

DUALITY OF MODEL SETS GENERATED BY SUBSTITUTIONS

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*Dedicated to Tudor Zamfirescu
on the occasion of his sixtieth birthday*

ABSTRACT. The nature of this paper is twofold: On one hand, we will give a short introduction and overview of the theory of model sets in connection with nonperiodic substitution tilings and generalized Rauzy fractals. On the other hand, we will construct certain Rauzy fractals and a certain substitution tiling with interesting properties, and we will use a new approach to prove rigorously that the latter one arises from a model set. The proof will use a duality principle which will be described in detail for this example. This duality is mentioned as early as 1997 in [GEL] in the context of iterated function systems, but it seems to appear nowhere else in connection with model sets.

1. INTRODUCTION

One of the essential observations in the theory of nonperiodic, but highly ordered structures is the fact that, in many cases, they can be generated by a certain projection from a higher dimensional (periodic) point lattice. This is true for the well-known Penrose tilings (cf. [GSH]) as well as for a lot of other substitution tilings, where most known examples are living in \mathbb{E}^1 , \mathbb{E}^2 or \mathbb{E}^3 . The appropriate framework for this arises from the theory of model sets ([MEY], [MOO]). Though the knowledge in this field has grown considerably in the last decade, in many cases it is still hard to prove rigorously that a given nonperiodic structure indeed is a model set.

This paper is organized as follows. Section 1.1 briefly collects some well-known facts about substitution tilings. These facts will be used later. Section 1.2 contains some basics about model sets, in particular in connection with substitution tilings. In Section 1.3 we give the definition of Rauzy fractals. This can be done completely in the framework of model sets; it is just a question of the terminology. In Section 2.1 we introduce a family of one-dimensional substitutions which are generalizations of the well-known Fibonacci sequences. Four of these possess a substitution factor which is a PV-number, so these are candidates for being model sets. Section 2.2 is dedicated to one of these substitutions. Therein, the corresponding Rauzy fractal is obtained. Section 2.3 shows the construction of the dual substitution tiling, which is a two-dimensional tiling with some interesting properties. Using the duality, we will show that this dual tiling is a model set. Hence, this is one of the few known cases of a two-dimensional model set, generated by a substitution, where the substitution factor is not an algebraic number of degree one or degree two (and the proof being rigorous). Section

2000 *Mathematics Subject Classification.* Primary 52C32.
Key words and phrases. Aperiodic tilings, model sets.

3 contains some additional remarks, and further examples of dual substitutions, where the duality principle can be used in order to prove that these are model sets, too.

Let us fix some notation. Throughout the paper, $\text{cl}(M)$ denotes the closure of a set M , $\text{int}(M)$ the interior of M , ∂M the boundary of M , and $\#M$ the cardinality of M . \mathbb{E}^d denotes the d -dimensional Euclidean space, i.e., \mathbb{R}^d equipped with the Euclidean metric $\|\cdot\|$. \mathbb{B}^d denotes the unit ball $\{\mathbf{x} \mid \|\mathbf{x}\| \leq 1\}$. If not stated otherwise, μ denotes the d -dimensional Lebesgue measure, where d will be clear from the context. A *tile* is a nonempty compact set $T \subset \mathbb{E}^d$ with the property $\text{cl}(\text{int}(T)) = T$. A *tiling* of \mathbb{E}^d is a collection of tiles $\{T_n\}_{n \geq 0}$ which covers \mathbb{E}^d and contains no *overlapping* tiles, i.e., $\text{int}(T_k) \cap \text{int}(T_n) = \emptyset$ for $k \neq n$. In other words: A tiling is both a covering and a packing of \mathbb{E}^d . A tiling \mathcal{T} is called *nonperiodic*, if the only solution of $\mathcal{T} + \mathbf{x} = \mathcal{T}$ is $\mathbf{x} = \mathbf{0}$.

1.1. SUBSTITUTION TILINGS. A convenient method to generate nonperiodic tilings is by a *substitution*: Choose a set of *prototiles*, that is, a set of tiles $\mathcal{F} := \{T_1, T_2, \dots, T_m\}$. Choose a *substitution factor* $\lambda \in \mathbb{R}, \lambda > 1$ and a rule how to dissect λT_i (for every $1 \leq i \leq m$) into tiles, such that any of these tiles is congruent to some tile $T_j \in \mathcal{F}$. It will be sufficient for this paper to think of a substitution like in Fig. 1. For completeness, we give the proper definition here.

Definition 1.1. Let $\mathcal{F} := \{T_1, T_2, \dots, T_m\}$ be a set of tiles, the *prototiles*, and $\lambda > 1$ a real number, the *substitution factor*. Let $\lambda T_i = \varphi_1(T_{i_1}) \cup \varphi_2(T_{i_2}) \cup \dots \cup \varphi_{n(i)}(T_{i_{n(i)}})$, such that every φ_j is an isometry of \mathbb{E}^d and the involved tiles do not overlap. Furthermore, let $\sigma(\{T_i\}) = \{\varphi_1(T_{i_1}), \varphi_2(T_{i_2}), \dots, \varphi_{n(i)}(T_{i_{n(i)}})\}$. Let \mathbf{S} be the set of all configurations of tiles congruent to tiles in \mathcal{F} . That is, \mathbf{S} is the set of all sets of the form $\{T_{j_i} + \mathbf{x}_i \mid T_{j_i} \in \mathcal{F}, i \in I, \mathbf{x}_i \in \mathbb{E}^d\}$, where I is some finite or infinite index set. By the requirement $\sigma(\{T_i + \mathbf{x}\}) = \sigma(\{T_i\}) + \lambda \mathbf{x}$ ($\mathbf{x} \in \mathbb{E}^d$), σ extends in a unique way to a well-defined map from \mathbf{S} to \mathbf{S} . Then $\sigma : \mathbf{S} \rightarrow \mathbf{S}$ is called a *substitution*. Every tiling of \mathbb{E}^d , where any finite part of it is congruent to a subset of some $\sigma^k(T_i)$ is called a *substitution tiling (with substitution σ)*.

More general, this definition makes sense if we replace λ by some expanding linear map. That is, replace λ in the definition above by a matrix Q such that all eigenvalues of Q are larger than 1 in modulus. In this full generality, many of the following results may break down. Anyway, in Section 2.2 we will use the generalization in one certain case, where it will cause no problem.

To a substitution σ we assign a *substitution matrix*: $A_\sigma := (a_{ij})_{1 \leq i, j \leq m}$, where a_{ij} is the number of tiles $T_j + \mathbf{x}$ (tiles of type $T_j \in \mathcal{F}$) in $\sigma(\{T_i\})$. In this paper we will consider substitutions with *primitive* substitution matrices only. Recall: A nonnegative matrix A is called *primitive*, if some power A^k is a strictly positive matrix. In this situation we can apply the following well-known theorem.

THEOREM 1.1 (Perron–Frobenius). *Let $A \in \mathbb{R}^{m \times m}$ be a primitive nonnegative matrix. Then A has an eigenvalue $0 < \lambda \in \mathbb{R}$, which is simple and larger in modulus than every other eigenvalue of A . This eigenvalue is called Perron–Frobenius-eigenvalue or shortly PF-eigenvalue.*

The corresponding eigenvector can be chosen such that it is strictly positive. Such an eigenvector is called PF-eigenvector of A . No other eigenvalue of A has a strictly positive eigenvector.

