Multidimensional skew reflected diffusions

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Summary. Let $G \subset \mathbb{R}^d$ be (non-empty) open with euclidean closure $\overline{G} \neq \mathbb{R}^d$. We assume that G is the interior of $\overline{G}$. Let $\rho \in L^1(\mathbb{R}^d, dx)$, $\rho > 0$ dx-a.e., $0 \leq \alpha \leq 1$, and $\rho_\alpha := (\alpha 1_G + (1-\alpha)1_{\mathbb{R}^d \setminus G})\rho$, $m_\alpha := \rho_\alpha dx$. Let $D_\alpha = \mathbb{R}^d \setminus \overline{G}$ if $\alpha = 0$, $D_\alpha = \mathbb{R}^d$ if $0 < \alpha < 1$, $D_\alpha = G$ if $\alpha = 1$. Let $A = (a_{ij})$ be symmetric, and globally uniformly strictly elliptic, ρ satisfying some mild regularity assumption. Non-symmetric conservative diffusion processes $X_t \in \overline{D}_\alpha$ which inside $\overline{D}_\alpha \setminus \partial G$ obey the generator L given in the following suggestive form

$$Lf = \frac{1}{2}\rho^{-1}\sum_{i,j=1}^{d} \partial_i(a_{ij}\rho\partial_j f) + \rho_\alpha^{-1}\sum_{i=1}^{d} \overline{B}_i\,\partial_i f$$

where $\rho_\alpha^{-1}(\overline{B}_1, ..., \overline{B}_d) \in L^2(D_\alpha; \mathbb{R}^d, m_\alpha)$ satisfies $\sum_{i=1}^d \int_{D_\alpha} \overline{B}_i \partial_i f\,dx = 0$ for all $f \in C_0^\infty(\overline{D}_\alpha)$, are constructed and analyzed. The coefficients are in general neither continuous nor locally bounded. For ∂G, ρ, a_{ij}, sufficiently regular, a Skorokhod type decomposition for X_t is given in Theorem 5.2. Indeed, due to the jump of ρ_α along ∂G, the generator L has to be regarded with boundary conditions, or as integration by parts show, with additional surface measure terms. To these surface measures, there are uniquely associated multidimensional local times similarly to the case of reflected Brownian motion, where the local time is associated to the Dirac measure in zero. Under the irreducibility assumption of the semigroup corresponding to the symmetric part of L we show in Theorem 6.5 that X_t is recurrent for q.e. starting point. Because of the non-symmetry $(\alpha, 1-\alpha)$ we can speak about skew reflection. The extreme cases are $\alpha = 1$, $\alpha = 0$. In particular, if $\alpha = \frac{1}{2}$ there is no reflection.

The constructed process reminds one of a multidimensional generalized analogue of the α-skew Brownian motion (see [5], [9]), which corresponds to the case $d = 1$, $\overline{G} = \mathbb{R}^+$, $\rho \equiv 1$, $\overline{B}_i \equiv 0$, $a_{ij} \equiv \delta_{ij}$.

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1 The generalized Dirichlet form and its capacity

The following is an extension result corresponding to the diffusion constructed in [13]. Let $G \subset \mathbb{R}^d$ be (non-empty) open with euclidean closure $\overline{G} \neq \mathbb{R}^d$. Since we are not interested in the geometry of G, we may assume that G is the interior of $\overline{G}$. Then $\partial G = \partial\overline{G} = \partial(\mathbb{R}^d \setminus \overline{G})$. Let dx be the Lebesgue measure on $\mathbb{R}^d$, $\rho \in L^1(\mathbb{R}^d, dx)$. Let $0 \le \alpha \le 1$, and put

$$D_\alpha := \begin{cases} \mathbb{R}^d \setminus \overline{G} & \text{for } \alpha = 0 \\ \mathbb{R}^d & \text{for } 0 < \alpha < 1 \\ G & \text{for } \alpha = 1 \end{cases}$$

We equip D_α with the usual Euclidean norm $|\cdot| = \langle \cdot, \cdot \rangle^{1/2}$. Let $\rho_\alpha = (\alpha 1_G \rho + (1-\alpha) 1_{\mathbb{R}^d \setminus G} \rho)$ with $\rho > 0$ dx-a.e., and $m_\alpha := \rho_\alpha dx$. Let $|\cdot|_1$, $|\cdot|_2$, $|\cdot|_\infty$, denote respectively the L^1-norm, L^2-norm, L^∞-norm, w.r.t. m_α on D_α. For $\mathcal{D} \subset L^1(D_\alpha, m_\alpha)$ let $\mathcal{D}_b := \mathcal{D} \cap L^\infty(D_\alpha, m_\alpha)$, $\mathcal{D}^+ := \{f \in \mathcal{D} | f \ge 0 \ m_\alpha - a.e.\}$.

Let $A = (a_{ij})_{1 \le i,j \le d}$ be measurable, symmetric, and uniformly globally strictly elliptic on D_α, i.e. $\exists \lambda > 0$ such that

$$\lambda^{-1} \sum_{i=1}^{d} \xi_i^2 \le \sum_{i,j=1}^{d} a_{ij}(x)\, \xi_i \xi_j \le \lambda \sum_{i=1}^{d} \xi_i^2 \quad \forall (\xi_1, ..., \xi_d) \in \mathbb{R}^d, \ m_\alpha\text{-a.e. } x \in D_\alpha. \tag{1}$$

For $k \in \mathbb{N} \cup \{\infty\}$, let $C^k(\mathbb{R}^d)$ be the k-times continuously differentiable functions on $\mathbb{R}^d$, and $C_0^k(\mathbb{R}^d)$ those functions in $C^k(\mathbb{R}^d)$ which have compact support. For $D \subset \mathbb{R}^d$ closed let

$$C_0^\infty(D) := \{u|_D : u \in C_0^\infty(\mathbb{R}^d)\}.$$

Denote by $\overline{D}_\alpha$ the euclidean closure of D_α. Let us from now on assume that

$$\mathcal{E}^r(u, v) := \frac{1}{2} \int_{D_\alpha} \langle A \nabla u, \nabla v \rangle \, dm_\alpha \, ; \quad u, v \in C_0^\infty(\overline{D}_\alpha),$$

is closable in $L^2(D_\alpha, m_\alpha)$. Conditions in order to have closability are very mild. For instance, if ρ satisfies the Hamza type condition in D_α, or $\partial_i \rho \in L^1_{loc}(\mathbb{R}^d, dx)$, $1 \le i \le d$, then $(\mathcal{E}^r, C_0^\infty(\overline{D}_\alpha))$ is closable in $L^2(D_\alpha, m_\alpha)$. This follows easily by adapting arguments from [13].

Let us denote the closure of $(\mathcal{E}^r, C_0^\infty(\overline{D}_\alpha))$ on $L^2(D_\alpha, m_\alpha)$ by $(\mathcal{E}^r, D(\mathcal{E}^r))$. $(\mathcal{E}^r, D(\mathcal{E}^r))$ is a symmetric Dirichlet form (cf. [4]). $(\mathcal{E}^r, D(\mathcal{E}^r))$ needs not to be regular (compare [4, p.6] for the definition) on D_α. Let for instance $\alpha = 1$, then $C_0(D_\alpha) \cap D(\mathcal{E}^r)$ is in general not dense in $D(\mathcal{E}^r)$. It becomes a regular Dirichlet form on $\overline{D}_\alpha$ with the identifications specified in the following remark.

