Multiple planar coincidences with N-fold symmetry

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Abstract. Planar coincidence site lattices and modules with N-fold symmetry are well understood in a formulation based on cyclotomic fields, in particular for the class number one case, where they appear as certain principal ideals in the corresponding ring of integers. We extend this approach to multiple coincidences, and give explicit results for spectral, combinatorial and asymptotic properties in terms of Dirichlet series generating functions.

Key Words: Lattices, Coincidence Ideals, Planar Modules, Cyclotomic Fields, Dirichlet Series, Asymptotic Properties

1 Introduction

Given a lattice $\Gamma \subset \mathbb{R}^d$, with $d \geq 2$, it is interesting to know its coincidence site lattices (CSLs), which originate from intersections of Γ with a rotated copy. In fact,

$$\mathrm{SOC}(\Gamma) \;:=\; \{R \in \mathrm{SO}(d) \mid [\Gamma : (\Gamma \cap R\Gamma)] < \infty\} \qquad (1)$$

is a *group*, whose structure in general (i.e., for $d > 2$) is not well understood. The coincidence index

$$\Sigma(R) \;=\; [\Gamma : (\Gamma \cap R\Gamma)] \qquad (2)$$

of a rotation R is defined as the number of cosets of $\Gamma \cap R\Gamma$ in Γ (which can be ∞), and the spectrum of finite Σ-values, $\Sigma\big(\mathrm{SOC}(\Gamma)\big)$, is often the first quantity considered, followed by counting all CSLs of a given index. Note that one can also consider the obvious extension to general orthogonal transformations $R \in \mathrm{O}(d)$ and the corresponding group $\mathrm{OC}(\Gamma)$. Coincidence lattices play an important role in the theory of grain boundaries [9, 11], and the indices can be determined in suitable experiments.

The classification of these elementary or *simple* coincidences is partly done, in particular for low-dimensional lattices with irreducible symmetries, compare [7, 2] and references given there, but also for certain generalizations to quasicrystals [15, 2]. The situation for various related problems [4, 5] is similar. Although interesting questions concerning the Bravais types of the possible CSLs are still open, the general theory in dimensions $d \leq 4$ is in rather good shape, see [2, 22, 23] and references given there.

More recent is the problem of optimal lattice quantizers [17], and connected with it is the question for *multiple* coincidences. Here, one would like to classify all lattices that can be obtained as multiple intersections of the form

$$\Gamma \cap R_1\Gamma \cap \ldots \cap R_m\Gamma \qquad (3)$$

with $R_\ell \in \mathrm{SO}(d)$. One defines the corresponding index as

$$\Sigma(R_1, \ldots, R_m) \;:=\; [\Gamma : (\Gamma \cap R_1\Gamma \cap \ldots \cap R_m\Gamma)], \qquad (4)$$

which is finite if and only if each $R_\ell \in \mathrm{SOC}(\Gamma)$, due to the mutual commensurability of the lattices $\Gamma \cap R_\ell\Gamma$ (we shall explain this in more detail below). Consequently, one attaches the group $\big(\mathrm{SOC}(\Gamma)\big)^m$ to the setting of m-fold coincidences. The corresponding spectrum is its image under Σ, while the full (or complete) coincidence spectrum of Γ is

$$\Sigma_\Gamma \;:=\; \bigcup_{m\geq 1} \Sigma\big((\mathrm{SOC}(\Gamma))^m\big). \qquad (5)$$

This is an inductive limit, with

$$\Sigma\big((\mathrm{SOC}(\Gamma))^m\big) \;\subset\; \Sigma\big((\mathrm{SOC}(\Gamma))^{m'}\big) \quad \text{for } m \leq m',$$

which often stabilizes: from a certain m on, the spectra are stable, i.e., they do not grow any more [24]. In our examples below, this actually happens at $m = 1$, so that the spectrum is $\Sigma_\Gamma = \Sigma\big(\mathrm{SOC}(\Gamma)\big)$.

The problem of multiple coincidences is considerably more involved than that of the simple ones, in particular in dimensions $d \geq 3$. However, for $d = 2$, one can rather easily extend the treatment of elementary coincidences to multiple ones, building on the powerful and well understood connection to the algebraic theory of cyclotomic fields and to the analytic theory of the corresponding zeta functions. This is precisely what we shall do below, both for lattices and dense modules of the plane. The latter step makes the results applicable to planar quasicrystals that are used to model the so-called T-phases, which show a periodic stacking of planar layers with non-crystallographic symmetries [18].

Before we expand on the mathematical tools, let us set the scene with a simple example. Afterwards, we shall treat all planar modules with N-fold symmetry, class number 1 (see below for an explanation), and rank $\phi(N)$, where ϕ denotes Euler's totient function (as defined below in Eq. (23)). To facilitate crystallographic applications, we also provide

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explicit results, both in terms of Dirichlet series generating functions and in terms of tables.

The following exposition is based upon previous results, and on references [15] and [6] in particular. In order to keep our presentation concise, we shall use the notation established there and frequently refer to these references.

2 Example: The triangular lattice

Let us consider the triangular lattice, often also called the hexagonal lattice. Since rotations in the plane are most easily written as multiplication with a complex number on the unit circle, $e^{i\varphi}$, a suitable representative of the lattice should be formulated accordingly. With $\xi_3 = e^{2\pi i/3}$, one defines

$$\Gamma \;=\; \{m + n\cdot\xi_3 \mid m,n\in\mathbb{Z}\} \;=\; \mathbb{Z}[\xi_3] \tag{6}$$

which is a triangular lattice, with minimal distance 1 and thus a fundamental cell of area $\frac{1}{2}\sqrt{3}$. At the same time, it is the set of *Eisenstein integers* (also called Eisenstein-Jacobi integers), which form the ring of integers [12, Ch. 12.2] of the cyclotomic field $\mathbb{Q}(\xi_3)$, a totally complex extension of the totally real field $\mathbb{Q}$ (and hence of degree 2 over $\mathbb{Q}$).

A remarkable property of $\mathbb{Z}[\xi_3]$ is its unique prime decomposition, up to units. The latter form the cyclic group

$$C_6 \;=\; \left\{\pm\xi_3^{\ell} \mid 0\le\ell\le 2\right\} \tag{7}$$

of rotation symmetries of Γ, with $-\xi_3$ being a possible generator for it. For any rational prime p (i.e., any prime of $\mathbb{Z}\subset\mathbb{Q}$), one of the following three possibilities applies, compare [12, Ch. 15.3]:

1. $p = 3$; this prime is called *ramified*, and factorizes as $3 = (1-\xi_3)(1-\bar{\xi}_3) = -\bar{\xi}_3(1-\xi_3)^2$. Up to a unit, 3 is thus the square of a prime in $\mathbb{Z}[\xi_3]$.
2. $p\equiv 2\;(3)$; these primes are called *inert*, because they stay prime in $\mathbb{Z}[\xi_3]$.
3. $p\equiv 1\;(3)$; these are the (complex) *splitting* primes, because they factorize as $p = \omega_p\cdot\bar{\omega}_p$ into a pair of complex conjugate primes of $\mathbb{Z}[\xi_3]$ that are not associated to one another (meaning that $\omega_p/\bar{\omega}_p$ is not a unit in $\mathbb{Z}[\xi_3]$).

The last type of primes is the key to solving the coincidence problem. As was shown in [15], any coincidence rotation of Γ of (6) is of the form

$$e^{i\varphi} \;=\; \varepsilon\cdot\prod_{p\equiv 1\,(3)}\left(\frac{\omega_p}{\bar{\omega}_p}\right)^{t_p} \tag{8}$$

where the product runs over all rational primes $\equiv 1\,(3)$, with all $t_p\in\mathbb{Z}$ (only *finitely* many of them differing from 0), and where ε is a unit in $\mathbb{Z}[\xi_3]$, i.e., $\varepsilon\in C_6$ from Eq. (7). An example with the smallest non-trivial index is shown in Figure 1.

The meaning (and the basic ingredient of the proof) is that $e^{i\varphi}$ of (8) rotates one Eisenstein integer (the denominator) into another (the numerator). The corresponding CSL is then generated by w and $\xi_3\cdot w$, where

$$w \;=\; \prod_{\substack{p\equiv 1\,(3)\\ t_p>0}}(\omega_p)^{t_p}\cdot\prod_{\substack{p\equiv 1\,(3)\\ t_p<0}}(\bar{\omega}_p)^{-t_p} \tag{9}$$

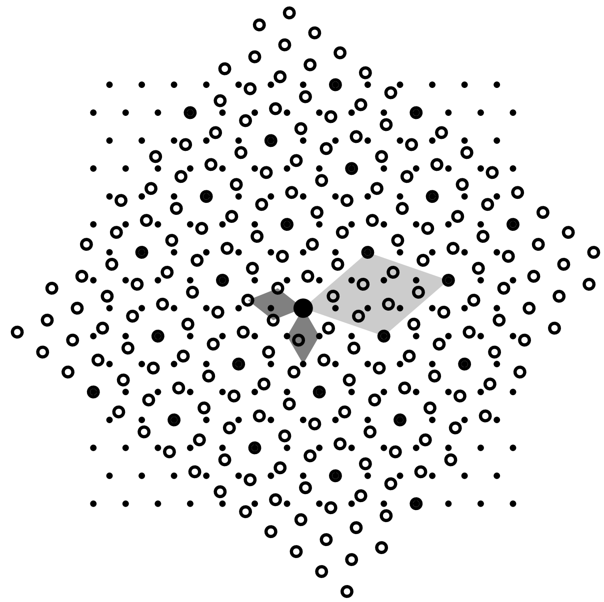

Fig. 1. Simple coincidence (with $\Sigma = 7$) for the triangular lattice. Small dots indicate the original lattice sites, while the circles correspond to the rotated copy. Large dots are the coinciding sites. Unit cells are shaded for the intersecting lattices (dark) and the CSL (light).

is the numerator of $e^{i\varphi}$ in (8).

Clearly, the CSL that emerges from Eq. (9) is the principal ideal $(w) = w\,\mathbb{Z}[\xi_3]$. Its index in $\mathbb{Z}[\xi_3]$, also called its total norm, is the corresponding coincidence index and reads

$$\Sigma(\varphi) \;=\; \prod_{p\equiv 1\,(3)} p^{|t_p|} \tag{10}$$

with $\Sigma = 1$ precisely for $e^{i\varphi} = \varepsilon\in C_6$.

This shows that the group of coincidence rotations has the structure

$$\mathrm{SOC}(\Gamma) \;=\; \mathrm{SOC}(\mathbb{Z}[\xi_3]) \;=\; C_6\times\mathbb{Z}^{(\aleph_0)}, \tag{11}$$

where $\mathbb{Z}^{(\aleph_0)}$ stands for the direct sum of countably many infinite cyclic groups, each with a generator of the form $\omega_p/\bar{\omega}_p$, and the restriction mentioned after Eq. (8). The set of possible indices, $\Sigma\big(\mathrm{SOC}(\mathbb{Z}[\xi_3])\big)$, forms a semigroup with unit, generated by 1 and the rational primes $p\equiv 1\,(3)$, the latter being called *basic indices*.

If $c_3(k)$ denotes the number of CSLs of Γ of index k, it is a multiplicative arithmetic function (i.e., $c_3(k\ell) = c_3(k)c_3(\ell)$ whenever k,ℓ are coprime). Consequently, its determination is most easily achieved by means of a Dirichlet series generating function. The result reads [15]

$$\begin{aligned}\Phi_3(s) \;&=\; \sum_{k=1}^{\infty}\frac{c_3(k)}{k^s} \;=\; \prod_{p\equiv 1\,(3)}\frac{1+p^{-s}}{1-p^{-s}} \\ &=\; 1+\tfrac{2}{7^s}+\tfrac{2}{13^s}+\tfrac{2}{19^s}+\tfrac{2}{31^s}+\tfrac{2}{37^s}+\tfrac{2}{43^s}+\tfrac{2}{49^s}+\dots\end{aligned} \tag{12}$$

This can also be expressed in terms of zeta functions,

$$\Phi_3(s) \;=\; \frac{\zeta_{\mathbb{Q}(\xi_3)}(s)}{(1+3^{-s})\,\zeta(2s)}, \tag{13}$$

where $\zeta(s)=\sum_{k=1}^{\infty}k^{-s}$ is Riemann's zeta function, and

$$\begin{aligned}\zeta_{\mathbb{Q}(\xi_3)}(s) \;&=\; \sum_{k=1}^{\infty}\frac{a_3(k)}{k^s} \qquad (14)\\ &=\; \frac{1}{1-3^{-s}}\prod_{p\equiv 1\,(3)}\frac{1}{(1-p^{-s})^2}\prod_{p\equiv 2\,(3)}\frac{1}{1-p^{-2s}}\end{aligned}$$

is Dedekind's zeta function of the cyclotomic field $\mathbb{Q}(\xi_3)$, with $a_3(k)$ the number of ideals of $\mathbb{Z}[\xi_3]$ of index k, see [20] for details. Both zeta functions converge absolutely on the right half-plane $\{s\in\mathbb{C}\mid \mathrm{Re}(s)>1\}$, simply written as $\{\mathrm{Re}(s)>1\}$ from now on.

