. The positive powers $A^\alpha = (A^{-\alpha})^{-1}$, $\alpha > 0$, of the operator A are defined. The operators A^α , $\alpha > 0$, are unbounded, and their domains $D(A^\alpha)$ are dense in E if A is unbounded. One has the continuous embeddings $D(A^\alpha) \subseteq D(A^\beta)$ if $\beta \leq \alpha$.

It is known (see, for example [26]) that the operator $A^{1/2}$ has better spectral properties than the positive operator A . In particular, the operator $\lambda I + \sqrt{A}$ has a bounded inverse for any complex number λ with $\operatorname{Re} \lambda \geq 0$, and the estimate

$$\|(\lambda I + \sqrt{A})^{-1}\| \leq M(|\lambda| + 1)^{-1}, \quad \operatorname{Re} \lambda \geq 0,$$

holds for some $M \geq 1$. Thus, the operator $-\sqrt{A}$ generates [26] analytic C_0 -semigroup on Banach space E , i.e. the following estimates hold:

$$(2.3) \quad \|e^{-t\sqrt{A}}\| \leq M e^{-\delta t}, \quad t \|\sqrt{A} e^{-t\sqrt{A}}\| \leq M e^{-\delta t}, \quad t > 0, \quad \delta > 0.$$

As was mentioned in Introduction in case of coercive well-posedness of (1.2) in $L^p([0, T]; E)$ it follows that the set $\{\lambda(\lambda I + A)^{-1} : \lambda > 0\}$ is R-bounded. If the set $\{\lambda(\lambda I + A)^{-1} : \lambda > 0\}$ is R-bounded one can directly show that the sets $\{\lambda(\lambda I \pm \sqrt{A})^{-1} : \lambda \in \mathbb{R}, \lambda \neq 0\}$ are also R-bounded. Indeed

$$\sqrt{\lambda}(\sqrt{\lambda}iI + \sqrt{A})^{-1} = -\sqrt{\lambda}(\sqrt{\lambda}iI - \sqrt{A})(\lambda I + A)^{-1} = -\lambda i(\lambda I + A)^{-1} + \sqrt{\lambda}\sqrt{A}(\lambda I + A)^{-1}$$

for any $\lambda > 0$. Since $\{\lambda(\lambda I + A)^{-1} : \lambda > 0\}$ is R-bounded it is enough to prove that

$$\sqrt{\lambda}\sqrt{A}(\lambda I + A)^{-1} = \frac{1}{2\pi i} \int_{\partial G_\varepsilon^-} \frac{\sqrt{\lambda}\sqrt{\rho}}{(\lambda + \rho)} \frac{\rho(\rho I + A)^{-1} d\rho}{\rho}$$

is R-bounded. After change of variables $\rho = \eta\lambda$ one gets

$$\int_{\partial G_\varepsilon} \frac{\eta\lambda(\eta\lambda I + A)^{-1} d\eta}{(1 + \eta)\sqrt{\eta}}$$

and the statemnet follows from Corollary 2.14 [22].

3 Coercive inequalities

In this section the notions of coercive inequalities are introduced. We give the complete picture of historical overview for elliptic case.

3.1 Coercive inequalities for differential elliptic problems

In a Banach space E we consider the boundary value problem (1.2) with positive operator A and $f(\cdot)$ is some function from some function space. The problem (1.2) can be considered in different functional spaces. A function $u(\cdot)$ is called a solution in classical sense of the problem (1.2) if the following conditions are satisfied:

- i) $u(\cdot)$ is a twice continuously differentiable on the interval $[0, T]$. The derivative at the endpoints of the segment are understood as the appropriate unilateral derivatives;
- ii) the element $u(t)$ belongs to $D(A)$ for all $t \in [0, T]$, and the function $Au(\cdot)$ is continuous on the interval $[0, T]$;
- iii) $u(\cdot)$ satisfies the equation and boundary conditions (1.2).

A solution of problem (1.2) defined in this manner will be called as coercive well-posedness of problem (1.2) in the space $C([0, T]; E)$ of all continuous functions $\varphi(\cdot)$ defined on $[0, T]$ with values in E equipped with the norm $\|\varphi(\cdot)\|_{C([0, T]; E)} = \max_{0 \leq t \leq T} \|\varphi(t)\|_E$. The coercive well-posedness in $C([0, T]; E)$ of the boundary value problem (1.2) means that for solution $u(\cdot) \in C([0, T]; E)$ coercive inequality

$$(3.1) \quad \|u''(\cdot)\|_{C([0, T]; E)} + \|Au(\cdot)\|_{C([0, T]; E)} \leq M \left(\|f(\cdot)\|_{C([0, T]; E)} + \|Au^0\|_E + \|Au^T\|_E \right)$$

holds with some constant M , which is independent on $u^0, u^T, f(\cdot) \in C([0, T]; E)$. It turns out [26] that this positivity property (2.1) of the operator A in E is necessary condition of well-posedness of the boundary value problem (1.2) in $C([0, T]; E)$. One can ask: does the positivity of the operator A in E imply the well-posedness of boundary value problem (1.2)? In general case (see [26]) the answer is no, so the coercive inequality does not take place in $C([0, T]; E)$ for the boundary value problem (1.2). It is easy to show that $u(\cdot)$, defined on $[0, T]$ by the formula

$$(3.2) \quad u(t) = (I - e^{-2T\sqrt{A}})^{-1} \{ (e^{-t\sqrt{A}} - e^{-(2T-t)\sqrt{A}})u^0 + (e^{-(T-t)\sqrt{A}} - e^{-(T+t)\sqrt{A}})u^T \}$$

$$\begin{aligned}
& +(e^{-(T-t)\sqrt{A}} - e^{-(T+t)\sqrt{A}})(2\sqrt{A})^{-1} \int_0^T (e^{-(T-s)\sqrt{A}} - e^{-(T+s)\sqrt{A}})f(s)ds\} \\
& - (2\sqrt{A})^{-1} \int_0^T (e^{-|t-s|\sqrt{A}} - e^{-(t+s)\sqrt{A}})f(s)ds,
\end{aligned}$$

is an unique solution in $C([0, T]; E)$ of problem (1.2) if, for example, $u^0, u^T \in D(A^2)$ and $Af(\cdot) \in C([0, T]; E)$ or $f'(\cdot) \in C([0, T]; E)$.

Remark 3.1 We assumed here that the operator $(I - \exp(-2T\sqrt{A}))^{-1}$ exists and is bounded. Meanwhile, it is just enough to assume that $(I - \exp(-t\sqrt{A}))^{-1} \in B(E)$ holds for $t \geq t_0$ with some $t_0 > 0$. This assumption is satisfied in our case, since C_0 -semigroup $\exp(-t\sqrt{A})$ is exponentially decreasing. It follows [12] that $(I - \exp(-t\sqrt{A}))^{-1} \in B(E)$ for any $t > 0$.

We recall that the Cauchy problem (1.3) is coercive well-posed in the space $C([0, T]; E)$, then [16] operator \sqrt{A} has to be bounded or the space E contains a subspace isomorphic to c_0 . A similar situation one gets for problem (1.2).

