

On Hunt processes and strict capacities associated with generalized Dirichlet forms¹

Gerald TRUTNAU

*Zentrum für interdisziplinäre Forschung der Universität Bielefeld
Wellenberg 1, 33615 Bielefeld, Germany
trutnau@math.uni-bielefeld.de*

Summary. Introducing the corresponding strict capacity, we give necessary and sufficient conditions for a generalized Dirichlet form to be associated with a Hunt process. We also show that Borel measurable sets with strict capacity zero can be checked-out by an appropriate subclass of smooth measures. In the last part of this article we present applications to three classes of examples.

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Preliminary note

Since the origin of quasi-regular Dirichlet forms on general state spaces, a fundamental question has arisen, namely whether the associated Markov process is not just standard, but moreover a so-called Hunt process. The difference lies mainly in the question, whether its paths have left limits at the life time ζ , i.e. whether $\lim_{t \nearrow \zeta} X_t$ exists, in the state space E enlarged by a cemetery Δ . Obviously, this question is crucial for localizing the process when “it dies”. In case of a regular symmetric Dirichlet form on a locally compact (separable metric) state space, i.e. in Fukushima’s classical framework (see [4]), and in the case of Oshima’s time-dependent Dirichlet forms (see [7]), the answer is always yes. In the framework of Ma/Röckner (see [5]), i.e. the case of a quasi-regular sectorial Dirichlet form on a general topological state space the answer is no in general. But one can characterize those Dirichlet forms for which the answer is yes by the notion of strict quasi-regularity (cf. Chapter 5 in [5]).

Our main result is to generalize this characterization to the class of (fully non-symmetric) generalized Dirichlet forms as introduced by Stannat in [10]. However, due to an apparent

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or existent lack of implicitly given structure in the domain of generalized Dirichlet forms we will similarly to [10, IV.2] be obliged to assume an extra condition SD3 (cf. below) in order to construct the associated Hunt process. This condition is an easy to check a priori assumption as can be seen in the three classes of examples that we treat as applications. It is trivially satisfied for any sectorial Dirichlet form.

Apart from additional theoretical interest, the achieved results are expected to have impact on the study of the asymptotic behavior of solutions to SPDEs with jump type noise which by nature are usually fully non-symmetric.

New definitions and first results

The following notations will be the common notations of [10], [12], [13], [14]. Therefore not all of them will be recalled explicitly.

Let E be a Hausdorff topological space such that $\sigma(C(E)) = \mathcal{B}(E)$, i.e. the Baire σ -algebra coincides with the Borel σ -algebra. Let m be a σ -finite measure on $(E, \mathcal{B}(E))$ and let $(\mathcal{E}, \mathcal{F})$ be a generalized Dirichlet form on $\mathcal{H} := L^2(E; m)$ with inner product (\cdot, \cdot) and coercive part $(\mathcal{A}, \mathcal{V})$ (see [10]). As usual we denote by $|\cdot|_{\mathcal{F}}$ the norm corresponding to $(\mathcal{E}, \mathcal{F})$, and by $|\cdot|_{\mathcal{V}}$ the norm corresponding to $(\mathcal{A}, \mathcal{V})$. Let $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ denote the coform. Adjoining an extra point to the measurable space $(E, \mathcal{B}(E))$ let $E_{\Delta} := E \cup \{\Delta\}$ and $\mathcal{B}(E_{\Delta}) = \mathcal{B}(E) \cup \{B \cup \{\Delta\} | B \in \mathcal{B}(E)\}$. We will consider different topologies on $E_{\Delta} := E \cup \{\Delta\}$. If E is a locally compact separable metric space but not compact, E_{Δ} will be the one point compactification of E , i.e. the open sets of E_{Δ} are the open sets of E together with the sets of the form $E_{\Delta} \setminus K$, $K \subset E$, K compact in E . Otherwise we adjoin the cemetery Δ to E as an isolated point. We extend m to $(E_{\Delta}, \mathcal{B}(E_{\Delta}))$ by setting $m(\{\Delta\}) = 0$. As usual, any function f on E is considered as a function on E_{Δ} with $f(\Delta) := 0$. For $A \subset E$ we set $A^c = E \setminus A$.

By \mathcal{B} , \mathcal{B}^+ , \mathcal{B}_b , \mathcal{B}_b^+ , we denote the Borel measurable, positive Borel measurable, bounded Borel measurable, bounded Borel measurable and positive functions on E . Throughout this article we fix $\varphi \in L^1(E, m) \cap \mathcal{B}$ with $0 < \varphi(z) \leq 1$ for every $z \in E$. Let $(G_{\alpha})_{\alpha>0}$ (resp. $(\widehat{G}_{\alpha})_{\alpha>0}$) be the resolvent (resp. the coresolvent) of \mathcal{E} . Let \mathcal{P} (resp. $\widehat{\mathcal{P}}$) denote the 1-excessive (resp. 1-coexcessive) elements of \mathcal{V} where $(\mathcal{A}, \mathcal{V})$ is the coercive part of $(\mathcal{E}, \mathcal{F})$. We let $\mathcal{V} = \mathcal{H}$ if $\mathcal{A} = 0$. Let $\mathcal{C}, \mathcal{D} \subset \mathcal{H}$. We define $\mathcal{D}_{\mathcal{C}} := \{u \in \mathcal{D} | \exists f \in \mathcal{C}, u \leq f\}$, $D_b := D \cap L^{\infty}(E, m)$. For an arbitrary Borel set $B \in \mathcal{B}(E)$ and an element $u \in \mathcal{H}$ such that $\{v \in \mathcal{H} | v \geq u \cdot 1_B\} \cap \mathcal{F} \neq \emptyset$ (resp. $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$) let $u_B := e_{u \cdot 1_B}$ be the 1-reduced function (resp. $\hat{u}_B := \hat{e}_{\hat{u} \cdot 1_B}$ be the 1-coreduced function) of $u \cdot 1_B$ (resp. $\hat{u} \cdot 1_B$) as defined in [10, III. Definition 1.8., p. 65]. Here we use the notation 1_B for the characteristic function of B . If $B = E$ we rather write e_u instead of u_E . In general only if B is open our definition of reduced function coincides with the one of [4, p. 92], [5, III. Exercise 3.10(ii), p. 84]. In particular, if $B \in \mathcal{B}(E)$ is such that $m(B) = 0$, then $u_B = 0$. By [10, III. Proposition 1.7.(ii)] $e_u = u$, $u_B \leq v_B$, $u_A \leq u_B$, $w_{C \cup D} \leq w_C + w_D$ for any $u, v, w \in \mathcal{P}_{\mathcal{F}}$, $u \leq v$, $A, B, C, D \in \mathcal{B}(E)$, $A \subset B$.

Definition 0.1 For $U \subset E$, U open, set

$$\text{Cap}_{1, \widehat{G}_1 \varphi}(U) := \lim_{k \rightarrow \infty} ((kG_1 \varphi \wedge 1)_U, \varphi).$$

If $A \subset E$ arbitrary then $\text{Cap}_{1, \widehat{G}_1 \varphi}(A) := \inf \{ \text{Cap}_{1, \widehat{G}_1 \varphi}(U) \mid U \supset A, U \text{ open} \}$.

We note that $\text{Cap}_{1, \widehat{G}_1 \varphi}(U)$ exists as a bounded and increasing limit for any open set U . The boundedness follows because of $\text{Cap}_{1, \widehat{G}_1 \varphi}(E) = \lim_{k \rightarrow \infty} (kG_1 \varphi \wedge 1, \varphi) \leq \int \varphi \, dm < \infty$.

Theorem 0.2 (i) $\text{Cap}_{1, \widehat{G}_1 \varphi}$ is a (finite) Choquet capacity, i.e., it has the following properties:

(a) If $(A_n)_{n \in \mathbb{N}}$ is an increasing sequence of arbitrary subsets of E then

$$\text{Cap}_{1, \widehat{G}_1 \varphi}(\cup_{n \in \mathbb{N}} A_n) = \sup_{n \in \mathbb{N}} \text{Cap}_{1, \widehat{G}_1 \varphi}(A_n).$$

(b) If $(K_n)_{n \in \mathbb{N}}$ is a decreasing sequence of compact subsets of E then

$$\text{Cap}_{1, \widehat{G}_1 \varphi}(\cap_{n \in \mathbb{N}} K_n) = \inf_{n \in \mathbb{N}} \text{Cap}_{1, \widehat{G}_1 \varphi}(K_n).$$

(ii) For any sequence $(A_n)_{n \in \mathbb{N}}$ of arbitrary subsets of E we have

$$\text{Cap}_{1, \widehat{G}_1 \varphi}(\cup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} \text{Cap}_{1, \widehat{G}_1 \varphi}(A_n).$$

(iii) For $U, V \subset E$ open we have (“strong subadditivity”)

$$\text{Cap}_{1, \widehat{G}_1 \varphi}(U \cap V) + \text{Cap}_{1, \widehat{G}_1 \varphi}(U \cup V) \leq \text{Cap}_{1, \widehat{G}_1 \varphi}(U) + \text{Cap}_{1, \widehat{G}_1 \varphi}(V).$$

Proof (iii) For fixed k it can be shown exactly as in the proof of [10, III.Lemma 2.7.(ii)] that $((kG_1 \varphi \wedge 1)_{U \cup V} + (kG_1 \varphi \wedge 1)_{U \cap V} - (kG_1 \varphi \wedge 1)_U - (kG_1 \varphi \wedge 1)_V, \varphi) \leq 0$. Letting $k \rightarrow \infty$ we obtain the desired inequality.

(i) To prove (i)(a) and (i)(b) let first $(U_n)_{n \in \mathbb{N}}$ be an increasing sequence of open sets in E and let $U := \cup_{n \in \mathbb{N}} U_n$. Similarly to the proof of [10, III.Lemma 2.7.(i)] we show $(kG_1 \varphi \wedge 1)_U = \lim_{n \rightarrow \infty} (kG_1 \varphi \wedge 1)_{U_n}$ in \mathcal{H} for any k . Hence $\text{Cap}_{1, \widehat{G}_1 \varphi}(U) = \sup_{k \in \mathbb{N}} \sup_{n \in \mathbb{N}} ((kG_1 \varphi \wedge 1)_{U_n}, \varphi) = \sup_{n \in \mathbb{N}} \text{Cap}_{1, \widehat{G}_1 \varphi}(U_n)$. Now (i)(a) and (i)(b) are an easy consequence of the just proven and (iii) (cf. [5, III.Theorem 2.8]).

(ii) This is an easy consequence of (iii) and (i)(a). □

Given an increasing sequence $(F_k)_{k \in \mathbb{N}}$ of closed subsets of E , we define

$$C(\{F_k\}) = \{f : A \rightarrow \mathbb{R} \mid \bigcup_{k \geq 1} F_k \subset A \subset E, f|_{F_k} \text{ is continuous } \forall k\},$$

$$C_\infty(\{F_k\}) = \{f : A \rightarrow \mathbb{R} \mid \bigcup_{k \geq 1} F_k \subset A \subset E, f|_{F_k \cup \{\Delta\}} \text{ is continuous } \forall k\}.$$

Obviously $C_\infty(\{F_k\}) \subset C(\{F_k\})$ and $C_\infty(\{F_k\})$ only differs from $C(\{F_k\})$ if E_Δ is the one point compactification since otherwise Δ is an isolated point of E . Recall that $C(\{F_k\})$ is a notion which is used in conjunction with quasi-regularity.

A subset $N \subset E$ is called strictly \mathcal{E} -exceptional if $\text{Cap}_{1, \widehat{G}_1 \varphi}(N) = 0$. An increasing sequence $(F_k)_{k \in \mathbb{N}}$ of closed subsets of E is called a strict \mathcal{E} -nest if $\text{Cap}_{1, \widehat{G}_1 \varphi}(F_k^c) \downarrow 0$ as $k \rightarrow \infty$. A property of points in E holds strictly \mathcal{E} -quasi-everywhere (s. \mathcal{E} -q.e.) if the property holds outside some strictly \mathcal{E} -exceptional set. A function f defined up to some strictly \mathcal{E} -exceptional set $N \subset E$ is called strictly \mathcal{E} -quasi-continuous (s. \mathcal{E} -q.c.) if there exists a strict \mathcal{E} -nest $(F_k)_{k \in \mathbb{N}}$, such that $f \in C_\infty(\{F_k\})$. For a subset $\mathcal{D} \subset \mathcal{H}$ denote by $\widetilde{\mathcal{D}}^{str}$ all the strictly \mathcal{E} -q.c. m -versions of elements in \mathcal{D} .

Definition 0.3 *The generalized Dirichlet form \mathcal{E} is called strictly quasi-regular if:*

- (i) *There exists a strict \mathcal{E} -nest $(E_k)_{k \geq 1}$ such that $E_k \cup \{\Delta\}$, $k \geq 1$, is compact in E_Δ .*
- (ii) *There exists a dense subset of \mathcal{F} whose elements have strictly \mathcal{E} -q.c. m -versions.*
- (iii) *There exist $u_n \in \mathcal{F}$, $n \in \mathbb{N}$, having strictly \mathcal{E} -q.c. m -versions \widetilde{u}_n , $n \in \mathbb{N}$, and a strictly \mathcal{E} -exceptional set $N \subset E$ such that $\{\widetilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E_\Delta \setminus N$.*

Proposition 0.4 *Let E be a locally compact separable metric space. Let the generalized Dirichlet form $(\mathcal{E}, \mathcal{F})$ be regular, i.e. $C_0(E) \cap \mathcal{F}$ is dense in \mathcal{F} w.r.t. $|\cdot|_{\mathcal{F}}$ as well as in $C_0(E)$ w.r.t. the uniform norm. Then it is strictly quasi-regular.*

Proof (cf. [5, V. Proposition 2.12.(ii)]) Recall that either E is already compact, or E_Δ is the one point compactification. In any case $E_k \equiv E$, $k \geq 1$, is a strict \mathcal{E} -nest fulfilling Definition 0.3(i). Since $C_0(E) \cap \mathcal{F}$ is dense in \mathcal{F} Definition 0.3(ii) holds. Definition 0.3(iii) holds because $C_0(E) \cap \mathcal{F}$ is dense in $C_0(E)$ and since E is separable. □

Remark 0.5 (i) *The notion of strictly quasi-regular in the above definition crucially depends on how Δ is adjoined to E .*

(ii) *When dealing with quasi-regular generalized Dirichlet forms \mathcal{E} -nests and \mathcal{E} -exceptional sets are determined by Cap_φ (cf. [10, III. Proposition 2.10.]) where $\text{Cap}_\varphi(U) := ((G_1 \varphi)_U, \varphi)$, U open. Obviously $\text{Cap}_\varphi \leq \text{Cap}_{1, \widehat{G}_1 \varphi}$. Hence, the above strict notions w.r.t. $\text{Cap}_{1, \widehat{G}_1 \varphi}$ are at least not weaker than the corresponding notions \mathcal{E} -q.e., \mathcal{E} -q.c., \mathcal{E} -nest, etc., w.r.t. Cap_φ . Furthermore, suppose that m is a positive Radon measure on $(E, \mathcal{B}(E))$, that \mathcal{E} is a regular Dirichlet form, and that Cap denotes the capacity associated to \mathcal{E} as defined in [4, p. 64]. By Proposition 0.4 \mathcal{E} is strictly quasi-regular and it is easy to see that $\text{Cap}_{1, \widehat{G}_1 \varphi} \leq \text{Cap}$ (see [5, III. Exercise 2.10.]).*

(iii) *Related statements in [10] (resp. [5]) may (and they do in nearly all cases) remain true if we replace Cap_φ (resp. $\text{Cap}_{h, g}$) by $\text{Cap}_{1, \widehat{G}_1 \varphi}$, $C(\{F_k\})$ by $C_\infty(\{F_k\})$, and add strict or strictly to quasi-regular, \mathcal{E} -nest, \mathcal{E} -quasi-uniformly, \mathcal{E} -quasi-continuous, etc.*

We shall refer to these statements as “strict versions”. For instance [10, III. Lemma 3.5.], [10, IV. Proposition 1.9.], are easily seen to have strict versions.