Using this theorem, it is a simple exercise to show the following facts (cf. [PYF], Thm. 1.2.7): We consider the *relative frequency* of the tiles of type T_i in a tiling \mathcal{T} , i.e., the ratio of the number of tiles of type T_i in \mathcal{T} and the number of all tiles in \mathcal{T} . Formally, this is defined by $\lim_{r \rightarrow \infty} \frac{\#\{T_i + \mathbf{x} \in \mathcal{T} \mid \mathbf{x} \in r\mathbb{B}^d\}}{\#\{T + \mathbf{x} \in \mathcal{T} \mid \mathbf{x} \in r\mathbb{B}^d, T \in \mathcal{F}\}}$. Since we consider primitive substitution tilings only, it is true that this limit exists. Moreover, this limit exists *uniformly*, i.e., for any translate $\mathcal{T} + \mathbf{x}$ we will obtain the same limit.

PROPOSITION 1.2. *Let σ be a primitive substitution in \mathbb{E}^d with substitution factor λ and prototiles T_1, \dots, T_m . Then the PF-eigenvalue of A_σ is λ^d . In particular, a substitution factor is always an algebraic number.*

The normalized (right) PF-eigenvector $\mathbf{v} = (v_1, \dots, v_m)^T$ of A_σ contains the relative frequencies of the tiles of different types in the tiling in the following sense: The entry v_i is the relative frequency of T_i in \mathcal{T} .

The left PF-eigenvector (resp. the PF-eigenvector of A^T) contains the d -dimensional volumes of the different prototiles, up to scaling.

Some people don't like the term 'left eigenvector'. Those may replace 'left eigenvector of A ' by 'eigenvector of A^T , written as a row vector' wherever it occurs. In the next section, we will associate with a tiling a related point set: By replacing every prototile T_i with a point in its interior, a tiling \mathcal{T} gives rise to a *Delone set* $V_{\mathcal{T}}$. (Note that this defines $V_{\mathcal{T}}$ not uniquely. It depends on the distinct choice of points in the prototiles. But the following is true for any Delone set $V_{\mathcal{T}}$ constructed out of a primitive substitution tiling \mathcal{T} in the described way.) A Delone set is a point set $V \subset \mathbb{E}^d$ which is *uniformly discrete* (there exists $r > 0$ such that $\#(V \cap r\mathbb{B}^d + \mathbf{x}) \leq 1$ for all $\mathbf{x} \in \mathbb{E}^d$) and *relatively dense* (there exists $R > 0$ such that $\#(V \cap R\mathbb{B}^d + \mathbf{x}) \geq 1$ for all $\mathbf{x} \in \mathbb{E}^d$). Since the frequencies of the prototiles (cf. remark after Theorem 1.1) are well-defined for all tilings \mathcal{T} considered in this paper, it makes sense to define the density of the associated Delone set $V_{\mathcal{T}}$: $\text{dens}(V_{\mathcal{T}}) = \lim_{r \rightarrow \infty} \frac{\#V_{\mathcal{T}} \cap r\mathbb{B}^d}{\mu(r\mathbb{B}^d)}$. It is not hard to show the following identity.

PROPOSITION 1.3. *Let \mathcal{T} be a substitution tiling with substitution σ , let \mathbf{v} be the right normalized PF-eigenvector of A_σ (containing the relative frequencies of the prototiles), let \mathbf{w}^T be the left PF-eigenvector of A_σ , such that \mathbf{w}^T contains the d -dimensional volumes of the prototiles, and $V_{\mathcal{T}}$ a Delone set constructed out of \mathcal{T} by replacing every prototile with a point in its interior. Then*

$$\text{dens}(V_{\mathcal{T}}) = (\mathbf{w}^T \cdot \mathbf{v})^{-1}$$

Example: The golden triangle substitution. The substitution rule, together with substitution factor, substitution matrix etc. is shown below.

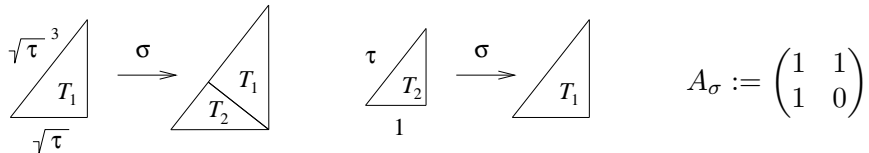


FIGURE 1. 'Golden triangles'. A substitution rule in \mathbb{E}^2 with two prototiles T_1, T_2 and substitution factor $\lambda = \sqrt{\tau}$, where $\tau = \frac{\sqrt{5}+1}{2}$ is the golden mean.

This substitution uses two prototiles T_1, T_2 . The PF-eigenvalue of the substitution matrix A_σ is $\tau = \lambda^2$. The normalized PF-eigenvector is $\mathbf{v} := (\tau^{-1}, \tau^{-2})^T$. Thus, the relative frequency of the tiles of type T_1 (resp. T_2) in the tiling is τ^{-1} (resp. τ^{-2}). A_σ is symmetric, thus \mathbf{v}^T is a left PF-eigenvector. The areas (2-dim volumes) of the two prototiles are $\mu(T_1) = \sqrt{\tau^3}, \mu(T_2) = \sqrt{\tau}$. If we replace every triangle in a corresponding substitution tiling by a point in its interior, then by Prop. 1.3 the density of the point set is $\left((\sqrt{\tau^3}, \sqrt{\tau}) \cdot (\tau^{-1}, \tau^{-2})^T \right)^{-1} = \frac{\tau^{3/2}}{(\tau^2+1)} = 0.56886448\dots$

1.2. MODEL SETS. Model sets are a special kind of Delone sets. They show a high degree of local and global order. For example, a model set V and its difference set $V - V := \{\mathbf{x} - \mathbf{y} \mid \mathbf{x}, \mathbf{y} \in V\}$ do not differ too much in the following sense (cf. [MEY]): There is a finite set F such that

$$V - V \subseteq V + F.$$

Lattices do have this property with $F = \{0\}$, and indeed lattices are special cases of model sets. Unfortunately, there is no reference known to the author which gives a comprehensive overview about model sets. A good reference is [MOO]. A comprehensive overview for the case of one-dimensional tilings is contained in [PYF], mainly chapter 7 and 8. Therein, many connections to combinatorics, ergodic theory and number theory are compiled (but the term 'model set' cannot be found in this book, the term 'geometric representation' is used instead).

A model set is defined by a collection of spaces and maps (a so-called cut-and-project scheme) as follows. In (1), let Λ be a lattice of full rank in \mathbb{E}^{d+e} , π_1, π_2 projections such that $\pi_1|_\Lambda$ is injective, and $\pi_2(\Lambda)$ is dense in \mathbb{E}^e . Let W be a nonempty compact set — the so-called *window set* — with the properties $\text{cl}(\text{int}(W)) = W$ and $\mu(\partial W) = 0$, where μ is the Lebesgue measure in \mathbb{E}^e .

$$(1) \quad \begin{array}{ccc} \mathbb{E}^d & \xleftarrow{\pi_1} \mathbb{E}^{d+e} & \xrightarrow{\pi_2} \mathbb{E}^e \\ \cup & & \cup \\ V & \Lambda & W \end{array}$$

Then the set $V = \{\pi_1(\mathbf{x}) \mid \mathbf{x} \in \Lambda, \pi_2(\mathbf{x}) \in W\}$ is called a *model set*¹. Obviously, V is a set of points in \mathbb{E}^d . A tiling can be obtained by applying certain rules to V , e.g., considering the Voronoi cells of V . In the case $d = 1$ it is quite simple to get a tiling: If $V = \{\mathbf{x}_i \mid i \in \mathbb{Z}\}$, such that $\mathbf{x}_i < \mathbf{x}_{i+1}$, then $\mathcal{T} = \{[\mathbf{x}_i, \mathbf{x}_{i+1}] \mid i \in \mathbb{Z}\}$ is a tiling of \mathbb{E}^1 .