Theorem 3.1 Let A be the positive operator on E . Assume that problem (1.2) is coercive well-posed in space $C([0, T]; E)$. Then either A is bounded or E contains a closed subspace which is isomorphic to c_0 .

Proof. The operator $-\sqrt{A}$ generates analytic C_0 -semigroup. So for unbounded operator \sqrt{A} one can find the sequences $\{t_j\}, \{y_j\}$ such that $\|y_j\| \leq 1, 0 \leq t_j \rightarrow 0$ and $\|t_j\sqrt{A}e^{-t_j\sqrt{A}}y_j\| \geq \frac{1}{2e} > 0$. Now follow [16] for the sequence $0 < t_j < \frac{1}{2}t_{j-1}, t_0 = 1$, we construct the sequence $x_j = \sqrt{A}e^{-t_j\sqrt{A}}y_j, \sqrt{A}e^{-\sqrt{A}}y_0 \neq 0$, where $\|y_j\| \leq 1$. The first, one has $\inf \|x_j\| > 0$. The second, we are going to show that there is a constant $\tilde{C} < \infty$ such that $\|x_0 \pm x_1 \pm x_2 \pm \dots \pm x_n\| \leq \tilde{C}$ for any n . Existence of such sequence $\{x_j\}$ is equivalent [16] to the statement of our theorem. From (3.1) for the case $T = 2$ and $u^0 = u^T = 0$ it follows that $\|Au(1)\| \leq C\|f(\cdot)\|_{C([0, 2]; E)}$ for any function $f(\cdot) \in C([0, 2]; E)$. Let us put $t = 1$ and consider in (3.2), after applying A to the function $u(\cdot)$, the term

$$\begin{aligned}
\frac{\sqrt{A}}{2} \int_0^2 e^{-|1-s|\sqrt{A}}f(s)ds &= \frac{\sqrt{A}}{2} \int_0^1 e^{-(1-s)\sqrt{A}}f(s)ds + \frac{\sqrt{A}}{2} \int_1^2 e^{-(s-1)\sqrt{A}}f(s)ds = \\
&= \sqrt{A} \int_0^1 e^{-(1-s)\sqrt{A}} \frac{f(s) + f(2-s)}{2} ds.
\end{aligned}$$

One can define now the function $f(\cdot)$ on $0 \leq s \leq 1$ exactly the same as in [16] and extend it as $f(2-s) = f(s)$ for $0 \leq s \leq 1$. For such choice of $f(\cdot)$ as it is proved in [16] one has

$$\|\sqrt{A}(e^{-t\sqrt{A}} * f(t))(1) - (\epsilon_0 x_0 + \epsilon_1 x_1 + \dots + \epsilon_n x_n)\| \leq \tilde{C} < \infty$$

for $|\epsilon_j| = 1, \epsilon_j \in \mathbb{R}$, and therefore

$$\begin{aligned}
\|\epsilon_0 x_0 + \epsilon_1 x_1 + \dots + \epsilon_n x_n\| &\leq \|\sqrt{A}(e^{-t\sqrt{A}} * f(t))(1) - (\epsilon_0 x_0 + \epsilon_1 x_1 + \dots + \epsilon_n x_n)\| + \\
&+ \|Au(1) - \sqrt{A}(e^{-t\sqrt{A}} * f(t))(1)\| + \|Au(1)\| \leq \tilde{C} < \infty.
\end{aligned}$$

In this way we constructed the sequence $\{x_j\}$ such that $\inf \|x_j\| > 0$ and $\|x_0 \pm x_1 \pm x_2 \dots \pm x_n\| \leq \tilde{C} < \infty$. Theorem is proved.

Consider $C^{\beta, \gamma}([0, T]; E), 0 \leq \gamma \leq \beta, 0 < \beta < 1$, the Banach space obtained by completion of the set of smooth E -valued functions $\varphi(\cdot)$ on $[0, T]$ in the norm

$$\|\varphi(\cdot)\|_{C^{\beta, \gamma}([0, T]; E)} = \max_{0 \leq t \leq T} \|\varphi(t)\|_E + \sup_{0 \leq t < t + \tau \leq T} \frac{(t + \tau)^\gamma (T - t)^\gamma \|\varphi(t + \tau) - \varphi(t)\|_E}{\tau^\beta}.$$

Theorem 3.2 [1], [7] *Let A be the positive operator in a Banach space E and $f(\cdot) \in C^{\beta,\gamma}([0, T]; E)$, $0 \leq \gamma \leq \beta$, $0 < \beta < 1$. Then the solution of the boundary value problem (1.2) belongs $u(\cdot) \in C^{\beta,\gamma}([0, T]; E)$ and the following coercive inequalities hold:*

$$\|u''(\cdot)\|_{C([0,T];E_{\beta-\gamma})} \leq M \left(\|Au^0 + f(0)\|_{E_{\beta-\gamma}} + \|Au^T + f(T)\|_{E_{\beta-\gamma}} + \beta^{-1}(1-\beta)^{-1} \|f\|_{C^{\beta,\gamma}([0,T];E)} \right)$$

for $Au^0 + f(0) \in E_{\beta-\gamma}$, $Au^T + f(T) \in E_{\beta-\gamma}$,

$$\begin{aligned} & \|u''(\cdot)\|_{C^{\beta,\gamma}([0,T];E)} + \|Au(\cdot)\|_{C^{\beta,\gamma}([0,T];E)} \\ & \leq M \left(|Au^0 + f(0)|_0^{\beta,\gamma} + |Au^T + f(T)|_0^{\beta,\gamma} + \beta^{-1}(1-\beta)^{-1} \|f(\cdot)\|_{C^{\beta,\gamma}([0,T];E)} \right) \end{aligned}$$

for $Au^0 + f(0) \in E_0^{\beta,\gamma}$, $Au^T + f(T) \in E_0^{\beta,\gamma}$, where M is independent on β, γ, u^0, u^T and $f(\cdot)$. Here, $|w|_0^{\beta,\gamma}$ denotes norm of a Banach space $E_0^{\beta,\gamma}$, which consists of those $w \in E$ for which the norm

$$|w|_0^{\beta,\gamma} = \max_{0 \leq z \leq T} \|e^{-z\sqrt{A}}w\|_E + \sup_{0 \leq z < z+\tau \leq T} \tau^{-\beta}(z+\tau)^\gamma(T-z)^\gamma \|(e^{-(z+\tau)\sqrt{A}} - e^{-z\sqrt{A}})w\|_E$$

is finite and a Banach space $E_\alpha = E_\alpha(\sqrt{A}, E)$, $0 < \alpha < 1$, consists of those $v \in E$ for which the norm $\|v\|_{E_\alpha} = \sup_{z>0} z^{1-\alpha} \|\sqrt{A} \exp(-z\sqrt{A})v\|_E + \|v\|_E$ is finite.