(iv) Under the assumption that each E_k in Definition 0.3(i) is metrizable it follows similarly to [5, Remark 3.2.(iv)] that $\sigma\{\tilde{u}_n | n \in \mathbb{N}\} \supset \mathcal{B}(E_\Delta \setminus N)$ where $N, \tilde{u}_n, n \in \mathbb{N}$, are as in Definition 0.3(iii).

Let us now give a technical condition which we will use in order to compensate a lack of implicitly given or a so far undisclosed structure in the domain \mathcal{F} of the generalized Dirichlet form. The following condition is trivially satisfied for any sectorial Dirichlet form. Indeed, in this case it is enough to consider the bounded elements in the domain of the sectorial form.

SD3 There exists an algebra of functions $\mathcal{G} \subset \mathcal{H}_b$ such that $\mathcal{G} \cap \mathcal{F}$ is dense in \mathcal{F} and $\lim_{\alpha \rightarrow \infty} e_{u - \alpha G_\alpha u} + e_{\alpha G_\alpha u - u} = 0$ in \mathcal{H} for every $u \in \mathcal{G}$.

We shall give here below among others the exact definition of a strictly m -tight Hunt process. Denote by $P(E_\Delta)$ the set of all probability measures on $(E_\Delta, \mathcal{B}(E_\Delta))$ and let $\mathcal{B}(E_\Delta)^*$ be the σ -algebra of universally measurable sets.

Definition 0.6 (i) $\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t \geq 0}, (P_x)_{x \in E_\Delta})$ is called a Hunt process with state space E , life time ζ , and corresponding filtration $(\mathcal{M}_t)_{t \geq 0}$, if

(M.1) $X_t : \Omega \rightarrow E_\Delta$ is $\mathcal{M}_t/\mathcal{B}(E_\Delta)$ -measurable for all $t \geq 0$, and $X_t(\omega) = \Delta \Leftrightarrow t \geq \zeta(\omega)$ for all $\omega \in \Omega$, where $(\mathcal{M}_t)_{t \geq 0}$ is a filtration on (Ω, \mathcal{M}) and $\zeta : \Omega \rightarrow [0, \infty]$.

(M.2) For all $t \geq 0$ there exists a map $\theta_t : \Omega \rightarrow \Omega$ such that $X_s \circ \theta_t = X_{s+t}$ for all $s \geq 0$.

(M.3) $(P_x)_{x \in E_\Delta}$ is a family of probability measures on (Ω, \mathcal{M}) , such that $x \mapsto P_x[B]$ is $\mathcal{B}(E_\Delta)^*$ -measurable for all $B \in \mathcal{M}$ and $\mathcal{B}(E_\Delta)$ -measurable for all $B \in \sigma(X_t | t \geq 0)$ and $P_\Delta[X_0 = \Delta] = 1$.

(M.4) For all $A \in \mathcal{B}(E_\Delta), s, t \geq 0$, and $x \in E_\Delta$

$$P_x[X_{t+s} \in A | \mathcal{M}_t] = P_{X_t}[X_s \in A] \text{ } P_x\text{-a.s.}$$

(M.5) $P_x[X_0 = x] = 1$ for all $x \in E_\Delta$.

(M.6) $t \mapsto X_t(\omega)$ is right continuous on $[0, \infty)$ for all $\omega \in \Omega$.

(M.7) $(\mathcal{M}_t)_{t \geq 0}$ is right continuous and for any $(\mathcal{M}_t)_{t \geq 0}$ -stopping time τ and $\mu \in \mathcal{P}(E_\Delta)$

$$P_\mu[X_{\tau+s} \in A | \mathcal{M}_\tau] = P_{X_\tau}[X_s \in A] \text{ } P_\mu\text{-a.s.}$$

for all $A \in \mathcal{B}(E_\Delta), s \geq 0$, where for a positive measure ν on $(E_\Delta, \mathcal{B}(E_\Delta))$ we set $P_\nu := \int P_x \nu(dx)$.

(M.8) $X_{t-} := \lim_{\substack{s \uparrow t \\ s < t}} X_s$ exists in E_Δ for all $t \in (0, \infty)$ P_μ -a.s. for all $\mu \in P(E_\Delta)$.

(M.9) $\lim_{n \rightarrow \infty} X_{\tau_n} = X_\tau$ P_μ -a.s. on $\{\tau < \infty\}$ and X_τ is $\bigvee_{n \geq 1} \mathcal{F}_{\tau_n}^{P_\mu}$ -measurable for every

increasing sequence $(\tau_n)_{n \geq 1}$ of $(\mathcal{F}_t^{P_\mu})_{t \geq 0}$ -stopping times with limit τ and for all $\mu \in P(E_\Delta)$, where for a sub- σ -algebra $\mathcal{A} \subset \mathcal{M}$ we let \mathcal{A}^{P_μ} be its P_μ -completion in \mathcal{M} .

(ii) \mathbb{M} is called a right process if it satisfies (M.1) – (M.7) above. \mathbb{M} is called a m -Hunt process if it is a right process and if it satisfies (M.8) – (M.9) but with “for all $\mu \in P(E_\Delta)$ ” replaced by “for $\mu = \varphi \cdot m$ ”. \mathbb{M} is called a m -special standard process if it is a right process and if it satisfies (M.8) – (M.9) but with “for all $\mu \in P(E_\Delta)$ ”, “ ∞ ”, “ E_Δ ”, replaced by “for $\mu = \varphi \cdot m$ ”, “ ζ ”, “ E ”.

(iii) A right process \mathbb{M} is said to be strictly m -tight if there exists an increasing sequence $(K_k)_{k \in \mathbb{N}}$ of compact metrizable sets in E_Δ such that

$$P_{\varphi \cdot m}[\lim_{k \rightarrow \infty} \sigma_{E_\Delta \setminus K_k} < \infty] = 0.$$

\mathbb{M} is said to be m -tight if it satisfies the above but with “ ∞ ”, “ E_Δ ”, replaced by “ ζ ”, “ E ”.

Given a right process $\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t \geq 0}, (P_x)_{x \in E_\Delta})$ with state space E . If μ is a positive measure on $(E_\Delta, \mathcal{B}(E_\Delta))$ let E_μ be the expectation w.r.t. P_μ . The family $(p_t)_{t \geq 0}$ (resp. $(R_\alpha)_{\alpha > 0}$) of kernels on $(E, \mathcal{B}(E))$ defined by $p_t f(x) := E_x[f(X_t)]$ (resp. $R_\alpha f(x) := \int_0^\infty e^{-\alpha t} p_t f(x) dt$), $x \in E$, $f \in \mathcal{B}_b$, is called the *transition semigroup* (resp. *resolvent*) of \mathbb{M} . Note that $R_\alpha f$ is well-defined and $\mathcal{B}(E)$ -measurable for all $f \in \mathcal{B}_b$ because of our assumption $\mathcal{B}(E) = \sigma(C(E))$. For a subset $A \subset E_\Delta$ let $\sigma_A := \inf\{t > 0 \mid X_t \in A\}$ (resp. $D_A = \inf\{t \geq 0 \mid X_t \in A\}$) be the *first hitting time* (resp. *first entry time*) w.r.t. \mathbb{M} .

Definition 0.7 A right process \mathbb{M} with resolvent $(R_\alpha)_{\alpha > 0}$ is called associated (in the resolvent sense) with \mathcal{E} if $R_\alpha f$ is an m -version of $G_\alpha f$ for all $\alpha > 0$ and $f \in \mathcal{B}_b \cap \mathcal{H}$. \mathbb{M} is called properly associated (in the resolvent sense) with \mathcal{E} (resp. strictly properly associated in the resolvent sense with \mathcal{E}) if in addition $R_\alpha f$ is \mathcal{E} -q.c. (resp. $s\mathcal{E}$ -q.c.) for $\alpha > 0$ and $f \in \mathcal{B}_b \cap \mathcal{H}$.

Lemma 0.8 (i) Assume that there exists a m -tight m -special standard process $\mathbb{M} = ((X_t)_{t \geq 0}, (P_z)_{z \in E_\Delta})$ that is associated with \mathcal{E} . Let $U \subset E$, U open. Then

$$Cap_{1, \widehat{G}_1 \varphi}(U) = E_{\varphi \cdot m}(e^{-\sigma_U}).$$

(ii) Let $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (P_z)_{z \in E_\Delta})$ be a strictly m -tight m -Hunt process which is associated with \mathcal{E} . Let $B \in \mathcal{B}(E)$. Then

$$Cap_{1, \widehat{G}_1 \varphi}(B) = E_{\varphi \cdot m}(e^{-D_B}) = E_{\varphi \cdot m}(e^{-S_B})$$

and $S_B := \inf\{t \geq 0 \mid \overline{X_0^t} \cap B \neq \emptyset\}$. Here $\overline{X_0^t}$ denotes the closure of $\{X_s \mid s \in [0, t]\}$ in E_Δ . In particular $D_B = S_B$ $P_{\varphi \cdot m}$ -a.e.

Moreover, if $(F_k)_{k \in \mathbb{N}}$ is a strict \mathcal{E} -nest then

$$P_z(\lim_{k \rightarrow \infty} \sigma_{F_k^c} = \infty) = 1 \quad \text{for } s\mathcal{E}\text{-q.e. } z \in E.$$

Proof (i) By [10, IV.Theorem 3.1.] \mathcal{E} is quasi-regular and $\mathbb{M} = ((X_t)_{t \geq 0}, (P_z)_{z \in E_\Delta})$ is properly associated in the resolvent sense with \mathcal{E} . Let $(R_\alpha)_{\alpha > 0}$ be the resolvent of \mathbb{M} . Let $v \in \mathcal{P}_{\mathcal{F}} \cap \mathcal{B}$ and $\bar{v} := \sup_{\alpha > 0} \alpha R_{\alpha+1} v$. We will first show that m -a.e. $\bar{v}_U := E. \left[e^{-\sigma_U} \bar{v}(X_{\sigma_U}) \right] = v_U$. Using the strong Markov property (M.7) it is easy to see that \bar{v}_U is 1-excessive, i.e. $\alpha R_{\alpha+1} \bar{v}_U \leq \bar{v}_U$. By (M.5), (M.6), we have $\bar{v}_U = \bar{v}$ on U . Hence, by [10, III.Proposition 1.7.(ii)] $\bar{v}_U \geq v_U$. By [10, IV.Proposition 3.4.] we know that $E. \left[e^{-\sigma_U} \alpha R_{\alpha+1} v(X_{\sigma_U}) \right]$ is a m -version of $(\alpha G_{\alpha+1} v)_U$. We obtain $\bar{v}_U = \sup_{\alpha > 0} E. \left[e^{-\sigma_U} \alpha R_{\alpha+1} v(X_{\sigma_U}) \right] = \sup_{\alpha > 0} (\alpha G_{\alpha+1} v)_U \leq v_U$ using again [10, III.Proposition 1.7.(ii)] for the last inequality. Therefore $\bar{v}_U = v_U$. Thus $(kG_1 \varphi \wedge 1)_U = E. \left[e^{-\sigma_U} \overline{kR_1 \varphi \wedge 1}(X_{\sigma_U}) \right]$ m -a.e. Note that $0 < \varphi(z) \leq 1$ for every $z \in E$. Hence $0 < R_1 \varphi(z) \leq 1$ for every $z \in E$ by (M.5), (M.6). It follows $kR_1 \varphi \wedge 1 \uparrow 1_E$ pointwise as $k \rightarrow \infty$. Noting that $\{\sigma_U < \zeta\} = \{\sigma_U < \infty\}$, we obtain

$$\begin{aligned} \text{Cap}_{1, \widehat{G}_1 \varphi}(U) &= \sup_{k \geq 1} \sup_{\alpha \geq 1} (E. \left[e^{-\sigma_U} \alpha R_{\alpha+1} (kR_1 \varphi \wedge 1)(X_{\sigma_U}) \right], \varphi) \\ &= \sup_{\alpha \geq 1} (E. \left[e^{-\sigma_U} \alpha R_{\alpha+1} 1_E(X_{\sigma_U}) \right], \varphi) = E_{\varphi \cdot m} \left[e^{-\sigma_U} \right]. \end{aligned}$$

(ii) By Theorem 0.2 the proof is the same as in [5, V. Lemma 2.19., Lemma 2.21.]. □

Proposition 0.9 *Let \mathcal{E} be strictly quasi-regular. Then:*

(i) *Every $g \in \mathcal{F}$ admits a s. \mathcal{E} -q.c. m -version \tilde{g} . In particular, we have for any s. \mathcal{E} -q.c. m -version \tilde{u} of $u \in \mathcal{F}$, $\varepsilon > 0$*

$$\text{Cap}_{1, \widehat{G}_1 \varphi}(\{|\tilde{u}| > \varepsilon\}) \leq \varepsilon^{-1} \|e_u + e_{-u}\|_{\mathcal{H}} \|\varphi\|_{\mathcal{H}}.$$

Let \mathcal{E} additionally satisfy SD3. Then every $g \in \mathcal{G}$ admits a s. \mathcal{E} -q.c. m -version \tilde{g} .