Remark 1.1 *If $0 < \alpha < 1$, then $C_0(\overline{D}_\alpha) \cap D(\mathcal{E}^r)$ is dense in $D(\mathcal{E}^r)$. Obviously, $C_0(\overline{D}_\alpha) \cap D(\mathcal{E}^r)$ is dense in $C_0(\overline{D}_\alpha)$. Therefore $(\mathcal{E}^r, D(\mathcal{E}^r))$ is regular on $\overline{D}_\alpha = D_\alpha$. For $\alpha = 0$, or $\alpha = 1$, we can make it a regular Dirichlet form on $\overline{D}_\alpha$ through the following identification. We first extend m_α to $\overline{D}_\alpha$ by setting $m_\alpha(\partial G) := 0$, thus identifying $L^p(D_\alpha, m_\alpha)$ with $L^p(\overline{D}_\alpha, m_\alpha)$, $p \ge 1$. We then may regard $(\mathcal{E}^r, D(\mathcal{E}^r))$ as a Dirichlet form on $L^2(\overline{D}_\alpha, m_\alpha)$. In this case $(\mathcal{E}^r, D(\mathcal{E}^r))$ is regular. Henceforth, we will make these identifications.*

Let $(G^r_\beta)_{\beta>0}$ be the resolvent associated to $(\mathcal{E}^r, D(\mathcal{E}^r))$ (see [4, Chapter 1.3]), and $(\cdot,\cdot)$ be the inner product in $L^2(D_\alpha, m_\alpha)$. For $\beta > 0$ let $\mathcal{E}^r_\beta(\cdot,\cdot) := \mathcal{E}^r(\cdot,\cdot) + \beta(\cdot,\cdot)$, $|\cdot|_{D(\mathcal{E}^r)} := \mathcal{E}^r_1(\cdot,\cdot)^{\frac{1}{2}}$ be the norm corresponding to $\mathcal{E}^r$. Then $D(\mathcal{E}^r)$ is the completion of $C_0^\infty(\overline{D}_\alpha)$ w.r.t. $|\cdot|_{D(\mathcal{E}^r)}$. We denote by $(L^r, D(L^r))$ the generator associated to $(\mathcal{E}^r, D(\mathcal{E}^r))$, i.e.

$$D(L^r) = \{f \in D(\mathcal{E}^r) : g \mapsto \mathcal{E}^r(f,g) \text{ is continuous w.r.t. } (\cdot,\cdot)^{\frac{1}{2}} \text{ on } D(\mathcal{E}^r)\},$$

and for $f \in D(L^r)$, $L^r f$ is the unique element in $L^2(\overline{D}_\alpha, m_\alpha)$ such that $(-L^r f, g)_m = \mathcal{E}^r(f,g)$ for all $g \in D(\mathcal{E}^r)$. Since $\rho \in L^1(\mathbb{R}^d, dx)$, one can easily see that $1_{\overline{D}_\alpha} \in D(\mathcal{E}^r)$, and $\mathcal{E}^r(1_{\overline{D}_\alpha}, f) = 0$ for any $f \in D(\mathcal{E}^r)$. Hence $1_{\overline{D}_\alpha} \in D(L^r)$ and $L^r 1_{\overline{D}_\alpha} = 0$, i.e. L^r is conservative. In particular $G^r_1 1_{\overline{D}_\alpha} = (1-L^r)^{-1} 1_{\overline{D}_\alpha} = (1-L^r)^{-1}(1-L^r)1_{\overline{D}_\alpha} = 1_{\overline{D}_\alpha}$.

Remark 1.2 *Suppose that as in Theorem 5.2 below, everything is sufficiently smooth. Let* $\widetilde{A} = (\widetilde{a}_{ij})_{1\le i,j\le d}$, *where* $\widetilde{a}_{ij}$ *is a q.c. version of* a_{ij}. *Let* $f \in C_0^\infty(\overline{D}_\alpha)$, $\alpha \neq \frac{1}{2}$. *Then* f *is in* $D(L^r)$, *if*

$$\langle \widetilde{A}\nabla f, \eta\rangle = 0 \quad Tr(\rho)d\sigma\text{-a.e.}$$

Indeed, it suffices to look at the proof of Theorem 5.2.

We consider a measurable vector field $B : D_\alpha \longrightarrow \mathbb{R}^d$, which is m_α-square integrable on D_α, i.e. $\int_{D_\alpha} |B|^2\, dm_\alpha < \infty$, and such that

$$\int_{D_\alpha} \langle B, \nabla u\rangle\, dm_\alpha = 0 \text{ for all } u \in C_0^\infty(\overline{D}_\alpha). \tag{2}$$

Note, that since $C_0^\infty(\overline{D}_\alpha) \subset D(\mathcal{E}^r)$ densely w.r.t. $|\cdot|_{D(\mathcal{E}^r)}$, and because of strict ellipticity (1), (2), extends to all of $D(\mathcal{E}^r)$. Furthermore,

$$\int_{D_\alpha} \langle B, \nabla u\rangle v\, dm_\alpha = -\int_{D_\alpha} \langle B, \nabla v\rangle u\, dm_\alpha \text{ for all } u, v \in D(\mathcal{E}^r)_b. \tag{3}$$

Exactly as in [13, Proposition 1.4] we have the following:

Proposition 1.3 *(i) The operator* $L^r u + \langle B, \nabla u\rangle$, $u \in D(L^r)_b$, *is dissipative, hence in particular closable in* $L^1(D_\alpha, m_\alpha)$. *The closure* $(\overline{L}, D(\overline{L}))$ *generates a sub-Markovian* C_0-*semigroup of contractions* $(\overline{T}_t)_{t\ge 0}$.

(ii) $D(\overline{L})_b \subset D(\mathcal{E}^r)$ *and*

$$\mathcal{E}^r(u,v) - \int_{D_\alpha} \langle B, \nabla u\rangle v\, dm_\alpha = -\int_{D_\alpha} \overline{L}u\, v\, dm_\alpha \textit{ for all } u \in D(\overline{L})_b, v \in D(\mathcal{E}^r)_b. \tag{4}$$

In particular,

$$\mathcal{E}^r(u,u) = -\int_{D_\alpha} \overline{L}u\, u\, dm_\alpha \textit{ for every } u \in D(\overline{L})_b. \tag{5}$$

(iii) $(\overline{T}_t)_{t\ge 0}$ *is Markovian.*

$(\overline{T}_t)_{t\geq 0}$ can be restricted to a C_0-semigroup $(T_t)_{t\geq 0}$ on $L^2(D_\alpha, m_\alpha)$. The corresponding generator $(L, D(L))$ is the part of $(\overline{L}, D(\overline{L}))$ on $L^2(D_\alpha, m_\alpha)$, i.e. $D(L) = \{u \in D(\overline{L}) \cap L^2(D_\alpha, m_\alpha) | \overline{L}u \in L^2(D_\alpha, m_\alpha)\}$, and $Lu := \overline{L}u$, $u \in D(L)$. Let $(L', D(L'))$ be the adjoint of $(L, D(L))$ in $L^2(D_\alpha, m_\alpha)$, and $(T_t')_{t\geq 0}$ the corresponding semigroup. According to [10, Examples 4.9(ii)], the *generalized Dirichlet form* corresponding to $(L, D(L))$ is

$$\mathcal{E}(u,v) := \left\{ \begin{array}{ll} (-Lu, v) & \text{for } u \in D(L),\ v \in L^2(D_\alpha, m_\alpha) \\ (-L'v, u) & \text{for } v \in D(L'),\ u \in L^2(D_\alpha, m_\alpha). \end{array} \right.$$

Define $\mathcal{E}_\beta(\cdot,\cdot) := \mathcal{E}(\cdot,\cdot) + \beta(\cdot,\cdot)$. As in [13] one extends $\mathcal{E}$ as

$$\mathcal{E}(u,v) = \mathcal{E}^r(u,v) - \int_{D_\alpha} \langle B, \nabla u\rangle v \, dm_\alpha \ \text{ for every } \ u, v \in D(\mathcal{E}^r)_b. \tag{6}$$

As a consequence of (3) we then have

$$\mathcal{E}^r(u,u) = \mathcal{E}(u,u) \ \text{ for every } \ u \in D(\mathcal{E}^r)_b. \tag{7}$$

