The linear part of an affine group acting properly discontinuously and leaving a quadratic form invariant

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A well known classical theorem due to Bieberbach says that every discrete group Γ of isometries of the n–dimensional Euclidean space $\mathbb{R}^n$ with compact quotient $\Gamma \diagdown \mathbb{R}^n$ contains a subgroup of finite index consisting of translations. In particular such a group Γ is virtually abelian, i.e. Γ contains an abelian subgroup of finite index.

Let us now consider the group $G_n = \mathrm{Aff}(\mathbb{R}^n)$ of affine transformations of $\mathbb{R}^n$ instead of the group of isometries of $\mathbb{R}^n$. The group G_n is the semidirect product $GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ where $\mathbb{R}^n$ is identified with its group of translations. A subgroup Γ of G_n is said to act *properly discontinuously* on $\mathbb{R}^n$ if for every compact subset K of $\mathbb{R}^n$ the set $\{g \in \Gamma : gK \cap K \neq \emptyset\}$ is finite. If a discrete group Γ consists of isometries then Γ acts properly on $\mathbb{R}^n$. But this is not true for an arbitrary discrete subgroup of G_n, e.g. for an infinite discrete subgroup of $GL_n(\mathbb{R})$.

A subgroup Γ of G_n is called *crystallographic* if Γ acts properly discontinuously on $\mathbb{R}^n$ and the orbit space $\Gamma \diagdown \mathbb{R}^n$ is compact. In [Au] Auslander conjectured that every crystallographic subgroup Γ of G_n is virtually solvable, i.e. contains a solvable subgroup of finite index. In [Mi] Milnor asked a question about the fundamental group of a complete locally flat affine manifold which is equivalent to the question whether the statement of

Auslander's conjecture is true without the assumption that the orbit space $\Gamma\diagdown\mathbb{R}^n$ is compact. Milnor's question has a positive answer for $n \leq 2$ (easy for $n = 2$, trivial for $n = 1$). But for $n = 3$ the answer to Milnor's question is negative. In fact, the second named author proved that there is a nonabelian free subgroup Γ of $\mathrm{Aff}(\mathbb{R}^3)$ acting properly discontinuously on $\mathbb{R}^3$ [Ma 1,2]. On the other hand, D. Fried and W. Goldman [FG] proved Auslander's conjecture for $n = 3$ using cohomological arguments. For higher dimensions the known results confirming the Auslander conjecture are proved under the assumption that the linear part $l(\Gamma)$ of Γ belongs to certain special subgroups of $GL_n(\mathbb{R})$ where $l : G_n \to GL_n(\mathbb{R})$ denotes the natural homomorphism [GK, GrM] (see survey [A]).

Let B be a non degenerate quadratic form on $\mathbb{R}^n$ with signature (p, q), $n = p + q$. Let $O(B)$ (resp. $SO(B)$) denote the orthogonal (resp. special orthogonal) group of B. The main results of this paper are the following theorems in which we assume that $l(\Gamma)$ belongs to $O(B)$:

Theorem A. *If $|p - q| \geq 2$ and the Zariski closure of $l(\Gamma)$ contains $SO(B)$, then Γ cannot act properly discontinuously on $\mathbb{R}^n$.*

We say that Γ is *co-compactly dense* in $SO(B)$ if the homogeneous space $SO(B)/G$ is compact where G is the Zariski closure of $l(\Gamma)$.

Theorem B. *If q is even and Γ is co-compactly dense in $SO(B)$ then Γ cannot act properly discontinuously on $\mathbb{R}^n$.*

As a corollary of these two theorems we prove the Auslander conjecture in the case $q = 2$.

Theorem C. *If $q = 2$ and Γ is a crystallographic group, then Γ is virtually solvable.*

Remark. The proof of Theorem A and Theorem B are based on the substantial extension of ideas using the dynamics of affine maps introduced in [Ma 1,2] and [AMS 2,3]. In the proof of Theorem C we use both dynamical and cohomological arguments. The dynamical arguments in the proofs of these theorems are crucial in handling the cases where the semisimple part of the Zariski closure of $l(\Gamma)$ contains simple non-abelian subgroups of real rank ≥ 2.

Note that we have proved in [AMS 3] that if $|p-q|=1$ and q is odd then there exists a subgroup Γ of the affine group $\mathrm{Aff}\mathbb{R}^n$ such that (I) the linear part $l(\Gamma)$ of Γ is Zariski dense in $SO(B)$, (II) Γ acts properly discontinuously on the affine space $\mathbb{R}^n$. By contrast if q is even there is no subgroup of $\mathrm{Aff}\mathbb{R}^n$ which fulfills (I) and (II).