Since all CSLs of $\varGamma$ are themselves triangular lattices (in particular, they are similarity sublattices of $\varGamma$), the problem of simple coincidences is thus solved. From Eq. (13), one can also extract the asymptotic behaviour of the arithmetic function $c_3(k)$. The result is [15]

$$\sum_{k\le x} c_3(k) \;\sim\; x\cdot\bigl(\mathrm{res}_{s=1}\,\varPhi_3(s)\bigr) \;=\; x\cdot\frac{\sqrt{3}}{2\pi}, \qquad (15)$$

where $f(x)\sim g(x)$ means $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$.

Let us now consider *multiple* coincidences (which are meant to include the simple ones, of course). The starting point is the rather obvious observation that

$$\varGamma\cap R_1\varGamma\cap\ldots\cap R_m\varGamma \;=\; \bigcap_{\ell=1}^{m}(\varGamma\cap R_\ell\varGamma)\,. \qquad (16)$$

If the result is still a sublattice of $\varGamma$ (and thus of finite index in it), each $(\varGamma\cap R_\ell\varGamma)$ itself must be a CSL. This necessary condition is also sufficient, because any two sublattices of $\varGamma$ are commensurate, i.e., their intersection is still a (full) sublattice of $\varGamma$. So, we get a CSL in (16) if and only if all R_ℓ are elements of $\mathrm{SOC}(\varGamma)$.

As follows immediately from Eq. (9), each simple CSL is an ideal of $\mathbb{Z}[\xi_3]$ of the form $(w):=w\,\mathbb{Z}[\xi_3]$, i.e., a principal ideal. Note that all ideals of $\mathbb{Z}[\xi_3]$ are principal, but not all of them appear as CSLs. Any multiple CSL is an intersection of simple CSLs. Consequently, one has

$$\bigcap_{i=1}^{m}(w_i) \;=\; \bigl(\mathrm{lcm}\{w_1,\ldots,w_m\}\bigr), \qquad (17)$$

which is indeed a principal ideal again, due to the unique prime factorization (up to units) in $\mathbb{Z}[\xi_3]$. Since each w_i is of the form (9), one quickly checks that multiple coincidences cannot enlarge the list of possible indices, though they *can* lead to more solutions for a given index, hence to genuinely multiple CSLs.

Grouping the solutions from Eq. (17) according to their coincidence indices, one finds the following

Proposition 1 *If $k=p_1^{r_1}\cdot\ldots\cdot p_t^{r_t}$ is the prime decomposition of an integer $k>1$ into rational primes, the triangular lattice $\varGamma$ has no CSL of index k unless* all $p_j\equiv 1\;(3)$. *In that case, the number of multiple CSLs of index k is given by the multiplicative arithmetic function*

$$b_3(k) \;=\; \prod_{j=1}^{t}(r_j+1).$$

Moreover, the Euler product representation of the corresponding Dirichlet series generating function reads

$$\varPsi_3(s) \;=\; \sum_{k=1}^{\infty}\frac{b_3(k)}{k^s} \;=\; \prod_{p\equiv 1\,(3)}\frac{1}{(1-p^{-s})^2},$$

with absolute convergence on $\{\mathrm{Re}(s)>1\}$.

PROOF. Clearly, each (w_i) in (17) must be a simple CSL of $\varGamma$ for the total coincidence index to be finite, hence all $p_j\equiv 1\,(3)$ from Eq. (8).

Then, each factor p^r of k, as $p=\omega\bar\omega$ is a complex splitting prime, can contribute a factor of the form $(\omega^{r-\ell}\bar\omega^{\ell})$ to the lcm of Eq. (17), with $0\le\ell\le r$. Any principal ideal $(\omega^{r-\ell}\bar\omega^{\ell})$, in turn, is either a simple CSL itself (if $\ell=0$ or $\ell=r$) or the intersection of two simple CSLs.

So, for the prime p, which enters as p^r to the decomposition of k, this amounts to $r+1$ different possibilities. Since the different primes give independent choices of this kind, one obtains $b_3(k)$ as the product stated above, and the multiplicativity of this arithmetic function is obvious.

The Dirichlet series has the form claimed, as one can see from its Euler factors, using the identity $(1-x)^{-2}=\sum_{\ell\ge 0}(\ell+1)x^{\ell}$ with $x=p^{-s}$; the convergence result is standard. □

A comparison of $\varPhi_3$ and $\varPsi_3$ shows that

$$\varPsi_3(s)-\varPhi_3(s) \;=\; \tfrac{1}{49^s}+\tfrac{1}{169^s}+\tfrac{2}{343^s}+\tfrac{1}{361^s}+\tfrac{2}{637^s}+\tfrac{2}{931^s}+\ldots$$

encapsulates the statistics of the genuinely multiple CSLs. In particular, a comparison with $\zeta_{\mathbb{Q}(\xi_3)}(s)$ shows that all ideals of norm k, with the additional condition that k factorizes into rational primes $p\equiv 1\;(3)$ only, are CSLs for multiple (in fact, single or double) intersections. One thus has

$$\begin{aligned}\Sigma_\varGamma \;&=\; \Sigma\bigl(\mathrm{SOC}(\varGamma)\bigr) \qquad (18)\\ &=\; \Bigl\{\prod_{i=1}^{t}p_i^{r_i}\mid \text{all } p_i\equiv 1\,(3),\, r_i\in\mathbb{N} \text{ and } t\ge 0\Bigr\}.\end{aligned}$$

To expand on the first difference between simple and double coincidences, consider $p=7$, which is the smallest rational prime $\equiv 1\;(3)$. It splits as $7=\omega\cdot\bar\omega$ with $\omega=2-\xi_3$, and both the ideals (ω^2) and $(\bar\omega^2)$ appear as simple CSLs (of index $\Sigma=49$), while $(\omega\bar\omega)=(7)$ is only possible as multiple CSL – namely as the intersection of the ideals (ω) and $(\bar\omega)$, both being simple CSLs (of index 7, as in Figure 1). Note that $(\omega\bar\omega)=(\omega)\cap(\bar\omega)$ is, at the same time, a simple CSL for the lattice (ω), because the latter is rotated into $(\bar\omega)$ by $e^{i\varphi}=\bar\omega/\omega$.

Asymptotically, the surplus of multiple CSLs leads to an additional factor in comparison to Eq. (15), so that

$$\sum_{k\le x} b_3(k) \;\sim\; x\cdot\bigl(\mathrm{res}_{s=1}\,\varPsi_3(s)\bigr) \;=\; x\cdot\frac{\sqrt{3}}{2\pi}\cdot q_3 \qquad (19)$$

with

$$q_3 \;:=\; \lim_{s\to 1}\frac{\varPsi_3(s)}{\varPhi_3(s)} \;=\; \prod_{p\equiv 1\,(3)}\frac{p^2}{p^2-1} \;\simeq\; 1.034\,015. \qquad (20)$$

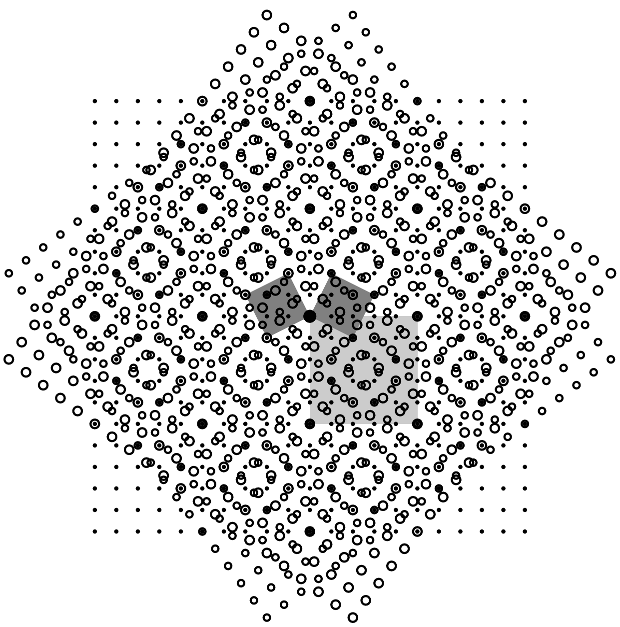

Fig. 2. Double coincidence (with $\Sigma = 25$) for the square lattice. Unit cells are shaded for simple (dark) and double (light) CSLs. Small dots indicate the original lattice sites, while the two types of circles correspond to the two rotated copies. Large dots are formed by double coincidences, i.e., at sites which belong to all three lattices.

The convergence of this product turns out to be rather slow (to put it mildly), whence one needs a different method to actually calculate q_3. As we shall show below in greater generality, this is achieved by means of the identity

$$q_3 \; = \; \prod_{\ell=1}^{\infty} \bigl(\Phi_3(2^\ell)\bigr)^{1/2^\ell} .$$

Although this looks just like another infinite product representation, its convergence is spectacularly fast[1]. Also, it is monotonically increasing, which is helpful to derive error bounds for the actual numerical calculations in this case.

3 Cyclotomic integers with class number one

The example of the triangular lattice was chosen because it is both simple and paradigmatic. The square lattice, when identified with the ring $\mathbb{Z}[i]$ of Gaussian integers [12], can be treated in the same way, leading to analogous expressions, with $C_4 = \{\pm 1, \pm i\}$ instead of C_6 and with the congruence condition $p \equiv 1 \; (4)$ rather than $p \equiv 1 \; (3)$, see [2] for details. For the square lattice, the first example of a genuinely multiple CSL occurs for $\Sigma = 25$. The corresponding situation is illustrated in Figure 2, and is completely analoguous to the situation met above for the triangular lattice, with $5 = (1 + 2i)(1 - 2i)$.

As we shall now show, the situation actually extends immediately to all rings of integers $\mathbb{Z}[\xi_n]$ in cyclotomic fields $\mathbb{Q}(\xi_n)$ with class number 1 (excluding $\mathbb{Q}$ itself). These emerge for the following 29 choices of n,

$$n \;\; \in \;\; \{3,4,5,7,8,9,11,12,13,15,16, \\ 17,19,20,21,24,25,27,28,32, \\ 33,35,36,40,44,45,48,60,84\}. \qquad (21)$$

Here, we view each set $\mathcal{O}_n = \mathbb{Z}[\xi_n]$ (i.e., the ring of polynomials in ξ_n) as a point set in $\mathbb{C}$, where ξ_n is a primitive n-th root of unity. To be explicit (which is not necessary), we choose $\xi_n = \exp(2\pi \mathrm{i}/n)$. Apart from $n = 1$ (where $\mathcal{O}_1 = \mathbb{Z}$ is one-dimensional), the values of n in (21) correspond to all cases where $\mathbb{Z}[\xi_n]$ is a principal ideal domain and thus has class number one, see [20, 4] for details. If n is odd, we have $\mathcal{O}_n = \mathcal{O}_{2n}$. Consequently, $\mathcal{O}_n$ has N-fold symmetry, where

$$N \; = \; N(n) \; = \; \mathrm{lcm}(2,n). \qquad (22)$$

To avoid duplication of results, integers $n \equiv 2 \; (4)$ do neither appear in the above list (21) nor in our further exposition.