(ii) *\mathcal{E} is quasi-regular. If \mathcal{E} additionally satisfies SD3, then there exists a strictly m -tight (m -)special standard process \mathbb{M} which is properly associated in the resolvent sense with \mathcal{E} .*

Proof (i) Assume that $u \in \mathcal{F}$ admits a s. \mathcal{E} -q.c. m -version $\tilde{u} \in C_\infty(\{F_k\})$ where $(F_k)_{k \in \mathbb{N}}$ is a strict \mathcal{E} -nest. Then $\{|\tilde{u}| > \varepsilon\} \cup F_k^c$ is open for each k and $\varepsilon^{-1}(e_u + e_{-u}) \geq lG_1 \varphi \wedge 1$ m -a.e. on $\{|\tilde{u}| > \varepsilon\}$ for each l . Hence (for intermediate steps cf. e.g. the implementations in [13, Section 2], or [14, Section 1])

$$\begin{aligned} \text{Cap}_{1, \widehat{G}_1 \varphi}(\{|\tilde{u}| > \varepsilon\}) &\leq \lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} ((lG_1 \varphi \wedge 1)_{\{|\tilde{u}| > \varepsilon\} \cup F_k^c}, \varphi) \\ &\leq \lim_{l \rightarrow \infty} ((lG_1 \varphi \wedge 1)_{\{|\tilde{u}| > \varepsilon\}}, \varphi) + \lim_{k \rightarrow \infty} \text{Cap}_{1, \widehat{G}_1 \varphi}(F_k^c) \\ &\leq \lim_{l \rightarrow \infty} \lim_{\alpha \rightarrow \infty} \mathcal{E}_1(lG_1 \varphi \wedge 1, (\widehat{G}_1 \varphi)_{\{|\tilde{u}| > \varepsilon\}}^\alpha) \\ &\leq \varepsilon^{-1} \|e_u + e_{-u}\|_{\mathcal{H}} \|\varphi\|_{\mathcal{H}}. \end{aligned} \tag{1}$$

Therefore, using 0.3(ii) and the strict version of [10, III. Proposition 3.7.] we can construct through a “locally” uniform approximation with s. \mathcal{E} -q.c. m -versions of functions in a dense subset of \mathcal{F} a s. \mathcal{E} -q.c. m -version for any $u \in \mathcal{F}$. Now let $u \in \mathcal{G}$ and let \mathcal{E} satisfy SD3. Since $\alpha G_\alpha u \in \mathcal{F}$, it admits, as we just proved, a s. \mathcal{E} -q.c. m -version for each α . Then, using again the strict version of [10, III. Proposition 3.7.] and (1) we obtain that every $u \in \mathcal{G}$ admits a s. \mathcal{E} -q.c. m -version.

(ii) We first check [10, IV.1.7. (i)-(iii)], i.e. the quasi-regularity of \mathcal{E} . Since $\text{Cap}_\varphi \leq \text{Cap}_{1, \widehat{G}_1 \varphi}$ 0.3(ii),(iii) immediately imply [10, IV.1.7. (ii),(iii)]. If $\{\Delta\}$ is an isolated point of E , then E_k is compact if and only if $E_k \cup \{\Delta\}$ is compact. Hence, in this case 0.3(i) also implies immediately [10, IV.1.7. (i)]. Thus, it remains to check [10, IV.1.7. (i)] if E_Δ is the one point compactification. By (i) $G_1 \varphi$ has a s. \mathcal{E} -q.c. m -version $\widetilde{G}_1 \varphi$. Let $\widetilde{G}_1 \varphi \in C_\infty(\{F_k\})$ for some s. \mathcal{E} -nest $(F_k)_{k \in \mathbb{N}}$. Because $\widetilde{G}_1 \varphi(\Delta) = 0$ we have $\Delta \notin \{\widetilde{G}_1 \varphi \geq k^{-1}\}$. Then, $E'_k := F_k \cap \{\widetilde{G}_1 \varphi \geq k^{-1}\} = (F_k \cup \{\Delta\}) \cap \{\widetilde{G}_1 \varphi \geq k^{-1}\}$, $k \in \mathbb{N}$, is compact in E . We have to check that $(E'_k)_{k \geq 1}$ is an \mathcal{E} -nest. But (cf. again e.g. [13, Section 2], or [14, Section 1] for intermediate steps)

$$\begin{aligned} \lim_{k \rightarrow \infty} \text{Cap}_\varphi(E'_k) &\leq \lim_{k \rightarrow \infty} \text{Cap}_{1, \widehat{G}_1 \varphi}(F_k) + \lim_{k \rightarrow \infty} \mathcal{E}_1(G_1 \varphi, (\widehat{G}_1 \varphi)_{\{\widetilde{G}_1 \varphi < k^{-1}\}}) \\ &\leq \lim_{k \rightarrow \infty} (G_1 \varphi \wedge k^{-1}, \varphi) = 0 \end{aligned}$$

and we have shown that \mathcal{E} is quasi-regular. Using the same arguments as in the proof of [10, Proposition 2.1.] it is easy to see that condition SD3 implies condition D3 in [10, IV.2]. Hence, by [10, IV. Theorem 2.2.] there exists a m -tight special standard process \mathbb{M} which properly associated in the resolvent sense with \mathcal{E} . Finally we show that \mathbb{M} is strictly m -tight. Let $(E_k)_{k \geq 1}$ be as in Definition 0.3. By Lemma 0.8(i) we have $\text{Cap}_{1, \widehat{G}_1 \varphi}(E_k) = E_{\varphi, m}(e^{-\sigma_{E_\Delta \setminus E_k \cup \{\Delta\}}})$. Hence $P_{\varphi, m}(\lim_{k \rightarrow \infty} \sigma_{E_\Delta \setminus K_k} < \infty) = 0$ where $K_k := E_k \cup \{\Delta\}$ and \mathbb{M} is strictly m -tight. □

Proposition 0.10 *Suppose that $(\mathcal{E}, \mathcal{F})$ is a quasi-regular generalized Dirichlet form on \mathcal{H} such that $1 \in \mathcal{F}$ and Δ is adjoined to E as an isolated point of E_Δ . Then $(\mathcal{E}, \mathcal{F})$ strictly quasi-regular.*

Proof In fact, comparing the definitions of quasi-regularity and strict quasi-regularity and taking into account that Δ is isolated it suffices to show that the φ -capacity which determines quasi-regularity is equivalent to the strict capacity. We have to show that strict \mathcal{E} -nests and \mathcal{E} -nests are the same. Of course any strict \mathcal{E} -nest is an \mathcal{E} -nest. So, let $(F_k)_{k \in \mathbb{N}}$ be an \mathcal{E} -nest. By definition $1_{F_k^c} \rightarrow 0$ in \mathcal{H} as $k \rightarrow \infty$ (here $1_{F_k^c}$ means “reduced” and not “indicator” function !). We obtain

$$\text{Cap}_{1, \widehat{G}_1 \varphi}(F_k) := \lim_{k \rightarrow \infty} ((kG_1 \varphi \wedge 1)_{F_k^c}, \varphi) \leq \lim_{k \rightarrow \infty} (1_{F_k^c}, \varphi) = 0,$$

hence $(F_k)_{k \in \mathbb{N}}$ is a strict \mathcal{E} -nest and the result follows. □

When we assume that our generalized Dirichlet form is strictly quasi-regular we let $Y := \cup_{k \in \mathbb{N}} E_k$ with E_k , $k \in \mathbb{N}$, as in Definition 0.3(i). We may assume that each E_k is metrizable w.r.t. the trace topology inherited from E (cf. [5, Remark 3.2.(iii)]). By [9, Corollary 2, p.102] Y is a Lusin space. Since $E \setminus Y$ is strictly \mathcal{E} -exceptional it is \mathcal{E} -exceptional, hence $m(E \setminus Y) = 0$ and we may identify $L^2(E; m)$ with $L^2(Y, m)$ canonically.

Lemma 0.11 *Let \mathcal{E} be strictly quasi-regular. Let $\alpha > 0$. There exists a kernel \tilde{R}_α from $(E, \mathcal{B}(E))$ to $(Y, \mathcal{B}(Y))$ such that*

$$(i) \quad \tilde{R}_\alpha(z, Y) \leq 1 \text{ for all } z \in E.$$

$$(ii) \quad \tilde{R}_\alpha f \text{ is a s.}\mathcal{E}\text{-q.c. } m\text{-version of } G_\alpha f \text{ for all } f \in \mathcal{H}.$$

The kernel \tilde{R}_α is unique in the sense that, if K is another kernel from $(E, \mathcal{B}(E))$ to $(Y, \mathcal{B}(Y))$ satisfying (i) and (ii), it follows that $K(z, \cdot) = \tilde{R}_\alpha(z, \cdot)$ s. \mathcal{E} -q.e.

Proof Fix $\alpha > 0$. Let $sq(C(E))$ denote the s. \mathcal{E} -q.c. functions defined s. \mathcal{E} -q.e. on E . Let $T : \mathcal{H} \rightarrow sq(C(E))$, $f \mapsto \tilde{G}_\alpha f$, where $\tilde{G}_\alpha f$ denotes a s. \mathcal{E} -q.c. m -version of $G_\alpha f$. $\tilde{G}_\alpha f$ exists by Proposition 0.9(i). We will show that T is quasi-linear in the sense of [1, Definition 1.2.]. Indeed, let $c_1, c_2 \in \mathbb{R}$, $f, g \in \mathcal{H}$. By Proposition 0.9(i) we have for any $\varepsilon > 0$

$$\text{Cap}_{1, \hat{G}_1 \varphi}(\{|\tilde{G}_\alpha(c_1 f + c_2 g) - c_1 \tilde{G}_\alpha f - c_2 \tilde{G}_\alpha g| > \varepsilon\}) \leq \varepsilon^{-1} \|e_0 + e_{-0}\|_{\mathcal{H}} \|\varphi\|_{\mathcal{H}} = 0,$$

hence $\tilde{G}_\alpha(c_1 f + c_2 g) = c_1 \tilde{G}_\alpha f + c_2 \tilde{G}_\alpha g$ s. \mathcal{E} -q.e. If $f_n \downarrow 0$, $f_n \in \mathcal{H}$, $n \in \mathbb{N}$, then $f_n \rightarrow 0$ in \mathcal{H} . Hence $e_{G_\alpha f_n} + e_{-G_\alpha f_n} \rightarrow 0$ in \mathcal{H} and therefore again by Proposition 0.9(i) $\text{Cap}_{1, \hat{G}_1 \varphi}(\{|\tilde{G}_\alpha f_n| > \varepsilon\}) \rightarrow 0$ for any $\varepsilon > 0$ as $n \rightarrow \infty$. Finally, if $f \in \mathcal{H}$, $f \geq 0$ m -a.e., then $\tilde{G}_\alpha f \geq 0$ s. \mathcal{E} -q.e. Indeed, since $G_\alpha f \geq 0$ m -a.e. and $\tilde{G}_\alpha f = G_\alpha f$ m -a.e. we have $m(\tilde{G}_\alpha f < 0) = 0$ and thus

$$\text{Cap}_{1, \hat{G}_1 \varphi}(\{\tilde{G}_\alpha f < 0\}) = \lim_{k \rightarrow \infty} \lim_{\beta \rightarrow \infty} \mathcal{E}_1(k G_1 \varphi \wedge 1, (\hat{G}_1 \varphi)_{\{\tilde{G}_\alpha f < 0\}}^\beta) = 0.$$

This completes the proof that T is quasi-linear. By [1, Theorem 4.2.] there exists then a unique (up to quasi-equivalence) kernel K from $(E, \mathcal{B}(E))$ to $(Y, \mathcal{B}(Y))$ such that $Kf = Tf$ s. \mathcal{E} -q.e. for all $f \in \mathcal{H}$. Since m is σ -finite it follows in particular $\alpha K 1_Y = \alpha T 1_Y = \alpha \tilde{G}_\alpha 1_Y \leq 1$ s. \mathcal{E} -q.e. and hence there exists a strictly \mathcal{E} -exceptional set $N \in \mathcal{B}(E)$ such that $\alpha K(z, Y) \leq 1$ for all $z \in E \setminus N$. It suffices now to let $\tilde{R}_\alpha(z, \cdot) := 1_{E \setminus N} K(z, \cdot)$. □

Lemma 0.12 *Let \mathcal{E} be strictly quasi-regular and satisfy SD3. Let \tilde{R}_α , $\alpha > 0$, be as in Lemma 0.11. Then there exists a countable family D of m -versions from elements in $\mathcal{G} \cup \mathcal{F}_b$ and a strict \mathcal{E} -nest $(F_k)_{k \in \mathbb{N}}$ with the following properties:*

$$(i) \quad \text{There exists a } \mathbb{Q}\text{-algebra } D_2 \text{ such that } D = D_2 \cup \{\tilde{R}_\alpha u \mid u \in D_2, \alpha \in \mathbb{Q}_+^*\}.$$

(ii) $F_k \subset E_k$ for all k .

(iii) $u \in C_\infty(\{F_k\})$ for all $u \in D$ and there exists $\alpha_n \in \mathbb{Q}_+^*$, $n \in \mathbb{N}$, such that $\lim_{n \rightarrow \infty} \alpha_n \tilde{R}_{\alpha_n} u(z) = u(z)$ for all $u \in D_2$, $z \in Y_1 := \cup_{k \geq 1} F_k$.

(iv) $\{\tilde{R}_1 u \mid u \in D_2\}$ separates the points of $Y_{1\Delta} := Y_1 \cup \{\Delta\}$ and $\sigma(D_2) \supset \mathcal{B}(Y_{1\Delta})$.