Since $V \subset \pi_1(\Lambda)$ and $\pi_1|_\Lambda$ is injective, every $\mathbf{x} \in V$ corresponds to exactly one point $\pi_1^{-1}(\mathbf{x}) \in \Lambda$, and thus exactly to one point $\pi_2(\pi_1^{-1}(\mathbf{x})) \in W$. Therefore, we are able to define the *star map*:

$$(2) \quad * : \pi_1(\Lambda) \rightarrow \mathbb{E}^e, \quad \mathbf{x}^* = \pi_2(\pi_1^{-1}(\mathbf{x}))$$

This allows for a convenient notation. E.g., we have

$$(3) \quad \text{cl}(V^*) = W,$$

where $V^* = \{\mathbf{x}^* \mid \mathbf{x} \in V\}$.

Remark: In general, \mathbb{E}^d and \mathbb{E}^e in (1) can be replaced by some locally compact abelian groups G and H . Then μ denotes the Haar measure of H . This is necessary to show that, for

¹To be precise, this defines a *regular* model set. Without the property $\mu(\partial W) = 0$, V is still a model set, but not a regular one.

example, the chair tiling is a model set (cf [BMSC]). If the substitution matrix A_σ of a given substitution tiling is *unimodular* (i.e., $\det(A_\sigma) = \pm 1$), then it suffices to consider Euclidean spaces only. If not, then one needs to consider more general groups (cf. [S]). In this paper, all occurring substitution matrices are unimodular, therefore we will consider Euclidean spaces only.

Given a substitution tiling, one may ask if this tiling can be obtained as a model set V . By the following theorem, this is only possible if the substitution factor is a PV number or a Salem number.

Definition. Let $\lambda \in \mathbb{R}$ be an algebraic integer. λ is called a *Pisot Vijayaraghavan number* or shortly *PV number*, if $|\lambda| > 1$, and for all its algebraic conjugates λ_i holds: $|\lambda_i| < 1$.

If for all conjugates λ_i of λ holds $|\lambda_i| \leq 1$, with equality for at least one i , then λ is called a *Salem number*.

THEOREM 1.4 (Meyer). *If V is a model set, $1 < \lambda \in \mathbb{R}$ and if $\lambda V \subseteq V$, then λ is a PV number or a Salem number.*

It is known that every substitution σ gives rise to at least one tiling \mathcal{T} with the property $\sigma(\mathcal{T}) = \mathcal{T}$. So, if V is a point set derived from \mathcal{T} (where the substitution factor is λ), say, V being the vertex set of this tiling, then by the definition of a substitution follows $\lambda V \subseteq V$. Now, if V is also a model set, by the theorem above the substitution factor must be a PV number or a Salem number.

The remaining things to check in order to show that V is a model set are the following (cf. (1)):

- Determine the lattice Λ ,
- determine the window set W ,
- show $\mu(W) > 0$,
- show $\mu(\partial W) = 0$,
- show $\text{dens}(V) = \text{dens}(\{\pi_1(\mathbf{x}) \mid \mathbf{x} \in \Lambda, \pi_2(\mathbf{x}) \in W\})$.

There are standard constructions for obtaining Λ and W . But the latter three items are hard to decide rigorously in general, or even in special cases. For an example how hard it may be to give a proper proof, cf. [BS]. In the next section, we will use a duality principle to prove that the three latter items are fulfilled for the considered tiling, thus proving that it arises from a model set (i.e., the associated point set is a model set). But first, we will discuss the above list in more detail.

Starting with a substitution tiling, or better a substitution point set V in \mathbb{E}^1 (cf. the remark after (1)), the standard construction for the lattice is as follows: Let λ be the substitution factor (where λ from now on is always assumed to be a PV number) and $\lambda_2, \dots, \lambda_m$ its algebraic conjugates. Then

$$\Lambda = \langle (1, 1, \dots, 1)^T, (\lambda, \lambda_2, \dots, \lambda_m)^T, (\lambda^2, \lambda_2^2, \dots, \lambda_m^2)^T, \dots, (\lambda^{m-1}, \lambda_2^{m-1}, \dots, \lambda_m^{m-1})^T \rangle_{\mathbb{Z}}.$$

Now, π_1 (resp. π_2) can be chosen as the canonical projections from $\mathbb{E}^m \rightarrow \mathbb{E}^1$ (resp. $\mathbb{E}^m \rightarrow \mathbb{E}^{m-1}$). To be precise, if $\mathbf{x} = (x_1, x_2, \dots, x_m)^T$, then $\pi_1(\mathbf{x}) = x_1, \pi_2(\mathbf{x}) = (x_2, \dots, x_m)^T$. At this point, we are already able to compute the window set W numerically: Since the left PF-eigenvector of the substitution matrix contains the lengths of the prototiles (cf. remark after Thm. 1.1 above), the lengths can be chosen such that $V \subset \mathbb{Z}[\lambda]$. That is, every $\mathbf{x} \in V$

can be expressed as $\mathbf{x} = \alpha_0 + \alpha_1\lambda + \cdots + \alpha_{m-1}\lambda^{m-1}$, where $\alpha_i \in \mathbb{Z}$. Then

$$\mathbf{x}^* = (\alpha_0 + \alpha_1\lambda_2 + \cdots + \alpha_{m-1}\lambda_2^{m-1}, \dots, \alpha_0 + \alpha_1\lambda_{m-1} + \cdots + \alpha_{m-1}\lambda_{m-1}^{m-1})^T.$$

More general, we may apply the star map to V , obtaining the (countable) set V^* , which in turn yields the window set $W = \text{cl}(V^*)$. Actually, some of the pictures in the next section were created in essentially in this way: Using the substitution to generate a large finite subset $F \subset V$, where the points are expressed as \mathbf{x} above, we plot all points of F^* . If this set F^* appears to be bounded, even if the number of points is increased, then this is an indication that the considered set V is a model set.

To prove rigorously that V is actually a model set, one proceeds by showing $\mu(W) > 0$ and $\mu(\partial W) = 0$. If W is a polyhedron, then this is an easy task. But in the generic case W is of fractal nature, and it may be even hard to prove that W is of positive measure.

The last thing to prove is $V = \{\pi_1(\mathbf{x}) \mid \mathbf{x} \in \Lambda, \pi_2(\mathbf{x}) \in W\}$, and not just a too small subset. More precisely: Let $U := \{\pi_1(\mathbf{x}) \mid \mathbf{x} \in \Lambda, \pi_2(\mathbf{x}) \in W\}$. Then it must hold $\text{dens}(V) = \text{dens}(U)$. In this context the following theorem turns out to be helpful [SC].

THEOREM 1.5. *Let V be a model set given by (1). Then*

$$\text{dens}(V) = \mu(W) / \det \Lambda,$$

where $\det \Lambda$ denotes the determinant of the generator matrix of Λ . Equivalently, $\det \Lambda$ is the volume of any measurable fundamental domain of Λ .

1.3. RAUZY FRACTALS. As mentioned above, 'Rauzy fractal' is essentially just another term for the window set W . The term 'Rauzy fractal' occurs in the context of dynamical systems and number theory, while the terms 'window' or 'window set' arose from the theory of quasicrystals. Originally, 'Rauzy fractal' denoted one special fractal of this kind, namely the one arising from the substitution

$$\sigma(1) = 1\ 2, \quad \sigma(2) = 1\ 3, \quad \sigma(3) = 1,$$

which is analyzed in [R]. Therefore, the term 'generalized Rauzy fractal' is sometimes used to emphasize the fact that one deals with the more general case. (Of course, this is just a symbolic substitution, but it is trivial to formulate the geometric substitution according to Def. 1.1, cf. comment after (4)).