The values of n from the list (21) are naturally grouped according to $\phi(n)$, which is Euler's totient function

$$\phi(n) \; = \; \bigl| \{1 \le k \le n \mid \gcd(k,n) = 1\} \bigr| . \qquad (23)$$

The set $\mathcal{O}_n$, which is the so-called maximal order [10] of the cyclotomic field $\mathbb{Q}(\xi_n)$, turns into a lattice in $\mathbb{R}^{\phi(n)}$ by means of Minkowski's embedding, compare [10], while it is a dense subset of $\mathbb{R}^2$ whenever $\phi(n) > 2$. Note that $\phi(n) = 2$ covers the two crystallographic cases $n = 3$ (triangular lattice) and $n = 4$ (square lattice), while $\phi(n) = 4$ means $n \in \{5, 8, 12\}$ which are the standard symmetries of planar quasicrystals, as they occur as layers in so-called T-phases, compare [18]. Again, $n = 6$ and $n = 10$ are covered implicitly, as explained above.

The rings of integers $\mathcal{O}_n$ arise in quasicrystal theory in several ways. One is in the form of the Fourier module that supports the Bragg peaks of X-ray diffraction [18]. Another is via the limit translation module of a (discrete) quasiperiodic tiling [3]. Characteristic points of the latter (e.g., vertex points) are then model sets (or cut and project sets) on the basis of the entire ring $\mathcal{O}_n$, viewed in parallel as a (dense) point set in $\mathbb{R}^2$ and as a lattice in $\mathbb{R}^{\phi(n)}$, see [2, 14] for details. Such sets are also called cyclotomic model sets.

To come to an efficient formulation, one needs another field, namely the maximal real subfield of $\mathbb{Q}(\xi_n)$, and its ring of integers. From now on, we shall thus use the following notation:

$$\begin{array}{lcllcl} \mathbb{K}_n & = & \mathbb{Q}(\xi_n), & \quad \mathbb{L}_n & = & \mathbb{Q}(\xi_n + \bar{\xi}_n), \\ \mathcal{O}_n & = & \mathbb{Z}[\xi_n], & \quad \scriptstyle\mathcal{O}_n & = & \mathbb{Z}[\xi_n + \bar{\xi}_n]. \end{array} \qquad (24)$$

The next result about the relation between these fields and rings of integers is standard [20].

Lemma 1 *Let ξ_n be a primitive n-th root of unity. The smallest field extension of $\mathbb{Q}$ that contains ξ_n is the cyclotomic field $\mathbb{K}_n$. It is an extension of $\mathbb{Q}$ of degree $\phi(n)$, with $\mathcal{O}_n$ as its ring of integers (and its maximal order).*

Moreover, $\mathbb{L}_n$ is the maximal real subfield of $\mathbb{K}_n$, with $\scriptstyle\mathcal{O}_n$ as its ring of integers. For $n \ge 3$, $\mathbb{K}_n$ is an extension of $\mathbb{L}_n$ of degree 2, while $\mathbb{L}_n$ has degree $\frac{1}{2}\phi(n)$ over $\mathbb{Q}$. □

[1] We thank Peter Pleasants for pointing this out to us.

In our example in Section 2, we had $n = 3$. Consequently, with $\xi_3 = e^{2\pi i/3}$, one finds $\xi_3 + \bar{\xi}_3 = -1$, so that $\mathbb{L}_3 = \mathbb{Q}$ and $\scriptstyle\mathcal{O}_3 = \mathbb{Z}$. Also, $\mathbb{L}_4 = \mathbb{Q}$ and $\scriptstyle\mathcal{O}_4 = \mathbb{Z}$, while for $\phi(n) = 4$, one finds the three cases

$$\begin{aligned} \mathbb{L}_5 &= \mathbb{Q}(\sqrt{5}), & \scriptstyle\mathcal{O}_5 &= \mathbb{Z}[\tau], \\ \mathbb{L}_8 &= \mathbb{Q}(\sqrt{2}), & \scriptstyle\mathcal{O}_8 &= \mathbb{Z}[\sqrt{2}], \\ \mathbb{L}_{12} &= \mathbb{Q}(\sqrt{3}), & \scriptstyle\mathcal{O}_{12} &= \mathbb{Z}[\sqrt{3}], \end{aligned}$$

with $\tau = (\sqrt{5}+1)/2$ being the golden ratio.

4 Simple coincidences for cyclotomic integers

One advantage of the number theoretic formulation is that the treatment of $\varGamma = \mathcal{O}_3$ from Section 2 can be extended to $\mathcal{O}_n$ for all n from the list (21). For simple coincidences, this was done by Pleasants et al. in [15]. This first requires the extension of the basic definitions from lattices to more general $\mathbb{Z}$-modules of finite rank, imbedded in Euclidean space. This is possible without difficulty in the natural group theoretic setting [2, 15], leading to the concept of a coincidence site module (CSM).

Here, we shall first summarise the main results of [15], and then completely work out all 29 cases. The extension to multiple coincidences follows in the next section.

Below, we shall always make the assumption that we deal with $\mathcal{O}_n$, where n is an element from our list (21). Since these are the cases with class number one, we shall abbreviate this assumption as **(CN1)**. In this case, also the maximal real subfields $\mathbb{L}_n$ have class number one [20, Prop. 11.19].

As in our example ($n = 3$) in Section 2, a prime $\mathfrak{p}$ in $\scriptstyle\mathcal{O}_n$, relative to $\mathcal{O}_n$, is either ramified, inert, or splits into a pair of complex conjugate primes of $\mathcal{O}_n$. However, in addition to our example, a rational prime p may or may not factorise into several primes of $\scriptstyle\mathcal{O}_n$, i.e., we need to look at the splitting pattern of rational primes with respect to the double extension $\mathbb{K}_n/\mathbb{L}_n/\mathbb{Q}$. Fortunately, the relevant contribution to the coincidence problem only comes from the so-called *complex splitting primes* $\mathcal{C}$. They form a subset of the rational primes $\mathcal{P}$, characterised by splitting (at least) in the extension step from $\mathbb{L}_n$ to $\mathbb{K}_n$. Various examples of the relevant splitting structure are given in [15], see also [5] for their appearance in a related problem.

It was shown in [15] that, under **(CN1)**, an integer $k \in \mathbb{N}$ is a non-trivial coincidence index if and only if it is a finite product of so-called basic indices [15]. They emerge from the structure of prime ideals in a way that we shall briefly recall below. The corresponding set of indices is thus a semigroup with unit, a property that is lost without **(CN1)**.

In our situation with **(CN1)**, the coincidence group is Abelian and has the structure

$$\mathrm{SOC}(\mathcal{O}_n) \;=\; C_N \times \mathbb{Z}^{(\aleph_0)},$$

with N according to Eq. (22), very much in analogy to (11), which is the case $n = 3$, though with slightly more complicated expressions for the generators, compare [15]. As above, $\mathbb{Z}^{(\aleph_0)}$ indicates the restriction that each element is a *finite* product of powers of generators.

Theorem 1 (CN1). *Let $\mathcal{C} \subset \mathcal{P}$ denote the subset of complex splitting primes for the field extension $\mathbb{K}_n/\mathbb{Q}$. If $c_n(k)$ denotes the number of simple CSMs of $\mathcal{O}_n$ of index k, its Dirichlet series generating function is given by*

$$\varPhi_n(s) \;=\; \sum_{k=1}^{\infty} \frac{c_n(k)}{k^s} \;=\; \prod_{p\in\mathcal{C}} \left(\frac{1+p^{-\ell_p s}}{1-p^{-\ell_p s}}\right)^{m_p/2},$$

with characteristic integers ℓ_p and m_p as given in Tables 1 and 2. This Dirichlet series converges absolutely on the half-plane $\{\mathrm{Re}(s) > 1\}$. It can be expressed as a ratio of zeta functions,

$$\varPhi_n(s) \;=\; \frac{\zeta_{\mathbb{K}_n}(s)}{\zeta_{\mathbb{L}_n}(2s)} \cdot \begin{cases} (1+p^{-s})^{-1}, & \text{if } n = p^r, \\ 1, & \text{otherwise,} \end{cases}$$

where $\zeta_{\mathbb{F}}(s)$ denotes the Dedekind zeta function of the field $\mathbb{F}$. The first terms of these Dirichlet series generating functions are given in Table 4. □

PROOF. Though a proof was given in [15], we briefly recall the relevant ingredients for the convenience of the reader.

A rotation $z = e^{i\varphi}$ produces a CSM of $\mathcal{O}_n$ if and only if it can be written as the quotient u/v of two integers $u, v \in \mathcal{O}_n$, meaning that v is rotated into u, and one has

$$\mathrm{SOC}(\mathcal{O}_n) \;=\; \bigl\{ z \in \mathbb{K}_n \;\big|\; |z| = 1 \bigr\}.$$

Moreover, due to **(CN1)**, the integers u and v can always be chosen coprime (up to units from $\mathcal{O}_n$), and the representation $z = u/v$ is unique in this sense.

Observe that the *relative norm* of $z \in \mathbb{K}_n$ over the maximal real subfield $\mathbb{L}_n$ is $\mathrm{norm}_{\mathbb{K}_n/\mathbb{L}_n}(z) = |z|^2$, where relative norms of integers (resp. units) in $\mathcal{O}_n$ are integers (resp. units) in $\scriptstyle\mathcal{O}_n$. For $z = u/v \in \mathrm{SOC}(\mathcal{O}_n)$, with u, v coprime, we clearly must have

$$\mathrm{norm}_{\mathbb{K}_n/\mathbb{L}_n}(u) \;=\; \mathrm{norm}_{\mathbb{K}_n/\mathbb{L}_n}(v) \;=\; \nu \;\in\; \scriptstyle\mathcal{O}_n.$$

Consequently, every prime factor of ν in $\scriptstyle\mathcal{O}_n$ must now factor into two non-associated primes of $\mathcal{O}_n$, one of which divides u only, while the other divides v only. This restricts us to primes $\mathfrak{p} \in \scriptstyle\mathcal{O}_n$ that split as $\mathfrak{p} = \omega_{\mathfrak{p}}\bar{\omega}_{\mathfrak{p}}$ in the extension from $\mathbb{L}_n$ to $\mathbb{K}_n$, with $\omega_{\mathfrak{p}}/\bar{\omega}_{\mathfrak{p}}$ *not* a unit in $\mathcal{O}_n$. They are precisely the ones lying over the rational complex splitting primes p for the extension $\mathbb{K}_n/\mathbb{Q}$. All other primes have cancelled out in the quotient u/v, possibly up to a unit. Each $z \in \mathrm{SOC}(\mathcal{O}_n)$ can thus be written as a finite product,

$$z \;=\; \varepsilon \prod_{\mathfrak{p}} \left(\frac{\omega_{\mathfrak{p}}}{\bar{\omega}_{\mathfrak{p}}}\right)^{t_{\mathfrak{p}}},$$

where $\mathfrak{p}$ runs over the primes of $\scriptstyle\mathcal{O}_n$ that divide ν and split as described, and ε is a unit in $\mathcal{O}_n$.

Given a complex splitting prime $p \in \mathcal{C}$, there are $m_p/2$ pairwise non-associated primes $\mathfrak{p}_i \in \scriptstyle\mathcal{O}_n$ that lie over p. Consequently, twice as many primes appear in $\mathcal{O}_n$, coming in complex conjugate pairs $\{\omega_{\mathfrak{p}_i}, \bar{\omega}_{\mathfrak{p}_i}\}$. Each single such prime has norm p^{ℓ_p} (meaning that the principal ideal defined by it has index p^{ℓ_p} in $\mathcal{O}_n$). Moreover, the CSM obtained from

Table 1. Complex splitting primes of $\mathbb{K}_n/\mathbb{Q}$, with n from the list (21), that originate from rational primes $p \nmid n$ (unramified). The symbol $p^{\ell_p}_{(k)}$ means that primes $p \equiv k \ (n)$ contribute via p^{ℓ_p} as basic index, where ℓ_p is the smallest integer such that $k^{\ell_p} \equiv 1 \ (n)$, and the integer m_p that appears in Theorems 1 and 2 is $m_p = \phi(n)/\ell_p$. Note that m_p is always even.