(7) suggests that the capacities related to $\mathcal{E}$ and $\mathcal{E}^r$ are equivalent. This will be shown further below.
Let $(G_\beta)_{\beta>0}$ (resp. $(G'_\beta)_{\beta>0}$) the strongly continuous contraction resolvents on $L^2(D_\alpha, m_\alpha)$ related to $(L, D(L))$ (resp. $(L', D(L'))$). As usual let f_U denote the 1-reduced function of f on U w.r.t. $\mathcal{E}$. By abuse of notation $(G_1'\varphi)_U$ will always denote the 1-coreduced of $G_1'\varphi$ on U. Let $e_f^{r,U}$, $f \in D(\mathcal{E}^r)$, denote the 1-reduced function of f on U. If $U = \overline{D}_\alpha$ we simply write e_f^r instead of $e_f^{r,\overline{D}_\alpha}$, and e_f instead of $f_{\overline{D}_\alpha}$.
Let Cap be the capacity associated to $\mathcal{E}^r$ as defined in [4, p.64]. Recall that an increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of $\overline{D}_\alpha$ is called an ($\mathcal{E}^r$-)*nest on* $\overline{D}_\alpha$, if $\lim_{k\to\infty} \text{Cap}(F_k^c) = 0$.

We fix throughout $\varphi \in L^1(D_\alpha, m_\alpha)$, $0 < \varphi \leq 1$. The φ-capacity related to $\mathcal{E}$ is determined by

$$\text{Cap}_\varphi(U) = \mathcal{E}_1(G_1\varphi, (G_1'\varphi)_U) \text{ for } U \subset \overline{D}_\alpha, U \text{ open}.$$

An increasing sequence of closed subsets $(F_k)_{k\geq 1}$ is called an $\mathcal{E}$-nest, if $\lim_{k\to\infty} \text{Cap}_\varphi(F_k^c) = 0$. A subset $N \subset \overline{D}_\alpha$ is called $\mathcal{E}$-exceptional if there is an $\mathcal{E}$-nest $(F_k)_{k\geq 1}$ such that $N \subset \cap_{k\geq 1} \overline{D}_\alpha \setminus F_k$. The $\mathcal{E}^r$-exceptional sets are defined similarly.

Theorem 1.4 *(i)* $(F_k)_{k\in\mathbb{N}}$ *is an* $\mathcal{E}^r$*-nest on* $\overline{D}_\alpha$ *in the sense of [4, p. 67] if, and only if* $(F_k)_{k\in\mathbb{N}}$ *is an* $\mathcal{E}$*-nest.*

(ii) A subset N *of* $\overline{D}_\alpha$ *is* $\mathcal{E}$*-exceptional if, and only if it is exceptional w.r.t.* $\mathcal{E}^r$ *in the sense of [4, p. 134].*

Proof It is enough to show (i). Let $(F_k)_{k\in\mathbb{N}}$ be an $\mathcal{E}^r$-nest. Then (see [4]) $e^{r,F_k^c}_{1_{\overline{D}_\alpha}} \to 0$ in $D(\mathcal{E}^r)$. Since $G_1\varphi \leq 1_{\overline{D}_\alpha}$ it follows $e^{r,F_k^c}_{G_1\varphi} \to 0$ in $D(\mathcal{E}^r)$. For $\beta > 0$ let $(G_1'\varphi)^\beta_{F_k^c}$ be the unique solution $f \in D(L')$ to $(1-L')f = \beta(f - G_1'\varphi \cdot 1_{F_k^c})^-$. It is known that $(G_1'\varphi)^\beta_{F_k^c} \uparrow (G_1'\varphi)_{F_k^c}$ and strongly in $L^2(D_\alpha, m_\alpha)$. It is easy to see that $(G_1'\varphi)^\beta_{F_k^c}$ also converges weakly in $D(\mathcal{E}^r)$ to $(G_1'\varphi)_{F_k^c}$ as $\beta \to \infty$. Then using in particular (1)

$$\begin{aligned}
\mathcal{E}_1\left(G_1\varphi, (G_1'\varphi)_{F_k^c}\right) &= \lim_{\beta\to\infty} \mathcal{E}_1\left(G_1\varphi, (G_1'\varphi)^\beta_{F_k^c}\right) \le \lim_{\beta\to\infty} \mathcal{E}_1\left(e^{r,F_k^c}_{G_1\varphi}, (G_1'\varphi)^\beta_{F_k^c}\right) \\
&= \lim_{\beta\to\infty} \left\{\mathcal{E}^r_1\left(e^{r,F_k^c}_{G_1\varphi}, (G_1'\varphi)^\beta_{F_k^c}\right) - \int_{D_\alpha} \langle B, \nabla e^{r,F_k^c}_{G_1\varphi}\rangle (G_1'\varphi)^\beta_{F_k^c}\, dm_\alpha\right\} \\
&\le \int_{D_\alpha} e^{r,F_k^c}_{G_1\varphi}\, dm_\alpha + \lambda |B|_2 \left| e^{r,F_k^c}_{G_1\varphi}\right|_{D(\mathcal{E}^r)} .
\end{aligned}$$

It follows that $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest. Conversely, suppose that $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest. Then by [10, III. Definition 2.3.(i)] $(1_{\overline{D}_\alpha})_{F_k^c} \to 0$ in $L^2(D_\alpha, m_\alpha)$. But since $|\beta G_{\beta+1}(1_{\overline{D}_\alpha})_{F_k^c}|^2_{D(\mathcal{E}^r)} \le \left((1_{\overline{D}_\alpha})_{F_k^c}, 1_{\overline{D}_\alpha}\right)$ we obtain that $|(1_{\overline{D}_\alpha})_{F_k^c}|^2_{D(\mathcal{E}^r)} \le \left((1_{\overline{D}_\alpha})_{F_k^c}, 1_{\overline{D}_\alpha}\right) \to 0$ as $k \to \infty$. Further

$$\begin{aligned}
\left((1_{\overline{D}_\alpha})_{F_k^c}, \varphi\right) &= \mathcal{E}^r_1\left((1_{\overline{D}_\alpha})_{F_k^c}, e^{r,F_k^c}_{G_1^r\varphi}\right) + \mathcal{E}^r_1\left((1_{\overline{D}_\alpha})_{F_k^c}, G_1^r\varphi - e^{r,F_k^c}_{G_1^r\varphi}\right) \\
&= \mathcal{E}^r_1\left(e^{r,F_k^c}_{1_{\overline{D}_\alpha}}, e^{r,F_k^c}_{G_1^r\varphi}\right) + \mathcal{E}^r_1\left((1_{\overline{D}_\alpha})_{F_k^c}, G_1^r\varphi - e^{r,F_k^c}_{G_1^r\varphi}\right),
\end{aligned}$$

where $(G^r_\alpha)_{\alpha>0}$ is the resolvent associated to $\mathcal{E}^r$. Since

$$\left|\mathcal{E}^r_1\left((1_{\overline{D}_\alpha})_{F_k^c}, G_1^r\varphi - e^{r,F_k^c}_{G_1^r\varphi}\right)\right| \le |(1_{\overline{D}_\alpha})_{F_k^c}|_{D(\mathcal{E}^r)} \left|G_1^r\varphi - e^{r,F_k^c}_{G_1^r\varphi}\right|_{D(\mathcal{E}^r)} \le (\varphi, G_1^r\varphi)|(1_{\overline{D}_\alpha})_{F_k^c}|_{D(\mathcal{E}^r)},$$

we obtain

$$0 = \lim_{k\to\infty} \left((1_{\overline{D}_\alpha})_{F_k^c}, \varphi\right) = \lim_{k\to\infty} \mathcal{E}^r_1\left(e^{r,F_k^c}_{1_{\overline{D}_\alpha}}, e^{r,F_k^c}_{G_1^r\varphi}\right) = \lim_{k\to\infty}\left(e^{r,F_k^c}_{1_{\overline{D}_\alpha}}, \varphi\right),$$

and consequently

$$\lim_{k\to\infty} \mathrm{Cap}(F_k^c) = \lim_{k\to\infty} \int_{D_\alpha} e^{r,F_k^c}_{1_{\overline{D}_\alpha}}\, dm_\alpha = 0.$$

□

Having shown that $\mathcal{E}^r$ and $\mathcal{E}$ have equivalent capacities, we can pronounce exceptional, quasi-continuous, quasi-everywhere, etc., unambiguously and without specifying whether it is meant w.r.t. $\mathcal{E}^r$ or $\mathcal{E}$, and so we will do.