According to [PYF], the definition of a Rauzy fractal is essentially as follows: if σ is a substitution for m prototiles in \mathbb{E}^1 , where the factor is a PV number of algebraic degree m , then the Rauzy fractal is just the window set W as in (3), where one does not care about $\mu(W) > 0$ or $\mu(\partial W) = 0$ or Theorem 1.5. Note that therefore it is much more difficult to show that a given substitution point set is a model set than it is to compute the corresponding Rauzy fractal.

Before we proceed, let us mention a theorem which is important in this context and which is necessary in the next section.

THEOREM 1.6 ([HUT]). *Let (X, d) be a complete metric space and $\{f_1, \dots, f_n\}$ a finite set of contractive maps (i.e., $\exists c < 1 : \forall x, y \in X : d(f(x), f(y)) \leq cd(x, y)$). Then there is a unique compact set $K \subset X$ such that $K = \bigcup_{i=1}^n f_i(K)$.*

This theorem is a consequence of the Banach Fixed Point Theorem, and is easily generalized to partitions of K : Let $\{f_1, \dots, f_n\}$ be as above and $I_j \subseteq \{1, \dots, n\}$ for $1 \leq j \leq \ell$, then

there is a unique tuple (K_1, \dots, K_ℓ) of compact sets K_j , such that $K_j = \bigcup_{i \in I_j} f_i(K_{j_i})$ for all $1 \leq j \leq \ell$ ([KLI]). In the next section, this theorem will help to show the compactness of the window set W .

2. TWO RAUZY FRACTALS AND THE DUALITY PRINCIPLE

We will describe the duality principle by choosing a certain one-dimensional substitution, constructing the corresponding Rauzy fractal and the associated dual tiling, and then we will use the duality principle to show that the dual tiling is a model set.

2.1. SOME RAUZY FRACTALS. It is easy to see, that for any nonnegative integer $n \times n$ -matrix A there is a substitution tiling in \mathbb{E}^1 with substitution matrix A . Actually, from any such matrix one can immediately deduce a substitution, since there are only few geometric restrictions for tilings in one dimension. (The same problem for higher dimensions becomes quite difficult!) Here we consider the following matrices for $n \geq 2$:

$$M_n := (m_{ij})_{1 \leq i, j \leq n}, \quad \text{where } m_{ij} = \begin{cases} 1 & : i + j > n \\ 0 & : i + j \leq n \end{cases}$$

Obviously all these matrices are primitive. A substitution with substitution matrix M_n for the prototiles $1, 2, \dots, n$ is given by

$$(4) \quad \begin{array}{rcl} 1 & \rightarrow & n \\ 2 & \rightarrow & n-1 \ n \\ & \vdots & \vdots \\ n & \rightarrow & 1 \ 2 \cdots n-1 \ n \end{array}$$

Note that this is just a symbolic substitution. But in one dimension the geometric realization is obtained from this scheme in a unique way: The prototiles are intervals of different lengths, and the lengths are given by the PF-eigenvector of M_n . There is a nice trick to deduce the substitution factor directly from this substitutions: For fixed $n \in \mathbb{Z}, n \geq 2$, let $s_k := \sin(\frac{k\pi}{2n+1})$. Because of the wonderful formula (see [ND])

$$(5) \quad \frac{s_k}{s_1} s_i = \sum_{\nu=0}^{k-1} s_{i+1-k+2\nu}$$

it follows $\frac{s_n}{s_1} s_1 = s_n, \frac{s_n}{s_1} s_2 = s_{n-1} + s_n, \dots, \frac{s_n}{s_1} s_n = s_1 + s_2 + \dots + s_n$. Therefore the substitution factor (which is the PF-eigenvalue of M_n) is $\frac{s_n}{s_1}$, and the length of tile i is s_i . Now, we determine which of these substitutions define a Rauzy-fractal. The first condition to be fulfilled is that the substitution factor must be a PV-number², cf. Theorem 1.4. This is the case for $n = 2, 3, 4$ and 7 , and for no other $n \leq 30$. (This has been checked by a Maple program, and we don't expect any more PV-numbers as PF-eigenvalues of M_n for larger values of n .) The corresponding characteristic polynomials of M_n are:

$$x^2 - x - 1, \quad x^3 - 2x^2 - x + 1, \quad x^4 - 2x^3 - 3x^2 + x + 1, \quad x^7 - 4x^6 - 6x^5 + 10x^4 + 5x^3 - 6x^2 - x + 1.$$

The latter two are reducible over \mathbb{Z} . The irreducible polynomials defining $\frac{s_2}{s_1}$ ($n = 2$), $\frac{s_3}{s_1}$ ($n = 3$), $\frac{s_4}{s_1}$ ($n = 4$) and $\frac{s_7}{s_1}$ ($n = 7$) are

$$(6) \quad x^2 - x - 1, \quad x^3 - 2x^2 - x + 1, \quad x^3 - 3x^2 + 1, \quad x^4 - 4x^3 - 4x^2 + x + 1.$$

²Or a Salem number, but these don't play a role here, so we don't mention them explicitly.

Therefore, the algebraic degree of these numbers is 2,3,3,4, resp. Here we will focus mainly on the third case ($n = 4$). The first case ($n = 2$) leads to the well known Fibonacci sequences (see Section 3, or [PYF], Sections 2.6 and 5.4). The fourth case ($n = 7$) will lead to a three-dimensional Rauzy fractal in the end. The third case ($n = 4$) will show the methods and the typical problems which may occur, and therefore we discuss this case in the following in detail. This part may be regarded as a recipe how to compute other Rauzy fractals explicitly.

2.2. THE CASE $n = 4$. In this case we have got four prototiles. Since we consider one-dimensional tilings here, the tiles can be chosen to be intervals. We denote the four prototiles by L ('large'), M ('medium'), S ('small') and X ('extra small'). Their lengths are already known: They are given by a left PV-eigenvector of the substitution matrix, or by (5), so the values can be chosen as $\frac{s_4}{s_1}, \frac{s_3}{s_1}, \frac{s_2}{s_1}, \frac{s_1}{s_1} = 1$, resp. (or any positive multiples of these values). In order to construct the Rauzy fractal, we need this values as expressions in $\lambda = \frac{s_4}{s_1}$ (cf. Section 1.2). W.l.o.g, we choose $X := [0, 1]$. The substitution now reads

$$(7) \quad X \rightarrow L, \quad S \rightarrow ML, \quad M \rightarrow SML, \quad L \rightarrow XSML.$$

Since $\lambda X = L$ it follows $L = [0, \lambda]$. Let s, m, ℓ denote the lengths of S, M, L , resp. Since $\ell = \lambda$, we conclude from (7) the equations

$$\begin{aligned} \lambda s &= m + \lambda \\ \lambda m &= s + m + \lambda = s(\lambda + 1) \\ \lambda^2 &= 1 + s + m + \lambda = s(\lambda + 1) + 1 = \lambda m + 1 \end{aligned}$$

and therefore $m = \frac{\lambda^2 - 1}{\lambda}$. From (6) follows $\lambda^3 = 3\lambda^2 - 1$, hence $\lambda^{-1} = 3\lambda - \lambda^2$, and we obtain

$$m = (\lambda^2 - 1) \cdot (3\lambda - \lambda^2) = \lambda^2 - 2\lambda.$$

Consequently,

$$s = \lambda^2 - \lambda - m - 1 = \lambda - 1.$$

To apply the standard construction of Rauzy fractals, the number of prototiles must equal the algebraic degree of the substitution factor. So we need to get rid of one letter. By (7), a tile X in the tiling is always followed by a tile S . Since $x + s = \lambda$, we can merge every pair XS into one tile L . It is easy to check that the new substitution