Proof Let $D_0 \subset \mathcal{G} \cap \mathcal{F}$ be a countable family which is both dense in \mathcal{H} and in \mathcal{F} . Using Proposition 0.9(i) let us choose for each $u \in D_0$ a s. \mathcal{E} -q.c. m -version to obtain a countable family D_1 consisting of s. \mathcal{E} -q.c. functions which are m -versions of elements in D_0 . Let D_2 be the smallest family of bounded s. \mathcal{E} -q.c. functions which is a \mathbb{Q} -algebra containing D_1 . Then D_2 and consequently $D := D_2 \cup \{\tilde{R}_\alpha u \mid u \in D_2, \alpha \in \mathbb{Q}_+^*\}$ is again a countable family. Obviously $\tilde{\mathcal{G}}^{str} \supset D_2$ because $\tilde{\mathcal{G}}^{str}$ is in particular also a \mathbb{Q} -algebra and contains D_1 . We know that $\lim_{\alpha \rightarrow \infty} e_{u-\alpha G_\alpha u} + e_{\alpha G_\alpha u-u} = 0$ in \mathcal{H} for every $u \in \mathcal{G}$. Hence as in the proof of Proposition 0.9(i) we can use the strict version of [10, III. Proposition 3.7.] and then apply the usual diagonal argument in order to obtain an increasing sequence $(\alpha_n)_{n \in \mathbb{N}} \subset \mathbb{Q}_+^*$ and a strictly \mathcal{E} -exceptional set N_1 such that $\lim_{n \rightarrow \infty} \alpha_n \tilde{R}_{\alpha_n} u(z) = u(z)$ for all $u \in D_2$ and all $z \in E \setminus N_1$. Since $D_1 \subset D_2$ and D_1 is dense in \mathcal{H} we have that $\{\tilde{R}_1 u \mid u \in D_2\}$ is dense in \mathcal{F} . Thus, we can use the strict version of [10, IV. Proposition 1.9.(i)] (cf. Remark 0.5(iii)) to obtain a strictly \mathcal{E} -exceptional set N_2 such that $\{\tilde{R}_1 u \mid u \in D_2\}$ separates the points of $E_\Delta \setminus N_2$. By Remark 0.5(iv) we can find a strictly \mathcal{E} -exceptional set N_3 such that $\sigma(D_2) \supset \mathcal{B}(E_\Delta \setminus N_3)$. Let $(F_{1k})_{k \in \mathbb{N}}$ be a strict \mathcal{E} -nest such that $N_1 \cup N_2 \cup N_3 \subset \cap_{k \in \mathbb{N}} F_{1k}^c$. By the strict version of [10, III. Lemma 3.5.] there exists a strict \mathcal{E} -nest $(F_{2k})_{k \in \mathbb{N}}$ such that $u \in C_\infty(\{F_{2k}\})$ for all $u \in D$. Then let $F_k := F_{1k} \cap F_{2k} \cap E_k$, $k \in \mathbb{N}$ (E_k as fixed right before Lemma 0.11). □

Main results

Theorem 0.13 *Assume that there exists a strictly m -tight m - Hunt process \mathbb{M} with state space E which is associated with \mathcal{E} . Then \mathcal{E} is strictly quasi-regular and \mathbb{M} is strictly properly associated with \mathcal{E} .*

Proof Using in particular Lemma 0.8(ii) the verification of Definition 0.3(i),(ii), as well as the verification that \mathbb{M} is strictly properly associated with \mathcal{E} , works in the same manner as in the proof of [5, V. Proposition 2.18., Claim 1, Claim 2]. One only has to see that the last argument in the proof of Claim 2 on p. 164 of [5] carries over because we use already association in the resolvent sense (see Definition 0.7), and because $\{G_\alpha f \mid f \in \mathcal{H}, \alpha > 0\}$ is dense in \mathcal{F} (see [10, I. Remark 3.5.]). It remains to verify Definition 0.3(iii): Since \mathbb{M} is strictly m -tight there exists an increasing sequence $(K_k)_{k \in \mathbb{N}}$ of compact metrizable sets in E_Δ such that

$$P_{\varphi \cdot m}(\lim_{k \rightarrow \infty} \sigma_{E_\Delta \setminus K_k} < \infty) = 0.$$

Hence $(F_k := K_k \setminus \{\Delta\})_{k \in \mathbb{N}}$ is a strict \mathcal{E} -nest of closed sets in E . Lemma 0.8(ii) implies that there exists a strictly \mathcal{E} -exceptional set N such that

$$P_z(\lim_{k \rightarrow \infty} \sigma_{F_k^c} = \infty) = 1 \quad \forall z \in E \setminus N.$$

Since K_k is compact and metrizable w.r.t. the topology inherited from E_Δ , there exist open sets $U_m^\Delta \subset E_\Delta$, $m \in \mathbb{N}$, such that $(U_m^\Delta \cap K_k)_{m \in \mathbb{N}}$ separates the points of K_k for each k (i.e. for $x, y \in K_k$, $x \neq y$, there exist $U, V \in \{U_m^\Delta | m \in \mathbb{N}\}$ such that $U \cap K_k \cap V = \emptyset$ and $x \in U, y \in V$). Of course $\mathcal{U} := \{U_m := U_m^\Delta \setminus \{\Delta\} | m \in \mathbb{N}\}$ is then a family of open subsets of E such that $(U_m \cap F_k)_{m \in \mathbb{N}}$ separates the points of F_k for each k .

Now, we can show exactly as in [10, IV.Lemma 3.11] that $(R_1\varphi - E.[\int_{\sigma_U}^\infty e^{-s}\varphi(X_s)ds])_{U \in \mathcal{U}}$ separates the points of $\bigcup_{n \in \mathbb{N}} F_n \cap (E \setminus N)$. Since

$$\lim_{n \rightarrow \infty} nR_{n+1}(R_1\varphi - E.[\int_{\sigma_U}^\infty e^{-s}\varphi(X_s)ds])(x) = R_1\varphi(x) - E_x[\int_{\sigma_U}^\infty e^{-s}\varphi(X_s)ds]$$

for all x we obtain that

$$\left\{ nR_{n+1}(R_1\varphi - E.[\int_{\sigma_U}^\infty e^{-s}\varphi(X_s)ds]) \mid U \in \mathcal{U}, n \in \mathbb{N} \right\}$$

separates also the points of $\bigcup_{n \in \mathbb{N}} F_n \cap (E \setminus N)$. Since \mathbb{M} is strictly properly associated with \mathcal{E} we obtain 0.3(iii). □

Theorem 0.14 *Let \mathcal{E} be a strictly quasi-regular generalized Dirichlet form satisfying SD3. Then there exists a strictly m -tight Hunt process which is strictly properly associated in the resolvent sense with \mathcal{E} .*

Proof By Proposition 0.9(ii) there exists a strictly m -tight m -special standard process \mathbb{M} which is properly associated in the resolvent sense with \mathcal{E} . \mathbb{M} is strictly properly associated with \mathcal{E} by [5, V. Proposition 2.18., Claim 2] which carries over to our more general situation. Using Lemma 0.12 one can show exactly as in [5, V. Proposition 2.25] that \mathbb{M} is a strictly m -tight m -Hunt process. One only has to replace the D in the proof of [5, V. Proposition 2.25] by our D_2 of Lemma 0.12. By Theorem 0.13 \mathbb{M} is then strictly properly associated in the resolvent sense with \mathcal{E} . Finally also [5, V. Proposition 2.26] can be carried over to our situation. We replace again the D in [5, V. Proposition 2.26] by our D_2 of Lemma 0.12. In that way we can show that \mathbb{M} is indeed a Hunt process up to a modification on a strictly \mathcal{E} -exceptional set. This completes the proof. □

Corollary 0.15 *Suppose that E is a locally compact separable metric space. Let $(\mathcal{E}, \mathcal{F})$ be a generalized Dirichlet form. Then a sufficient condition for $(\mathcal{E}, \mathcal{F})$ to be associated with a Hunt process is given by:*

(i) $C_0(E) \cap \mathcal{F}$ contains an algebra of functions \mathcal{G} which is dense in \mathcal{F} .

(ii) There exists a countable set $\{u_n | n \in \mathbb{N}\} \subset C_0(E) \cap \mathcal{F}$ separating the points of $E_\Delta \setminus N$ for some strictly \mathcal{E} -exceptional set $N \subset E$.

Proof It is easy to see that (i), (ii), imply (ii), (iii), of Definition 0.3. Clearly $E_k := E$, $k \geq 1$, is a strict \mathcal{E} -nest such that $E_k \cup \{\Delta\}$, $k \geq 1$, is compact in E_Δ . Indeed, this holds if E_Δ is the one-point compactification. If E is already compact, then Δ is an isolated point and $E \cup \{\Delta\}$ is also compact. Hence Definition 0.3(i) is fulfilled. Furthermore, \mathcal{E} satisfies SD3 with $\mathcal{G} \subset C_0(E) \cap \mathcal{F}$ by (i). We then apply Theorem 0.14. □

Remark 0.16 If $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form then $C_0(E) \cap \mathcal{F}$ is always an algebra. Furthermore, already in this case the conditions (i), (ii), of Corollary 0.15 are really weaker than assuming the regularity of $(\mathcal{E}, \mathcal{F})$, i.e. $C_0(E) \cap \mathcal{F}$ is dense both in \mathcal{F} and $C_0(E)$ (see [5, V.Exercise 2.17.]).

We will from now on up to the end of this section assume that our generalized Dirichlet form $(\mathcal{E}, \mathcal{F})$ is *strictly quasi-regular*. By strict quasi-regularity every element in \mathcal{F} admits a strictly \mathcal{E} -q.c. m -version (see Proposition 0.9). For a subset $\mathcal{D} \subset \mathcal{H}$ recall that $\widetilde{\mathcal{D}}^{str}$ denotes the strictly \mathcal{E} -q.c. m -versions of elements in \mathcal{D} . In particular $\widetilde{\mathcal{P}}_{\mathcal{F}}^{str}$ denotes the set of all strictly \mathcal{E} -q.c. m -versions of 1-excessive elements in \mathcal{V} which are dominated by elements of \mathcal{F} . Note that $\widetilde{\mathcal{F}} \cap \widetilde{\mathcal{P}}^{str} \subset \widetilde{\mathcal{P}}_{\mathcal{F}}^{str}$ and that $\widetilde{\mathcal{P}}_{\mathcal{F}}^{str} - \widetilde{\mathcal{P}}_{\mathcal{F}}^{str}$ is a linear lattice, that is $\tilde{u} \wedge \alpha \in \widetilde{\mathcal{P}}_{\mathcal{F}}^{str} - \widetilde{\mathcal{P}}_{\mathcal{F}}^{str}$ for all $\alpha \geq 0$ and all $\tilde{u} \in \widetilde{\mathcal{P}}_{\mathcal{F}}^{str} - \widetilde{\mathcal{P}}_{\mathcal{F}}^{str}$. We emphasize that an element in $\mathcal{P}_{\mathcal{F}}$ not necessarily admits an strictly \mathcal{E} -q.c. m -version.

Exactly as in [12, Theorem 2.3] we show the following integral representation theorem for elements in $\widehat{\mathcal{P}}_{\mathcal{F}}$. The only difference lies in the test functions which are now strictly quasi-continuous instead of quasi-continuous.

Theorem 0.17 Let $\hat{u} \in \widehat{\mathcal{P}}_{\mathcal{F}}$. Then there exists a unique σ -finite and positive measure $\mu_{\hat{u}}^{str}$ on $(E, \mathcal{B}(E))$ charging no strictly \mathcal{E} -exceptional set, such that

$$\int_E \tilde{f} d\mu_{\hat{u}}^{str} = \lim_{\alpha \rightarrow \infty} \mathcal{E}_1(f, \alpha \widehat{G}_{\alpha+1} \hat{u}) \quad \forall \tilde{f} \in \widetilde{\mathcal{P}}_{\mathcal{F}}^{str} - \widetilde{\mathcal{P}}_{\mathcal{F}}^{str}.$$

Since $\lim_{\alpha \rightarrow \infty} \alpha R_{\alpha+1}(kG_1\varphi \wedge 1) = kR_1\varphi \wedge 1$, $\lim_{k \rightarrow \infty} kR_1\varphi \wedge 1 = 1_E$, s. \mathcal{E} -q.e, and everything is s. \mathcal{E} -q.e. bounded, we obtain (for intermediate steps cf. e.g. [13, (9) on p. 416])

$$\begin{aligned} \text{Cap}_{1, \widehat{G}_1\varphi}(U) &= \lim_{k \rightarrow \infty} \lim_{\alpha \rightarrow \infty} \mathcal{E}_1((kG_1\varphi \wedge 1)_U^\alpha, \widehat{G}_1\varphi) \\ &= \lim_{k \rightarrow \infty} \lim_{\alpha \rightarrow \infty} \mathcal{E}_1(kG_1\varphi \wedge 1, (\widehat{G}_1\varphi)_U^\alpha) \\ &= \lim_{k \rightarrow \infty} \lim_{\alpha \rightarrow \infty} \mathcal{E}_1(\alpha G_{\alpha+1}(kG_1\varphi \wedge 1), (\widehat{G}_1\varphi)_U) \\ &= \lim_{k \rightarrow \infty} \lim_{\alpha \rightarrow \infty} \int_E \alpha R_{\alpha+1}(kR_1\varphi \wedge 1) d\mu_{(\widehat{G}_1\varphi)_U}^{str} = \mu_{(\widehat{G}_1\varphi)_U}^{str}(E). \end{aligned}$$

It is quite easy to see that the support of $\mu_{(\widehat{G}_1\varphi)_U}^{str}$ is located in the topological closure \overline{U} of U (use e.g. the strict version of [12, Lemma 2.4]). Thus

$$\text{Cap}_{1,\widehat{G}_1\varphi}(U) = \mu_{(\widehat{G}_1\varphi)_U}^{str}(\overline{U}). \quad (2)$$

We assume up to the end of this section that there exists a strictly m -tight Hunt process \mathbf{M} which is strictly properly associated in the resolvent sense with \mathcal{E} . By 0.14 this is the case if \mathcal{E} satisfies SD3. In analogy to [12] we introduce the following class of measures

$$\widehat{S}_{00}^{str} := \{\mu_{\hat{u}}^{str} \mid \hat{u} \in \widehat{\mathcal{P}}_{\widehat{G}_1\mathcal{H}_b^+} \text{ and } \mu_{\hat{u}}^{str}(E) < \infty\}$$

where $\widehat{G}_1\mathcal{H}_b^+ := \{\widehat{G}_1h \mid h \in \mathcal{H}_b^+\}$.

Theorem 0.18 *For $B \in \mathcal{B}(E)$ the following conditions are equivalent:*

(i) B is strictly \mathcal{E} -exceptional.

(ii) $\mu(B) = 0 \forall \mu \in \widehat{S}_{00}^{str}$.

Proof (i) \Rightarrow (ii) is clear. We show $\neg(\text{i}) \Rightarrow \neg(\text{ii})$. If B is not strictly \mathcal{E} -exceptional then $\text{Cap}_{1,\widehat{G}_1\varphi}(B) > 0$. Since $\text{Cap}_{1,\widehat{G}_1\varphi}$ is regular by Choquet's capacity theorem there exists a compact $K \subset B$ with $\text{Cap}_{1,\widehat{G}_1\varphi}(K) > 0$.