2 Construction of an associated diffusion

The results obtained in [13] are derived by using the following ingredients: strict ellipticity (1), finiteness of the reference measure m_α, and m_α-square integrability of the vectorfield B. This is the reason why the results of [13] carry over to our situation. We will list below the results corresponding to [13]. The interested reader may then refer to [13].

In order to show the existence of an associated process we have to check quasi-regularity of $\mathcal{E}$ and condition D3 of [10, IV.2]. Doing the identifications of Remark 1.1, the quasi-regularity of the generalized Dirichlet form $\mathcal{E}$ on $\overline{D}_\alpha$ is in view of Theorem 1.4 a consequence of the regularity of the classical Dirichlet form $\mathcal{E}^r$ on $\overline{D}_\alpha$. Since exactly as in [13] $\mathcal{Y} := D(\overline{L})_b$ is a linear space such that $\mathcal{Y} \subset L^\infty(D_\alpha, m_\alpha)$, $\mathcal{Y} \cap D(L) \subset D(L)$ dense, $\lim_{\beta\to\infty} e_{f-\beta G_\beta f} = 0$ in $L^2(D_\alpha, m_\alpha)$ for all $f \in \mathcal{Y}$, and such that $f \wedge \beta \in \overline{\mathcal{Y}}$(=the closure of $\mathcal{Y}$ in $L^\infty(D_\alpha, m_\alpha)$) if $f \in \overline{\mathcal{Y}}$, $\beta \geq 0$, the existence of an associated process follows from a general result in [10]. In particular one also shows that $D(\overline{L})_b$ is an algebra.
Finally, we can also show that $\mathcal{E}$ is local in the sense of generalized Dirichlet forms and obtain the following.

Theorem 2.1 *There exists a Hunt process* $\mathbb{M} = (\Omega, (\mathcal{F})_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in\overline{D}_\alpha})$ *such that the resolvent* $R_\beta f(x) := E_x[\int_0^\infty e^{-\beta t} f(X_t)dt]$ *is a quasi-continuous* m_α*-version of* $G_\beta f$ *for any* $f \in L^2(D_\alpha, m_\alpha)_b$, $\beta > 0$. *In particular* $\mathbb{M}$ *is conservative, i.e.* $\beta R_\beta 1_{\overline{D}_\alpha}(x) = 1$ *for quasi every* $x \in \overline{D}_\alpha$, *and there exists an exceptional set* $N \subset \overline{D}_\alpha$ *such that*

$$P_x\left(t \mapsto X_t \textit{ is continuous on } [0,\infty[\right) = 1 \textit{ for every } x \in \overline{D}_\alpha \setminus N.$$

Remark 2.2 *Since* $-\overline{B}$ *satisfies the same assumptions as* $\overline{B}$, *exactly as we have constructed the process* $\mathbb{M}$ *associated to* L *one can construct the coprocess* $\widehat{\mathbb{M}}$ *associated to* L'. *The coprocess will have exactly the same properties than* $\mathbb{M}$.

3 The Revuz correspondence

Definition 3.1 *A positive measure* ν *on* $(\overline{D}_\alpha, \mathcal{B}(\overline{D}_\alpha))$ *charging no exceptional set is called smooth if there exists a nest* $(F_k)_{k\in\mathbb{N}}$ *of compact subsets of* $\overline{D}_\alpha$, *such that*

$$\nu(F_k) < \infty \quad \textit{for all} \quad k \in \mathbb{N}.$$

The smooth measures are denoted by S.

The positive continuous additive functionals (PCAF's) of $\mathbb{M}$ are defined as in [12]. The following theorem accomplishes together with [12, Theorem 3.1] the so-called Revuz correspondence for the generalized Dirichlet form $\mathcal{E}$. It can be shown directly and exactly as in [13] (please see also the following Remark 3.3).

Theorem 3.2 *Let* $\nu \in S$. *Then there exists one, and only one PCAF* $(A_t)_{t\geq 0}$ *of* $\mathbb{M}$ *such that for any positive, Borel measurable* f, *and any quasi-continuous* m_α*-version* $\widetilde{v}$ *of* $v \in \widehat{\mathcal{P}}_b = \{u \in L^2(D_\alpha, m_\alpha)_b \,|\, \beta G'_{\beta+1} u \leq u \ \forall \beta \geq 0\}$, *we have*

$$\int_{\overline{D}_\alpha} \widetilde{v} f d\nu = \lim_{\beta\to\infty} \beta \int_{D_\alpha} \widetilde{v}(x) E_x\left[\int_0^\infty e^{-(\beta+1)t} f(X_t) dA_t\right] m_\alpha(dx). \qquad (8)$$

In this case we write $\nu = \nu_A$.

Remark 3.3 *A consequence of Remark 2.2 is that the process* $\mathbb{M}$ *is in weak duality with the coprocess* $\widehat{\mathbb{M}}$. *In particular* m_α *is an excessive measure for the process* $\mathbb{M}$. *Theorem 3.2 could then also follow from results in [6, §9] where properties of Revuz measures are discussed in the weak duality context. The interested reader is invited to consult the given reference, and to compare carefully the fine (process) topology of [6], with our analytic capacity.*

4 Semimartingale characterization

From [12, Theorem 4.5(i)] we know that for any $f \in D(L)$ with quasi-continuous m_α-version $\widetilde{f}$ we have a unique decomposition

$$A_t^{[f]} := \widetilde{f}(X_t) - \widetilde{f}(X_0) = M_t^{[f]} + N_t^{[f]}, \quad t \geq 0, \tag{9}$$

where $M^{[f]}$ is a martingale additive functional (MAF) of $\mathbb{M}$ of finite energy and $N^{[f]}$ is a continuous additive functional (CAF) of $\mathbb{M}$ of zero energy. The energy of an additive functional A of $\mathbb{M}$ is defined by

$$e(A) \;=\; \frac{1}{2}\lim_{\beta\to\infty}\beta^2\int_{D_\alpha} E_x\left[\int_0^\infty e^{-\beta t}A_t^2 dt\right] m_\alpha(dx)$$

whenever this limit exists in $[0,\infty]$. The equality (9) is to be understood in the sense of equivalence of additive functionals of $\mathbb{M}$.
If $A^{[f]}$ decomposes in the sense of (9) for some $\widetilde{f}$ with $f \in L^2(D_\alpha, m_\alpha)$ we write $f \in L^2(D_\alpha, m_\alpha)^{dec}$. Using the extension theorem [12, Theorem 4.5(ii)] it can be shown exactly as in [13] that $D(\mathcal{E}^r)_b \cup D(L) \subset L^2(D_\alpha, m_\alpha)^{dec}$.