$$(8) \quad S \rightarrow ML, \quad M \rightarrow SML, \quad L \rightarrow LML$$

gives rise to the same tilings as (7) does, up to merging pairs XS into L . We identify these tilings with a point set V , where V contains all (right) endpoints of the tiles. If $0 \in V$, then $V \subset \mathbb{Z}[\lambda]$, and every point $x \in V$ can be expressed by $x = a + b\lambda + c\lambda^2$ ($a, b, c \in \mathbb{Z}$). The lattice $\Lambda \in \mathbb{E}^3$ now is given by

$$(9) \quad \Lambda = \langle (1, 1, 1)^T, (\lambda, \lambda_2, \lambda_3)^T, (\lambda^2, \lambda_2^2, \lambda_3^2)^T \rangle_{\mathbb{Z}},$$

where λ_2, λ_3 are the algebraic conjugates of λ . From $\lambda\lambda_2\lambda_3 = -1$ and the fact that the values s_k occur frequently above, one easily deduces $\lambda_2 = -\frac{s_1}{s_2}, \lambda_3 = \frac{s_2}{s_4}$. This allows us to write down the star map now (cf. (2)). Let $0 \in V$. If $x \in V$, then $x = a + b\lambda + c\lambda^2$ for some $a, b, c \in \mathbb{Z}$, and

$$(10) \quad x^* = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} + c \begin{pmatrix} \lambda_2^2 \\ \lambda_3^2 \end{pmatrix}.$$

By (3), the window set W is just the closure of V^* . A short computation (using $\lambda_i^3 = 3\lambda_i^2 - 1$ for $i = 2, 3$) shows

$$(\lambda x)^* = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix} x^* =: Qx^*.$$

In order to show the compactness of W , and possibly to draw further conclusions about W (which is the Rauzy fractal wanted), we consider a substitution set V such that $\sigma(V) = V$. (As mentioned above, the considered substitution σ can be easily formulated for the point set V as well as for the tiling \mathcal{T} .) For example, let

$$(11) \quad V = \dots S \underline{M L L M} \underline{L L M L S} \underline{M L L M L M L S} \underline{M L L M L L M L S} \underline{M L L} \dots$$

where the right endpoint of the underlined L is located at zero. (Note, that the condition $\sigma(V) = V$ together with (11) defines V uniquely.) Denote by V_L (V_M, V_S , resp.) the set of all points in V which are right endpoints of intervals L (M, S , resp.). Then obviously

$$V = V_L \cup V_M \cup V_S.$$

Moreover, from (8) and $\sigma(V) = V$ follows

$$(12) \quad \begin{aligned} V_L &= \lambda V \cup \lambda V_L - \lambda^2 + \lambda, \\ V_M &= \lambda V - \lambda, \\ V_S &= \lambda V_M - \lambda^2 + \lambda. \end{aligned}$$

This can be seen as follows: One tile of type M and one tile of type L occur in *every* substituted tile T , not depending on the type of T . If $T = [a, b]$, then $\sigma(T)$ contains tiles $[\lambda b - \lambda - m, \lambda b - \lambda]$ (of type M) and $[\lambda b - \lambda, \lambda b]$ (of type L). Therefore, from $b \in V$ follows $\lambda b - \lambda \in V_M$, yielding the second equation, and $\lambda b \in V_L$, yielding the first part of the first equation. The other parts of (12) are obtained analogously.

Applying the star map to (12), we obtain the following equations for V^* :

$$(13) \quad \begin{aligned} V^* &= V_L^* \cup V_M^* \cup V_S^* \\ V_L^* &= QV^* \cup QV_L^* - (\lambda_2^2, \lambda_3^2)^T + (\lambda_2, \lambda_3)^T \\ V_M^* &= QV^* - (\lambda_2, \lambda_3)^T \\ V_S^* &= QV_M^* - (\lambda_2^2, \lambda_3^2)^T + (\lambda_2, \lambda_3)^T \end{aligned}$$

Obviously, Q defines a contracting linear map, thus all maps appearing in (13) are contracting. Therefore, Theorem 1.6 applies: There is one unique compact $W = W_L \cup W_M \cup W_S$ which fulfills (13). This assures the existence of this set, so we obtained a Rauzy fractal W .

To visualize W and its fractal nature, W can be approximated numerically. In fact, we just produce a finite subset of V and apply the star map to it. The resulting points are plotted into a diagram, which is shown in Fig. 2 (left). If we distinguish the points in the diagram according to their type (S, M or L), the result is shown in Fig. 2 (middle). The set W is the union of the three sets W_L, W_M, W_S , where $W_i = \text{cl}(V_i^*)$. From (13) follows that W_S is an affine image of W_M , but not similar to W_M , and that W_M is an affine image of W .

In the same way one can compute the Rauzy fractal for the case $n = 3$ in (4). The resulting pattern is shown in Fig. 2, right. Note, that we did not claim that one of the original substitution sets V is a model set until now. In order to prove this, one needs to prove $\mu(W) > 0$, $\mu(\partial W) = 0$ and $\text{dens}(V) = \text{dens}(\{\pi_1(\mathbf{x}) \mid \mathbf{x} \in \Lambda, \pi_2(\mathbf{x}) \in W\})$.

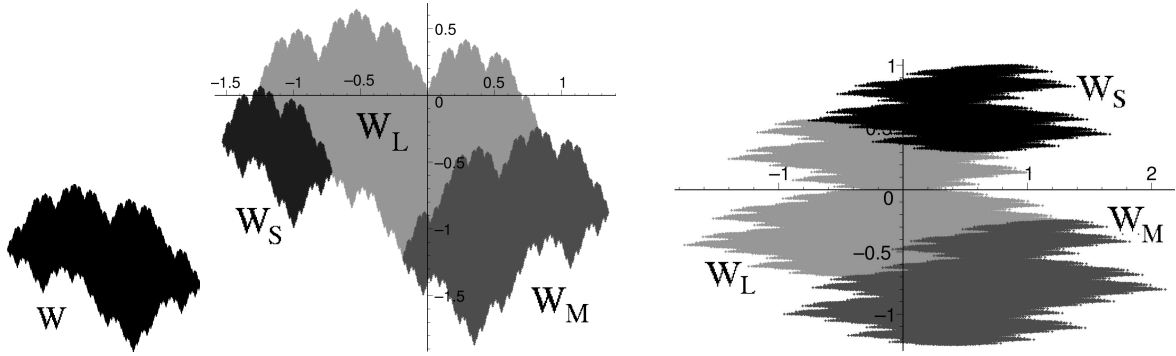


FIGURE 2. Left: The Rauzy fractal W arising from the substitution (8). Middle: the same set, partitioned according to the distinction of tiles S , M and L . Right: The Rauzy fractal arising from the substitution (4) for $n = 3$: $S \rightarrow L$, $M \rightarrow ML$, $L \rightarrow SML$.

2.3. THE DUAL TILING. Recall that Q in (13) is a contraction. Therefore $Q' := Q^{-1}$ is an expansion. Applying Q' to (13) and replacing V_i^* with its closure W_i ($i \in \{S, M, L\}$) yields the following equation system:

$$(14) \quad \begin{aligned} Q'W_L &= W_L \cup W_M \cup W_S \cup W_L - (\lambda_2, \lambda_3)^T + (1, 1)^T \\ Q'W_M &= W_L - (1, 1)^T \cup W_M - (1, 1)^T \cup W_S - (1, 1)^T \\ Q'W_S &= W_M - (\lambda_2, \lambda_3)^T + (1, 1)^T \end{aligned}$$

This defines the substitution σ' in \mathbb{E}^2 with three prototiles W_L, W_M, W_S :

$$(15) \quad \begin{aligned} \sigma'(\{W_L\}) &= \{W_L, W_M, W_S, W_L - (\lambda_2, \lambda_3)^T + (1, 1)^T\} \\ \sigma'(\{W_M\}) &= \{W_L - (1, 1)^T, W_M - (1, 1)^T, W_S - (1, 1)^T\} \\ \sigma'(\{W_S\}) &= \{W_M - (\lambda_2, \lambda_3)^T + (1, 1)^T\} \end{aligned}$$

In contrary to Def. 1.1, the expansion by Q' is not a similarity, but an affinity. Anyway, these substitution rules will yield a tiling of \mathbb{E}^2 . A part of it is shown in Fig. 3. (To be precise, we need to show that there will be no overlaps and no gaps, but this is not essential for the following, since we will consider the corresponding substitution point set, or a modified version where all tiles are polygons.)