Fix a strict \mathcal{E} -nest $(F_k)_{k \in \mathbb{N}}$ consisting of metrizable sets as in 0.3(i) and $k_0 \in \mathbb{N}$ such that

$$\text{Cap}_{1,\widehat{G}_1\varphi}(K \cap F_{k_0}) - \text{Cap}_{1,\widehat{G}_1\varphi}(F_{k_0}^c) > 0. \quad (3)$$

Let $\rho_{F_{k_0}}$ be a metric on F_{k_0} which is compatible with the relative topology on F_{k_0} inherited from E . Define for $n \in \mathbb{N}$

$$B_n := \{z \in F_{k_0} \mid \rho_{F_{k_0}}(z, K \cap F_{k_0}) < \frac{1}{n}\}, \quad \overline{B}_n := \{z \in F_{k_0} \mid \rho_{F_{k_0}}(z, K \cap F_{k_0}) \leq \frac{1}{n}\}.$$

Then $B_n \cup F_{k_0}^c \subset E$ is open and contains $K \cap F_{k_0}$ for all n . Thus by (2)

$$\begin{aligned} \text{Cap}_{1,\widehat{G}_1\varphi}(K \cap F_{k_0}) &\leq \inf_{n \geq 1} \text{Cap}_{1,\widehat{G}_1\varphi}(B_n \cup F_{k_0}^c) \\ &\leq \inf_{n \geq 1} \mu_{(\widehat{G}_1\varphi)_{\overline{B}_n}}^{str}(E) + \text{Cap}_{1,\widehat{G}_1\varphi}(F_{k_0}^c) \end{aligned}$$

Let $(\widehat{G}_1\varphi)_\infty$ be the weak limit of $((\widehat{G}_1\varphi)_{\overline{B}_n})_{n \in \mathbb{N}}$ in \mathcal{V} . Note that $(\widehat{G}_1\varphi)_\infty \in \widehat{\mathcal{P}}_{\widehat{G}_1\mathcal{H}_b^+}$, $(\widehat{G}_1\varphi)_\infty \leq (\widehat{G}_1\varphi)_{\overline{B}_n}$ for all n , and thus $\mu_{(\widehat{G}_1\varphi)_\infty}^{str} \in \widehat{S}_{00}^{str}$ exists uniquely by Theorem 0.17.

If Δ is an isolated point then each F_k is compact in E hence \overline{B}_1 as a closed subspace of F_{k_0} is also compact in E . If E is locally compact but not compact and E_Δ is the one point compactification of E then \overline{B}_1 is compact as a bounded closed neighborhood of the compact set $K \cap F_{k_0}$. Hence, in any case \overline{B}_1 is compact in E . Now, similarly to the proof

of [12, Theorem 2.5] we can show that there exists a subsequence $(\mu_{(\widehat{G}_1\varphi)_{\overline{B}_{n_k}}}^{str})_{k \in \mathbb{N}}$ which converges weakly on \overline{B}_1 to $\mu_{(\widehat{G}_1\varphi)_\infty}^{str}$ and that $\text{supp}(\hat{\mu}_\infty^{str}) \subset K \cap F_{k_0}$. It now follows from (3) that

$$0 < \mu_{(\widehat{G}_1\varphi)_\infty}^{str}(K \cap F_{k_0}).$$

□

Remark 0.19 *It is now quite easy to understand why statements in [12] which are based on the non strict version of Theorem 0.18 might be reformulated w.r.t. the strict capacity. As before we refer to these reformulations as strict versions. For instance Theorem 3.1 and Theorem 4.5 in [12] have strict versions.*

Applications

We apply here our previously achieved theoretical results to three different classes of examples. In example (a) we will use Theorem 0.14. In this case the \mathcal{G} of condition SD3 will not be a subset of \mathcal{F} . Example (b), resp. example (c), is in the spirit of Corollary 0.15, resp. Theorem 0.14. In both last cases \mathcal{G} is a subset of \mathcal{F} .

(a) Non-symmetric perturbations through divergence free vector fields

Let U be an open subset of \mathbb{R}^d and $\langle \cdot, \cdot \rangle$ the Euclidean inner product in \mathbb{R}^d with norm $|\cdot| = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$. Let $H_0^{1,2}(U)$ be the closure of $C_0^\infty(U)$ in $L^2(U, dx)$ w.r.t. the norm given by $\int_U (|\nabla u|^2 + u^2) dx$. Let $H_{loc}^{1,2}(U)$ be the space of all elements u such that $u\chi \in H_0^{1,2}(U)$ for all $\chi \in C_0^\infty(U)$. Let $\rho \in H_{loc}^{1,2}(U)$ such that the measure $m := \rho^2 dx$ on U has full support. Similarly to the above $H_{loc}^{1,2}(U, m)$ is the space of all elements u such that $u\chi \in H_0^{1,2}(U, m)$ for all $\chi \in C_0^\infty(U)$ and $H_0^{1,2}(U, m)$ is given as the closure of $C_0^\infty(U)$ in $L^2(U, m)$ w.r.t. the norm given by $\int_U (|\nabla u|^2 + u^2) dm$.

Let $A = (a_{ij})_{1 \leq i, j \leq d}$ with $a_{ij} \in H_{loc}^{1,2}(U, m)$ be a symmetric matrix and locally uniformly strictly elliptic, i.e. for any V relatively compact in U there exists $\nu_V > 0$ such that

$$\nu_V^{-1}|h|^2 \leq \langle A(x)h, h \rangle \leq \nu_V|h|^2 \text{ for all } h \in \mathbb{R}^d, x \in V.$$

Consider the closure (cf. [5, II.2b] for the closability) of

$$\mathcal{E}^0(u, v) = \frac{1}{2} \int \langle A \nabla u, \nabla v \rangle dm ; u, v \in C_0^\infty(U)$$

on $L^2(U, m)$ which we denote by $(\mathcal{E}^0, D(\mathcal{E}^0))$. Let $(L^0, D(L^0))$ be the associated generator. By construction we have that $C_0^\infty(U) \subset D(L^0)$ and $L^0 u = \frac{1}{2} \sum_{i,j=1}^d a_{ij} \partial_i \partial_j u + \frac{1}{2} \sum_{i=1}^d \beta_{A,i}^m \partial_i u$, $u \in C_0^\infty(U)$, where $\beta_A^m = (\beta_{A,1}^m, \dots, \beta_{A,d}^m)$, $\beta_{A,i}^m = \sum_{j=1}^d (\partial_j a_{ij} + 2a_{ij} \partial_j \rho / \rho)$.

Let $\bar{B} = (\bar{b}_1, \dots, \bar{b}_d) \in L^2_{loc}(U, \mathbb{R}^d, m)$, i.e. $\int_V \langle \bar{B}, \bar{B} \rangle dm < \infty$ for all V relatively compact in U , and such that

$$\int \langle \bar{B}, \nabla u \rangle dm = 0 \text{ for all } u \in C_0^\infty(U). \quad (4)$$

For a subspace $W \subset L^2(U, m)$ let W_0 denote the space of all $u \in W$ such that $\text{supp}(|u|m)$ is compact in U and let $W_{0,b} = W_0 \cap W_b$. Define

$$Lu := L^0u + \sum_{i,j=1}^d \bar{b}_i \partial_i u; \quad u \in D(L^0)_{0,b}$$

There is no symmetric bilinear form associated with L . By [11, Theorem 1.5] there exists a closed extension $(\bar{L}, D(\bar{L}))$ of $(L, D(L^0)_{0,b})$ on $L^1(U, m)$ generating a strongly continuous resolvent $(\bar{G}_\alpha)_{\alpha>0}$ on $L^1(U, m)$ which is sub-Markovian. Furthermore we have $D(\bar{L})_b \subset D(\mathcal{E}^0)$ and

$$\mathcal{E}^0(u, v) - \int \langle \bar{B}, \nabla u \rangle v dm = - \int \bar{L}u v dm; \quad u \in D(\bar{L})_b, \quad v \in D(\mathcal{E}^0)_{0,b}.$$

Moreover

$$\mathcal{E}^0(u, u) \leq - \int \bar{L}u u dm; \quad u \in D(\bar{L})_b.$$

Remark 0.20 *Since $-\bar{B}$ satisfies the same assumptions as \bar{B} similarly to $(\bar{L}, D(\bar{L}))$ we can construct a closed extension $(\bar{L}', D(\bar{L}'))$ of $L^0u - \langle \bar{B}, \nabla u \rangle$ generating a strongly continuous resolvent $(\bar{G}'_\alpha)_{\alpha>0}$ which is sub-Markovian. It follows that*

$$\int \bar{G}_\alpha u v dm = \int u \bar{G}'_\alpha v dm \text{ for all } u, v \in L^1(U, m)_b.$$

Note also that similarly to the case of symmetric Dirichlet operators which admit a carré du champ (cf. [2, I.4]) $D(\bar{L})_b$ is an algebra.

Let $(L, D(L))$ be the part of $(\bar{L}, D(\bar{L}))$ in $L^2(U, m)$, i.e. $D(L) = \{f \in D(\bar{L}) \cap L^2(G; m) \mid \bar{L}f \in L^2(G; m)\}$, $Lf := \bar{L}f$, $f \in D(L)$. Let $(\bar{L}', D(\bar{L}'))$ be the adjoint operator of $(L, D(L))$ in $L^2(U, m)$. Let $(G_\alpha)_{\alpha>0}$ (resp. $(G'_\alpha)_{\alpha>0}$) be the associated resolvent to $(L, D(L))$ (resp. $(L', D(L'))$). $(\bar{G}_\alpha)_{\alpha>0}$ (resp. $(\bar{G}'_\alpha)_{\alpha>0}$) and $(G_\alpha)_{\alpha>0}$ (resp. $(G'_\alpha)_{\alpha>0}$) coincide on $L^1(U, m) \cap L^2(U, m)$. Let (\cdot, \cdot) be the inner product in $L^2(U, m)$. According to [10, I.Example 4.9](ii) $(L, D(L))$ is associated to a generalized Dirichlet form on $D(L) \times L^2(U, m) \cup L^2(U, m) \times D(L')$ by

$$\mathcal{E}(u, v) := \begin{cases} (-Lu, v) & \text{for } u \in D(L), \quad v \in L^2(U, m); \\ (u, -L'v) & \text{for } u \in L^2(U, m), \quad v \in D(L'). \end{cases}$$

In this case $\mathcal{F} = D(L)$. Let U_Δ be the one point compactification of U . In the following we will show that \mathcal{E} is a strictly quasi-regular generalized Dirichlet form satisfying SD3.

Hence by Theorem 0.14 there will exist a strictly m -tight Hunt process which is strictly properly associated in the resolvent sense with \mathcal{E} . Since by Remark 0.20 the co-form $\widehat{\mathcal{E}}$ satisfies the same assumptions than \mathcal{E} , analogously one can show the existence of a strictly m -tight Hunt process which is strictly properly associated in the resolvent sense with $\widehat{\mathcal{E}}$.

Theorem 0.21 *There exists a strictly m -tight Hunt process which is strictly properly associated in the resolvent sense with \mathcal{E} .*

Proof By Theorem 0.14 it is enough to show that \mathcal{E} is strictly quasi-regular and satisfies SD3. The latter holds with $\mathcal{G} := \{u \in D(\overline{L})_b \mid \exists f, g \in D(L) \text{ with } u \leq f, -u \leq g\}$. Indeed, by Remark 0.20 \mathcal{G} is an algebra of bounded functions. Furthermore $\mathcal{G} \cap D(L) = D(L)_b$ because $(L, D(L))$ is the part of $(\overline{L}, D(\overline{L}))$ in $L^2(U, m)$. Clearly $D(L)_b$ is dense in $D(L)$ because $(L, D(L))$ is a Dirichlet operator. Furthermore, by [11, Lemma 3.2(iii)] and the strong continuity of $(G_\alpha)_{\alpha>0}$ on $L^1(U, m)$ it follows $\lim_{\alpha \rightarrow \infty} e_{u-\alpha G_\alpha u} + e_{\alpha G_\alpha u-u} = 0$ in \mathcal{H} for every $u \in \mathcal{G}$. It remains hence to show the strict quasi-regularity of \mathcal{E} .

Put $E_k \equiv U \cup \{\Delta\}$, $k \geq 1$. Then $(E_k)_{k \geq 1}$ is a strict \mathcal{E} -nest such that $E_k \cup \{\Delta\}$, $k \geq 1$, is compact in U_Δ . Hence 0.3(i) is satisfied. Since $C_0^\infty(U) \subset D(L)$ we can easily see that 0.3(iii) is satisfied. In [10, Theorem 4.6.] the existence of a m -tight special standard process $\mathbb{M} = (X_t, P_z)$ with lifetime ζ such that $R_\alpha f(z) := E_z \left[\int_0^\infty e^{-s} f(X_s) ds \right]$ is an \mathcal{E} -q.c. m -version of $G_\alpha f$ for any $f \in \mathcal{B}_b \cap L^2(E; m)$, $\alpha > 0$, has already been proved. In order to show 0.3(ii) we will proceed in two steps:

step 1: Let $(V_n)_{n \in \mathbb{N}}$ be an increasing sequence of open subsets relatively compact in U such that $\overline{V}_n \subset V_{n+1}$, $n \in \mathbb{N}$, and $U = \bigcup_{n \in \mathbb{N}} V_n$. Then $R_1^{V_n} f(z) := E_z \left[\int_0^{\sigma_{V_n^c}} e^{-s} f(X_s) ds \right]$ admits a s. \mathcal{E} -q.c. m -version for all $f \in \mathcal{B}_b \cap L^2(E; m)$, $n \in \mathbb{N}$.