Remark 4.1 *For* $f \in D(\mathcal{E}^r)$ *we obtain also the decomposition (9) except that we do not know whether* $e(N^{[f]}) = 0$ *(cf. Remark 4.2 in [13]).*

If $u \in D(L)$ then clearly $N_t^{[u]} = \int_0^t Lu(X_s)ds$ is of bounded variation, i.e. it can be represented as the difference of two PCAF's of $\mathbb{M}$. For general $u \in D(\mathcal{E}^r)_b$ we have roughly the following. $N_t^{[u]}$ is of bounded variation, if and only if there exist ν_1, $\nu_2 \in S$, such that

$$\int_{\overline{D}_\alpha} w\, d(\nu_1 - \nu_2) \;=\; \mathcal{E}^r(u, w) - \int_{D_\alpha} \langle B, \nabla u\rangle w\, dm_\alpha \tag{10}$$

for "enough"quasi-continuous $w \in D(\mathcal{E}^r)_b$. In this case $N_t^{[u]} = A_t^2 - A_t^1$, where A^1, A^2, are the PCAF's associated with ν_1, ν_2. For the precise meaning of "enough"we refer to Theorem 4.5. in [13].

5 Identification of the process

Throughout this section we assume that $G \subset \mathbb{R}^d$ is a bounded Lipschitz domain, i.e. G is open, bounded, and its boundary ∂G is locally the graph of a Lipschitz function. Let $\overline{G}$ be the (compact) closure of G in $\mathbb{R}^d$ equipped with the usual Euclidean norm $|\cdot| = \langle \cdot, \cdot \rangle^{1/2}$. Let $\Omega \subset \mathbb{R}^d$ open. Let $H^{1,p}(\Omega)$, $p \in [1, \infty[$, denote the classical Sobolev spaces of order one in $L^p(\Omega, dx)$, i.e. $H^{1,p}(\Omega) := \{u \in L^p(\Omega, dx) | \partial_i u \in L^p(\Omega, dx), 1 \leq i \leq d\}$. We will give some kind of Skorokhod representation of X_t when $\rho \in H^{1,1}(\mathbb{R}^d)$, $a_{ij} \in D(\mathcal{E}^r)$. Note that $\rho \in H^{1,1}(\mathbb{R}^d)$ implies the closability of $(\mathcal{E}^r, C_0^\infty(\overline{D}_\alpha))$ in $L^2(D_\alpha, m_\alpha)$.
Let σ be the surface measure on ∂G. Recall that since G is a bounded Lipschitz domain there exists a bounded linear operator

$$Tr : H^{1,p}(G) \to L^p(\partial G, \sigma), \tag{11}$$

called the trace on ∂G, and $Tr(f) = f$ on ∂G for any $f \in H^{1,p}(G) \cap C(\overline{G})$. Furthermore, the weak Gauss-Green theorem holds, i.e. if $f \in H^{1,1}(G)$, $1 \leq i \leq d$, then

$$\int_G \partial_i f \, dx = -\int_{\partial G} Tr(f) \eta_i \, d\sigma$$

where $\eta = (\eta_1, ..., \eta_d)$ is the inward normal of $\overline{G}$ on ∂G (see for instance [3]). Let $\nu = (\nu_1, ..., \nu_d)$ denote the inward normal of $\mathbb{R}^d \setminus G$ on ∂G. We have $\nu_i = -\eta_i$, $1 \leq i \leq d$. Let $g \in C_0^\infty(\overline{D}_\alpha)$, and $f \in C_0^\infty(\mathbb{R}^d)$ such that $f|_{\overline{D}_\alpha} = g$. Let $\rho_m \in C_0^\infty(\mathbb{R}^d)$ such that $\rho_m \to \rho$ in $H^{1,1}(\mathbb{R}^d)$. Then by (11) and Gauss-Green theorem

$$\begin{aligned} \int_{\mathbb{R}^d \setminus G} \partial_i (g\rho) \, dx &= \lim_{m \to \infty} \int_{\mathbb{R}^d \setminus G} \partial_i (f\rho_m) \, dx \\ &= \lim_{m \to \infty} \left\{ \int_{\mathbb{R}^d} \partial_i (f\rho_m) \, dx + \int_{\partial G} Tr(f\rho_m) \eta_i \, d\sigma \right\} \\ &= -\int_{\partial G} g Tr(\rho) \nu_i \, d\sigma. \end{aligned} \tag{12}$$

For $\rho \in H^{1,1}(\mathbb{R}^d)$ we show that the weighted surface measure $Tr(\rho)d\sigma$, or equivalently $1_{\{Tr(\rho)>0\}} d\sigma$, on ∂G is smooth.The proof is similar to the corresponding proof in [13] but we include because of some subtle differences.

Theorem 5.1 *Let $\rho \in H^{1,1}(\mathbb{R}^d)$. Then $Tr(\rho)d\sigma \in S$. In particular for any $w \in D(\mathcal{E}^r)_b$ and quasi-continuous m_α- version $\widetilde{w}$ we have*

$$-\int_{\partial G} \widetilde{w} Tr(\rho) \eta_i \, d\sigma = \int_G \partial_i (w\rho) \, dx = -\int_{\mathbb{R}^d \setminus G} \partial_i (w\rho) \, dx = \int_{\partial G} \widetilde{w} Tr(\rho) \nu_i \, d\sigma. \tag{13}$$

Proof If $h \in C^1(\overline{G})$ it is well-known (see e.g. [3, p.134, 3. $(\star\star\star)$, $(\star\star\star\star)$]) that there exists a universal constant C depending only on the Lipschitz domain such that

$$\int_{\partial G} |h|^p \, d\sigma \leq C \int_G |\nabla h|^p + |h|^p \, dx \tag{14}$$

for any $p \in [1, \infty[$. Let us choose $(\rho_k)_{k\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ with $\rho_k \to \rho$ in $H^{1,1}(\mathbb{R}^d)$ as $k \to \infty$. By (11) $\rho_k \to Tr(\rho)$ in $L^1(\partial G, \sigma)$ as $k \to \infty$. Let $K \subset \overline{D}_\alpha$ be a compact set. Let $f \in C_0^\infty(\overline{D}_\alpha)$, $f \geq 1$ everywhere on K. Then, using (14), Lebesgue's theorem, and the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_{K\cap\partial G} Tr(\rho)\, d\sigma &\leq \lim_{k\to\infty}\int_{\partial G}|f\rho_k|\, d\sigma \\ &\leq \lim_{k\to\infty} C\int_G |\nabla(f\rho_k)| + |f\rho_k|\, dx \\ &= C\left\{\int_G (|\nabla f| + |f|)\rho dx + \int_G |f\nabla\rho| dx\right\} \\ &\leq \frac{C\sqrt{2\lambda m_\alpha(G)}}{\max(\alpha, 1-\alpha)}|f|_{D(\mathcal{E}^r)} + C\int_G |f||\nabla\rho| dx\end{aligned}$$

Assume $\mathrm{Cap}(K) = 0$. By [4, Lemma 2.2.7(ii)]

$$\mathrm{Cap}(K) = \inf_{u\in\mathcal{C}_K} \mathcal{E}_1^r(u,u),$$

where $\mathcal{C}_K = \{u \in C_0^\infty(\overline{D}_\alpha) \,|\, u(x) \geq 1, \forall x \in K\}$. Hence, there exists $(f_n)_{n\in\mathbb{N}} \subset C_0^\infty(\overline{D}_\alpha)$, $f_n(x) \geq 1$, for every $n \in \mathbb{N}$, $x \in K$, such that $|f_n|_{D(\mathcal{E}^r)} \to 0$ as $n \to \infty$. Since normal contractions operate on $D(\mathcal{E}^r)$ we may assume that $\sup_{n\in\mathbb{N}}\sup_{x\in\mathbb{R}^d}|f_n(x)| \leq C$. Selecting a subsequence if necessary we may also assume that $\lim_{n\to\infty}|f_n| = 0$ m_α-a.e., hence dx-a.e. on D_α. Suppose $\alpha \neq 1$, then $D_\alpha \supset G$. Consequently, using Lebesgue's theorem we obtain

$$\lim_{n\to\infty}\int_G |f_n||\nabla\rho| dx = 0,$$

and therefore $\int_{K\cap\partial G} Tr(\rho)\, d\sigma = 0$. Since $Tr(\rho)d\sigma$, as well as Cap are inner regular the first assertion now follows for $\alpha \neq 1$. If $\alpha = 1$, we choose a compact domain $\overline{V}$ with smooth boundary such that $\overline{G} \subset V$ and $\partial G \cap \partial V = \emptyset$. Then $U := V \setminus G$ is a bounded Lipschitz domain which is contained in D_α. Let κ denote its surface measure. As before we can then show that $Tr(\rho)d\kappa$ is smooth. Note that $1_{\partial G}Tr(\rho)d\kappa = Tr(\rho)d\sigma$. Thus $Tr(\rho)d\sigma$ is also smooth. The second assertion is clear by (13) and since $Tr(\rho)d\sigma$ is finite and smooth as we just have shown.