$\phi(n)$	n	basic indices
2	3	$p^1_{(1)}$
	4	$p^1_{(1)}$
4	5	$p^1_{(1)}$
	8	$p^1_{(1)}, p^2_{(3)}, p^2_{(5)}$
	12	$p^1_{(1)}, p^2_{(5)}, p^2_{(7)}$
6	7	$p^1_{(1)}, p^3_{(2)}, p^3_{(4)}$
	9	$p^1_{(1)}, p^3_{(4)}, p^3_{(7)}$
8	15	$p^1_{(1)}, p^2_{(4)}, p^2_{(11)}, p^4_{(2)}, p^4_{(7)}, p^4_{(8)}, p^4_{(13)}$
	16	$p^1_{(1)}, p^2_{(7)}, p^2_{(9)}, p^4_{(3)}, p^4_{(5)}, p^4_{(11)}, p^4_{(13)}$
	20	$p^1_{(1)}, p^2_{(9)}, p^2_{(11)}, p^4_{(3)}, p^4_{(7)}, p^4_{(13)}, p^4_{(17)}$
	24	$p^1_{(1)}, p^2_{(5)}, p^2_{(7)}, p^2_{(11)}, p^2_{(13)}, p^2_{(17)}, p^2_{(19)}$
10	11	$p^1_{(1)}, p^5_{(3)}, p^5_{(4)}, p^5_{(5)}, p^5_{(9)}$
12	13	$p^1_{(1)}, p^3_{(3)}, p^3_{(9)}$
	21	$p^1_{(1)}, p^2_{(8)}, p^2_{(13)}, p^3_{(4)}, p^3_{(16)}, p^6_{(2)}, p^6_{(10)}, p^6_{(11)}, p^6_{(19)}$
	28	$p^1_{(1)}, p^2_{(13)}, p^2_{(15)}, p^3_{(9)}, p^3_{(25)}, p^6_{(5)}, p^6_{(11)}, p^6_{(17)}, p^6_{(23)}$
	36	$p^1_{(1)}, p^2_{(17)}, p^2_{(19)}, p^3_{(13)}, p^3_{(25)}, p^6_{(5)}, p^6_{(7)}, p^6_{(29)}, p^6_{(31)}$
16	17	$p^1_{(1)}$
	32	$p^1_{(1)}, p^2_{(15)}, p^2_{(17)}, p^4_{(7)}, p^4_{(9)}, p^4_{(23)}, p^4_{(25)}, p^8_{(3)}, p^8_{(5)}, p^8_{(11)}, p^8_{(13)}, p^8_{(19)}, p^8_{(21)}, p^8_{(27)}, p^8_{(29)}$
	40	$p^1_{(1)}, p^2_{(9)}, p^2_{(11)}, p^2_{(19)}, p^2_{(21)}, p^2_{(29)}, p^2_{(31)}, p^4_{(3)}, p^4_{(7)}, p^4_{(13)}, p^4_{(17)}, p^4_{(23)}, p^4_{(27)}, p^4_{(33)}, p^4_{(37)}$
	48	$p^1_{(1)}, p^2_{(7)}, p^2_{(17)}, p^2_{(23)}, p^2_{(25)}, p^2_{(31)}, p^2_{(41)}, p^4_{(5)}, p^4_{(11)}, p^4_{(13)}, p^4_{(19)}, p^4_{(29)}, p^4_{(35)}, p^4_{(37)}, p^4_{(43)}$
	60	$p^1_{(1)}, p^2_{(11)}, p^2_{(19)}, p^2_{(29)}, p^2_{(31)}, p^2_{(41)}, p^2_{(49)}, p^4_{(7)}, p^4_{(13)}, p^4_{(17)}, p^4_{(23)}, p^4_{(37)}, p^4_{(43)}, p^4_{(47)}, p^4_{(53)}$
18	19	$p^1_{(1)}, p^3_{(7)}, p^3_{(11)}, p^9_{(4)}, p^9_{(5)}, p^9_{(6)}, p^9_{(9)}, p^9_{(16)}, p^9_{(17)}$
	27	$p^1_{(1)}, p^3_{(10)}, p^3_{(19)}, p^9_{(4)}, p^9_{(7)}, p^9_{(13)}, p^9_{(16)}, p^9_{(22)}, p^9_{(25)}$
20	25	$p^1_{(1)}, p^5_{(6)}, p^5_{(11)}, p^5_{(16)}, p^5_{(21)}$
	33	$p^1_{(1)}, p^2_{(10)}, p^2_{(23)}, p^5_{(4)}, p^5_{(16)}, p^5_{(25)}, p^5_{(31)}, p^{10}_{(5)}, p^{10}_{(7)}, p^{10}_{(13)}, p^{10}_{(14)}, p^{10}_{(19)}, p^{10}_{(20)}, p^{10}_{(26)}, p^{10}_{(28)}$
	44	$p^1_{(1)}, p^2_{(21)}, p^2_{(23)}, p^5_{(5)}, p^5_{(9)}, p^5_{(25)}, p^5_{(37)}, p^{10}_{(3)}, p^{10}_{(13)}, p^{10}_{(15)}, p^{10}_{(17)}, p^{10}_{(27)}, p^{10}_{(29)}, p^{10}_{(31)}, p^{10}_{(41)}$
24	35	$p^1_{(1)}, p^2_{(6)}, p^2_{(29)}, p^3_{(11)}, p^3_{(16)}, p^4_{(8)}, p^4_{(13)}, p^4_{(22)}, p^4_{(27)}, p^6_{(4)}, p^6_{(9)}, p^6_{(26)}, p^6_{(31)},$ $p^{12}_{(2)}, p^{12}_{(3)}, p^{12}_{(12)}, p^{12}_{(17)}, p^{12}_{(18)}, p^{12}_{(23)}, p^{12}_{(32)}, p^{12}_{(33)}$
	45	$p^1_{(1)}, p^2_{(19)}, p^2_{(26)}, p^3_{(16)}, p^3_{(31)}, p^4_{(8)}, p^4_{(17)}, p^4_{(28)}, p^4_{(37)}, p^6_{(4)}, p^6_{(11)}, p^6_{(34)}, p^6_{(41)},$ $p^{12}_{(2)}, p^{12}_{(7)}, p^{12}_{(13)}, p^{12}_{(22)}, p^{12}_{(23)}, p^{12}_{(32)}, p^{12}_{(38)}, p^{12}_{(43)}$
	84	$p^1_{(1)}, p^2_{(13)}, p^2_{(29)}, p^2_{(41)}, p^2_{(43)}, p^2_{(55)}, p^2_{(71)}, p^3_{(25)}, p^3_{(37)}, p^6_{(5)}, p^6_{(11)}, p^6_{(17)}, p^6_{(19)},$ $p^6_{(23)}, p^6_{(31)}, p^6_{(53)}, p^6_{(61)}, p^6_{(65)}, p^6_{(67)}, p^6_{(73)}, p^6_{(79)}$

Table 2. Contribution to splitting primes by ramified rational primes with corresponding integers ℓ_p and m_p for $\mathbb{Z}[\xi_n]$, with n from the list (21). Here, r is the p-free part of n, and $\ell_p\, m_p = \phi(r)$. The corresponding basic index for the coincidence spectrum is always p^{ℓ_p}.

$\phi(n)$	n	p	r	$\phi(r)$	ℓ_p	m_p
8	20	5	4	2	1	2
	24	3	8	4	2	2
12	21	7	3	2	1	2
	28	2	7	6	3	2
16	40	5	8	4	2	2
	48	3	16	8	4	2
	60	2	15	8	4	2
		3	20	8	4	2
		5	12	4	2	2
20	33	3	11	10	5	2
24	84	2	21	12	6	2
		7	12	4	2	2

a rotation z as above is the principal ideal generated by its numerator, which has index

$$\prod_{\mathfrak{p}} \bigl(\mathrm{norm}_{\mathbb{K}_n/\mathbb{Q}}(\omega_{\mathfrak{p}})\bigr)^{|t_{\mathfrak{p}}|} \;=\; \prod_{\mathfrak{p}} p^{\ell_p |t_{\mathfrak{p}}|}.$$

Collect now all possibilities that result in the same index. For each single factor, the corresponding rotation can either be clockwise or anticlockwise, which give different CSMs. This contributes to the overall generating function a factor of the form

$$\frac{1+x}{1-x} \;=\; 1 + 2x + 2x^2 + 2x^3 + \dots$$

with x replaced by p^{ℓ_p}, which then appears precisely $m_p/2$ times. Taking the product over all complex splitting primes, the first claim follows.

The convergence statement is standard, as is the representation by means of zeta functions. The latter will become clearer from our further discussion, too. □

The integers ℓ_p and m_p are characteristic quantities that depend on n and on the prime p. In particular, m_p is the number of prime ideal divisors of p in $\mathcal{O}_n$, and ℓ_p is their degree (also called the residue class degree) [20]. With the usual approach of algebraic number theory, its determination (and that of the corresponding m_p) depends on whether p divides n or not. If $p \nmid n$, ℓ_p is the smallest integer such that $p^{\ell_p} \equiv 1 \;(n)$, and $m_p = \phi(n)/\ell_p$. Clearly, the result only depends on the residue class of p modulo n. The numbers ℓ_p and m_p are contained in Table 1. If, however, $p \mid n$, one writes $n = p^t r$, where r is the p-free part of n. Now, ℓ_p is the smallest integer such that $p^{\ell_p} \equiv 1 \;(r)$ and $m_p = \phi(r)/\ell_p$, see Table 2. For a step-by-step derivation of these claims, see Facts 1–3 and Appendix B of [15].

Referring back to [6, 20], and to the theory of Dirichlet characters summarized there, one can express the zeta functions in terms of L-series as follows,

$$\zeta_{\mathbb{K}_n}(s) \;=\; \prod_{\chi\in\widehat{G}_n} L(s,\chi), \tag{25}$$

$$\zeta_{\mathbb{L}_n}(s) \;=\; \prod_{\substack{\chi\in\widehat{G}_n\\ \chi \text{ even}}} L(s,\chi), \tag{26}$$

where a character χ is called *even* if $\chi(-1) = 1$ and the L-series read

$$L(s,\chi) \;=\; \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s}.$$

As in [6, 20], $\widehat{G}_n$ denotes the set of primitive Dirichlet characters for the field extension $\mathbb{K}_n/\mathbb{Q}$, and thus contains $\phi(n)$ elements. The principal character in this formulation, $\chi_0 \equiv 1$, always leads to $L(s,\chi_0) = \zeta(s)$, i.e., to Riemann's zeta function itself. This is a meromorphic function on $\mathbb{C}$, with a single simple pole of residue 1 at $s = 1$, and no zeros in the closed half-plane $\{s\in\mathbb{C} \mid \mathrm{Re}(s) \ge 1\}$. All remaining L-series are entire functions on $\mathbb{C}$, without any zeros in this half-plane either, compare [1, Thm. 12.5] or [20, Ch. 4].

Note that

$$\zeta_{\mathbb{K}_n}(s) \;=\; \sum_{k=1}^{\infty} \frac{a_n(k)}{k^s},$$

under the assumption (**CN1**), counts the *principal* ideals of $\mathcal{O}_n$, so that $a_n(k)$ is the number of similarity sublattices or submodules of $\mathcal{O}_n$ of index k. It was shown in [6, Thm. 2] that

$$A_n(x) \;:=\; \sum_{k\le x} a_n(k) \;\sim\; \alpha_n x,$$

with the growth constant

$$\alpha_n \;=\; \mathrm{res}_{s=1}\bigl(\zeta_{\mathbb{K}_n}(s)\bigr) \;=\; \prod_{1\not\equiv\chi\in\widehat{G}_n} L(1,\chi).$$

Examples are given in Table 3.

Let us next explain how to calculate the coefficients $c_n(k)$ defined in Theorem 1 explicitly. Under our assumption (**CN1**), the arithmetic function $c_n(k)$ is multiplicative, whence it is sufficient to know the values $c_n(p^r)$ for all primes p and powers $r > 0$. On the level of the generating function, this corresponds to expanding the factors $E(p^{-s})$ of the Euler products $\prod_p E(p^{-s})$, where p runs over the primes of $\mathbb{Z}$.