Consider $C_{0T}^{\beta,\gamma}([0, T], E_{\alpha-\beta})$, $0 \leq \gamma \leq \beta \leq \alpha$, $0 < \alpha < 1$. To these there correspond the spaces of traces $E_{\alpha-\beta}^{\beta,\gamma}$, which consist of elements $w \in E$ for which the norm

$$|w|_{\alpha-\beta}^{\beta,\gamma} = \max_{0 \leq z \leq 1} \|e^{-z\sqrt{A}}w\|_{E_{\alpha-\beta}} + \sup_{0 \leq z < z+\tau \leq 1} \tau^{-\beta}(z+\tau)^\gamma(1-z)^\gamma \|(e^{-(z+\tau)\sqrt{A}} - e^{-z\sqrt{A}})w\|_{E_{\alpha-\beta}}$$

is finite.

Theorem 3.3 [1] *Let A is the positive operator in a Banach space E , $Au^0 + f(0) \in E_{\alpha-\beta}^{\beta,\gamma}$, $Au^T + f(T) \in E_{\alpha-\beta}^{\beta,\gamma}$ and $f(\cdot) \in C^{\beta,\gamma}([0, T]; E_{\alpha-\beta})$, $0 \leq \gamma \leq \beta \leq \alpha$, $0 < \alpha < 1$. Then for the solution $u(\cdot)$ in $C^{\beta,\gamma}([0, T]; E_{\alpha-\beta})$ of the boundary value problem (1.2) the coercive inequality*

$$\begin{aligned} & \|u''(\cdot)\|_{C^{\beta,\gamma}([0,T];E_{\alpha-\beta})} + \|Au(\cdot)\|_{C^{\beta,\gamma}([0,T];E_{\alpha-\beta})} + \|u''(\cdot)\|_{C([0,T];E_{\alpha-\beta}^{\beta,\gamma})} \\ & \leq M \left(|Au^0 + f(0)|_{\alpha-\beta}^{\beta,\gamma} + |Au^T + f(T)|_{\alpha-\beta}^{\beta,\gamma} + \alpha^{-1}(1-\alpha)^{-1} \|f(\cdot)\|_{C^{\beta,\gamma}([0,T];E_{\alpha-\beta})} \right) \end{aligned}$$

holds, where M is independent on $\alpha, \beta, \gamma, u^0, u^T$ and $f(\cdot)$.

Theorem 3.4 [1] *Let A is the positive operator in a Banach space E , $Au^0 + f(0) \in E_{\alpha-\gamma}$, $Au^T + f(T) \in E_{\alpha-\beta}$ and $f(\cdot) \in C^{\beta,\gamma}([0, T]; E_{\alpha-\gamma})$, $0 \leq \gamma \leq \beta \leq \alpha$, $0 < \alpha < 1$. Then for the solution $u(\cdot)$ in $C^{\beta,\gamma}([0, T]; E_{\alpha-\beta})$ of the boundary value problem (1.2) the coercive inequality*

$$\begin{aligned} & \|u''(\cdot)\|_{C^{\beta,\gamma}([0,T];E_{\alpha-\beta})} + \|Au(\cdot)\|_{C^{\beta,\gamma}([0,T];E_{\alpha-\beta})} + \|u''(\cdot)\|_{C([0,T];E_{\alpha-\gamma})} \\ & \leq M\alpha^{-1}(1-\alpha)^{-1} \left(\|Au^0 + f(0)\|_{E_{\alpha-\gamma}} + \|Au^T + f(T)\|_{E_{\alpha-\gamma}} + \|f(\cdot)\|_{C^{\beta,\gamma}([0,T];E_{\alpha-\beta})} \right) \end{aligned}$$

holds, where M is independent on $\alpha, \beta, \gamma, u^0, u^T$ and $f(\cdot)$.

Let us consider the problem (1.2) in the spaces $L^p([0, T]; E)$, $1 \leq p < \infty$, of all strongly measurable E -valued functions $v(\cdot)$ on $[0, T]$ with the norm $\|v\|_{L^p([0, T]; E)} = \left(\int_0^T \|v(t)\|_E^p dt\right)^{\frac{1}{p}}$. A function $v(\cdot)$ is said to be absolutely continuous if it has a derivative $v'(t)$ for almost every t such that $v'(\cdot) \in L^1([0, T]; E)$, and if the Newton-Leibniz formula $v(t) - v(\tau) = \int_{\tau}^t v'(s) ds$ holds for all $t, \tau \in [0, T]$. Here the integral is understood in the sense of Bochner. A function $u(\cdot)$ is said to be a solution of the problem (1.2) in $L^p([0, T]; E)$ if it is absolutely continuous, the functions $u''(\cdot)$ and $Au(\cdot)$ belong to $L^p([0, T]; E)$, equation (1.2) is satisfied for almost every t , and $u(0) = u^0$, $u(T) = u^T$. From this definition it follows that a necessary condition for the solvability of problem (1.2) in $L^p([0, T]; E)$ is that $f(\cdot) \in L^p([0, T]; E)$. One can show that in certain cases this condition is also sufficient for the solvability of problem (1.2). As concerns the boundary elements, in contrast to the situation considered earlier, from the solvability of problem (1.2) in $L^p([0, T]; E)$ it follows only that $u^0, u^T \in E$. From the unique solvability of (1.2) it follows that the operator $u(t; f(t), u^0, u^T)$ is bounded in $L^p([0, T]; E)$ and one has coercive inequality

$$\|u''\|_{L^p([0, T]; E)} + \|Au(t)\|_{L^p([0, T]; E)} \leq M_C \left(\|f(\cdot)\|_{L^p([0, T]; E)} + \|Au^0\|_E + \|Au^T\|_E \right),$$

where $1 \leq M_C < +\infty$ is independent on u^0, u^T and $f(\cdot)$. From that one can obtain the positivity of A under the stronger assumption that the operator A^{-1} is compact in E [28].

Theorem 3.5 [28] *Let A is the positive operator in a Banach space E . Suppose the problem (1.2) is coercive well-posed in $L^{p_0}([0, T]; E)$ for some $1 < p_0 < \infty$. Then it is coercive well-posed in $L^p([0, T]; E)$ for any $1 < p < \infty$ and the coercivity inequality holds:*

$$\begin{aligned} & \|u''(\cdot)\|_{L^p([0, T]; E)} + \|Au(\cdot)\|_{L^p([0, T]; E)} + \|u(\cdot)\|_{C(E_{1-1/p, p}(\sqrt{A}, D(\sqrt{A})))} \\ & \leq \frac{M(p_0)p^2}{p-1} \|f(\cdot)\|_{L^p([0, T]; E)} + M \left(\|u^0\|_{E_{1-1/p, p}(\sqrt{A}, D(\sqrt{A}))} + \|u^T\|_{E_{1-1/p, p}(\sqrt{A}, D(\sqrt{A}))} \right), \end{aligned}$$

where $M(p_0)$ and M are independent on p, u^0, u^T and $f(\cdot)$.