□

We present here below the identification of the process in Theorem 2.1 for a special class of ρ, a_{ij}, ∂G.

Theorem 5.2 *Let $\rho \in H^{1,1}(\mathbb{R}^d)$, $\rho > 0$ dx-a.e. Let $A = (a_{ij})$ satisfy* (1) *with $a_{ij} \in D(\mathcal{E}^r)$, and q.c. m_α-versions $\widetilde{a}_{ij}$, $1 \leq i,j \leq d$. Let $\sqrt{A} = (\sigma_{ij})_{1\leq i,j\leq d}$ be the positive square root of the matrix A. Let $B = \rho^{-1}(\overline{B}_1, ..., \overline{B}_1)$ be a m_α-square integrable vector field satisfying* (2). *Let G be a bounded Lipschitz domain. Let $\eta = (\eta_1, ..., \eta_d)$ (resp. $\nu = (\nu_1, ..., \nu_d)$) be the unit inward normal vector field of $\overline{G}$ on ∂G (resp. of $\mathbb{R}^d \setminus G$ on ∂G). Let $\widetilde{A}(\eta)_k := \sum_{j=1}^d \widetilde{a}_{kj}\eta_j$, (resp. $\widetilde{A}(\nu)_k := \sum_{j=1}^d \widetilde{a}_{kj}\nu_j$), $1 \leq k \leq d$, be the inward normal of $\overline{G}$ (resp.*

$\mathbb{R}^d \setminus G$) associated with A. The conservative diffusion $X = (X^1, ..., X^d)$ of Theorem 2.1 *is a semimartingale and has the following Skorokhod decomposition for $1 \le k \le d$:*

$$\begin{aligned} X_t^k &= z_k + \sum_{j=1}^d \int_0^t \sigma_{kj}(X_s)dW_s^j + \int_0^t \frac{1}{2}\sum_{j=1}^d \left(\partial_j a_{kj} + a_{kj}\frac{\partial_j \rho}{\rho}\right)(X_s)ds \\ &\quad + \int_0^t \rho_\alpha^{-1}\overline{B}_k(X_s)ds + \frac{\alpha}{2}\int_0^t \widetilde{A}(\eta)_k(X_s)d\ell_s^\rho + \frac{1-\alpha}{2}\int_0^t \widetilde{A}(\nu)_k(X_s)d\ell_s^\rho, \end{aligned}$$

$t \ge 0$, P_z-a.s. for q.e. $z = (z_1, ..., z_d) \in \overline{D}_\alpha$, where $W = (W^1, ..., W^d)$ is a d-dimensional standard BM starting from zero, $(\ell_t^\rho)_{t\ge 0}$ denotes the unique PCAF associated to the weighted surface measure $Tr(\rho)\,d\sigma \in S$ through Theorem 3.2. *In particular*

$$\ell_t^\rho = \int_0^t 1_{\partial G \cap \{Tr(\rho)>0\}}(X_s)d\ell_s^\rho, \;\; t \ge 0. \tag{15}$$

Proof The coordinate functions $p_k(x_1, ..., x_d) := x_k$ are not in $D(\mathcal{E}^r)_b$. But they are locally in in $D(\mathcal{E}^r)_b$, i.e. $p_k f \in C_0^\infty(\overline{D}_\alpha) \subset D(\mathcal{E}^r)_b$ for any $f \in C_0^\infty(\overline{D}_\alpha)$. Let $f \in C_0^\infty(\overline{D}_\alpha)$, $\langle M^{[f]}\rangle_t$ be the square bracket of $M_t^{[f]}$. Then an easy calculation gives that the energy measure of $M_t^{[f]}$, i.e. the Revuz measure of $\langle M^{[f]}\rangle_t$, is

$$\mu_{\langle M^{[f]}\rangle} = \langle A\nabla f, \nabla f\rangle dm_\alpha.$$

Thus $\langle M^{[f]}\rangle_t = \int_0^t \langle A\nabla f(X_s), \nabla f(X_s)\rangle ds$ by Theorem 3.2. Let $w \in D(\mathcal{E}^r)_b$. Then by the previous results of this section

$$\begin{aligned} &-\mathcal{E}^r(f,w) + \int_{D_\alpha} \langle B, \nabla f\rangle w\, dm_\alpha \\ &= \int_{D_\alpha} \left(\frac{1}{2}\sum_{i,j=1}^d a_{ij}\partial_i\partial_j f + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d \left\{\partial_j a_{ij} + a_{ij}\frac{\partial_j\rho}{\rho}\right\}\partial_i f + \sum_{i=1}^d \rho_\alpha^{-1}\overline{B}_i\,\partial_i f\right) w\, dm_\alpha \\ &\quad + \frac{\alpha}{2}\sum_{i,j=1}^d \int_{\partial G} \widetilde{w}\partial_i f\, \widetilde{a}_{ij}\eta_j Tr(\rho)d\sigma + \frac{1-\alpha}{2}\sum_{i,j=1}^d \int_{\partial G} \widetilde{w}\,\partial_i f\, \widetilde{a}_{ij}\nu_j Tr(\rho)d\sigma, \end{aligned}$$

Let $f_l \in C_0^\infty(\overline{D}_\alpha)$, $f_l = 1$ on $K_l(0) := \{x \in \overline{D}_\alpha \,:\, |x| \le l\}$, $l \ge 1$. Using the above, Theorem 5.1, (10), Theorem 3.2, we easily derive the decomposition (9) for $p_k f_l$. One can also easily see that $1_{K_l(0)}\mu_{\langle M^{[p_k f_l - p_k f_m]}\rangle} = 0$ for any $m \ge l$. This further implies that $M_t^{[p_k f_l]} = M_t^{[p_k f_m]}$ $\forall t \le \sigma_{K_l(0)^c} := \inf\{t > 0 : X_t \in K_l(0)^c\}$. Since obviously $A_t^{[p_k f_l]} = A_t^{[p_k f_m]}$ $\forall t \le \sigma_{K_l(0)^c}$, and $\sigma_{K_l(0)^c} \uparrow \infty$, by letting $l \to \infty$ we get the identification of the process. Since by (10) $N_t^{[p_k f_l]}$ is of bounded variation for any l, the process is a semimartingale. Of course $\ell_t^\rho = \int_0^t 1_{\partial G\cap\{Tr(\rho)>0\}}(X_s)d\ell_s^\rho$, $t \ge 0$, by Theorem 3.2, because $1_{\partial G\cap\{Tr(\rho)>0\}}Tr(\rho)d\sigma = Tr(\rho)d\sigma$ since $supp(\sigma) \subset \partial G$.