For $V \subset U$, V arbitrary but open, let $(\mathcal{E}^{0,V}, D(\mathcal{E}^{0,V}))$ denote the closure of $(\mathcal{E}^0, C_0^\infty(V))$ in $L^2(V; m)$. We first show $R_1^{V_n} f \in D(\mathcal{E}^{0,U}) = D(\mathcal{E}^0)$. Indeed, since

$$\mathcal{E}_1^0(\alpha G_{\alpha+1} R_1^{V_n} f, \alpha G_{\alpha+1} R_1^{V_n} f) \leq \mathcal{E}_1(\alpha G_{\alpha+1} R_1^{V_n} f, R_1^{V_n} f) \leq \mathcal{E}_1(\alpha G_{\alpha+1} R_1 f, R_1 f)$$

is uniformly bounded in α this follows from the usual arguments (see [5, I. Lemma 2.12.]). Hence, there exists $(u_k)_{k \in \mathbb{N}} \subset C_0^\infty(U)$ such that $u_k \rightarrow R_1^{V_n} f$ in $D(\mathcal{E}^0)$ as $k \rightarrow \infty$. Denote by $\|\cdot\|_\infty$ (resp. $|\cdot|_\infty$) the sup-norm (resp. m -essential sup-norm). Let $\xi \in C_0^\infty(\mathbb{R})$, $\xi(x) = x$ on $[-|f|_\infty, |f|_\infty]$, $w \in C_0^\infty(V_{n+2})$, $w(x) = 1$ on \overline{V}_n , $\text{supp}(w) \subset \overline{V}_{n+1}$. Then $w_k := \xi(u_k)w \in C_0^\infty(V_{n+2})$ and $w_k \rightarrow R_1^{V_n} f$ in $L^2(V_{n+2}; m)$ as $k \rightarrow \infty$. By [4, Theorem 1.4.2.(ii)] and since normal contractions operate on $D(\mathcal{E}^{0,V_{n+2}})$

$$\sqrt{\mathcal{E}^{0,V_{n+2}}(w_k, w_k)} = \sqrt{\mathcal{E}^0(w_k, w_k)} \leq \|\xi\|_\infty \sqrt{\mathcal{E}^0(w, w)} + \|w\|_\infty \|\xi'\|_\infty \sqrt{\mathcal{E}^0(u_k, u_k)}$$

is uniformly bounded in k . Hence $R_1^{V_n} f \in D(\mathcal{E}^{0,V_{n+2}})$. For $V \subset U$, V arbitrary but open, let $\text{Cap}^{0,V}$ denote the capacity associated to the regular Dirichlet form $(\mathcal{E}^{0,V}, D(\mathcal{E}^{0,V}))$ as defined in [4, p. 64]. Let Cap_φ^0 be the φ -capacity associated to the quasi-regular Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$. By [4, Theorem 2.1.3.] there exists a $\text{Cap}^{0,V_{n+2}}$ -nest $(F_k)_{k \in \mathbb{N}}$ and a m -version $\widetilde{R}_1^{V_n} f$ of $R_1^{V_n} f$ such that $\widetilde{R}_1^{V_n} f \in C_\infty(\{F_k\})$, i.e.

$$\widetilde{R}_1^{V_n} f|_{F_k \cup \{\Delta\}} \text{ is continuous for all } k.$$

By [4, Theorem 4.4.3.(ii)] $(F_k \cup V_{n+2}^c)_{k \in \mathbb{N}}$ is a $\text{Cap}^{0,U}$ -nest and hence by Remark 0.5(ii) a Cap_φ^0 -nest. Since by [11, Lemma 4.5.] a Cap_φ^0 -nest is an \mathcal{E} -nest (i.e. a Cap_φ -nest) and vice versa we obtain that $(F_k \cup V_{n+2}^c)_{k \in \mathbb{N}}$ is a Cap_φ -nest. Using in particular (2) and Theorem 0.17 we obtain

$$\text{Cap}_{1, \widehat{G}_1 \varphi}(F_k^c \cap V_{n+2}) \leq \int g d\mu_{(\widehat{G}_1 \varphi)_{F_k^c \cap V_{n+2}}}^{str} = \mathcal{E}_1 \left(g, (\widehat{G}_1 \varphi)_{F_k^c \cap V_{n+2}} \right)$$

for any $g \in C_0^\infty(U)$, $g(x) = 1$ on \overline{V}_{n+2} . Since $(F_k \cup V_{n+2}^c)_{k \in \mathbb{N}}$ is an \mathcal{E} -nest $\mathcal{E}_1(g, (\widehat{G}_1 \varphi)_{F_k^c \cap V_{n+2}})$ tends to zero as $k \rightarrow \infty$. Thus $\lim_{k \rightarrow \infty} \text{Cap}_{1, \widehat{G}_1 \varphi}(F_k \cup V_{n+2}^c) = 0$ and $(F_k \cup V_{n+2}^c)_{k \in \mathbb{N}}$ is a strict \mathcal{E} -nest. Since $R_1^{V_n} f$ is a m -version of $\widetilde{R}_1^{V_n} f \in C_\infty(\{F_k\})$, and $R_1^{V_n} f(z) = 0$ for every $z \in \overline{V}_n^c$, by [4, Lemma 2.1.4.] we may assume that $\widetilde{R}_1^{V_n} f(z) = 0$ for every $z \in \overline{V}_n^c$. Hence

$$\widetilde{R}_1^{V_n} f|_{F_k \cup V_{n+2}^c \cup \{\Delta\}} \text{ is continuous for all } k.$$

The last just means that $\widetilde{R}_1^{V_n} f \in C_\infty(\{F_k \cup V_{n+2}^c\})$, thus $\widetilde{R}_1^{V_n} f$ is strictly \mathcal{E} -q.c.

step 2: $R_1 f$ admits a s. \mathcal{E} -q.c. m -version for all $f \in \mathcal{B}_b \cap L^2(E; m)$.

Let $m \geq n$, $U_{n,m,\varepsilon} := \{|\widetilde{R}_1^{V_n} f - \widetilde{R}_1^{V_m} f| > \varepsilon\}$. Then

$$\begin{aligned} \text{Cap}_{1, \widehat{G}_1 \varphi}(U_{n,m,\varepsilon}) &\leq \varepsilon^{-1} \lim_{\alpha \rightarrow \infty} \mathcal{E}_1(|\widetilde{R}_1^{V_n} f - \widetilde{R}_1^{V_m} f|, (\widehat{G}_1 \varphi)_{U_{n,m,\varepsilon}}^\alpha) \\ &\leq \varepsilon^{-1} \mathcal{E}_1(\alpha G_{\alpha+1}(E.[\int_{\sigma_{V_n^c}}^\infty e^{-s}|f|(X_s) ds]), (\widehat{G}_1 \varphi)_{U_{n,m,\varepsilon}}) \\ &\leq \varepsilon^{-1} (E.[\int_{\sigma_{V_n^c}}^\infty e^{-s}|f|(X_s) ds], \varphi). \end{aligned}$$

Since $(\overline{V}_n)_{n \in \mathbb{N}}$ is a Cap_φ^0 -nest it is an \mathcal{E} -nest by [11, Lemma 4.5.]. Hence $P_{\varphi m}(\lim_{n \rightarrow \infty} \sigma_{V_n^c} \geq \zeta) = 1$. Thus, we may choose a subsequence such that

$$\text{Cap}_{1, \widehat{G}_1 \varphi}(\{|\widetilde{R}_1^{V_{n_i+1}} f - \widetilde{R}_1^{V_{n_i}} f| > 2^{-i}\}) \leq 2^{-i} \text{ for every } i.$$

Set $A_k := \bigcup_{i \geq k} \{|\widetilde{R}_1^{V_{n_i+1}} f - \widetilde{R}_1^{V_{n_i}} f| > 2^{-i}\}$. Let $(F'_k)_{k \in \mathbb{N}}$ be a strict \mathcal{E} -nest such that $\widetilde{R}_1^{V_n} f \in C_\infty(\{F'_k\})$, $n \in \mathbb{N}$. Then $F_k := F'_k \cap A_k^c$, $k \in \mathbb{N}$, is a strict \mathcal{E} -nest and $\widetilde{R}_1^{V_{n_i}} f$ converges uniformly on $F_k \cup \{\Delta\}$ as $i \rightarrow \infty$. It then suffices to set

$$\widetilde{R}_1 f(z) := \begin{cases} \lim_{i \rightarrow \infty} \widetilde{R}_1^{V_{n_i}} f(z) & \text{for } z \in \bigcup_{k \in \mathbb{N}} F_k, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\widetilde{R}_1 f$ is a s. \mathcal{E} -q.c. m -version of $G_1 f$. Since $G_1 \mathcal{H}_b \subset D(L)$ dense we obtain 0.3(ii). \square

(b) Monotonely moving domains

For the reader's convenience let us here summarize what is developed in [8]. Anything

corresponding to this framework that is not proved here is rigorously shown in [8].

If U is an open set in some Euclidean space let $C^k(U)$ (resp. $C_0^k(U)$) denote the k -times continuously differentiable functions (resp. with compact support) in U . For $F \subset \mathbb{R}^n$ closed let $C^k(F)$ (resp. $C_0^k(F)$) denote the restrictions to F of functions in $C^k(\mathbb{R}^n)$ (resp. $C_0^k(\mathbb{R}^n)$). For $V \subset \mathbb{R}^{d+1}$ with usual Euclidean metric let $C_0(V)$ be the continuous functions on V with compact support in V .

Let $A \subset \mathbb{R}^n$, $n \geq 1$, be Borel measurable. We denote by 1_A the characteristic function of A . For a Borel measure ν on A , $p \in [1, \infty]$, we denote by $L_{loc}^p(A, \nu)$ the space of all Borel measurable F on A for which $1_K F \in L^p(A, \nu)$ for any in compact subset K of A , and $L^p(A, \nu)$ denotes as usual the p -fold integrable functions on A w.r.t. ν if $p < \infty$, and the ν -essentially bounded functions on A if $p = \infty$. A function is called locally bounded, if it is bounded on compact sets. We denote by $|\cdot|$ the Euclidean norm, by $|\cdot|_\infty$ the sup norm in the corresponding space, i.e. either the point-wise sup norm or the essential sup norm w.r.t. a measure. Let $A \subset E$. We set $A^c := E \setminus A$, i.e. the complement of A in E .

Let $d \geq 1$, and $dy = dx ds$ be the Lebesgue measure on \mathbb{R}^{d+1} , where dx is the Lebesgue measure on \mathbb{R}^d , ds the Lebesgue measure on \mathbb{R} . Let $E \subset [0, \infty) \times \mathbb{R}^d$ be a closed space-time domain where each $E_t := \{x \in \mathbb{R}^d | (t, x) \in E\}$, $t \in [0, \infty)$, is a closed, not necessarily bounded subset of the Euclidean space \mathbb{R}^d . We assume that E_t moves monotonely, i.e. either $E_s \subset E_t$, $\forall s \leq t$, or $E_s \supset E_t$, $\forall s \leq t$, $s, t \in [0, \infty)$. Let finally the boundary ∂E of E have zero $(d+1)$ -dimensional Lebesgue measure.

Let $a_{ij} = a_{ji}, \rho : E \rightarrow \mathbb{R}$, $1 \leq i, j \leq d$, be measurable functions with $\rho \in L_{loc}^1(E, dy)$, $\rho > 0$ dy -a.e. on E . We let $A = (a_{ij})_{1 \leq i, j \leq d}$ be the symmetric matrix given by the functions a_{ij} . For a function F we denote by $\partial_i F$ the i -th partial distributional derivative in space direction, by ∇F the space gradient of F . Because $dy(\partial E) = 0$ we will identify $L^p(E, dy)$ and $L^p(E \setminus \partial E, dy)$. In connection with this we will consider a function $F : E \rightarrow \mathbb{R}$ as a function $F : E \setminus \partial E \rightarrow \mathbb{R}$ and vice versa. We assume that either for some constant $C \geq 1$

$$(A1) \quad C^{-1} \sum_{i=1}^d \xi_i^2 \leq \sum_{i,j=1}^d a_{ij}(y) \xi_i \xi_j \leq C \sum_{i=1}^d \xi_i^2; \quad \forall (\xi_1, \dots, \xi_d) \in \mathbb{R}^d, y \in E,$$

or

$$(A1') \quad E = [0, \infty) \times \mathbb{R}^d, a_{ij} \text{ is locally bounded, } \partial_j a_{ij} \in L_{loc}^2(E, \rho dy), 1 \leq i, j \leq d, \text{ and}$$

$$\sum_{i,j=1}^d a_{ij}(y) \xi_i \xi_j \geq 0 \quad \forall (\xi_1, \dots, \xi_d) \in \mathbb{R}^d, y \in E.$$

Let $\mathcal{H} := L^2(E, \rho dy)$ with norm $|F|_{\mathcal{H}} := \sqrt{\int_E F(y)^2 \rho(y) dy}$, and inner product $(\cdot, \cdot)_{\mathcal{H}}$. Let $c \in L^\infty(E, \rho dy)$, $c \geq 0$ ρdy -a.e, and consider the following bilinear form

$$\begin{aligned} \mathcal{A}(F, G) &:= \frac{1}{2} \sum_{i,j=1}^d \int_0^\infty \int_{E_s} a_{ij}(s, x) \partial_i F(s, x) \partial_j G(s, x) \rho(s, x) dx ds \\ &\quad + \int_0^\infty \int_{E_s} c(s, x) F(s, x) G(s, x) \rho(s, x) dx ds; \quad F, G \in C_0^1(E) \end{aligned}$$

Obviously $C_0^1(E) \subset \mathcal{H}$ dense, so \mathcal{A} is densely defined on \mathcal{H} . We assume from now on that it is closable in \mathcal{H} , i.e. if $(F_n)_{n \in \mathbb{N}} \subset C_0^1(E)$ is an \mathcal{A} -Cauchy sequence converging to 0 in \mathcal{H} , then we must have $\lim_{n \rightarrow \infty} \mathcal{A}(F_n, F_n) = 0$. We remark that conditions on ρ in order to obtain closability are very mild. In case of (A1) it is for instance enough to have a Hamza type condition for ρ . In case of (A1') it is enough to have $\rho = \varphi^2$ with $\partial_i \varphi \in L_{loc}^2([0, \infty) \times \mathbb{R}^d, dx ds)$, $1 \leq i \leq d$. The closure $(\mathcal{A}, \mathcal{V})$ of $(\mathcal{A}, C_0^1(E))$ in \mathcal{H} is a regular symmetric Dirichlet form on \mathcal{H} (cf. [4]). Let $|F|_{\mathcal{V}}$ be the corresponding Dirichlet norm, i.e. $|F|_{\mathcal{V}} := (\mathcal{A}(F, F) + |F|_{\mathcal{H}}^2)^{\frac{1}{2}}$, and $\mathcal{A}_\alpha(\cdot, \cdot) := \mathcal{A}(\cdot, \cdot) + \alpha(\cdot, \cdot)_{\mathcal{H}}$.

At this point it will be convenient to recall some basic definitions of semigroups and generators. A family $(T_t)_{t \geq 0}$ of bounded linear operators from a Banach space \mathcal{W} into \mathcal{W} is called a C_0 -semigroup on \mathcal{W} if $T_0 = id_{\mathcal{W}}$, $T_t T_s = T_{t+s}$ for all $t, s \geq 0$, and $\lim_{t \rightarrow 0} T_t F = F$ in \mathcal{W} for all $F \in \mathcal{W}$. It is called a C_0 -semigroup of contractions on \mathcal{W} if additionally $|T_t F|_{\mathcal{W}} \leq |F|_{\mathcal{W}}$ for any $F \in \mathcal{W}$, $t \geq 0$.

Let $(T_t)_{t \geq 0}$ be a C_0 -semigroup on \mathcal{W} . The linear operator $\mathcal{L}F := \lim_{t \rightarrow 0} \frac{1}{t}(T_t F - F)$ with domain

$$D(\mathcal{L}, \mathcal{W}) = \{F \in \mathcal{W} \mid \exists \lim_{t \rightarrow 0} \frac{1}{t}(T_t F - F) \text{ in } \mathcal{W}\}$$

is called the (*infinitesimal*) generator of $(T_t)_{t \geq 0}$.