Theorem 3.6 [14] *Let $1 < p < \infty$ and $0 < \alpha < 1$. Suppose that A is the positive operator in a Banach space E . Then problem (1.2) is coercive well-posed in $L^p([0, T]; E_{\alpha, p})$ and the coercivity inequality holds:*

$$\begin{aligned} & \|u''(\cdot)\|_{L^p([0, T]; E_{\alpha, p})} + \|Au(\cdot)\|_{L^p([0, T]; E_{\alpha, p})} \\ & \leq \frac{M}{\alpha(1-\alpha)} \|f(\cdot)\|_{L^p([0, T]; E_{\alpha, p})} + M \left(\|Au^0\|_{E_{\alpha, p}} + \|Au^T\|_{E_{\alpha, p}} \right), \end{aligned}$$

where M is independent on α, p, u^0, u^T and $f(\cdot)$.

From these theorems it follows

Theorem 3.7 *Let $1 < p, q < \infty$ and $0 < \alpha < 1$. Suppose that A is the positive operator in a Banach space E . Then problem (1.2) is coercive well-posed in $L^p([0, T]; E_{\alpha, q})$ and the coercivity inequality holds:*

$$\begin{aligned} & \|u''(\cdot)\|_{L^p([0, T]; E_{\alpha, q})} + \|Au(\cdot)\|_{L^p([0, T]; E_{\alpha, q})} \leq \\ & \leq \frac{M(q)}{\alpha(1-\alpha)} \|f(\cdot)\|_{L^p([0, T]; E_{\alpha, q})} + M \left(\|Au^0\|_{E_{\alpha, q}} + \|Au^T\|_{E_{\alpha, q}} \right), \end{aligned}$$

where $M(q)$ and M are independent on α , p , u^0 , u^T and $f(\cdot)$. Here a Banach space $E_{\alpha,q} = E_{\alpha,q}(\sqrt{A}, E)$, $0 < \alpha < 1, 1 < q < \infty$, consists of those $v \in E$ for which the norm

$$\|v\|_{E_{\alpha,q}} = \left(\int_0^\infty \lambda^{1-\alpha} \|\sqrt{A} \exp(-\lambda\sqrt{A})v\|_E^q \frac{d\lambda}{\lambda} \right)^{1/q}$$

is finite.

3.2 Coercive inequalities in $C_{\tau_n}([0, T]; E_n)$ spaces

The semidiscrete approximation of (1.2) are the boundary value problems in Banach spaces E_n :

$$(3.3) \quad u_n''(t) = A_n u_n(t) + f_n(t), \quad t \in [0, T], \quad u_n(0) = u_n^0, \quad u_n(T) = u_n^T,$$

with strongly positive operators A_n , A_n and A are compatible, $u_n^0 \rightarrow u^0$, $u_n^T \rightarrow u^T$ and $f_n(\cdot) \rightarrow f(\cdot)$ in appropriate sense of \mathcal{P} -convergence.

The investigation of discretization of the boundary value problem (1.2) with respect to one variable t only includes, in view of the arbitrariness of A , the general approximation method in applications, if the differential operator in other variables A , is replaced by the approximation operators A_n which are acting in a Banach spaces E_n and are uniformly positive in n (see Theorem ABC). Therefore Theorems 3.2 - 3.7 permit us to establish the coercive stability inequalities for the solutions of the boundary value problem (3.3) under the condition of positivity of A_n uniformly in n .

Here we are going to describe the discretization of (3.3) in variable t . Let us denote $\bar{U}_n = (U_n^0, \dots, U_n^K)$ and $A_n \bar{U}_n = \{A_n U_n^k\}_{k=1}^{K-1}$. One of the simplest difference scheme is

$$(3.4) \quad \frac{U_n^{k+1} - 2U_n^k + U_n^{k-1}}{\tau_n^2} = A_n U_n^k + \varphi_n^k, \quad k \in \{1, \dots, [\frac{T}{\tau_n}] - 1\}, \quad U_n^0 = u_n^0, \quad U_n^K = u_n^T,$$

where, for example, in the case of $f_n(\cdot) \in C([0, T]; E_n)$ one can put

$$\varphi_n^k = f_n(k\tau_n), \quad k \in \{1, \dots, K-1\}, \quad K = [\frac{T}{\tau_n}],$$

and in the case $f_n(\cdot) \in L^1([0, T]; E_n)$ one can put

$$\varphi_n^k = \frac{1}{2\tau_n} \int_{t_{k-1}}^{t_{k+1}} f_n(s) ds, \quad t_k = k\tau_n, \quad k \in \{1, \dots, K-1\}.$$

We have that

$$(3.5) \quad U_n^k = (I - R_n^{2K})^{-1} (R_n^k - R_n^{2K-k}) \left(u_n^0 + \sum_{j=1}^{K-1} B_n^{-1} R_n^j (I + \tau_n B_n) (2 + \tau_n B_n)^{-1} \varphi_n^j \tau_n \right) \\ - \sum_{j=1}^{K-1} B_n^{-1} R_n^{|k-j|} (I + \tau_n B_n) (2 + \tau_n B_n)^{-1} \varphi_n^j \tau_n \\ + (I - R_n^{2K})^{-1} (R_n^{K-k} - R_n^{K+k}) \left(u_n^T + \sum_{j=1}^{K-1} B_n^{-1} R_n^{K-j} (I + \tau_n B_n) (2 + \tau_n B_n)^{-1} \varphi_n^j \tau_n \right),$$

where

$$(3.6) \quad R_n = (I_n + \tau_n B_n)^{-1}, \quad B_n = \frac{A_n \tau_n + \sqrt{A_n} \sqrt{\tau_n^2 A_n + 4}}{2}.$$

Denote by $C_{\tau_n}([0, T]; E_n)$ the space of the elements $\bar{\varphi}_n = \{\varphi_n^k\}_{k=1}^{K-1}$ such that $\varphi_n^k \in E_n, k \in \{1, \dots, K-1\}$, with the norm $\|\bar{\varphi}_n\|_{C_{\tau_n}([0, T]; E_n)} = \max_{1 \leq k \leq K-1} \|\varphi_n^k\|_{E_n}$.

Theorem 3.8 [27] *Let A_n be strongly positive operator in a Banach space E_n uniformly in n . Then the problem (3.4) is stable in the space $C_{\tau_n}([0, T]; E_n)$, i.e.*

$$\|\bar{U}_n\|_{C_{\tau_n}([0, T]; E_n)} \leq M \left(\|\bar{\varphi}_n\|_{C_{\tau_n}([0, T]; E_n)} + \|u_n^0\|_{E_n} + \|u_n^T\|_{E_n} \right),$$

where M is independent on n, τ_n, u_n^0, u_n^T and $\bar{\varphi}_n$.

In general problem (3.4) is not coercive well-posed in $C_{\tau_n}([0, T]; E_n)$ space.

Theorem 3.9 [27] *Let A_n be strongly positive operator in a Banach space E_n . Then the problem (3.4) is almost coercive stable in the space $C_{\tau_n}([0, T]; E_n)$, i.e.*

$$\|A_n \bar{U}_n\|_{C_{\tau_n}([0, T]; E_n)} \leq M \left(\|A_n u_n^0\|_{E_n} + \|A_n u_n^T\|_{E_n} + \min \left(|\ln(1/\tau_n)|, 1 + \left| \ln \|B_n\|_{B(E_n)} \right| \right) \|\bar{\varphi}_n\|_{C_{\tau_n}([0, T]; E_n)} \right),$$

where M is independent on n, τ_n, u_n^0, u_n^T and $\bar{\varphi}_n$.