Here and below, each Euler factor is either of the form

$$\begin{aligned} E(p^{-s}) \;&=\; \frac{1}{(1-p^{-\ell s})^m} \\ &=\; \sum_{j=0}^{\infty} \binom{j+m-1}{m-1} \frac{1}{(p^s)^{\ell j}} \end{aligned} \tag{27}$$

or, as for $\varPhi(s)$ above, reads

$$\begin{aligned} E(p^{-s}) \;&=\; \left(\frac{1+p^{-\ell s}}{1-p^{-\ell s}}\right)^m \\ &=\; \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} \binom{m}{i}\binom{m-1+j-i}{m-1}\right) \frac{1}{(p^s)^{\ell j}}. \end{aligned} \tag{28}$$

Table 3. Numerical values of residues (at $s = 1$) of Dedekind zeta functions and related Dirichlet series generating functions.

$\phi(n)$	n	α_n	β_n	γ_n
2	3	0.604 600	0.285 041	0.275 664
	4	0.785 398	0.336 193	0.318 310
4	5	0.339 837	0.249 136	0.243 785
	8	0.543 676	0.258 663	0.252 584
	12	0.361 051	0.235 129	0.231 117
6	7	0.287 251	0.240 733	0.235 393
	9	0.333 685	0.216 086	0.213 490
8	15	0.215 279	0.202 672	0.200 651
	16	0.464 557	0.230 565	0.226 884
	20	0.288 769	0.267 051	0.255 192
	24	0.299 995	0.222 573	0.218 534
10	11	0.239 901	0.217 016	0.214 504
12	13	0.213 514	0.195 868	0.194 563
	21	0.227 271	0.225 794	0.220 216
	28	0.251 795	0.245 523	0.239 718
	36	0.220 933	0.192 881	0.191 637
16	17	0.153 708	0.143 107	0.142 875
	32	0.278 432	0.137 913	0.137 709
	40	0.210 946	0.197 390	0.195 931
	48	0.222 767	0.166 373	0.165 820
	60	0.195 032	0.194 455	0.192 773
18	19	0.121 018	0.113 726	0.113 651
	27	0.212 854	0.141 292	0.141 081
20	25	0.181 458	0.144 724	0.144 466
	33	0.159 226	0.157 420	0.156 974
	44	0.126 912	0.124 934	0.124 741
24	35	0.166 239	0.166 011	0.165 506
	45	0.118 121	0.116 387	0.116 302
	84	0.116 090	0.115 471	0.115 336

These two formulas follow easily from expanding

$$\left(\frac{1}{1-x}\right)^m \quad \text{and} \quad \left(\frac{1+x}{1-x}\right)^m$$

into power series, followed by inserting $p^{-\ell s}$ for x. From (27) and (28), one then quickly extracts the values of $c_n(p^r)$ for $r \geq 0$.

The Dirichlet series generating function of an arithmetic function permits the derivation of asymptotic properties (here, of $c_n(k)$) or, more precisely, of the corresponding summatory function, $\sum_{k \leq x} c_n(k)$, as $x \to \infty$. Here, Delange's theorem (see [19, Ch. II.7, Thm. 15] for the general formulation, and [6, Prop. 4] for the reduction to the situation at hand) leads to the following result.

Corollary 1 (CN1). *The number of simple CSMs of index $\leq x$ is given by*

$$\sum_{k \leq x} c_n(k) \;\sim\; x \cdot \bigl(\operatorname{res}_{s=1} \Phi_n(s)\bigr) \;=\; x \cdot \gamma_n,$$

with the residue

$$\gamma_n \;=\; \frac{\alpha_n}{\zeta_{\mathbb{L}_n}(2)} \cdot \begin{cases} p/(p+1), & \text{if } n = p^r, \\ 1, & \text{otherwise.} \end{cases}$$

Here, $\alpha_n = \operatorname{res}_{s=1} \zeta_{\mathbb{K}_n}(s) = \prod_{1 \not\equiv \chi \in \widehat{G}_n} L(1,\chi)$, with $\widehat{G}_n$ denoting the set of primitive Dirichlet characters of $\mathbb{K}_n/\mathbb{Q}$.

PROOF. The coefficients $c_n(k)$ are non-negative numbers, and their generating function $\Phi_n(s)$, by Theorem 1, is a meromorphic function on the half-plane $\{\mathrm{Re}(s) > 1/2\}$, with a single simple pole at $s = 1$. Delange's theorem then results in the asymptotic linear growth as claimed, with the residue of $\Phi_n(s)$ at $s = 1$ as growth constant. Its calculation is obvious from Theorem 1 together with formulas (25) and (26). □

One way to calculate the residues uses the factorization of our zeta functions into L-series, followed by rewriting the latter in terms of the Hurwitz zeta function [1, Ch. 12.1]

$$\zeta(s,a) \;=\; \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}, \tag{29}$$

which is well defined for $0 < a \leq 1$ and absolutely convergent on the half-plane $\{\mathrm{Re}(s) > 1\}$. On $\mathbb{C}$, analytic continuation defines $\zeta(s,a)$ as a meromorphic function with a single simple pole of residue 1 at $s = 1$, see [1, Thm. 12.4]. In particular, $\zeta(s,1) = \zeta(s)$. Furthermore, if $\Gamma(s)$ denotes the ordinary gamma function, one has

$$\lim_{s \to 1} \Bigl(\zeta(s,a) - \frac{1}{s-1}\Bigr) \;=\; -\frac{\Gamma'(a)}{\Gamma(a)}, \tag{30}$$

see [21, P. 271]. For further connections to the generalised Euler constants and the Laurent series of the Hurwitz zeta function at $s = 1$, see [8].

Next, if a Dirichlet character has period q (e.g., if $q = f_\chi$ is the conductor of χ, which is its minimal period), one has the representation [1]

$$L(s,\chi) \;=\; q^{-s} \sum_{r=1}^{q} \chi(r)\, \zeta(s, \tfrac{r}{q}). \tag{31}$$

Moreover, if χ is not the principal character, one can rewrite this equation as

$$L(s,\chi) \;=\; q^{-s} \sum_{r=1}^{q} \chi(r) \Bigl(\zeta(s, \tfrac{r}{q}) - \frac{1}{s-1}\Bigr), \tag{32}$$

because $\sum_{r=1}^{q} \chi(r) = 0$ for non-principal characters by the orthogonality relation. This shows regularity at $s = 1$, and with Eq. (30) one obtains

$$L(1,\chi) \;=\; -\frac{1}{q} \sum_{r=1}^{q} \chi(r)\, \frac{\Gamma'(\frac{r}{q})}{\Gamma(\frac{r}{q})}. \tag{33}$$

In [6], we additionally used the class number formula to transform the calculation of $\mathrm{res}_{s=1}\bigl(\zeta_{\mathbb{K}_n}(s)\bigr)$ into an algebraic problem. However, given the fact that $\zeta(s,a)$ and $\Gamma(s)$ are accessible by means of algebraic program packages to arbitrary numerical precision, it seems easier to directly use formulas (25) and (26) together with (31) for these calculations. Some results[2] are given in Table 3.

5 Multiple coincidences

In generalisation of what we explained in Section 2, we are now interested in multiple intersections of the form

$$\mathcal{O}_n \cap e^{i\varphi_1}\mathcal{O}_n \cap \ldots \cap e^{i\varphi_m}\mathcal{O}_n,$$

provided that this is a submodule of full rank, and the corresponding index

$$\Sigma(\varphi_1,\ldots,\varphi_m) \; := \; [\mathcal{O}_n : (\mathcal{O}_n \cap e^{i\varphi_1}\mathcal{O}_n \cap \ldots \cap e^{i\varphi_m}\mathcal{O}_n)].$$

We call such intersections *multiple coincidence site modules*, or multiple CSMs for short.

Theorem 2 (CN1). *Let $b_n(k)$ denotes the number of multiple CSMs of $\mathcal{O}_n$ of index k, including the simple ones. Then, its Dirichlet series generating function is given by*

$$\Psi_n(s) \; = \; \sum_{k=1}^{\infty} \frac{b_n(k)}{k^s} \; = \; \prod_{p\in\mathcal{C}} \left(\frac{1}{1-p^{-\ell_p s}}\right)^{m_p},$$

with $\mathcal{C}$ once again the complex splitting primes of the field extension $\mathbb{K}_n/\mathbb{Q}$ and ℓ_p, m_p the characteristic integers introduced above. This Dirichlet series converges absolutely on the half-plane $\{\mathrm{Re}(s)>1\}$. The first terms of the difference $\Psi_n(s)-\Phi_n(s)$ are given in Table 5.

PROOF. Observe as before that

$$\mathcal{O}_n \cap e^{i\varphi_1}\mathcal{O}_n \cap \ldots \cap e^{i\varphi_m}\mathcal{O}_n \; = \; \bigcap_{\ell=1}^{m} (\mathcal{O}_n \cap e^{i\varphi_\ell}\mathcal{O}_n).$$

This shows that the multiple intersection is a submodule of $\mathcal{O}_n$ of full rank $\phi(n)$ if and only if each $\mathcal{O}_n \cap e^{i\varphi_\ell}\mathcal{O}_n$ is a simple CSM of $\mathcal{O}_n$, each of which is a principal ideal of $\mathcal{O}_n$ due to **(CN1)**. Consequently, the multiple intersection is the lcm of the simple CSMs involved, as in Eq. (17).

For each prime ideal $\mathfrak{p}$ of $\mathcal{O}_n$ that lies over the complex splitting prime p, the combinatorial argument used in the proof of Proposition 1 applies *independently*. This modifies each factor of the Euler product decomposition of Theorem 1 in the same way as in our previous example (the triangular lattice), i.e., each term of the form $(1+x)$ in the numerator is replaced by a term $(1-x)^{-1}$, with $x=p^{-\ell_p s}$. This gives the new Euler product, which clearly converges as claimed. □

Comparing Ψ_n with Φ_n, one notices that the two Dirichlet series have precisely the same terms non-vanishing, i.e., $b_n(k)\neq 0$ if and only if $c_n(k)\neq 0$. This implies

Corollary 2 (CN1). *The spectrum of possible coincidence indices remains unchanged by the addition of multiple coincidences. In particular, the total spectrum $\Sigma_{\mathcal{O}_n}$ is the semigroup generated by 1 and the basic indices given in Tables 1 and 2.* □

Clearly, there is an ordering, $\Psi_n(s) \succcurlyeq \Phi_n(s)$, in the sense that $b_n(k) \geq c_n(k)$ for all $k\geq 1$. In Table 5, we list the first terms of the difference, $\Psi_n(s)-\Phi_n(s)$. In fact, one quickly checks that

$$\zeta_{\mathbb{K}_n}(s) \; \succcurlyeq \; \Psi_n(s) \; \succcurlyeq \; \Phi_n(s) \tag{34}$$

in this sense, because every CSM of $\mathcal{O}_n$ is a principal ideal of this ring, but not necessarily vice versa, so that

$$a_n(k) \; \geq \; b_n(k) \; \geq \; c_n(k)$$

for all $k\in\mathbb{N}$. The summatory functions for the coefficients of $\Phi_n(s)$ and of $\zeta_{\mathbb{K}_n}(s)$ both show linear growth, see Corollary 1 and [6, Thm. 4]. Consequently, by monotonicity, the same behaviour shows up for the coefficients of $\Psi_n(s)$.

To gain a better understanding of $\Psi_n(s)$, and to prepare for our later analysis of asymptotic properties, we observe the fundamental relation

$$\Psi_n(s) \; = \; \Phi_n(s)\cdot\bigl\{\Psi_n(2s)\bigr\}^{1/2}, \tag{35}$$

which is certainly valid for all $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$. This relation can directly be derived from the Euler product expansions of Ψ_n and Φ_n, compare [16, Sec. 2] for a similar situation and [13] for a historical perspective.