A bounded linear operator $U : \mathcal{W} \rightarrow \mathcal{W}$ is called *sub-Markovian* if $0 \leq UF \leq 1$ whenever $0 \leq F \leq 1$, $F \in \mathcal{W}$. A C_0 -semigroup $(T_t)_{t \geq 0}$ on \mathcal{H} is called *sub-Markovian* if T_t is sub-Markovian for all $t \geq 0$.

Let $\kappa : [0, \infty) \times \mathbb{R}^+ \rightarrow [0, \infty)$, $\kappa \in C^1([0, \infty) \times \mathbb{R}^+)$, $\kappa(s, 0) = s$. We assume $(\kappa(s, t), x) \in E$ whenever $(s, x) \in E$, $t \geq 0$, and

$$(A2) \quad \forall s \in [0, \infty), r, t \geq 0 : \kappa(s, r+t) = \kappa(\kappa(s, r), t).$$

We will now define the C_0 -semigroup corresponding to the perturbation of \mathcal{A} . For $G \in C_0^1(E)$, $t \geq 0$, let first

$$U_t G(y) := G(\kappa(s, t), x) \text{ for every } y = (s, x) \in E, t \geq 0.$$

(A2) implies the semigroup property of $(U_t)_{t \geq 0}$. In order to have $U_t C_0^1(E) \subset C_0^1(E)$, $t \geq 0$, we assume that $G(\kappa(\cdot, t), \cdot) \in C_0^1(E)$ whenever $G \in C_0^1(E)$, $t \geq 0$. Let $G \in C_0^1(E)$. We suppose for any $T > 0$ the existence of a compact set K such that the support of $U_t G - G$, $t \in [0, T]$, is located in K . We assume further

$$(A3) \quad \int_E G(\kappa(s, t), x) \rho(s, x) dx ds \leq \int_E G(s, x) \rho(s, x) dx ds \quad \forall G \in C_0^1(E)^+, t \geq 0.$$

and the existence of constants $M \geq 1, \omega \geq 0$, such that

$$(A4) \quad \mathcal{A}(U_t G, U_t G) \leq M e^{\omega t} \mathcal{A}(G, G) \quad \forall G \in C_0^1(E), t \geq 0.$$

Note that (A3), resp. (A4), is in particular a condition on ρ , resp. c . Concrete examples for $(a_{ij})_{1 \leq i, j \leq d}, E, \rho, \kappa, c$ satisfying (A1), (A2), (A3), (A4), or (A1'), (A2), (A3), and (A4), can be found in [8].

Using (A3) we obtain $|U_t G|_{\mathcal{H}} \leq |G|_{\mathcal{H}}$ for any $G \in C_0^1(E)$. Since $C_0^1(E) \subset \mathcal{H}$ dense, U_t induces a (linear) contraction on \mathcal{H} which we also denote by U_t . For $s \in [0, \infty)$ let

$$\kappa'(s, 0) := \lim_{t \rightarrow 0} \frac{\kappa(s, t) - \kappa(s, 0)}{t}.$$

It can easily be shown that $(U_t)_{t \geq 0}$ is a sub-Markovian C_0 -semigroup of contractions on \mathcal{H} , and that the corresponding generator $(\Lambda, D(\Lambda, \mathcal{H}))$ is an extension of $\Lambda F = \kappa'(\cdot, 0) \partial_t F$, $F \in C_0^1(E)$. Furthermore, by (A4) $(U_t)_{t \geq 0}$ can be restricted to a C_0 -semigroup on \mathcal{V} .

We denote by $(\Lambda, D(\Lambda, \mathcal{V}))$ the generator corresponding to the restriction of $(U_t)_{t \geq 0}$ on \mathcal{V} , and let $(\widehat{U}_t)_{t \geq 0}$ be the adjoint semigroup of $(U_t)_{t \geq 0}$ in \mathcal{H} , i.e. \widehat{U}_t is the adjoint operator (on \mathcal{H}) of U_t for every $t \geq 0$.

Although in general $(\widehat{U}_t)_{t \geq 0}$ cannot be restricted to a C_0 -semigroup on \mathcal{V} we have that $(\widehat{U}_t)_{t \geq 0}$ is sub-Markovian. Indeed, this is an easy consequence of (A3). Thus $(\Lambda, D(\Lambda, \mathcal{H}))$ is the generator of a C_0 -semigroup of contractions on \mathcal{H} that can be restricted to a C_0 -semigroup on \mathcal{V} . Hence [10, I.Lemma 2.3.] implies that

$$\Lambda : D(\Lambda, \mathcal{H}) \cap \mathcal{V} \rightarrow \mathcal{V}'$$

is closable as an operator from \mathcal{V} to \mathcal{V}' . Let (Λ, \mathcal{F}) denote its closure. For the following up to the definition of the generalized Dirichlet form see [10]. \mathcal{F} is a real Hilbert space with norm

$$|F|_{\mathcal{F}} := \left(|F|_{\mathcal{V}}^2 + |\Lambda F|_{\mathcal{V}'}^2 \right)^{\frac{1}{2}}.$$

$(\widehat{U}_t)_{t \geq 0}$ can be extended to a C_0 -semigroup on \mathcal{V}' . The corresponding generator $(\widehat{\Lambda}, D(\widehat{\Lambda}, \mathcal{V}'))$ is the dual operator of $(\Lambda, D(\Lambda, \mathcal{V}))$. $\widehat{\mathcal{F}} := D(\widehat{\Lambda}, \mathcal{V}') \cap \mathcal{V}$ is a real Hilbert space with norm

$$|F|_{\widehat{\mathcal{F}}} := \left(|F|_{\mathcal{V}}^2 + |\widehat{\Lambda} F|_{\mathcal{V}'}^2 \right)^{\frac{1}{2}}.$$

Let $\langle \cdot, \cdot \rangle$ be the dualization between \mathcal{V}' and \mathcal{V} . The *generalized Dirichlet form* is now given through

$$\mathcal{E}(F, G) := \begin{cases} \mathcal{A}(F, G) - \langle \Lambda F, G \rangle & \text{for } F \in \mathcal{F}, G \in \mathcal{V} \\ \mathcal{A}(F, G) - \langle \widehat{\Lambda} G, F \rangle & \text{for } G \in \widehat{\mathcal{F}}, F \in \mathcal{V}. \end{cases}$$

Note that $\langle \cdot, \cdot \rangle$ when restricted to $\mathcal{H} \times \mathcal{V}$ coincides with $(\cdot, \cdot)_{\mathcal{H}}$. Let $\mathcal{E}_\alpha(F, G) := \mathcal{E}(F, G) + \alpha(F, G)_{\mathcal{H}}$ for $\alpha > 0$. It follows, from [10, I. Proposition 3.4, p.19], that for all $\alpha > 0$ there exist continuous, linear bijections $W_\alpha : \mathcal{V}' \rightarrow \mathcal{F}$ and $\widehat{W}_\alpha : \mathcal{V}' \rightarrow \widehat{\mathcal{F}}$ such that $\mathcal{E}_\alpha(W_\alpha F, G) = \langle F, G \rangle = \mathcal{E}_\alpha(G, \widehat{W}_\alpha F)$, $\forall F \in \mathcal{V}', G \in \mathcal{V}$. Furthermore $(W_\alpha)_{\alpha>0}$ and $(\widehat{W}_\alpha)_{\alpha>0}$ satisfy the resolvent equation

$$W_\alpha - W_\beta = (\beta - \alpha)W_\alpha W_\beta \quad \text{and} \quad \widehat{W}_\alpha - \widehat{W}_\beta = (\beta - \alpha)\widehat{W}_\alpha \widehat{W}_\beta.$$

Restricting W_α to \mathcal{H} we get a strongly continuous contraction resolvent $(G_\alpha)_{\alpha>0}$ on \mathcal{H} satisfying $\lim_{\alpha \rightarrow \infty} \alpha G_\alpha F = F$ in \mathcal{V} for all $F \in \mathcal{V}$. The resolvent $(G_\alpha)_{\alpha>0}$ is called the *resolvent associated with \mathcal{E}* . Let $(\widehat{G}_\alpha)_{\alpha>0}$ be the adjoint of $(G_\alpha)_{\alpha>0}$ in \mathcal{H} . $(\widehat{G}_\alpha)_{\alpha>0}$ is called the *coresolvent associated with \mathcal{E}* .

The resolvent $(G_\alpha)_{\alpha>0}$ and the coresolvent $(\widehat{G}_\alpha)_{\alpha>0}$ associated with \mathcal{E} are sub-Markovian, i.e. $0 \leq \alpha G_\alpha F \leq 1$ (resp. $0 \leq \alpha \widehat{G}_\alpha F \leq 1$) whenever $0 \leq F \leq 1$, $F \in \mathcal{H}$, $\alpha > 0$. This follows from [10, I. Proposition 4.3] since $(U_t)_{t \geq 0}$, $(\widehat{U}_t)_{t \geq 0}$, are sub-Markovian, thus semigroups corresponding to Dirichlet operators. We have now the following:

Lemma 0.22 *We have $C_0^1(E) \subset \mathcal{F}$ dense.*

Proof Since $D(\Lambda, \mathcal{V}) \subset \mathcal{F}$ dense by [10, I.2 Lemma 2.5.] it is enough to show that $C_0^1(E) \subset D(\Lambda, \mathcal{V})$ dense w.r.t. the graph norm. Since $(\Lambda, D(\Lambda, \mathcal{V}))$ is the generator of the C_0 -semigroup $(U_t)_{t \geq 0}$ on \mathcal{V} , $U_t F \in C_0^1(E)$, for any $F \in C_0^1(E)$, $t \geq 0$, i.e. $C_0^1(E)$ is invariant under $(U_t)_{t \geq 0}$, and $C_0^1(E) \subset \mathcal{V}$ dense, we can apply [3, 1.7 Proposition] in order to obtain that $C_0^1(E) \subset D(\Lambda, \mathcal{V})$ dense w.r.t. the graph norm. □

Since $(\mathcal{E}, \mathcal{F})$ is regular, i.e. $C_0(E) \cap \mathcal{F}$ is dense in $C_0(E)$ w.r.t. the uniform norm as well as in \mathcal{F} , it follows that $(\mathcal{E}, \mathcal{F})$ is a strictly quasi-regular generalized Dirichlet form on E (see Proposition 0.4). On the other hand we can see from Lemma 0.22 that $\mathcal{G} := C_0^1(E)$ satisfies the condition SD3. Hence, the following theorem is an immediate consequence of Corollary 0.15 or Theorem 0.14.

Theorem 0.23 *There exists a Hunt process $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t \geq 0}, (Y_t)_{t \geq 0}, (P_y)_{y \in E_\Delta})$ with state space E , life time ζ , such that $R_\alpha F(y) := \int_0^\infty \int_\Omega e^{-\alpha t} F(Y_t(\omega)) dP_y dt$ is a s. \mathcal{E} -q.c. p.d.y-version of $G_\alpha F$ for any $\alpha > 0$ and any $F \in \mathcal{H}_b$.*

Using Theorem 0.18 the proof of the following theorem is similar to [13, Theorem 3.3], or [14, Theorem 2.5]. We therefore omit it. Note however that due to the here refined potential theory (see Remark 0.19) we obtain statements w.r.t. the strict capacity defined above.

Theorem 0.24 *There exists a strictly \mathcal{E} -exceptional set $N \subset E$ such that*

$$P_y(t \mapsto Y_t \text{ is continuous on } [0, \zeta]) = 1 \text{ for every } y \in E \setminus N.$$

The generator corresponding to the diffusion Y_t has the following informal expression in the interior of E :

$$LF = \frac{1}{2}\rho^{-1} \sum_{i,j=1}^d \partial_i(a_{ij}\rho\partial_j F) + cF + \kappa'(\cdot, 0)\partial_t F.$$

(c) Time inhomogenous parametrized diffusions on infinite dimensional space

Let E be a separable real Banach space with norm $\|\cdot\|_E$. Let $\mathcal{B}(E)$ denote its Borel σ -algebra and let E' with operator norm $\|\cdot\|_{E'}$ denote the dual of E . In particular $\mathcal{B}(E) = \sigma(E')$. Let $(H, \langle \cdot, \cdot \rangle_H)$ a separable real Hilbert space such that $H \subset E$ densely and continuously. Identifying H with its topological dual H' we obtain that $E' \subset H \subset E$ densely and continuously. The corresponding dualization ${}_{E'}\langle \cdot, \cdot \rangle_E : E' \times E \rightarrow \mathbb{R}$ restricted to $E' \times H$ coincides with $\langle \cdot, \cdot \rangle_H$.

Let $C_{0,b}^1(\mathbb{R} \times \mathbb{R}^m)$ denote the one times continuously differentiable functions on $\mathbb{R} \times \mathbb{R}^m$ with all partial derivatives in space bounded and compact support in time. Let $C_{0,b}^1([0, \infty) \times \mathbb{R}^m)$ denote the restrictions to $[0, \infty) \times \mathbb{R}^m$ of the functions in $C_{0,b}^1(\mathbb{R} \times \mathbb{R}^m)$. Let us define the finitely based time-dependent functions as

$$\mathcal{FT}C_{0,b}^1 := \{f(t, l_1(z), \dots, l_m(z)) \mid m \in \mathbb{N}, f \in C_{0,b}^1([0, \infty) \times \mathbb{R}^m), l_1, \dots, l_m \in E'\}.$$

For $u \in \mathcal{FT}C_{0,b}^1$, $k \in E$, let

$$\frac{\partial u}{\partial k}(t, z) := \frac{d}{ds}u(t, z + sk) \Big|_{s=0}, (t, z) \in [0, \infty) \times E.$$

If $u(t, z) = f(t, l_1(z), \dots, l_m(z))$, then

$$\frac{\partial u}{\partial k}(t, z) = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(t, l_1(z), \dots, l_m(z))_{E'} \langle l_i, k \rangle_E.$$

Thus, if $k \in H$, there exists by the Riesz lemma a unique element $\nabla_H u(t, z) \in H$ with

$$\langle \nabla_H u(t, z), k \rangle_H = \frac{\partial u}{\partial k}(t, z).$$

Let ds denote the Lebesgue measure on $[0, \infty)$. Let μ be finite positive measure with full support on $(E, \mathcal{B}(E))$, i.e. we assume that $\mu(U) > 0$ for any non-empty open subset $U \subset E$. Let $\rho : [0, \infty) \times E \rightarrow \mathbb{R}$ be $\mathcal{B}([0, \infty)) \otimes \mathcal{B}(E)$ -measurable, $\rho > 0$ $d\mu ds$ -a.e, and $\int_K \int_E \rho(s, z)\mu(dz)ds < \infty$ for any $K \subset [0, \infty)$, K compact.