Now, we will prove that the dual tiling is a model set. In analogy to the construction of V out of the original substitution tiling arising from (8), we choose in every prototile a *control point*. A good choice is (cf. (13)): $c_L := 0 \in W_L$, $c_M := Qc_L - (\lambda_2, \lambda_3)^T = (\lambda_2, \lambda_3)^T \in W_M$, $c_S := Qc_M - (\lambda_2^2, \lambda_3^2)^T + (\lambda_2, \lambda_3)^T \in W_S$. It is not too hard to check that the set $\{W_L, W_M - (\lambda_2^2, \lambda_3^2)^T + 2(\lambda_2, \lambda_3)^T + (1, 1)^T\}$ gives rise to a point set V' such that $\sigma'(V') = V'$, and V' is a Delone set. To be precise, let $V'_0 := \{c_L, c_M - (\lambda_2^2, \lambda_3^2)^T + 2(\lambda_2, \lambda_3)^T + (1, 1)^T\}$, and σ' be defined for points as well as for tiles in the canonical way. Then $V'_0 \subset \sigma'(V'_0)$, and inductively $(\sigma')^k(V'_0) \subset (\sigma')^{k+1}(V'_0)$. Therefore,

$$(16) \quad V' = \bigcup_{k \geq 1} (\sigma')^k(V'_0)$$

is well defined, and it holds $\sigma'(V') = V'$.

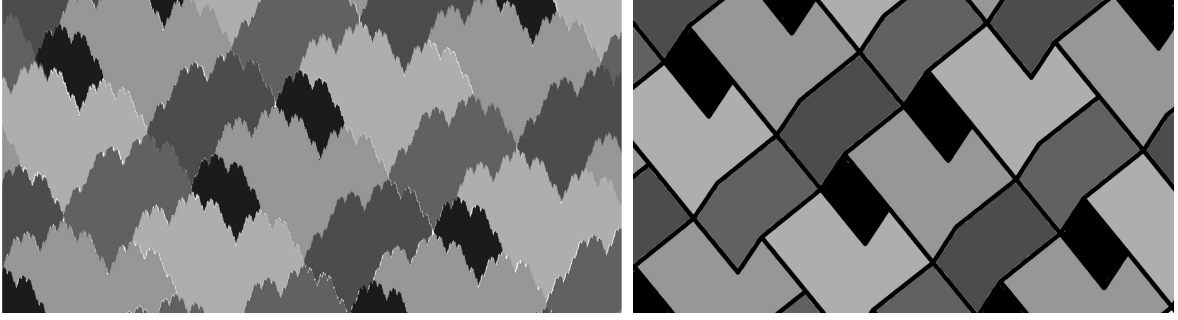


FIGURE 3. Left: A patch of the tiling arising from (15). Tiles of type W_S are black, the other tiles come in two slightly different shadings, such that the boundary between tiles of the same type remains visible. Right: A polygonalized version. Tiles and colours correspond to the tiling on the left.

Remark: As a first idea, one would try to start with the set $\{c_L\}$ instead of V'_0 . In this case, the property $(\sigma')^k(\{c_L\}) \subset (\sigma')^{k+1}(\{c_L\})$ is fulfilled, but the resulting point set would be no Delone set, since it possesses arbitrary large holes.

By the construction, it holds $V \subset \langle (1, 1)^T, (\lambda_2, \lambda_3)^T, (\lambda_2^2, \lambda_3^2)^T \rangle_{\mathbb{Z}}$. In particular, this shows a curious property of V' : If one coordinate of some $\mathbf{z} = (x, y)^T \in V'$ is known, say, $x = a + b\lambda_2 + c\lambda_2^2$ ($a, b, c \in \mathbb{Z}$), then the other coordinate y is known, too: $y = a + b\lambda_3 + c\lambda_3^2$. Another unusual property is that each prototile occurs in just one orientation. This is not true for most known examples of nonperiodic substitution tilings in dimension larger than one.

In a similar way as above (cf. (12)) we obtain from (15) the following equations, where $V' = V'_S \cup V'_L \cup V'_M$ and V'_i is the set of all points of type c_i in V' ($i \in \{S, M, L\}$).

$$(17) \quad \begin{aligned} V'_L &= Q'V'_L \cup Q'V'_M \cup Q'V'_L - (\lambda_2, \lambda_3)^T + (1, 1)^T \\ V'_M &= Q'V'_L - (\lambda_2, \lambda_3)^T \cup Q'V'_M - (\lambda_2, \lambda_3)^T \cup Q'V'_S \\ V'_S &= Q'V'_L + (\lambda_2, \lambda_3)^T - 2(\lambda_2^2, \lambda_3^2)^T \cup Q'V'_M + (\lambda_2, \lambda_3)^T - 2(\lambda_2^2, \lambda_3^2)^T \end{aligned}$$

In order to show that V' is a model set, we have to determine the lattice Λ' and the window set W' for V' . By the construction of V' follows $\Lambda' = \Lambda$. Since $0 \in V'$, we know that every $\mathbf{x} \in V'$ is of the form $a(1, 1)^T + b(\lambda_2, \lambda_3)^T + c(\lambda_2^2, \lambda_3^2)^T$. Thus our new star map is just the inverse of the original one in (10), namely

$$\star : \pi_2(\Lambda) \rightarrow \mathbb{E}^1, \quad \mathbf{x}^\star = \pi_1(\pi_2^{-1}(\mathbf{x})) = a + b\lambda + c\lambda^2.$$

Applying \star to (17) yields:

$$(18) \quad \begin{aligned} V'_L{}^\star &= \lambda^{-1}V'_L{}^\star \cup \lambda^{-1}V'_M{}^\star \cup \lambda^{-1}V'_L{}^\star - \lambda + 1 \\ V'_M{}^\star &= \lambda^{-1}V'_L{}^\star - \lambda \cup \lambda^{-1}V'_M{}^\star - \lambda \cup \lambda^{-1}V'_S{}^\star \\ V'_S{}^\star &= \lambda^{-1}V'_L{}^\star + \lambda - 2\lambda^2 \cup \lambda^{-1}V'_M{}^\star + \lambda - 2\lambda^2 \end{aligned}$$

Since all occurring maps are contractions, Theorem 1.6 applies, and the unique compact solution is

$$\text{cl}(V'_L{}^\star) = [-\lambda, 0] = L, \quad \text{cl}(V'_M{}^\star) = [\lambda - \lambda^2, -\lambda] = M, \quad \text{cl}(V'_S{}^\star) = [-2\lambda^2 + 1, -2\lambda^2 + \lambda] = S.$$