Moreover, iterating Eq. (35) leads to

Proposition 2 *The two Dirichlet series generating functions $\Phi_n(s)$ and $\Psi_n(s)$ of Theorems 1 and 2 are related by*

$$\Psi_n(s) \; = \; \bigl(\Psi_n(2^{L+1}s)\bigr)^{1/2^{L+1}} \cdot \prod_{\ell=0}^{L} \bigl(\Phi_n(2^{\ell}s)\bigr)^{1/2^{\ell}},$$

for any integer $L\geq 0$. Moreover, one also has

$$\Psi_n(s) \; = \; \prod_{\ell=0}^{\infty} \bigl(\Phi_n(2^{\ell}s)\bigr)^{1/2^{\ell}} \; = \; \Phi_n(s)\cdot\prod_{\ell=1}^{\infty} \bigl(\Phi_n(2^{\ell}s)\bigr)^{1/2^{\ell}},$$

which is absolutely convergent on $\{\mathrm{Re}(s)>1\}$.

PROOF. The first claim follows by simple induction (in L) from Eq. (35).

If $\mathrm{Re}(s)>1$, it is clear that $\mathrm{Re}(2^L s)\longrightarrow\infty$ as $L\to\infty$. In this limit, by well-known properties of convergent Dirichlet series, one has

$$\Psi_n(2^L s)\longrightarrow b_n(1)=1 \quad\text{and}\quad \Phi_n(2^L s)\longrightarrow c_n(1)=1\,.$$

If we write $s=\sigma+it$, these limits are uniform in $t\in\mathbb{R}$, see [1, Thm. 11.2]. One obvious consequence is

$$\lim_{L\to\infty}\bigl(\Psi_n(2^L s)\bigr)^{1/2^L} \; = \; 1\,,$$

[2] Please note that the exact value of γ_7 shown in [15] should be replaced by $\gamma_7=\frac{21\sqrt{7}R}{16\pi^3}$, with the regulator R as given there.

Table 4. First terms of the Dirichlet series from Theorem 1 for simple coincidences, for integers from the list (21).

n	$\Phi_n(s)$
3	$1+\frac{2}{7^s}+\frac{2}{13^s}+\frac{2}{19^s}+\frac{2}{31^s}+\frac{2}{37^s}+\frac{2}{43^s}+\frac{2}{49^s}+\frac{2}{61^s}+\frac{2}{67^s}+\frac{2}{73^s}+\frac{2}{79^s}+\frac{4}{91^s}+\frac{2}{97^s}+\frac{2}{103^s}+\frac{2}{109^s}+\frac{2}{127^s}+\frac{4}{133^s}+\dots$
4	$1+\frac{2}{5^s}+\frac{2}{13^s}+\frac{2}{17^s}+\frac{2}{25^s}+\frac{2}{29^s}+\frac{2}{37^s}+\frac{2}{41^s}+\frac{2}{53^s}+\frac{2}{61^s}+\frac{4}{65^s}+\frac{2}{73^s}+\frac{4}{85^s}+\frac{2}{89^s}+\frac{2}{97^s}+\frac{2}{101^s}+\frac{2}{109^s}+\frac{2}{113^s}+\dots$
5	$1+\frac{4}{11^s}+\frac{4}{31^s}+\frac{4}{41^s}+\frac{4}{61^s}+\frac{4}{71^s}+\frac{4}{101^s}+\frac{8}{121^s}+\frac{4}{131^s}+\frac{4}{151^s}+\frac{4}{181^s}+\frac{4}{191^s}+\frac{4}{211^s}+\frac{4}{241^s}+\frac{4}{251^s}+\frac{4}{271^s}+\frac{4}{281^s}+\dots$
7	$1+\frac{2}{8^s}+\frac{6}{29^s}+\frac{6}{43^s}+\frac{2}{64^s}+\frac{6}{71^s}+\frac{6}{113^s}+\frac{6}{127^s}+\frac{6}{197^s}+\frac{6}{211^s}+\frac{12}{232^s}+\frac{6}{239^s}+\frac{6}{281^s}+\frac{6}{337^s}+\frac{12}{344^s}+\frac{6}{379^s}+\frac{6}{421^s}+\dots$
8	$1+\frac{2}{9^s}+\frac{4}{17^s}+\frac{2}{25^s}+\frac{4}{41^s}+\frac{4}{73^s}+\frac{2}{81^s}+\frac{4}{89^s}+\frac{4}{97^s}+\frac{4}{113^s}+\frac{2}{121^s}+\frac{4}{137^s}+\frac{8}{153^s}+\frac{2}{169^s}+\frac{4}{193^s}+\frac{4}{225^s}+\frac{4}{233^s}+\dots$
9	$1+\frac{6}{19^s}+\frac{6}{37^s}+\frac{6}{73^s}+\frac{6}{109^s}+\frac{6}{127^s}+\frac{6}{163^s}+\frac{6}{181^s}+\frac{6}{199^s}+\frac{6}{271^s}+\frac{6}{307^s}+\frac{2}{343^s}+\frac{18}{361^s}+\frac{6}{379^s}+\frac{6}{397^s}+\frac{6}{433^s}+\frac{6}{487^s}+\dots$
11	$1+\frac{10}{23^s}+\frac{10}{67^s}+\frac{10}{89^s}+\frac{10}{199^s}+\frac{2}{243^s}+\frac{10}{331^s}+\frac{10}{353^s}+\frac{10}{397^s}+\frac{10}{419^s}+\frac{10}{463^s}+\frac{50}{529^s}+\frac{10}{617^s}+\frac{10}{661^s}+\frac{10}{683^s}+\frac{10}{727^s}+\frac{10}{859^s}+\dots$
12	$1+\frac{4}{13^s}+\frac{2}{25^s}+\frac{4}{37^s}+\frac{2}{49^s}+\frac{4}{61^s}+\frac{4}{73^s}+\frac{4}{97^s}+\frac{4}{109^s}+\frac{4}{157^s}+\frac{8}{169^s}+\frac{4}{181^s}+\frac{4}{193^s}+\frac{4}{229^s}+\frac{4}{241^s}+\frac{4}{277^s}+\frac{2}{289^s}+\dots$
13	$1+\frac{4}{27^s}+\frac{12}{53^s}+\frac{12}{79^s}+\frac{12}{131^s}+\frac{12}{157^s}+\frac{12}{313^s}+\frac{12}{443^s}+\frac{12}{521^s}+\frac{12}{547^s}+\frac{12}{599^s}+\frac{12}{677^s}+\frac{8}{729^s}+\frac{12}{859^s}+\frac{12}{911^s}+\frac{12}{937^s}+\dots$
15	$1+\frac{2}{16^s}+\frac{8}{31^s}+\frac{8}{61^s}+\frac{4}{121^s}+\frac{8}{151^s}+\frac{8}{181^s}+\frac{8}{211^s}+\frac{8}{241^s}+\frac{2}{256^s}+\frac{8}{271^s}+\frac{8}{331^s}+\frac{4}{361^s}+\frac{8}{421^s}+\frac{16}{496^s}+\frac{8}{541^s}+\frac{8}{571^s}+\dots$
16	$1+\frac{8}{17^s}+\frac{4}{49^s}+\frac{2}{81^s}+\frac{8}{97^s}+\frac{8}{113^s}+\frac{8}{193^s}+\frac{8}{241^s}+\frac{8}{257^s}+\frac{32}{289^s}+\frac{8}{337^s}+\frac{8}{353^s}+\frac{8}{401^s}+\frac{8}{433^s}+\frac{8}{449^s}+\frac{4}{529^s}+\frac{8}{577^s}+\dots$
17	$1+\frac{16}{103^s}+\frac{16}{137^s}+\frac{16}{239^s}+\frac{16}{307^s}+\frac{16}{409^s}+\frac{16}{443^s}+\frac{16}{613^s}+\frac{16}{647^s}+\frac{16}{919^s}+\frac{16}{953^s}+\frac{16}{1021^s}+\frac{16}{1123^s}+\frac{16}{1259^s}+\frac{16}{1327^s}+\frac{16}{1361^s}+\dots$
19	$1+\frac{18}{191^s}+\frac{18}{229^s}+\frac{6}{343^s}+\frac{18}{419^s}+\frac{18}{457^s}+\frac{18}{571^s}+\frac{18}{647^s}+\frac{18}{761^s}+\frac{18}{1103^s}+\frac{18}{1217^s}+\frac{6}{1331^s}+\frac{18}{1483^s}+\frac{18}{1559^s}+\frac{18}{1597^s}+\dots$
20	$1+\frac{2}{5^s}+\frac{2}{25^s}+\frac{8}{41^s}+\frac{8}{61^s}+\frac{2}{81^s}+\frac{8}{101^s}+\frac{4}{121^s}+\frac{2}{125^s}+\frac{8}{181^s}+\frac{16}{205^s}+\frac{8}{241^s}+\frac{8}{281^s}+\frac{16}{305^s}+\frac{8}{401^s}+\frac{4}{405^s}+\frac{8}{421^s}+\dots$
21	$1+\frac{2}{7^s}+\frac{12}{43^s}+\frac{2}{49^s}+\frac{2}{64^s}+\frac{12}{127^s}+\frac{6}{169^s}+\frac{12}{211^s}+\frac{24}{301^s}+\frac{12}{337^s}+\frac{2}{343^s}+\frac{12}{379^s}+\frac{12}{421^s}+\frac{4}{448^s}+\frac{12}{463^s}+\frac{12}{547^s}+\frac{12}{631^s}+\dots$
24	$1+\frac{2}{9^s}+\frac{4}{25^s}+\frac{4}{49^s}+\frac{8}{73^s}+\frac{2}{81^s}+\frac{8}{97^s}+\frac{4}{121^s}+\frac{4}{169^s}+\frac{8}{193^s}+\frac{8}{225^s}+\frac{8}{241^s}+\frac{4}{289^s}+\frac{8}{313^s}+\frac{8}{337^s}+\frac{4}{361^s}+\frac{8}{409^s}+\dots$
25	$1+\frac{20}{101^s}+\frac{20}{151^s}+\frac{20}{251^s}+\frac{20}{401^s}+\frac{20}{601^s}+\frac{20}{701^s}+\frac{20}{751^s}+\frac{20}{1051^s}+\frac{20}{1151^s}+\frac{20}{1201^s}+\frac{20}{1301^s}+\frac{20}{1451^s}+\frac{20}{1601^s}+\frac{20}{1801^s}+\dots$
27	$1+\frac{18}{109^s}+\frac{18}{163^s}+\frac{18}{271^s}+\frac{18}{379^s}+\frac{18}{433^s}+\frac{18}{487^s}+\frac{18}{541^s}+\frac{18}{757^s}+\frac{18}{811^s}+\frac{18}{919^s}+\frac{18}{1297^s}+\frac{18}{1459^s}+\frac{18}{1567^s}+\frac{18}{1621^s}+\frac{18}{1783^s}+\dots$
28	$1+\frac{2}{8^s}+\frac{12}{29^s}+\frac{2}{64^s}+\frac{12}{113^s}+\frac{6}{169^s}+\frac{12}{197^s}+\frac{24}{232^s}+\frac{12}{281^s}+\frac{12}{337^s}+\frac{12}{421^s}+\frac{12}{449^s}+\frac{2}{512^s}+\frac{12}{617^s}+\frac{12}{673^s}+\frac{12}{701^s}+\frac{12}{757^s}+\dots$
32	$1+\frac{16}{97^s}+\frac{16}{193^s}+\frac{16}{257^s}+\frac{8}{289^s}+\frac{16}{353^s}+\frac{16}{449^s}+\frac{16}{577^s}+\frac{16}{641^s}+\frac{16}{673^s}+\frac{16}{769^s}+\frac{16}{929^s}+\frac{16}{1153^s}+\frac{16}{1217^s}+\frac{16}{1249^s}+\frac{16}{1409^s}+\dots$
33	$1+\frac{20}{67^s}+\frac{20}{199^s}+\frac{2}{243^s}+\frac{20}{331^s}+\frac{20}{397^s}+\frac{20}{463^s}+\frac{10}{529^s}+\frac{20}{661^s}+\frac{20}{727^s}+\frac{20}{859^s}+\frac{20}{991^s}+\frac{20}{1123^s}+\frac{20}{1321^s}+\frac{20}{1453^s}+\frac{20}{1783^s}+\dots$
35	$1+\frac{24}{71^s}+\frac{24}{211^s}+\frac{24}{281^s}+\frac{24}{421^s}+\frac{24}{491^s}+\frac{24}{631^s}+\frac{24}{701^s}+\frac{12}{841^s}+\frac{24}{911^s}+\frac{24}{1051^s}+\frac{8}{1331^s}+\frac{24}{1471^s}+\frac{12}{1681^s}+\frac{24}{2311^s}+\frac{24}{2381^s}+\dots$
36	$1+\frac{12}{37^s}+\frac{12}{73^s}+\frac{12}{109^s}+\frac{12}{181^s}+\frac{6}{289^s}+\frac{6}{361^s}+\frac{12}{397^s}+\frac{12}{433^s}+\frac{12}{541^s}+\frac{12}{577^s}+\frac{12}{613^s}+\frac{12}{757^s}+\frac{12}{829^s}+\frac{12}{937^s}+\frac{12}{1009^s}+\dots$
40	$1+\frac{2}{25^s}+\frac{16}{41^s}+\frac{4}{81^s}+\frac{8}{121^s}+\frac{16}{241^s}+\frac{16}{281^s}+\frac{8}{361^s}+\frac{16}{401^s}+\frac{16}{521^s}+\frac{16}{601^s}+\frac{2}{625^s}+\frac{16}{641^s}+\frac{16}{761^s}+\frac{8}{841^s}+\frac{16}{881^s}+\dots$
44	$1+\frac{20}{89^s}+\frac{20}{353^s}+\frac{20}{397^s}+\frac{10}{529^s}+\frac{20}{617^s}+\frac{20}{661^s}+\frac{20}{881^s}+\frac{20}{1013^s}+\frac{20}{1277^s}+\frac{20}{1321^s}+\frac{20}{1409^s}+\frac{20}{1453^s}+\frac{20}{2069^s}+\frac{20}{2113^s}+\dots$
45	$1+\frac{24}{181^s}+\frac{24}{271^s}+\frac{12}{361^s}+\frac{24}{541^s}+\frac{24}{631^s}+\frac{24}{811^s}+\frac{24}{991^s}+\frac{24}{1171^s}+\frac{24}{1531^s}+\frac{24}{1621^s}+\frac{24}{1801^s}+\frac{24}{2161^s}+\frac{24}{2251^s}+\frac{24}{2341^s}+\dots$
48	$1+\frac{8}{49^s}+\frac{2}{81^s}+\frac{16}{97^s}+\frac{16}{193^s}+\frac{16}{241^s}+\frac{8}{289^s}+\frac{16}{337^s}+\frac{16}{433^s}+\frac{8}{529^s}+\frac{16}{577^s}+\frac{4}{625^s}+\frac{16}{673^s}+\frac{16}{769^s}+\frac{8}{961^s}+\frac{16}{1009^s}+\dots$
60	$1+\frac{2}{16^s}+\frac{2}{25^s}+\frac{16}{61^s}+\frac{2}{81^s}+\frac{8}{121^s}+\frac{16}{181^s}+\frac{16}{241^s}+\frac{2}{256^s}+\frac{8}{361^s}+\frac{4}{400^s}+\frac{16}{421^s}+\frac{16}{541^s}+\frac{16}{601^s}+\frac{2}{625^s}+\frac{16}{661^s}+\frac{8}{841^s}+\dots$
84	$1+\frac{2}{49^s}+\frac{2}{64^s}+\frac{12}{169^s}+\frac{24}{337^s}+\frac{24}{421^s}+\frac{24}{673^s}+\frac{24}{757^s}+\frac{12}{841^s}+\frac{24}{1009^s}+\frac{24}{1093^s}+\frac{24}{1429^s}+\frac{24}{1597^s}+\frac{12}{1681^s}+\frac{12}{1849^s}+\frac{24}{1933^s}+\dots$