3.3 Coercive inequalities in $C_{\tau_n}^{\beta, \gamma}([0, T]; E_n)$ spaces

Denote by $C_{\tau_n}^{\beta, \gamma}([0, T]; E_n)$ for $0 \leq \gamma \leq \beta < 1$ the space of the elements $\bar{\varphi}_n = \{\varphi_n^k\}_{k=1}^{K-1}$ with the norm

$$\|\bar{\varphi}_n\|_{C_{\tau_n}^{\beta, \gamma}([0, T]; E_n)} = \max_{1 \leq k \leq K-1} \|\varphi_n^k\|_{E_n} + \max_{1 \leq k < k+l \leq K-1} \|\varphi_n^{k+l} - \varphi_n^k\|_{E_n} (\tau_n(k+l))^\gamma (T - \tau_n k)^\gamma (l\tau_n)^{-\beta}.$$

Theorem 3.10 [2], [3] *Let A_n be strongly positive operator in a Banach space E_n and $A_n u_n^0 + \varphi_n^1, A_n u_n^T + \varphi_n^{K-1} \in \widetilde{E}_n^{\beta, \gamma}$. Then the problem (3.4) is coercive well-posed in the space $C_{\tau_n}^{\beta, \gamma}([0, T]; E_n)$ with $0 \leq \gamma \leq \beta < 1$, i.e.*

$$\|A_n \bar{U}_n\|_{C_{\tau_n}^{\beta, \gamma}([0, T]; E_n)} \leq \frac{M}{\beta(1-\beta)} \left(|A_n u_n^0 + \varphi_n^1|_{\widetilde{E}_n^{\beta, \gamma}} + |A_n u_n^T + \varphi_n^{K-1}|_{\widetilde{E}_n^{\beta, \gamma}} + \|\bar{\varphi}_n\|_{C_{\tau_n}^{\beta, \gamma}([0, T]; E_n)} \right),$$

where M is independent on $n, \tau_n, \beta, \gamma, u_n^0, u_n^T$ and $\bar{\varphi}_n$.

Here, $|w_n|_n^{\beta, \gamma}$ denotes the norm of a Banach space $\widetilde{E}_n^{\beta, \gamma}$, which consists of those $w_n \in E_n$ for which the norm $|w_n|_n^{\beta, \gamma} = \max_{1 \leq k \leq K-1} \|R_n^k w_n\|_{E_n} + \sup_{1 \leq k < k+r \leq K-1} t_r^{-\beta} (t_k + t_r)^\gamma (T - t_k)^\gamma \|(R_n^{k+r} - R_n^k)w_n\|_{E_n}$ is finite. Note that the spaces of smooth functions $C_{\tau_n}^{\beta, \gamma}([0, T]; E_n)$ in which well-posedness has been established, depend on the parameters β and γ . However, the constants in the last coercive inequality depend only on β . Hence, γ can be chosen freely in $[0, \beta]$, which increases the number of function spaces in which the scheme (3.4) is coercive well-posed. For example, it is important that the scheme (3.4) is coercive well-posed in the Holder space without a weight ($\gamma = 0$). The Holder degree β must belong to $(0, 1)$ which does not allow us to establish the well-posedness of the scheme (3.4) in spaces $C_{\tau_n}([0, T]; E_n)$. These conclusions refer to the case of an arbitrary Banach space E_n . But for some restrictions of the arbitrary space E_n we are able to establish the well-posedness of the scheme (3.4) in spaces $C_{\tau_n}([0, T]; E_{n, \alpha})$. Here the Banach space $E_{n, \alpha}, 0 < \alpha < 1$, consists of those $v_n \in E_n$ for which the norm $\|v_n\|_{E_{n, \alpha}} = \sup_{z > 0} z^\alpha \|\sqrt{A_n}(zI_n + \sqrt{A_n})^{-1}v_n\|_{E_n} + \|v_n\|_{E_n}$ is finite.

Theorem 3.11 [2], [3] *Let A_n be strongly positive operator in a Banach space E_n and $A_n u_n^0 + \varphi_n^1, A_n u_n^T + \varphi_n^{K-1} \in E_{n, \alpha - \gamma}$. Then the problem (3.4) is coercive well-posed in the space $C_{\tau_n}^{\beta, \gamma}([0, T]; E_{n, \alpha - \beta})$ with $0 \leq \gamma \leq \beta \leq \alpha < 1$, i.e.*

$$\|A_n \bar{U}_n\|_{C_{\tau_n}^{\beta, \gamma}([0, T]; E_{n, \alpha - \beta})} \leq \frac{M}{\alpha(1-\alpha)} \left(\|A_n u_n^0 + \varphi_n^1\|_{E_{n, \alpha - \gamma}} + \|A_n u_n^T + \varphi_n^{K-1}\|_{E_{n, \alpha - \gamma}} + \|\bar{\varphi}_n\|_{C_{\tau_n}^{\beta, \gamma}([0, T]; E_{n, \alpha - \beta})} \right),$$

where M is independent on $n, \tau_n, \alpha, \beta, \gamma, u_n^0, u_n^T$ and $\bar{\varphi}_n$.

3.4 Coercive inequalities in $L^p_{\tau_n}([0, T]; E_n)$ spaces

Denote by $L^p_{\tau_n}([0, T]; E_n)$ for $1 \leq p < \infty$ the space of the elements $\bar{\varphi}_n$ with the norm

$$\|\bar{\varphi}_n\|_{L^p_{\tau_n}([0, T]; E_n)} = \left(\sum_{j=1}^{K-1} \|\varphi_n^j\|_{E_n}^p \tau_n \right)^{1/p}.$$

Theorem 3.12 [27] *Let A_n be strongly positive operator in a Banach space E_n . Assume that problem (3.4) is coercive well-posed in $L^{p_0}_{\tau_n}([0, T]; E_n)$ for some $1 < p_0 < \infty$. Then it is also coercive well-posed in $L^p_{\tau_n}([0, T]; E_n)$ for any $1 < p < \infty$, i.e.*

$$\|A_n \bar{U}_n\|_{L^p_{\tau_n}([0, T]; E_n)} + \max_{0 \leq k \leq K} \|U_n^k\|_{E_{n, 1-1/p}} \leq \frac{M(p_0)p^2}{p-1} \|\bar{\varphi}_n\|_{L^p_{\tau_n}([0, T]; E_n)} + M \left(\|U_n^0\|_{E_{n, 1-1/p}} + \|U_n^K\|_{E_{n, 1-1/p}} \right),$$

where $M(p_0)$ and M are independent on $n, \tau_n, p, u_n^0, u_n^T$ and $\bar{\varphi}_n$.

The space $E_{n, 1-1/p}$ coincides with equivalent norm with the space $E_{1-1/p, p}(\sqrt{A_n}, D(\sqrt{A_n}))$.