Let $C_0^1([0, \infty))$ consist of the restrictions to $[0, \infty)$ of the one times continuously differentiable functions with compact support on \mathbb{R} . Note that $\mathcal{FT}C_{0,b}^1$ contains functions of the form fg , $f \in C_0^1([0, \infty))$, $g \in \mathcal{FC}_b^\infty$, where \mathcal{FC}_b^∞ are the so-called finitely based smooth

functions (see [5, II.3, p.54]). Since \mathcal{FC}_b^∞ separates by Hahn-Banach theorem the points of E , it is clear that $\mathcal{FTC}_{0,b}^1$ separates the points of $[0, \infty) \times E$. Furthermore $\mathcal{FTC}_{0,b}^1$ is an algebra of functions. Thus, by monotone class arguments

$$\mathcal{FTC}_{0,b}^1 \subset \mathcal{H} := L^2([0, \infty) \times E, \rho d\mu ds) \text{ densely.}$$

Let $c \in L^\infty([0, \infty) \times E, \rho d\mu ds)$, $c \geq 0$, and assume that the following densely defined positive definite symmetric bilinear form

$$\begin{aligned} \mathcal{A}(u, v) &:= \frac{1}{2} \int_0^\infty \int_E \langle \nabla_H u(s, z), \nabla_H v(s, z) \rangle_H \rho(s, z) \mu(dz) ds \\ &+ \int_0^\infty \int_E c(s, z) u(s, z) v(s, z) \rho(s, z) \mu(dz) ds; \quad u, v \in \mathcal{FTC}_{0,b}^1 \end{aligned}$$

is closable on \mathcal{H} . The closure $(\mathcal{A}, \mathcal{V})$ is then a symmetric Dirichlet form.

Remark 0.25 *The closability of $(\mathcal{A}, \mathcal{FTC}_{0,b}^1)$ in \mathcal{H} is fulfilled, if there exists an orthonormal basis $K_0 \subset H$, and $\beta_k \in \mathcal{H}$, $k \in K_0$, such that for all u, v in $\mathcal{FTC}_{0,b}^1$, and $k \in K_0$,*

$$\int_0^\infty \int_E \frac{\partial u}{\partial k}(s, z) \rho(s, z) \mu(dz) ds = - \int_0^\infty \int_E u(s, z) \beta_k(s, z) \rho(s, z) \mu(dz) ds.$$

Indeed, this is an easy consequence of the results obtained in [5, II.3].

Let us now define the semigroup corresponding to the perturbation of \mathcal{A} . Let $d \geq 0$ be a constant. For $u(s, z) = f(s, l_1(z), \dots, l_m(z))$ let first either

$$U_t u(s, z) := f(se^{dt}, l_1(z), \dots, l_m(z)); \quad t \geq 0,$$

or

$$U_t u(s, z) := u(s + t, z); \quad t \geq 0.$$

It is easily seen that $(U_t)_{t \geq 0}$ defined in such a way has the semigroup property on $\mathcal{FTC}_{0,b}^1$, i.e. U_0 is the identity operator, and

$$U_{t'+t} u = U_{t'} U_t u; \quad u \in \mathcal{FTC}_{0,b}^1, \quad t', t \geq 0.$$

Also $(U_t)_{t \geq 0}$ leaves $\mathcal{FTC}_{0,b}^1$ invariant, i.e. $U_t u \in \mathcal{FTC}_{0,b}^1$ whenever $u \in \mathcal{FTC}_{0,b}^1$, $t \geq 0$. Furthermore $0 \leq U_t u \leq 1$ whenever $u \in \mathcal{FTC}_{0,b}^1$, $0 \leq u \leq 1$, $t \geq 0$ ("submarkovianity"). Let us make the assumptions

$$\rho(s, \cdot) \leq \rho(t, \cdot) \quad \mu\text{-a.e.} \quad \forall s \leq t. \quad (5)$$

and

$$c(s, \cdot) \rho(s, \cdot) \leq c(t, \cdot) \rho(t, \cdot) \quad \mu\text{-a.e.} \quad \forall s \leq t. \quad (6)$$

In case of choice of the first semigroup we have the following contraction property for $u \in \mathcal{FTC}_{0,b}^1$

$$\begin{aligned} \int_0^\infty \int_E U_t u(s, z)^2 \rho(s, z) \mu(dz) ds &= \int_0^\infty \int_E f(se^{dt}, l_1(z), \dots, l_m(z))^2 \rho(s, z) \mu(dz) ds \\ &= \int_0^\infty \int_E u(s, z)^2 e^{-dt} \rho(se^{-dt}, z) \mu(dz) ds \\ &\leq \int_0^\infty \int_E u(s, z)^2 \rho(s, z) \mu(dz) ds \end{aligned}$$

where we used in particular (5) for the last inequality. In the second case $U_t u(s, z) := u(s + t, z)$ the contraction property is even easier to see. Since $\mathcal{FTC}_{0,b}^1 \subset \mathcal{H}$ densely, $(U_t)_{t \geq 0}$ above induces a sub-Markovian semigroup of contractions on \mathcal{H} which we also denote by $(U_t)_{t \geq 0}$. This semigroup is easily seen to be a C_0 -semigroup on \mathcal{H} . Indeed, in the first case we have

$$|U_t u(s, z) - u(s, z)| = |f(se^{dt}, l_1(z), \dots, l_m(z)) - f(s, l_1(z), \dots, l_m(z))|$$

and $|U_t u(s, z) - u(s, z)|$ converges pointwise to zero as $t \rightarrow 0$. Let the support of f be contained in $K \times E$, with $K \subset [0, \infty)$, K compact. For any $t \geq 0$, the support of $f(se^{dt}, l_1(z), \dots, l_m(z))$ is also contained in $K \times E$ since d is a positive constant. Therefore $|U_t u - u|$ is bounded by the integrable function $2|f|_\infty \mathbf{1}_{K \times E}$. As a consequence $(U_t)_{t \geq 0}$ is strongly continuous on \mathcal{H} . In the case $U_t u(s, z) := u(s + t, z)$ the strong continuity is again even easier to see. One can easily see that the corresponding generator $(\Lambda, D(\Lambda, \mathcal{H}))$ on \mathcal{H} is an extension of

$$\Lambda u(t, z) = td\partial_t u(t, z)$$

on $\mathcal{FTC}_{0,b}^1$ in the first case and of

$$\Lambda u(t, z) = \partial_t u(t, z)$$

in the second case.

Furthermore, $(U_t)_{t \geq 0}$ can be restricted to a C_0 -semigroup on \mathcal{V} . The last clearly follows from

$$|U_t u|_{\mathcal{V}}^2 \leq |u|_{\mathcal{V}}^2$$

which is easily derived from (5) and (6). As in example (b) we obtain a generalized Dirichlet form corresponding to \mathcal{A} and Λ . We maintain the notations of example (b). Let us show that the adjoint semigroup $(\widehat{U}_t)_{t \geq 0}$ on \mathcal{H} is sub-Markovian. Therefore, it suffices to show that for $u \in \mathcal{FTC}_{0,b}^1$, $0 \leq u \leq 1$, and $t \geq 0$, we have $0 \leq \widehat{U}_t u \leq 1$. Clearly $0 \leq \widehat{U}_t u$, because for any $v \in \mathcal{FTC}_{0,b}^1$, $v \geq 0$, we have $U_t v \geq 0$, and hence

$$\int_0^\infty \int_E \widehat{U}_t u v \rho d\mu ds = \int_0^\infty \int_E u U_t v \rho d\mu ds \geq 0.$$

On the other hand for any $v \in \mathcal{FTC}_{0,b}^1$, $v \geq 0$,

$$\begin{aligned} & \int_0^\infty \int_E (1 - \widehat{U}_t u) v \rho d\mu ds \geq \int_0^\infty \int_E (v - U_t v) \rho d\mu ds \\ & = \int_0^\infty \int_E v (\rho(s, z) - \rho(se^{-dt}, z)) \mu(dz) ds \geq 0 \end{aligned}$$

by (5), and the result follows. Exactly as in example (b) we obtain $\mathcal{FTC}_{0,b}^1 \subset \mathcal{F}$ densely, where \mathcal{F} is the domain of the corresponding generalized Dirichlet form.

Proposition 0.26 *Let $\Lambda = \partial_t$. Then $(\mathcal{E}, \mathcal{F})$ is quasi-regular.*

Proof Since $\mathcal{FTC}_{0,b}^1 \subset \mathcal{F}$ densely, property (ii) of Definition 0.3 is satisfied. Using in particular Hahn-Banach theorem, there exists $l_n \in E'$, $n \in \mathbb{N}$, such that $\{\sin(l_n) | n \in \mathbb{N}\}$ separates the points of E . Let $i(s) = s$, and $f_n \in C_0^\infty[0, \infty)$, $f_n \equiv 1$ on $[0, n]$. Let $g_n := i \cdot f_n$. Then $\{g_m \cdot \sin(l_n) | m, n \in \mathbb{N}\}$ is a countable set of functions in \mathcal{F} which separates the points of $[0, \infty) \times E$. Thus property (iii) of Definition 0.3 is satisfied. In order to show that property (i) of Definition 0.3 is satisfied we proceed as follows:

Since E is separable we may choose a countable dense set $\{y_m | m \in \mathbb{N}\}$ in E . By the Hahn-Banach theorem we can find $l_m \in E'$ so that $\|l_m\|_{E'} = 1$ and $l_m(y_m) = \|y_m\|_E$. Then $\|z\|_E = \sup_{m \in \mathbb{N}} l_m(z)$. Let $\varphi \in C_b^\infty(\mathbb{R})$, φ increasing with $\varphi(t) = t$ for all $t \in [-1, 1]$, $|\varphi'|_\infty \leq 1$, and for any $m \in \mathbb{N}$ define $v_m : E \rightarrow \mathbb{R}$ by $v_m := \varphi(\|z - y_m\|_E)$. Then

$$w_n := \inf_{m \leq n} v_m, \quad z \in E, n \in \mathbb{N},$$

is continuous and positive. In the same way as before choose $\{s_m | m \in \mathbb{N}\}$ which is dense in $[0, \infty)$. Put $g_n(t) := \inf_{m \leq n} \varphi(|t - s_m|)$, and $u_n := \max(g_n, w_n) = \frac{1}{2}(g_n + w_n + |g_n - w_n|)$. We fix $0 < \varepsilon < 1$. Then

$$F_n := \{u_n \leq \varepsilon\} = \{g_n \leq \varepsilon\} \times \{w_n \leq \varepsilon\} = \bigcup_{m \leq n} \{|t - s_m| \leq \varepsilon\} \times \bigcup_{m \leq n} \{\|z - y_m\|_E \leq \varepsilon\}$$

is compact and increasing in n since $g_n, w_n \downarrow 0$. One can show $u_n G_1 \varphi \in \mathcal{F}$ for any $n \in \mathbb{N}$. We have $u_n G_1 \varphi \rightarrow 0$ in \mathcal{H} . Additionally $|u_n G_1 \varphi|_{\mathcal{V}}$, $|\partial_t(u_n G_1 \varphi)|_{\mathcal{V}'}$, stays bounded in n . Hence by Banach/Alaoglu/Saks there exists a subsequence n_k , such that $\frac{1}{n} \sum_{k=1}^n u_{n_k} G_1 \varphi$ converges strongly to zero in \mathcal{F} . But since u_n is decreasing we obtain

$$u_{n_n} G_1 \varphi \leq \frac{1}{n} \sum_{k=1}^n u_{n_k} G_1 \varphi.$$

Note further that $|\widehat{G}_1 \varphi - (\widehat{G}_1 \varphi)_{\{u_n > \varepsilon\}}|_{\mathcal{V}}^2 \leq \mathcal{E}_1(\widehat{G}_1 \varphi, \widehat{G}_1 \varphi)$, hence $|(\widehat{G}_1 \varphi)_{\{u_n > \varepsilon\}}|_{\mathcal{V}}$ is bounded in n . Therefore

$$\begin{aligned} \text{Cap}_\varphi(F_{n_n}^c) & \leq \varepsilon^{-1} \mathcal{E}_1(u_{n_n} G_1 \varphi, (\widehat{G}_1 \varphi)_{\{u_{n_n} > \varepsilon\}}) \\ & \leq \varepsilon^{-1} 2 \left| \frac{1}{n} \sum_{k=1}^n u_{n_k} G_1 \varphi \right|_{\mathcal{F}} \left| (\widehat{G}_1 \varphi)_{\{u_n > \varepsilon\}} \right|_{\mathcal{V}} \end{aligned}$$

converges to zero as $n \rightarrow \infty$. It follows that $(F_n)_{n \in \mathbb{N}}$ is an \mathcal{E} -nest of compacts. □

Proposition 0.27 *Let $\Lambda = \partial_t$, and $1 \in \mathcal{F}$. Then $(\mathcal{E}, \mathcal{F})$ is strictly quasi-regular and there exists a Hunt process $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t \geq 0}, (Y_t)_{t \geq 0}, (P_{(s,z)})_{(s,z) \in E_\Delta})$ with state space $[0, \infty) \times E$, life time ζ , such that $R_\alpha F(s, z) := \int_0^\infty \int_\Omega e^{-\alpha t} F(Y_t(\omega)) dP_{(s,z)} dt$ is a s. \mathcal{E} -q.c. $\rho d\mu ds$ -version of $G_\alpha F$ for any $\alpha > 0$ and any $F \in \mathcal{H}_b$.*

Proof If $\Lambda = \partial_t$, and $1 \in \mathcal{F}$, then $(\mathcal{E}, \mathcal{F})$ is strictly quasi-regular by Proposition 0.26 and 0.10. Since furthermore $\mathcal{G} := \mathcal{FTC}_{0,b}^1$ satisfies the condition SD3 the result follows from Theorem 0.14. □

Assume that the condition assuring closability in Remark 0.25 is fulfilled. Let as before $u(t, z) = f(t, l_1(z), \dots, l_m(z)) \in \mathcal{FTC}_{0,b}^1$. Let l_1, \dots, l_m , be in the linear span of $k_1, \dots, k_n \in K_0$. The generator L related to \mathcal{E} has the expression

$$Lu(t, z) = \frac{1}{2} \sum_{i=1}^n \left(\frac{\partial}{\partial k_i} \left(\frac{\partial u}{\partial k_i}(t, z) \right) + \beta_{k_i}(t, z) \frac{\partial u}{\partial k_i}(t, z) \right) + (\Lambda - c)u(t, z).$$

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