Table 5. Additional terms in the Dirichlet series of Theorem 2 for truly multiple coincidences.

n	$\Psi_n(s) - \Phi_n(s)$
3	$\frac{1}{49^s} + \frac{1}{169^s} + \frac{2}{343^s} + \frac{1}{361^s} + \frac{2}{637^s} + \frac{2}{931^s} + \frac{1}{961^s} + \frac{2}{1183^s} + \frac{1}{1369^s} + \frac{2}{1519^s} + \frac{2}{1813^s} + \frac{1}{1849^s} + \frac{2}{2107^s} + \frac{2}{2197^s} + \frac{3}{2401^s} + \dots$
4	$\frac{1}{25^s} + \frac{2}{125^s} + \frac{1}{169^s} + \frac{1}{289^s} + \frac{2}{325^s} + \frac{2}{425^s} + \frac{3}{625^s} + \frac{2}{725^s} + \frac{1}{841^s} + \frac{2}{845^s} + \frac{2}{925^s} + \frac{2}{1025^s} + \frac{2}{1325^s} + \frac{1}{1369^s} + \frac{2}{1445^s} + \dots$
5	$\frac{2}{121^s} + \frac{2}{961^s} + \frac{8}{1331^s} + \frac{2}{1681^s} + \frac{2}{3721^s} + \frac{8}{3751^s} + \frac{8}{4961^s} + \frac{2}{5041^s} + \frac{8}{7381^s} + \frac{8}{8591^s} + \frac{2}{10201^s} + \frac{8}{10571^s} + \frac{8}{12221^s} + \frac{19}{14641^s} + \dots$
7	$\frac{1}{64^s} + \frac{2}{512^s} + \frac{3}{841^s} + \frac{3}{1849^s} + \frac{6}{1856^s} + \frac{6}{2752^s} + \frac{3}{4096^s} + \frac{6}{4544^s} + \frac{3}{5041^s} + \frac{6}{6728^s} + \frac{6}{7232^s} + \frac{6}{8128^s} + \frac{6}{12608^s} + \frac{3}{12769^s} + \dots$
8	$\frac{1}{81^s} + \frac{2}{289^s} + \frac{1}{625^s} + \frac{2}{729^s} + \frac{4}{1377^s} + \frac{2}{1681^s} + \frac{2}{2025^s} + \frac{4}{2601^s} + \frac{4}{3321^s} + \frac{8}{4913^s} + \frac{2}{5329^s} + \frac{2}{5625^s} + \frac{4}{5913^s} + \frac{3}{6561^s} + \dots$
9	$\frac{3}{361^s} + \frac{3}{1369^s} + \frac{3}{5329^s} + \frac{18}{6859^s} + \frac{3}{11881^s} + \frac{18}{13357^s} + \frac{3}{16129^s} + \frac{18}{26011^s} + \frac{18}{26353^s} + \frac{3}{26569^s} + \frac{3}{32761^s} + \frac{18}{39349^s} + \frac{3}{39601^s} + \dots$
11	$\frac{5}{529^s} + \frac{5}{4489^s} + \frac{5}{7921^s} + \frac{50}{12167^s} + \frac{50}{35443^s} + \frac{5}{39601^s} + \frac{50}{47081^s} + \frac{1}{59049^s} + \frac{50}{103247^s} + \frac{50}{105271^s} + \frac{5}{109561^s} + \frac{5}{124609^s} + \dots$
12	$\frac{2}{169^s} + \frac{1}{625^s} + \frac{2}{1369^s} + \frac{8}{2197^s} + \frac{1}{2401^s} + \frac{2}{3721^s} + \frac{4}{4225^s} + \frac{2}{5329^s} + \frac{8}{6253^s} + \frac{4}{8125^s} + \frac{4}{8281^s} + \frac{2}{9409^s} + \frac{8}{10309^s} + \frac{2}{11881^s} + \dots$
13	$\frac{2}{729^s} + \frac{6}{2809^s} + \frac{6}{6241^s} + \frac{6}{17161^s} + \frac{8}{19683^s} + \frac{6}{24649^s} + \frac{24}{38637^s} + \frac{24}{57591^s} + \frac{24}{75843^s} + \frac{24}{95499^s} + \frac{6}{97969^s} + \frac{24}{114453^s} + \frac{72}{148877^s} + \dots$
15	$\frac{1}{256^s} + \frac{4}{961^s} + \frac{4}{3721^s} + \frac{2}{4096^s} + \frac{8}{7936^s} + \frac{2}{14641^s} + \frac{8}{15376^s} + \frac{8}{15616^s} + \frac{4}{22801^s} + \frac{32}{29791^s} + \frac{4}{30976^s} + \frac{4}{32761^s} + \frac{8}{38656^s} + \dots$
16	$\frac{4}{289^s} + \frac{2}{2401^s} + \frac{32}{4913^s} + \frac{1}{6561^s} + \frac{4}{9409^s} + \frac{4}{12769^s} + \frac{16}{14161^s} + \frac{8}{23409^s} + \frac{32}{28033^s} + \frac{32}{32657^s} + \frac{4}{37249^s} + \frac{16}{40817^s} + \frac{32}{55777^s} + \dots$
17	$\frac{8}{10609^s} + \frac{8}{18769^s} + \frac{8}{57121^s} + \frac{8}{94249^s} + \frac{8}{167281^s} + \frac{8}{196249^s} + \frac{8}{375769^s} + \frac{8}{418609^s} + \frac{8}{844561^s} + \frac{8}{908209^s} + \frac{8}{1042441^s} + \dots$
19	$\frac{9}{36481^s} + \frac{9}{52441^s} + \frac{3}{117649^s} + \frac{9}{175561^s} + \frac{9}{208849^s} + \frac{9}{326041^s} + \frac{9}{418609^s} + \frac{9}{579121^s} + \frac{9}{1216609^s} + \frac{9}{1481089^s} + \frac{3}{1771561^s} + \dots$
20	$\frac{1}{25^s} + \frac{2}{125^s} + \frac{3}{625^s} + \frac{8}{1025^s} + \frac{8}{1525^s} + \frac{4}{1681^s} + \frac{2}{2025^s} + \frac{8}{2525^s} + \frac{4}{3025^s} + \frac{4}{3125^s} + \frac{4}{3721^s} + \frac{8}{4525^s} + \frac{16}{5125^s} + \frac{8}{6025^s} + \dots$
21	$\frac{1}{49^s} + \frac{2}{343^s} + \frac{6}{1849^s} + \frac{12}{2107^s} + \frac{3}{2401^s} + \frac{2}{3136^s} + \frac{1}{4096^s} + \frac{12}{6223^s} + \frac{6}{8281^s} + \frac{12}{10339^s} + \frac{12}{12943^s} + \frac{24}{14749^s} + \frac{6}{16129^s} + \frac{12}{16513^s} + \dots$
24	$\frac{1}{81^s} + \frac{2}{625^s} + \frac{2}{729^s} + \frac{4}{2025^s} + \frac{2}{2401^s} + \frac{4}{3969^s} + \frac{4}{5329^s} + \frac{4}{5625^s} + \frac{8}{5913^s} + \frac{3}{6561^s} + \frac{8}{7857^s} + \frac{4}{9409^s} + \frac{4}{9801^s} + \frac{4}{13689^s} + \dots$
25	$\frac{10}{10201^s} + \frac{10}{22801^s} + \frac{10}{63001^s} + \frac{10}{160801^s} + \frac{10}{361201^s} + \frac{10}{491401^s} + \frac{10}{564001^s} + \frac{200}{1030301^s} + \frac{10}{1104601^s} + \frac{10}{1324801^s} + \frac{10}{1442401^s} + \dots$
27	$\frac{9}{11881^s} + \frac{9}{26569^s} + \frac{9}{73441^s} + \frac{9}{143641^s} + \frac{9}{187489^s} + \frac{9}{237169^s} + \frac{9}{292681^s} + \frac{9}{573049^s} + \frac{9}{657721^s} + \frac{9}{844561^s} + \frac{162}{1295029^s} + \dots$
28	$\frac{1}{64^s} + \frac{2}{512^s} + \frac{6}{841^s} + \frac{12}{1856^s} + \frac{3}{4096^s} + \frac{12}{6728^s} + \frac{12}{7232^s} + \frac{6}{10816^s} + \frac{12}{12608^s} + \frac{6}{12769^s} + \frac{24}{14848^s} + \frac{12}{17984^s} + \frac{12}{21568^s} + \frac{72}{24389^s} + \dots$
32	$\frac{8}{9409^s} + \frac{8}{37249^s} + \frac{8}{66049^s} + \frac{4}{83521^s} + \frac{8}{124609^s} + \frac{8}{201601^s} + \frac{8}{332929^s} + \frac{8}{410881^s} + \frac{8}{452929^s} + \frac{8}{591361^s} + \frac{8}{863041^s} + \frac{128}{912673^s} + \dots$
33	$\frac{10}{4489^s} + \frac{10}{39601^s} + \frac{1}{59049^s} + \frac{10}{109561^s} + \frac{10}{157609^s} + \frac{10}{214369^s} + \frac{5}{279841^s} + \frac{200}{300763^s} + \frac{10}{436921^s} + \frac{10}{528529^s} + \frac{10}{737881^s} + \frac{200}{893311^s} + \dots$
35	$\frac{12}{5041^s} + \frac{12}{44521^s} + \frac{12}{78961^s} + \frac{12}{177241^s} + \frac{12}{241081^s} + \frac{288}{357911^s} + \frac{12}{398161^s} + \frac{12}{491401^s} + \frac{6}{707281^s} + \frac{12}{829921^s} + \frac{288}{1063651^s} + \dots$
36	$\frac{6}{1369^s} + \frac{6}{5329^s} + \frac{6}{11881^s} + \frac{6}{32761^s} + \frac{72}{50653^s} + \frac{3}{83521^s} + \frac{72}{99937^s} + \frac{3}{130321^s} + \frac{72}{149221^s} + \frac{6}{157609^s} + \frac{6}{187489^s} + \frac{72}{197173^s} + \dots$
40	$\frac{1}{625^s} + \frac{8}{1681^s} + \frac{2}{6561^s} + \frac{4}{14641^s} + \frac{2}{15625^s} + \frac{16}{25625^s} + \frac{16}{42025^s} + \frac{4}{50625^s} + \frac{8}{58081^s} + \frac{128}{68921^s} + \frac{8}{75625^s} + \frac{8}{78961^s} + \dots$
44	$\frac{10}{7921^s} + \frac{10}{124609^s} + \frac{10}{157609^s} + \frac{5}{279841^s} + \frac{10}{380689^s} + \frac{10}{436921^s} + \frac{200}{704969^s} + \frac{10}{776161^s} + \frac{10}{1026169^s} + \frac{10}{1630729^s} + \frac{10}{1745041^s} + \dots$
45	$\frac{12}{32761^s} + \frac{12}{73441^s} + \frac{6}{130321^s} + \frac{12}{292681^s} + \frac{12}{398161^s} + \frac{12}{657721^s} + \frac{12}{982081^s} + \frac{12}{1371241^s} + \frac{12}{2343961^s} + \frac{12}{2627641^s} + \frac{12}{3243601^s} + \dots$
48	$\frac{4}{2401^s} + \frac{1}{6561^s} + \frac{8}{9409^s} + \frac{8}{37249^s} + \frac{8}{58081^s} + \frac{4}{83521^s} + \frac{8}{113569^s} + \frac{32}{117649^s} + \frac{8}{187489^s} + \frac{8}{194481^s} + \frac{64}{232897^s} + \frac{4}{279841^s} + \dots$
60	$\frac{1}{256^s} + \frac{1}{625^s} + \frac{8}{3721^s} + \frac{2}{4096^s} + \frac{2}{6400^s} + \frac{1}{6561^s} + \frac{2}{10000^s} + \frac{4}{14641^s} + \frac{16}{15616^s} + \frac{2}{15625^s} + \frac{2}{20736^s} + \frac{8}{30976^s} + \frac{8}{32761^s} + \dots$
84	$\frac{1}{2401^s} + \frac{1}{4096^s} + \frac{6}{28561^s} + \frac{12}{113569^s} + \frac{2}{117649^s} + \frac{2}{153664^s} + \frac{12}{177241^s} + \frac{2}{200704^s} + \frac{2}{262144^s} + \frac{12}{405769^s} + \frac{12}{452929^s} + \frac{12}{573049^s} + \dots$