Theorem 3.13 [2], [3] *Let A_n be strongly positive operator in a Banach space E_n . Then the problem (3.4) is coercive well-posed in $L^p_{\tau_n}([0, T]; E_{n, \alpha, p})$ for any $1 \leq p \leq \infty, 0 < \alpha < 1$, i.e.*

$$\|A_n \bar{U}_n\|_{L^p_{\tau_n}([0, T]; E_{n, \alpha, p})} \leq \frac{M}{\alpha(1-\alpha)} \left(\|\bar{\varphi}_n\|_{L^p_{\tau_n}([0, T]; E_{n, \alpha, p})} + \|A_n U_n^0\|_{E_{n, \alpha, p}} + \|A_n U_n^K\|_{E_{n, \alpha, p}} \right),$$

where M is independent on $n, \tau_n, p, \alpha, u_n^0, u_n^T$ and $\bar{\varphi}_n$.

Theorem 3.14 [2], [3] *Let A_n be strongly positive operator in a Banach space E_n . Then the problem (3.4) is coercive well-posed in $L^p_{\tau_n}([0, T]; E_{n, \alpha, q}), 1 < p, q < \infty, 0 < \alpha < 1$, i.e.*

$$\|A_n \bar{U}_n\|_{L^p_{\tau_n}([0, T]; E_{n, \alpha, q})} \leq \frac{M(q)p^2}{(p-1)\alpha(1-\alpha)} \|\bar{\varphi}_n\|_{L^p_{\tau_n}([0, T]; E_{n, \alpha, q})} + M \left(\|A_n U_n^0\|_{E_{n, \alpha, q}} + \|A_n U_n^K\|_{E_{n, \alpha, q}} \right),$$

where $M(q)$ and M are independent on $n, \tau_n, \alpha, p, u_n^0, u_n^T$ and $\bar{\varphi}_n$.

Here, the interpolation space $E_{n, \alpha, q} = E_{\alpha, q}(\sqrt{A_n}, E_n), 0 < \alpha < 1, 1 < q < \infty$, consists of those $v_n \in E_n$ for which the norm

$$\|v_n\|_{E_{n, \alpha, q}} = \left(\int_0^\infty \|\lambda^\alpha \sqrt{A_n} (\lambda I_n + \sqrt{A_n})^{-1} v_n\|_{E_n}^q \frac{d\lambda}{\lambda} \right)^{1/q}$$

is finite and $E_{n, \alpha, \infty} = E_{n, \alpha}$.

Of course, in Subsections 3.2-3.4 well-posedness could be also proved for the high order of accuracy two step difference schemes generated by an exact difference scheme or Taylor's decomposition function on three points for the approximate solutions of partial differential equations of the elliptic type (see [2], [3], [9]). In this case for the formulation of coercive statements of Subsections 3.2-3.4 for the schemes a high order of accuracy we just need to substitute in the statements $A_n \bar{U}_n$ for $\left\{ \frac{U_n^{k+1} - 2U_n^k + U_n^{k-1}}{\tau_n^2} \right\}$.

Finally, the well-posedness of the various elliptic differential and difference problems in Banach spaces has been studied extensively by many researches, see [4], [8], [10], [11], [13], [17], [25], [29] and the references therein.

4 Main results

The purpose of this paper to complete the picture of well-posedness of the difference schemes in $L^p_{\tau_n}([0, T]; E_n)$ space for general Banach space E_n and also for the case of UMD Banach spaces E_n .

4.1 Coercive inequality in general Banach space E_n

In the following Theorem we assume that operators $A_n \in B(E_n)$ are bounded for any fixed n , so $B_n \in B(E_n)$ as one can see from definition (3.6). As was mentioned in Section 2.1 one has $\|A_n\| \rightarrow \infty$, $\|B_n\| \rightarrow \infty$ as $n \rightarrow \infty$, since A and \sqrt{A} are unbounded in general.

Theorem 4.1 *Let A_n be strongly positive operator in a Banach space E_n . Then the problem (3.4) is almost coercive stable in the space $L^p_{\tau_n}([0, T]; E_n)$ for any $1 \leq p \leq \infty$, i.e.*

$$\|A_n \bar{U}_n\|_{L^p_{\tau_n}([0, T]; E_n)} \leq M \left(\|A_n U_n^0\|_{E_n} + \|A_n U_n^K\|_{E_n} + \min \left(|\ln(1/\tau_n)|, 1 + \left| \ln \|B_n\|_{B(E_n)} \right| \right) \|\bar{\varphi}_n\|_{L^p_{\tau_n}([0, T]; E_n)} \right).$$

holds for any $1 \leq p \leq \infty$, where M is independent on n, τ_n, u_n^0, u_n^T , and $\bar{\varphi}_n$.

Proof. The proof of this theorem in the case $p = \infty$ was obtained by Sobolevskii P.E. in [27]. It is based on the estimate

$$(4.7) \quad \sum_{j=1}^{K-1} \|B_n(1 + \tau_n B_n)^{-j}\| \tau_n \leq M \min \left(|\ln(1/\tau_n)|, 1 + \left| \ln \|B_n\|_{B(E_n)} \right| \right)$$

for any strongly positive operator B_n . Now, the proof of our theorem in the case $p = 1$ is carried out according to the same scheme and is based on the estimate (4.7). So, we will consider the proof of this theorem for any $1 < p < \infty$. Applying formula (3.5), we can write

$$\begin{aligned} U_n^k &= (I - R_n^{2K})^{-1} (R_n^k - R_n^{2K-k}) \left(u_n^0 + \sum_{j=1}^{K-1} B_n^{-1} R_n^j (I + \tau_n B_n) (2 + \tau_n B_n)^{-1} \varphi_n^j \tau_n \right) \\ &\quad - \sum_{j=1}^{k-1} B_n^{-1} R_n^{k-j} (I + \tau_n B_n) (2 + \tau_n B_n)^{-1} \varphi_n^j \tau_n \\ &+ (I - R_n^{2K})^{-1} (R_n^{K-k} - R_n^{K+k}) \left(u_n^T + \sum_{j=1}^{K-1} B_n^{-1} R_n^{K-j} (I + \tau_n B_n) (2 + \tau_n B_n)^{-1} \varphi_n^j \tau_n \right) \\ &\quad - \sum_{j=k}^{K-1} B_n^{-1} R_n^{j-k} (I + \tau_n B_n) (2 + \tau_n B_n)^{-1} \varphi_n^j \tau_n = U_{n,1}^k + U_{n,2}^k, \quad 1 \leq k \leq K-1, \end{aligned}$$

We get almost coercive inequality proving the estimates

$$(4.8) \quad \|A_n \bar{U}_{n,1}\|_{L^p_{\tau_n}([0, T]; E_n)} \leq M \left(\|A_n u_n^0\|_{E_n} + \min \left(|\ln(1/\tau_n)|, 1 + \left| \ln \|B_n\|_{B(E_n)} \right| \right) \|\bar{\varphi}_n\|_{L^p_{\tau_n}([0, T]; E_n)} \right),$$

$$(4.9) \quad \|A_n \bar{U}_{n,2}\|_{L^p_{\tau_n}([0, T]; E_n)} \leq M \left(\|A_n u_n^T\|_{E_n} + \min \left(|\ln(1/\tau_n)|, 1 + \left| \ln \|B_n\|_{B(E_n)} \right| \right) \|\bar{\varphi}_n\|_{L^p_{\tau_n}([0, T]; E_n)} \right).$$