□

Remark 5.3 *We would like to describe shortly the decomposition of Theorem 5.2. The diffusion has a symmetric drift part corresponding to the logarithmic derivative of* ρ *associated to the diffusion matrix* A*. It has a purely non-symmetric part* $\rho_\alpha^{-1}\overline{B}$*, and two reflection parts. (15) tells us that* ℓ^ρ *behaves like a multidimensional local time on some part of the boundary, i.e.* ℓ^ρ *only grows when* X_t *meets the boundary* ∂G *at those points where* $Tr(\rho) > 0$*. At that time,* X_t *is reflected at, or passes through (see Remark 6.6),* ∂G *in normal direction associated with* A*. Finally note that if* $\alpha = \frac{1}{2}$ *there is no reflection.*

6 Recurrence

Let $f \in L^1(D_\alpha, m_\alpha)^+$. Then

$$Gf(x) := E_x\left[\int_0^\infty f(X_t)dt\right] = \lim_{\beta\downarrow 0} G_\beta f(x) \le \infty$$

is uniquely determined at least for m_α-a.e. $x \in \overline{D}_\alpha$. Recall that $1_{\overline{D}_\alpha} \in D(L)_b$ and $\beta G_\beta 1_{\overline{D}_\alpha} = 1_{\overline{D}_\alpha}$ for any $\beta > 0$. Therefore

$$G1_{\overline{D}_\alpha} = \infty \ \ m_\alpha\text{-a.e.}$$

We will need the following Hopf's maximal ergodic inequality (cf. [4, Lemma 1.5.2], [8, Lemma 1.5.4]). It can be shown exactly as in the sectorial case.

Lemma 6.1 *Let* $h \in L^1(D_\alpha, m_\alpha)$*,* $\beta > 0$*, and let* $E_\beta := \{x \in \overline{D}_\alpha \mid \sup_{n\ge 1} G_{\frac{\beta}{n}} h(x) > 0\}$*. Then*

$$\int_{E_\beta} G_\beta h \, dm_\alpha \ge 0. \tag{16}$$

Hopf's maximal ergodic inequality is essentially sufficient in order to prove the following recurrence lemma.

Lemma 6.2 *Let* $f \in L^1(D_\alpha, m_\alpha)^+$*. Then* $\{Gf = \infty\} \cup \{Gf = 0\} = \overline{D}_\alpha$*, and* $\{f > 0\} \subset \{Gf = \infty\}$*, up to an* m_α*-negligible set.*

Proof For arbitrary $f \in L^1(D_\alpha, m_\alpha)^+$, $a > 0$, we set $h := 1_{\overline{D}_\alpha} - af$ in (16). Since $B := \{Gf < \infty\} = \{G1_{\overline{D}_\alpha} = \infty\} \cap \{Gf < \infty\} \subset E_\beta$ for any $\beta > 0$, up to a m_α-negligible set we obtain

$$\infty > \int_{D_\alpha} 1_{\overline{D}_\alpha} \, dm_\alpha \ \ \ge \ \int_{E_\beta} \beta G_\beta 1_{\overline{D}_\alpha} \, dm_\alpha \ge a \int_{E_\beta} \beta G_\beta f \, dm_\alpha \ge a \int_B \beta G_\beta (f \wedge N) \, dm_\alpha$$

for any $N \ge 1$. Dividing by a, and letting $a \to \infty$, $N \to \infty$, we get $\int_B \beta G_\beta f \, dm_\alpha = 0$, as well as $\int_B G_\beta f \, dm_\alpha = 0$. Letting in the first case $\beta \to \infty$, and in the second $\beta \to 0$, we get

$$1_{\{Gf<\infty\}} \cdot f = 0 \quad \text{as well as} \quad 1_{\{Gf<\infty\}} \cdot Gf = 0.$$

$1_{\{Gf<\infty\}} \cdot Gf = 0$ implies $(\{Gf < \infty\} \cap \{Gf > 0\})^c = \{Gf = \infty\} \cup \{Gf = 0\} = \overline{D}_\alpha$ up to m_α-negligible set. $1_{\{Gf<\infty\}} \cdot f = 0$ implies $\{f > 0\} \subset \{Gf = \infty\}$. This concludes the proof.

□

Lemma 6.3 *Let $f \in L^1(D_\alpha, m_\alpha)^+$. Then:*
(i) $\{Gf < \infty\}$ is invariant, $\{Gf = \infty\}$ is co-invariant, i.e. for any $\beta > 0$, $t > 0$, $h \in L^2(D_\alpha, m_\alpha)$, we have

$$1_{\{Gf<\infty\}} G_\beta(1_{\{Gf=\infty\}} h) = 1_{\{Gf<\infty\}} T_t(1_{\{Gf=\infty\}} h) = 0,$$

and

$$1_{\{Gf=\infty\}} G'_\beta(1_{\{Gf<\infty\}} h) = 1_{\{Gf=\infty\}} T'_t(1_{\{Gf<\infty\}} h) = 0.$$

(ii) $1_{\{Gf=\infty\}}$ is excessive, $1_{\{Gf<\infty\}}$ is co-excessive, and $1_{\{Gf<\infty\}}$, $1_{\{Gf=\infty\}} \in D(\mathcal{E}^r)$.
(iii) Let $u, v \in D(\mathcal{E}^r)_b$. Set $B := \{Gf < \infty\}$. Then $1_B u, 1_B v \in D(\mathcal{E}^r)_b$, and

$$\mathcal{E}^r(u, v) = \mathcal{E}^r(1_B u, 1_B v) + \mathcal{E}^r(1_{B^c} u, 1_{B^c} v). \tag{17}$$

In particular

$$\mathcal{E}^r(1_B u, v) = \mathcal{E}^r(1_B u, 1_B v) = \mathcal{E}^r(u, 1_B v),$$

and B, hence also B^c, is an invariant set w.r.t. $\mathcal{E}^r$ in the sense of [4, Lemma 1.6.1.].

Proof (i) We may assume that f is bounded. Put $B = \{Gf < \infty\}$, $B^c = \{Gf = \infty\}$. Let $h \in L^2(D_\alpha, m_\alpha)^+$. Then for any $n \geq 1$

$$\begin{aligned}
\int_{D_\alpha} 1_{B^c} Gf\, T'_t(h 1_{\{Gf \leq n\}})\, dm_\alpha &= \lim_{k\to\infty} \int_{D_\alpha} 1_{B^c} G_{\frac{1}{k}} f\, T'_t(h 1_{\{Gf\leq n\}})\, dm_\alpha \\
&= \lim_{k\to\infty} \int_{D_\alpha} T_t(1_{B^c} G_{\frac{1}{k}} f)\, h 1_{\{Gf\leq n\}}\, dm_\alpha \\
&\leq \lim_{k\to\infty} \int_{D_\alpha} T_t(G_{\frac{1}{k}} f)\, h 1_{\{Gf\leq n\}}\, dm_\alpha \\
&\leq \lim_{k\to\infty} \int_{D_\alpha} e^{\frac{t}{k}} G_{\frac{1}{k}} f\, h 1_{\{Gf\leq n\}}\, dm_\alpha \\
&= \int_{D_\alpha} Gf\, h 1_{\{Gf\leq n\}}\, dm_\alpha < \infty.
\end{aligned}$$

Thus $1_{B^c} T'_t(h 1_{\{Gf\leq n\}}) = 0$, and therefore $1_{B^c} T'_t(g 1_B) = 0$ for any $g \in L^2(D_\alpha, m_\alpha)$. It follows $(1_B T_t(1_{B^c} h), g) = (h, 1_{B^c} T'_t(1_B g)) = 0$ and therefore $1_B T_t(1_{B^c} h) = 0$. We conclude that (i) holds. We now show (ii). By (i) $\beta G_\beta 1_{B^c} = 1_{B^c} \beta G_\beta 1_{B^c} \leq 1_{B^c} \beta G_\beta 1_{\overline{D}_\alpha} = 1_{B^c}$, and

$$\mathcal{E}^r(\beta G_\beta 1_{B^c}, \beta G_\beta 1_{B^c}) = \mathcal{E}(\beta G_\beta 1_{B^c}, \beta G_\beta 1_{B^c}) \leq \mathcal{E}(\beta G_\beta 1_{\{Gf=\infty\}}, 1_{\overline{D}_\alpha}) = 0.$$

Hence $1_{B^c} \in D(\mathcal{E}^r)$ easily follows from [7, I. Lemma 2.12]. The proof corresponding to 1_B works similarly.