but the above observation also gives the absolute convergence of the infinite product involved, because its logarithm,

$$\sum_{\ell=0}^{\infty} \frac{1}{2^\ell} \log\bigl(\Phi_n(2^\ell s)\bigr),$$

converges absolutely on $\{\mathrm{Re}(s) > 1\}$. □

Observe that, by analytic continuation, the product

$$\prod_{\ell=1}^{\infty} \bigl(\Phi_n(2^\ell s)\bigr)^{1/2^\ell},$$

starting with $\Phi_n(2s)^{1/2}$, defines an analytic function on $\{\mathrm{Re}(s) > 1/2\}$ without any zeros in this half-plane (the absence of zeros can actually be extended to its closure, $\{\mathrm{Re}(s) \geq 1/2\}$, by known properties of $\zeta(s)$ and the L-series involved). This shows that $\Phi_n(s)$ and $\Psi_n(s)$ share the positions of their zeros and poles, when defined by analytic continuation on this half-plane. In particular, they both have a single simple pole in $\{\mathrm{Re}(s) > 1/2\}$, at $s = 1$, and they share the position (if any) and orders of zeros there. This remains true on the critical line $\{\mathrm{Re}(s) = 1/2\}$, where the only zeros of $\Phi_n(s)$ (and hence of $\Psi_n(s)$) in the entire set $\{\mathrm{Re}(s) \geq 1/2\}$ are expected on the basis of the Riemann hypothesis for $\zeta(s)$ and its generalisation to L-series.

Without any reference to this (still unproved) hypothesis, one has, by standard arguments on the basis of compact convergence,

Corollary 3 *On $\{\mathrm{Re}(s) > 1/2\}$, by unique analytic continuation, one has the representation*

$$\frac{\Psi_n}{\Phi_n}(s) \;=\; \prod_{\ell=1}^{\infty} \bigl(\Phi_n(2^\ell s)\bigr)^{1/2^\ell}$$

which is an analytic and zero-free function in this half-plane.

In particular, Ψ_n/Φ_n has a well-defined positive value at $s = 1$. □

At this point, we can state the asymptotic result for the summatory function of the coefficients $b_n(k)$.

Corollary 4 (CN1). *The number of multiple CSMs of index $\leq x$ shows the asymptotic behaviour*

$$\sum_{k\leq x} b_n(k) \;\sim\; x\cdot\bigl(\mathrm{res}_{s=1}\,\Psi_n(s)\bigr) \;=\; x\cdot\beta_n,$$

with the growth constant

$$\beta_n \;=\; q_n\cdot\bigl(\mathrm{res}_{s=1}\,\Phi_n(s)\bigr) \;=\; q_n\cdot\gamma_n .$$

Here, the constant q_n has the monotonically increasing and rapidly converging product representation

$$q_n \;:=\; \lim_{s\to 1}\frac{\Psi_n(s)}{\Phi_n(s)} \;=\; \prod_{\ell=1}^{\infty}\bigl(\Phi_n(2^\ell)\bigr)^{1/2^\ell}.$$

PROOF. The monotonicity w.r.t. $\succcurlyeq$ stated in (34), together with the analyticity properties of $\Psi_n(s)$, implies that $\Psi_n(s)$ also has a simple pole at $s = 1$, and no other singularity in $\{\mathrm{Re}(s) \geq 1\}$. Delange's theorem then yields the claim on the asymptotic behaviour.

The second statement is almost immediate from our above discussion. Clearly, $\Phi_n(2^\ell) > 1$ for all $\ell \in \mathbb{N}$, hence also $\Phi_n(2^\ell)^{1/2^\ell} > 1$, which gives the monotonicity, while convergence follows from Corollary 3 and Proposition 2. □

In Table 3, we give the numerical values of the residues of the three types of generating functions we have encountered. Some values for α_n and γ_n, including exact expressions (except for γ_7, see a previous footnote), are also contained in [6] and [15], respectively.

6 Outlook

It is desirable to extend the above analysis to higher dimensions, which is significantly more involved due to non-commutativity of the SOC-groups in these cases. Nevertheless, quite a bit is known for simple coincidences [2, 22, 23], and first steps are in sight for multiple ones [24].

Also, in the planar case, one would like to get rid of the assumption (**CN1**). This is already pretty tricky for simple coincidences, see [15] for an example ($n = 23$), but some further results seem possible.

Finally, there are many open questions concerning the structure of the SOC-groups, particularly for dimensions $d \geq 3$. As they are a natural extension of point symmetry groups, they certainly deserve further attention.

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References

[1] T. M. Apostol, *An Introduction to Analytic Number Theory*, 5th corr. printing (Springer, New York, 1998).

[2] M. Baake, Solution of the coincidence problem in dimensions $d \leq 4$, in: *The Mathematics of Longe-Range Aperiodic Order*, ed. R. V. Moody, NATO ASI Series C 489 (Kluwer, Dordrecht, 1997) pp. 9–44.

[3] M. Baake, A guide to mathematical quasicrystals, in: *Quasicrystals. An Introduction to Structure, Physical Properties and Applications*, eds. J.-B. Suck, M. Schreiber and P. Häussler (Springer, Berlin, 2002) pp. 17–48; `math-ph/9901014`.

[4] M. Baake and U. Grimm, Combinatorial problems of (quasi)-crystallography, in: *Quasicrystals: Structure and Physical Properties*, ed. H.-R. Trebin (Wiley-VCH, Weinheim, 2003) pp. 160–171; `math-ph/0212015`.

[5] M. Baake and U. Grimm, A note on shelling, *Discr. Comput. Geom.* **30** (2003) 573–589; `math.MG/0203025`.

[6] M. Baake and U. Grimm, Bravais colourings of planar modules with N-fold symmetry, *Z. Kristallographie* **219** (2004) 72–80; `math.CO/0301021`.

[7] M. Baake and P. A. B. Pleasants, Algebraic solution of the coincidence problem in two and three dimensions, *Z. Naturf.* **50a** (1995) 711–717.

[8] B. C. Berndt, On the Hurwitz zeta-function, *Rocky Mountain J. Math.* **2** (1972) 151–157.

[9] W. Bollmann, *Crystal Defects and Crystalline Interfaces* (Springer, Berlin, 1970).
[10] Z. I. Borevich and I. R. Shafarevich, *Number Theory* (Academic Press, New York, 1966).
[11] H. Grimmer, Coincidence orientations of grains in rhombohedral materials, *Acta Cryst.* **A 45** (1989) 505–523.
[12] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed. (Clarendon Press, Oxford 1979).
[13] P. Moree and J. Cazaran, On a claim of Ramanujan in his first letter to Hardy, *Exp. Math.* **17** (1999) 289–312.
[14] P. A. B. Pleasants, Designer quasicrystals: Cut-and-project sets with pre-assigned properties, in: *Directions in Mathematical Quasicrystals*, eds. M. Baake and R. V. Moody, CRM Monograph Series, vol. 13 (AMS, Providence, RI, 2000) pp. 95–141.
[15] P. A. B. Pleasants, M. Baake and J. Roth, Planar coincidences for N-fold symmetry, *J. Math. Phys.* **37** (1996) 1029–1058; `math.MG/0511147`.
[16] D. Shanks, The second-order term in the asymptotic expansion of $B(x)$, *Math. Comp.* **18** (1964) 78–86.
[17] N. J. E. Sloane and B. Beferull-Lozano, Quantizing using lattice intersections, in: *Discrete and Computational Geometry*, eds. B. Aronov, S. Basu, J. Pach and M. Sharir (Springer, Berlin, 2003) pp. 799–824; `math.CO/0207147`.
[18] W. Steurer, Twenty years of structure research on quasicrystals. Part I. Pentagonal, octagonal, decagonal and dodecagonal quasicrystals, *Z. Kristallographie* **219** (2004) 391–446.
[19] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory* (Cambridge University Press, Cambridge, 1995).
[20] L. C. Washington, *Introduction to Cyclotomic Fields*, 2nd ed. (Springer, New York, 1997).
[21] E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis*, 4th ed. (reprinted) (Cambridge University Press, Cambridge, 1965).
[22] P. Zeiner, Symmetries of coincidence site lattices of cubic lattices, *Z. Kristallographie* **220** (2005) 915–925.
[23] P. Zeiner, Coincidences of hypercubic lattices in 4 dimensions *Z. Kristallographie* **220** (2005), in press.
[24] P. Zeiner, Multiple CSLs for the body centered cubic lattice, preprint (2005).