We have that

$$\begin{aligned} U_{n,1}^k &= (I - R_n^{2K})^{-1} \left(R_n^k - R_n^{2K-k} \right) \left(u_n^0 + \sum_{j=1}^{K-1} B_n^{-1} R_n^j (I + \tau_n B_n) (2 + \tau_n B_n)^{-1} \varphi_n^j \tau_n \right) \\ &\quad - \sum_{j=1}^{k-1} B_n^{-1} R_n^{k-j} (I + \tau_n B_n) (2 + \tau_n B_n)^{-1} \varphi_n^j \tau_n = W_n(k) + G_n(k) + V_n(k). \end{aligned}$$

It is known (see, for example [26]) that if operators A_n are strongly positive, i.e. (4.11) is satisfied, then for operators B_n the following estimates hold:

$$(4.10) \quad \|(I_n - R_n^{2K})^{-1}\| \leq M, \|R_n^j\| \leq M(1 + \delta\tau_n)^{-j}, j\tau_n\|B_n R_n^j\| \leq M, \quad j \geq 1,$$

with $\delta > 0$. Using the estimate (4.10), we obtain

$$\begin{aligned} \left(\sum_{k=1}^{K-1} \|A_n W_n(k)\|_{E_n}^p \tau_n\right)^{\frac{1}{p}} &\leq \left(\sum_{k=1}^{K-1} (\|(I - R_n^{2K})^{-1}(I - R_n^{2K-2k})\| \|R_n^k\| \|A_n u_n^0\|_{E_n})^p \tau_n\right)^{\frac{1}{p}} \\ &\leq M \|A_n u_n^0\|_{E_n} \left(\sum_{k=1}^{K-1} \|R_n^k\|^p \tau_n\right)^{\frac{1}{p}} \leq M_1 \|A_n u_n^0\|_{E_n} (K\tau_n)^{\frac{1}{p}} = M_2 \|A_n u_n^0\|_{E_n}. \end{aligned}$$

Now let us estimate $\{A_n G_n(k)\}$ in the norm of $L_{\tau_n}^p([0, T]; E_n)$. Set $f_n^{*,j} = \varphi_n^j$ if $j = 1, \dots, K-1$, and $f_n^{*,j} = 0$ otherwise. We have that

$$\begin{aligned} A_n G_n(k) &= A_n (I - R_n^{2K})^{-1} (R_n^k - R_n^{2K-k}) \sum_{j=1}^{K-1} B_n^{-1} R_n^j (I + \tau_n B_n) (2 + \tau_n B_n)^{-1} \varphi_n^j \tau_n \\ &= (I - R_n^{2K})^{-1} (I - R_n^{2K-2k}) \sum_{j=k+1}^{k+K-1} B_n R_n^j (2 + \tau_n B_n)^{-1} \varphi_n^{j-k} \tau_n \\ &= (I - R_n^{2K})^{-1} (I - R_n^{2K-2k}) \sum_{j=1}^{2K-1} B_n R_n^j (2 + \tau_n B_n)^{-1} f_n^{*,j-k} \tau_n. \end{aligned}$$

Using this formula, Minkowski sum inequality and the estimate (4.10) it follows that

$$\begin{aligned} &\|A_n G_n(\cdot)\|_{L_{\tau_n}^p([0, T]; E_n)} \\ &= \left(\sum_{k=1}^{K-1} \|(I - R_n^{2K})^{-1} (I - R_n^{2K-2k}) \sum_{j=1}^{2K-1} B_n R_n^j (2 + \tau_n B_n)^{-1} f_n^{*,j-k} \tau_n\|_{E_n}^p \tau_n\right)^{\frac{1}{p}} \\ &\leq \sum_{j=1}^{2K-1} \left(\sum_{k=1}^{K-1} \left(\|(I - R_n^{2K})^{-1} (I - R_n^{2K-2k}) (2 + \tau_n B_n)^{-1}\|\right)^p \|B_n R_n^j\|^p \|f_n^{*,j-k}\|_{E_n}^p \tau_n\right)^{\frac{1}{p}} \\ &\leq M \sum_{j=1}^{2K-1} \|B_n R_n^j\| \left(\sum_{k=1}^{K-1} \|f_n^{*,j-k}\|_{E_n}^p \tau_n\right)^{\frac{1}{p}} \tau_n. \end{aligned}$$

By the definition of the grid function $\{f_n^{*,k}\}_{k=1}^{K-1}$ we obtain that

$$\|A_n G_n(\cdot)\|_{L_{\tau_n}^p([0, T]; E_n)} \leq M \sum_{j=1}^{2K-1} \|B_n R_n^j\| \cdot \|\overline{\varphi}_n\|_{L_{\tau_n}^p([0, T]; E_n)} \tau_n.$$

Applying estimates (4.7) and (4.10), we obtain

$$\|A_n G_n(\cdot)\|_{L_{\tau_n}^p([0, T]; E_n)} \leq M_1 \min\left(|\ln(1/\tau_n)|, 1 + \left|\ln \|B_n\|_{B(E_n)}\right|\right) \|\overline{\varphi}_n\|_{L_{\tau_n}^p([0, T]; E_n)}.$$

Now let us estimate $\{A_n V_n(\cdot)\}$ in the norm of $L_{\tau_n}^p([0, T]; E_n)$. Set $f_n^{*,j} = \varphi_n^j$ if $j = 1, \dots, k-1, k = 1, \dots, K-1$ and $f_n^{*,j} = 0$ otherwise. We have that

$$A_n V_n(k) = -A_n \sum_{j=1}^{k-1} B_n^{-1} R_n^{k-j} (I + \tau_n B_n) (2 + \tau_n B_n)^{-1} \varphi_n^j \tau_n$$

$$\begin{aligned}
&= - \sum_{j=1}^{k-1} B_n R_n^j (2 + \tau_n B_n)^{-1} \varphi_n^{k-j} \tau_n \\
&= - \sum_{j=1}^{k-1} B_n R_n^j (2 + \tau_n B_n)^{-1} f_n^{*,k-j} \tau_n.
\end{aligned}$$

Using this formula, Minkowski sum inequality and the estimate (4.10) it follows that

$$\begin{aligned}
\|A_n V_n(\cdot)\|_{L_{\tau_n}^p([0,T];E_n)} &= \left(\sum_{k=1}^{K-1} \left\| \sum_{j=1}^{K-1} B_n R_n^j (2 + \tau_n B_n)^{-1} f_n^{*,k-j} \tau_n \right\|_{E_n}^p \tau_n \right)^{\frac{1}{p}} \\
&\leq \sum_{j=1}^{K-1} \|(2 + \tau_n B_n)^{-1}\| \left(\sum_{k=1}^{K-1} \|B_n R_n^j\|^p \|f_n^{*,k-j}\|_{E_n}^p \tau_n \right)^{\frac{1}{p}} \tau_n \\
&\leq M \sum_{j=1}^{K-1} \|B_n R_n^j\| \left(\sum_{k=1}^{K-1} \|f_n^{*,k-j}\|_{E_n}^p \tau_n \right)^{\frac{1}{p}} \tau_n.
\end{aligned}$$