(iii) Let $\mu_{\langle M^{[w]}\rangle}$, be the energy measure related to $\langle M^{[w]}\rangle_t$, $w \in D(\mathcal{E}^r)_b$. Suppose that w is constant $\mu_{\langle M^{[w]}\rangle}$-a.e. on some Borel set A. Then it is well-known that $\int_A d\mu_{\langle M^{[w]}\rangle} = 0$ (see [11, Lemma 3.8. (iii)], but also [1, Corollaire 6], [2, equation (8)] in the symmetric case). Note that $\mu_{\langle M^{[w]}\rangle} = \langle A\nabla w, \nabla w\rangle \, dm_\alpha$. It follows that $\mathcal{E}^r(1_B u, 1_{B^c} v) = \mathcal{E}^r(1_{B^c} u, 1_B v) = 0$. Indeed, by Cauchy-Schwarz

$$\begin{aligned} 2|\mathcal{E}^r(1_B u, 1_{B^c} v)| &\leq \left|\int_B d\mu_{\langle M^{[1_B u]}, M^{[1_{B^c} v]}\rangle}\right| + \left|\int_{B^c} d\mu_{\langle M^{[1_B u]}, M^{[1_{B^c} v]}\rangle}\right| \\ &\leq \sqrt{\int_B d\mu_{\langle M^{[1_B u]}\rangle} \int_B d\mu_{\langle M^{[1_{B^c} v]}\rangle}} + \sqrt{\int_{B^c} d\mu_{\langle M^{[1_B u]}\rangle} \int_{B^c} d\mu_{\langle M^{[1_{B^c} v]}\rangle}} = 0. \end{aligned}$$

$\mathcal{E}^r(1_{B^c} u, 1_B v) = 0$ follows in the same way. Thus $\mathcal{E}^r(u, v) = \mathcal{E}^r(1_B u, 1_B v) + \mathcal{E}^r(1_{B^c} u, 1_{B^c} v)$. Replacing first u by $1_B u$, and then v by $1_B v$ in (17), the second assertion of (iii) follows. For the last assertion, exactly as in the proof of [4, Theorem 1.6.1.], we use the second assertion of (iii) in order to see that for any $h, g \in L^2(D_\alpha, m_\alpha)_b$,

$$(G^r_\beta(1_B h), g) = (h, 1_B G^r_\beta g) = \mathcal{E}^r_\beta(G^r_\beta h, 1_B G^r_\beta g) = \mathcal{E}^r_\beta(1_B G^r_\beta h, G^r_\beta g) = (1_B G^r_\beta h, g).$$

From this, the invariance of B follows immediately.

□

Proposition 6.4 *Suppose $\mathcal{E}^r$ is irreducible (in the sense of [4, p. 48]). Then $Gf = \infty$ m_α-a.e. for any $f \in L^1(D_\alpha, m_\alpha)^+$ with $m_\alpha(\{f > 0\}) > 0$.*

Proof By our assumption $\int_{D_\alpha} f \, dm_\alpha > 0$. Since $\{f > 0\} \subset \{Gf = \infty\}$ by Lemma 6.2, it follows $\int_{D_\alpha} f \, dm_\alpha = \int_{D_\alpha} f 1_{\{Gf=\infty\}} \, dm_\alpha$, hence $m_\alpha(\{Gf = \infty\}) > 0$. Since $\{Gf = \infty\}$ is $\mathcal{E}^r$-invariant by Lemma 6.3 (iii), and $\mathcal{E}^r$ is irreducible, we must have $m_\alpha(\{Gf = \infty\}^c) = 0$. Therefore the assertion follows.

□

Theorem 6.5 *Suppose $\mathcal{E}^r$ is irreducible (in the sense of [4, p. 48]). For $r > 0$ let $D_r(x) := \{y \in \overline{D}_\alpha : |x - y| < r\}$, and $\sigma_{D_r(x)} := \inf\{t > 0 | X_t \in D_r(x)\}$. Let $(\vartheta_t)_{t>0}$ be the shift operator correponding to X_t. Then*

$$P_x(\sigma_{D_r(x)} \circ \vartheta_n < \infty, \ \forall n \geq 0) = 1 \quad \textit{for q.e. } x \in \overline{D}_\alpha. \tag{18}$$

Proof The proof is similar to the proof of Theorem 4.6.6.(ii) in [4], but we include it in order to point excactly out the subtle differences. Let $B \subset \overline{D}_\alpha$ be an arbitrary Borel set, and $(p_t)_{t\geq 0}$ the transition semigroup of X_t. The Markov property implies that $f(x) := P_x(\sigma_B < \infty)$ is excessive, since $p_t f(x) = P_x(\sigma_B \circ \vartheta_t + t < \infty) \leq P_x(\sigma_B < \infty)$. In particular due to the boundedness of f, standard arguments then imply that $f \in D(\mathcal{E}^r)_b$, and f is q.c. On the other hand for any positive $g \in C_0^\infty(\overline{D}_\alpha)$ with $\int_{D_\alpha} g \, dm_\alpha > 0$ we have $G'g = \lim_{\beta\to 0} G'_\beta g = \infty$ m_α-a.e. Indeed, this follows immediately from Remark 2.2 and the co-version of Proposition 6.4. Then, using the resolvent equation

$$0 \leq (G'_\beta g, f - \alpha R_\alpha f) = (g, R_\beta f - \alpha R_\beta R_\alpha f) = (g, R_\alpha f - \beta R_\beta R_\alpha f) \leq (g, R_\alpha f) < \infty.$$

Letting $\beta \to 0$ we conclude $\alpha R_\alpha f = f$, hence $Lf = 0$ and thus $f(X_t)$ in (9) is a P_x-martingale for every $x \in \overline{D}_\alpha \setminus N$ where N is some exceptional set. Let $E := \{f = 1\}$, $\gamma \in [0,1)$, and $E := \{f = 1\}$, $\sigma_\gamma := \sigma_{E_\gamma}$ be the first hitting time of $E_\gamma := \{f \geq \gamma\}$. Note that E_γ is finely closed. Now, for any $x \in E \setminus N$, $T > 0$, we have by the optional sampling theorem

$$\begin{aligned} 1 &= f(x) = E_x[f(X_{\sigma_\gamma \wedge T})] \\ &= E_x[f(X_{\sigma_\gamma}); \sigma_\gamma \leq T] + E_x[f(X_T); \sigma_\gamma > T] \\ &\leq \gamma P_x(\sigma_\gamma \leq T) + P_x(\sigma_\gamma > T), \end{aligned}$$

which means that $P_x(\sigma_\gamma \leq T) = 0$. Thus

$$P_x(\sigma_{E^c} < \infty) = 0 \ \text{ for any } x \in E \setminus N,$$

and E is invariant w.r.t. $\mathcal{E}$. Exactly as in the proof of Lemma 6.3(iii) one shows that E is then invariant w.r.t. $\mathcal{E}^r$. Owing to the irreducibility of $\mathcal{E}^r$ we must have either $m_\alpha(E) = 0$ or $m_\alpha(E^c) = 0$. Finally we let $B = D_r(x)$. Then $E \supset D_r(x)$ and $m_\alpha(E) \geq m_\alpha(D_r(x)) > 0$, thus $m_\alpha(E^c) = 0$. It follows $f(x) = 1$ for m_α-a.e. x. But f is quasi-continuous, and therefore $f = 1$ q.e. (18) now follows from the Markov property.

□

Remark 6.6 *We have seen in Theorem 6.5 that if $\mathcal{E}^r$ is irreducible, then X_t is recurrent q.e. in the classical sense. In particular, if additionally $0 < \alpha < 1$ one can conclude that the process passes infinitely often through ∂G.*

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