The multiplication of (18) by λ yields a distinct geometric realization of our original substitution (8). So, the dual tiling of the dual tiling is the one we started with. In particular, the

window set of V' is $W' = S \cup M \cup L = [-2\lambda^2 + 1, -2\lambda^2 + \lambda] \cup [\lambda - \lambda^2, 0]$. Therefore W' is compact,

$$(19) \quad \mu(W') = \lambda^2 - 1 = 7.290859374 \dots > 0$$

and $\mu(\partial W') = 0$. All we are left with is to compute $\text{dens}(V')$ and use Theorem 1.5. If the tiles W_i of the substitution σ' in (15) would be polygons, this would be an easy task, using the remark after Theorem 1.1. Therefore we replace the fractal tiles W_S, W_M, W_L by appropriate polygons P_S, P_M, P_L . Let

$$\begin{aligned} a &:= (\lambda_2^2 - 1, \lambda_3^2 - 1)^T, & b &:= (\lambda_2^2, \lambda_3^2)^T, \\ c &:= (\lambda_2^2 - \lambda_2, \lambda_3^2 - \lambda_3)^T, & d &:= (\lambda_2^2 - \lambda_2 - 1, \lambda_3^2 - \lambda_3 - 1)^T, \\ e &:= (2\lambda_2^2 - \lambda_2 - 1, 2\lambda_3^2 - \lambda_3 - 1)^T, & f &:= (2\lambda_2^2 - 1, 2\lambda_3^2 - 1)^T, \\ g &:= (\lambda_2, \lambda_3)^T, & h &:= (\lambda_2^2 + \lambda_2 - 1, \lambda_3^2 + \lambda_3 - 1)^T. \end{aligned}$$

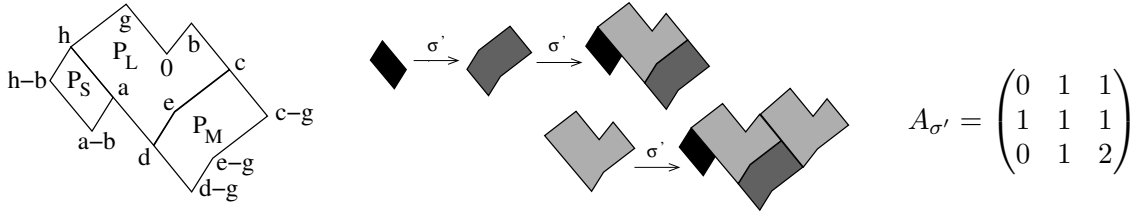


FIGURE 4. Left: The construction of the polygonal prototiles. Middle: A sketch of the substitution for them. Right: The corresponding substitution matrix $A_{\sigma'}$. Note that $A_{\sigma'} = (A_{\sigma})^T$.

Let $\text{conv}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ denote the convex hull of $\mathbf{x}_1, \dots, \mathbf{x}_n$. Then (see Fig. 4)

$$(20) \quad \begin{aligned} P_S &:= \text{conv}(a - b, a, h, h - b), \\ P_M &:= \text{conv}(e - g, e, d, d - g) \cup \text{conv}(c - g, c, e, e - g), \\ P_L &:= \text{conv}(0, e, c, b) \cup \text{conv}(0, g, h, d, e) \end{aligned}$$

It is quite simple (but lengthy) to show that by replacing W_i with P_i in (15), the substitution σ' leads to a tiling, without any gaps or overlaps. (Essentially, one uses the definition of a substitution tiling in Definition 1.1 and the primitivity of the substitution matrix. To show that there are no gaps and no overlaps it suffices to consider all vertex configurations in the first n substitutions $\sigma'(T), (\sigma')^2(T), \dots, (\sigma')^n(T)$ for appropriate n and some prototile T . So, in any certain case this is a finite problem. Here $n = 4$ and $T = P_L$ will do.) It is also quite simple (but lengthy again) to compute the areas of the prototiles P_S, P_M, P_L . Unfortunately, we cannot use the fact that the left eigenvector of the corresponding substitution matrix reflects the ratio of the areas, since the substitution for the polygonal tiles does not preserve the shape any longer. E.g., $\sigma'(W_S)$ is congruent to $Q'W_S$, but $\sigma'(P_S)$ is *not* congruent to $Q'P_S$. Altogether, we obtain

$$(21) \quad \begin{aligned} \mu(P_S) &= |\lambda_2| \lambda_3 (\lambda_3 - \lambda_2) = \frac{s_1}{s_4} \left(\frac{s_2}{s_4} + \frac{s_1}{s_2} \right), \\ \mu(P_M) &= \lambda_3 - \lambda_2 = \frac{s_2}{s_4} + \frac{s_1}{s_2}, \\ \mu(P_L) &= \lambda_2^2 - \lambda_3^2 + 2|\lambda_2| + 2\lambda_3 - |\lambda_2| \lambda_3^2 - \lambda_3 \lambda_2^2 = 1 + \frac{s_4 s_1}{s_2^2}, \end{aligned}$$

where the last equality requires an excessive use of (5). The vector $(\mu(P_S), \mu(P_M), \mu(P_L))$ indeed is a left eigenvector of the substitution matrix A (cf. Fig. 4) corresponding to the PF eigenvalue λ . A right eigenvector corresponding to λ is $(s_2, s_3, s_4)^T$. Therefore, the normalized right eigenvector of A — which entries, by Theorem 1.2, are the relative frequencies p_i of tile types P_i ($i \in \{S, M, L\}$) in the tiling — is $(p_S, p_M, p_L)^T := \frac{1}{s_2+s_3+s_4}(s_2, s_3, s_4)$. Since every point of V' corresponds to exactly one tile in the polygonal tiling, by Prop. 1.3 the relative density of V' is

$$\begin{aligned}
 \text{dens}(V') &= ((\mu(P_S), \mu(P_M), \mu(P_L)) \cdot (p_S, p_M, p_L)^T)^{-1} \\
 (22) \quad &= (s_2 + s_3 + s_4) \left(\frac{s_1 s_2^2}{s_4^2} + \frac{s_1^2}{s_4} + \frac{s_2 s_3}{s_4} + \frac{s_1 s_3}{s_2} + s_4 + \frac{s_1 s_4^2}{s_2^2} \right)^{-1} \\
 &= 0.8100954858\dots
 \end{aligned}$$

In order to apply Theorem 1.5 we need to compute $\det \Lambda$ (cf. (9)). A rather lengthy computation (or the usage of a computer algebra system, here: Maple) shows $\det \Lambda = \frac{27}{64s_1^2 s_2^2 s_4^2}$. Using elementary trigonometric formulas yields

$$\begin{aligned}
 s_1 s_2 s_4 &= \frac{1}{4}(s_3 + s_4 - s_2 - s_1) \\
 &= \frac{1}{4} \left(s_3 + 2 \cos \left(\frac{3\pi}{9} \right) s_1 - s_1 \right) = \frac{1}{4} s_3 = \frac{\sqrt{3}}{8},
 \end{aligned}$$

and therefore $\det \Lambda = 9$. Putting together this with (19) and (22) in view of Theorem 1.5 we obtain

$$\mu(W') / \text{dens}(V') = 9 = \det(\Lambda).$$

The leftmost equality requires again a very lengthy computation, or the usage of Maple. This completes the proof, and we obtain the following theorem.

THEOREM 2.1. *The set V' in (16), arising from the substitution σ' in (15), is a model set.*

One may ask why we do not state that the original set V arising from the substitution (8) is a model set. The reason is that there is one gap, namely, the value of $\mu(W)$. For example, one needs to show that the tiling by the fractal tiles W_S, W_M, W_L is not overlapping. This seems to be clear from the pictures, but we did not give a rigorous proof here. Indeed, this will require some more work, and we refer to further publications. Anyway, there are good reasons to assume that the areas of the fractal tiles W_S, W_M, W_L are equal to the areas of the polygonal tiles P_S, P_M, P_L , resp. Proving this will complete the proof that the original set V is a model set, too.

3. FURTHER REMARKS

The question whether a given structure is a model set is motivated for example by the question for the diffraction property of this structure. It is known that model sets have a pure point diffraction image, without any continuous part. For details, cf. [BMRS].