By the definition of the grid function $\{f_n^{*,k}\}_{k=1}^{K-1}$ we obtain that

$$\|A_n V_n(\cdot)\|_{L_{\tau_n}^p([0,T];E_n)} \leq M \sum_{j=1}^{K-1} \|B_n R_n^j\| \tau_n \|\overline{\varphi}_n\|_{L_{\tau_n}^p([0,T];E_n)}.$$

Applying the estimates (4.7), one gets

$$\|A_n V_n(\cdot)\|_{L_{\tau_n}^p([0,T];E_n)} \leq M_1 \min \left(|\ln(1/\tau_n)|, 1 + \left| \ln \|B_n\|_{B(E_n)} \right| \right) \|\overline{\varphi}_n\|_{L_{\tau_n}^p([0,T];E_n)}.$$

Finally, using the estimates for $\|A_n W_n(\cdot)\|_{L_{\tau_n}^p([0,T];E_n)}$, $\|A_n G_n(\cdot)\|_{L_{\tau_n}^p([0,T];E_n)}$, and $\|A_n V_n(\cdot)\|_{L_{\tau_n}^p([0,T];E_n)}$ and using the triangle inequality we obtain the estimate (4.8). The proof of the estimate (4.9) is carried out according to the same scheme as in (4.8). Theorem is proved.

4.2 Coercive inequalities for UMD Banach spaces E_n

In this Subsection we assume that spaces E_n are UMD Banach spaces with the Hilbert transform operators H_n which are uniformly bounded in n . Moreover, we assume that operators A_n uniformly positive, i.e. the following estimate holds

$$(4.11) \quad \|(\lambda I_n + A_n)^{-1}\| \leq \frac{M}{1 + \lambda} \text{ for any } \lambda \geq 0$$

for some $1 \leq M < \infty$.

Lemma 4.1 *Assume that the sets $\{s(sI_n + A_n)^{-1} : s > 0\}$ are R -bounded uniformly in n and (4.11) is satisfied. Then the sets $\{\lambda(\lambda I_n \pm B_n)^{-1} : \lambda \in \mathbb{R}, \lambda \neq 0\}$, are uniformly in n R -bounded.*

Proof. Let us consider

$$\lambda(\lambda i I_n + B_n)^{-1} = \frac{1}{2\pi i} \int_G \frac{\lambda}{\lambda i + \frac{\rho \tau_n + \sqrt{\rho} \sqrt{\rho \tau_n^2 + 4}}{2}} (\rho I_n + A_n)^{-1} d\rho$$

$$= \frac{1}{2\pi i} \int_{|\rho| \leq 1} \frac{\lambda}{\lambda i + \frac{\rho\tau_n + \sqrt{\rho}\sqrt{\rho\tau_n^2 + 4}}{2}} (\rho I_n + A_n)^{-1} d\rho + \frac{1}{2\pi i} \int_{|\rho| \geq 1} \frac{\lambda}{\lambda i + \frac{\rho\tau_n + \sqrt{\rho}\sqrt{\rho\tau_n^2 + 4}}{2}} \frac{\rho(\rho I_n + A_n)^{-1} d\rho}{\rho}.$$

Since the set $\{\rho : |\rho| \leq 1, \rho \in G\}$ is compact and resolvent according to (4.11) is analytic in the neighborhood of $\{\rho : |\rho| \leq 1, \rho \in G\}$, then by Corrolary 2.17 in [22] the first part is R-bounded set. The second part after change of variables $\rho = \lambda^2\eta$ is also R-bounded because of Corrolary 2.14 in [22]. Lemma is proved.

Theorem 4.2 *Let operators A_n be strongly positive operators in a UMD Banach spaces E_n and the sets $\{s(sI_n + A_n)^{-1} : s > 0\}$ be uniformly in n R-bounded. Then the problem (3.4) is coercive stable in the space $L^p_{\tau_n}([0, T]; E_n)$ for any $1 < p < \infty$, i.e.*

$$\|A_n \bar{U}_n\|_{L^p_{\tau_n}([0, T]; E_n)} \leq M \left(\|A_n U_n^0\|_{E_n} + \|A_n U_n^K\|_{E_n} + \|\bar{\varphi}_n\|_{L^p_{\tau_n}([0, T]; E_n)} \right)$$

holds for any $1 < p < \infty$, where M is independent on n, τ_n, u_n^0, u_n^T , and $\bar{\varphi}_n$.

Proof. The same way as in Theorem 4.1 we write $U_n^k = U_{n,1}^k + U_{n,2}^k$, $1 \leq k \leq K-1$, and consider for example just one term $U_{n,1}^k = W_n(k) + G_n(k) + V_n(k)$. To estimate the term $A_n V_n(k)$ we use the fact that $\{R_n^j\}$ is a discrete semigroup with generator $\frac{R_n - I_n}{\tau_n} = -B_n(I_n + \tau_n B_n)^{-1}$. Since by Lemma 4.1 the sets $\{\lambda(\lambda I_n + B_n)^{-1} : \lambda \in \mathbb{R}, \lambda \neq 0\}$ are uniformly in n R-bounded one gets the result from Theorem 5.4 [5]. To do this one has to use identity $A_n = B_n^2(I_n + \tau_n B_n)^{-1}$, inequalities (4.10) and check that $\{\check{A}_n(\lambda I_n - R_n)^{-1} Q_n \tau_n : |\lambda| = 1, \lambda \neq \pm 1\}$ and $\{(\lambda - 1)(\lambda + 1)A_n(\lambda I_n - R_n)^{-2} Q_n \tau_n : |\lambda| = 1, \lambda \neq \pm 1\}$ are R-bounded for $\check{A}_n = -B_n R_n$ and $Q_n = B_n^{-1}(2I_n + \tau_n B_n)^{-1}$. This can be done exactly the same way as in Theorem 5.4 [5].

The term $A_n W_n(k)$ can be estimated the same way as in Theorem 4.1. Let us consider the term $A_n G_n(k)$. Since by (4.10) $\|B_n R_n^K\| \leq \frac{M}{K\tau_n}$, it is enough to estimate just $R_n^k \sum_{j=1}^{K-1} B_n R_n^j (2 + \tau_n B_n)^{-1} \varphi_n^j \tau_n$. Operator $\sum_{j=1}^{K-1} \frac{1}{(k+j)\tau_n} \|\varphi_n^j\| \tau_n$ is a discrete version of operator $\int_0^T \frac{1}{t+s} \|f(s)\| ds$, which is bounded from $L^p([0, T]; \mathbb{R})$ to $L^p([0, T]; \mathbb{R})$, see [15]. Theorem is proved.

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