We did not emphasize it throughout the text, but all tilings considered here are nonperiodic. For the one-dimensional tilings this is a consequence of the fact that the relative frequencies of the tiles are irrational numbers. If a one-dimensional tiling is periodic, then the relative

frequencies are rational numbers. Anyway, all occurring two-dimensional tilings are nonperiodic, too. This can be shown with standard tools from the theory of nonperiodic substitution tilings (cf. statement 10.1.1 in [GSH]).

An important point in proving Theorem 2.1 is that we know the occurring values in terms of trigonometric expressions, thus it is possible to make exact computations of $\mu(W')$, $\text{dens}(V')$, $\det(\Lambda)$ etc. In general, the fractal nature of the occurring window sets may forbid this. But if all occurring sets are polytopes the things become easier. In the following, some examples are listed where the duality principle provides short and simple proofs that these considered substitution tilings are model sets.

Fibonacci sequences: The symbolic substitution $S \rightarrow L$, $L \rightarrow LS$ gives rise to the widely examined Fibonacci sequences. (Not to be confused with the Fibonacci numbers 1,1,2,3,5,8,13,..., sometimes also called Fibonacci sequence. If one considers the first substitutions of S , then the relation becomes clear: L , LS , LLS , $LSLLS$, $LSLLSLSL$, ...).

A geometric representation as a substitution tiling (Fibonacci tiling) is for example given by $L := [0, 1]$, $S = [1, \tau]$ (τ the golden ratio) and

$$\sigma(\{S\}) = \{L + \tau\}, \quad \sigma(\{L\}) = \{L, S\}.$$

By the standard construction (cf. Section 1.2), we obtain for some V which fulfills $\sigma(V) = V$ (namely, $V = \bigcup_{k \geq 0} \sigma^{2k}(\{L, L - 1\})$, up to one point this will do) the equations

$$\begin{aligned} V_L^* &= -\tau^{-1}V_L^* \cup -\tau^{-1}V_S^* \\ V_S^* &= -\tau^{-1}V_L^* + 1 \end{aligned},$$

to be read as equations in \mathbb{E}^1 . The unique compact solution is

$$\text{cl}(V_L^*) = W_L = [-1, \tau^{-1}], \quad \text{cl}(V_S^*) = W_S = [\tau^{-1}, \tau].$$

The dual tiling is given by the following substitution:

$$\begin{aligned} \sigma'(\{W_L\}) &= \{W_L, W_S\} \\ \sigma'(\{W_S\}) &= \{W_L - \tau\} \end{aligned}$$

Note that the substitution factor is $-\tau$. Nevertheless, the resulting family of substitution tilings for σ' is equal to these for σ . That means the dual tilings of Fibonacci tilings are Fibonacci tilings again; or shortly: Fibonacci tilings are self-dual. Moreover, a short computation yields that the equation in Theorem 1.5 is fulfilled, and therefore the Fibonacci sequences are model sets.

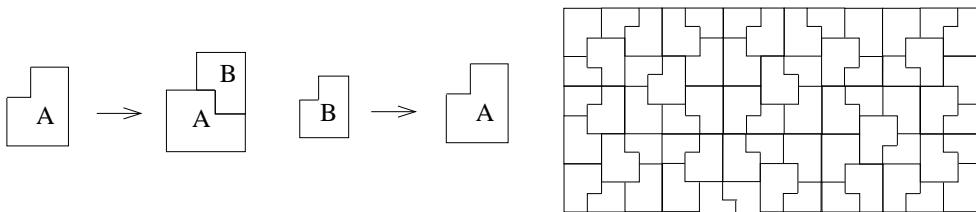


FIGURE 5. Left: The substitution for the Ammann chair. Right: A part of a corresponding substitution tiling.

Golden triangles and Ammann chair: The substitution for the golden triangles (cf. Fig. 1) defines a model set, if one replaces every triangle with the vertex at its right angle. This was essentially shown in [DVO] in detail³. It turns out that the golden triangle tilings are dual to the so-called Ammann chair tilings (or 'A2 tilings', cf. [GSH]). A sketch of the substitution of the latter is shown in Fig. 5, together with a part of the tiling. The substitution factor and the substitution matrix are equal to those of the golden triangle substitution. All occurring tiles, and all occurring window sets are polygons. Therefore, it is simple to compute the areas and the densities, thus proving — with the help of Theorem 1.5 — that both substitution tilings (resp. the corresponding point sets) are model sets.

The duality of the Ammann chair tilings and the golden triangle tilings was already indicated in [GEL] briefly. But a proof of this needs some care. For example, one has to pay attention to the fact, that each prototile occurs in four different orientations, not just in one, as it is true for the tilings in Fig. 3.

Variants of the tilings in Section 2: The tilings considered in the last section are a rich source of other substitutions, or other Rauzy fractals. If we restrict to the case $n = 4$, then the underlying lattice Λ will always be the same (namely, as in (9)). One possible modification is to permute the letters in the substitution. E.g., the substitution

$$(23) \quad S \rightarrow ML, M \rightarrow MLS, L \rightarrow LML$$

instead of (8) yields the Rauzy fractal shown in Fig. 6 (left). Another variant is to use the matrix $A_{\sigma'}$ of Fig. 4 (the transpose of the original substitution matrix A_{σ}) as a substitution matrix for a one-dimensional substitution with substitution factor λ . One possibility is the substitution

$$(24) \quad S \rightarrow M, M \rightarrow SML, L \rightarrow SMLL,$$

where S, M, L are intervals of length $\frac{s_1 s_2}{s_4}, s_2, s_4$, resp. This substitution yields the Rauzy fractal shown in Fig. 6 (right). Note that in this case the set W_M is not connected, it consists of two parts which are similar to each other. The substitution tiling arising from this Rauzy fractal (resp. from the corresponding equation system, multiplied by Q' , cf. (14)) would possess a prototile, namely, W_M , which is disconnected. Note, that we did not exclude disconnected (proto-)tiles in our definition of (substitution) tilings.

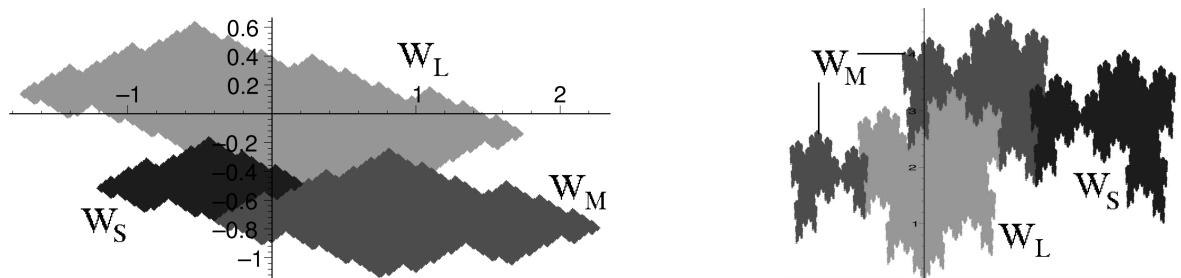


FIGURE 6. Two more Rauzy fractals, the left one arising from the substitution (23), the right one arising from the substitution (24)

³The authors used *all* vertices in the tiling instead of one vertex of each triangle, but their result is easily transferred to the latter case.

Combining these two modifications leads to even more examples, all of them based on the lattice Λ . All the two-dimensional tilings are model sets, which is easily proven by using the results of Section 2. From the case $n = 3$ one can also construct different tilings by permuting the letters. In contrary, all variants of the case $n = 2$ will yield Fibonacci sequences. Anyway, in the cases $n = 3, n = 4$ it may be interesting to examine the different possibilities with respect to common properties of the corresponding Rauzy fractals.

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