We will explain now the main idea of our work . Let γ be a hyperbolic element of the group $\mathrm{Aff}\mathbb{R}^n$ whose linear part $\ell(\gamma)$ leaves a quadratic form B invariant (for definitions see chapter 1). Then there is a unique maximal γ–invariant affine subspace C_γ such that the restriction B_γ of B to C_γ is positive definite. Let $e(\gamma)$ be the restriction of γ to C_γ. Let us denote by $\Omega(\Gamma)$ the set of all hyperbolic elements of Γ. We thus have the following two sets. The set of Euclidean subspaces $\{C_\gamma\}_{\{\gamma\in\Omega(\Gamma)\}}$ and their isometries $\{e(\gamma)\}_{\{\gamma\in\Omega(\Gamma)\}}$. The main idea of our work is to construct isometries $C_\gamma \to C_{\tilde{\gamma}}$ between the various Euclidean subspaces so that we can think of the transformations $(e(\gamma))$ as acting on the same Euclidean subspace. In this way, studying the dynamics of our affine group Γ is reduced to studying the dynamics of affine Euclidean maps, which is well understood. First steps toward this reduction were done in [M] and then in [AMS 3]. These papers carried out the case when the corresponding Euclidean subspaces are lines and their Euclidean transformations are just translations along these lines.

We now give an outline of the paper. In chapter one we consider a finite subset S of ε–hyperbolic, pairwise ε–transversal elements of Γ. The main goal of this chapter is to show some geometrical estimations. Namely, let r_0 be the smallest eigenvalue greater then 1 of the elements of S and let $d_0 = \max\{d(p_0, C_\gamma),\, \gamma \in S\}$ where p_0 is a point in $\mathbb{R}^n$. Then for any element γ from the group generated by S the distance from C_γ to the given point $p_0 \in \mathbb{R}^n$ is bounded by $d(\varepsilon, d_0, r_0)$ and the length of the projection of $\gamma p_0 - p_0$ onto the Euclidean subspace C_γ is bounded by a linear function of the word length of γ with respect to the generating set S. Some of these results are generalizations of our previous results (see [AMS 3]). An important step lies in finding an orthogonal transformation $o(\gamma)$ and an anisotropic subspace $A^0(\gamma)$ for any element γ in the subgroup generated by S. In chapter three we show how this can be read off for every γ in the group generated by S from the orthogonal transformations $o(\gamma)$ and anisotropic subspaces $A^0(\gamma)$ corresponding to the elements γ of the generating set. In order to do that we define a topology on the set of pairs (W, g) where W is a maximal B–anisotropic subspace in $\mathbb{R}^n$ and g is a B–orthogonal transformation of W. We can thus speak of the limit set and describe in it points in general position (Lemma 3.5 and Lemma 3.6). This is crucial in particular when dealing with co-compactly dense subgroup Γ (for definition see chapter 1, n 1.1). In chapter four we extend this construction to affine groups. We aim at information about directions of translational parts of affine transformations in $< S >$. We obtain that we get enough directions, which means that the zero-vector is contained in their convex hull, lemma 4.2 and lemma 4.7. The most substantial part of our work is to show that there exists a (θ, ε)– system (see chapter 4, definition 4.4) in our group Γ. The purpose of a (θ, ε)–system is to have a collection of elements of Γ with enough directions of translational parts so that we can fetch back far away points to a fixed compact set and thus obtain a contradiction to proper discontinuity.

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1. Geometric properties of hyperbolic and transversal elements

1.1 Notation and terminology. In this section we introduce the terminology we will use throughout the whole paper. We also prove some basic lemmas about the geometry of hyperbolic elements. They are close in spirit to the methods of [AMS 3].

Let g be a semisimple element in $GL(\mathbb{R}^n)$. Then the space $\mathbb{R}^n$ can be decomposed into the direct sum of three subspaces $A^+(g),\ A^-(g),\ A^0(g)$ determined by the condition that all eigenvalues of the restriction $g \mid A^+(g)$ (resp. $g \mid A^-(g),\ g \mid A^0(g)$) have absolute value more than 1 (resp. less than 1, equal to 1). Let now $D^+(g) = A^+(g) \oplus A^0(g)$ and $D^-(g) = A^-(g) \oplus A^0(g)$, then obviously $D^+(g) \cap D^-(g) = A^0(g)$. Let $s^-(g) = \max\{|\lambda \mid :\ \lambda$ is an eigenvalue of g of absolute value less than 1 $\}$. Let $s^+(g) = s^-(g^{-1})$ and $s(g) = \max\{s^+(g), s^-(g)\}$.

Let B be a quadratic form of signature $(p, q),\ q \leq p$, and $H_B = SO(B)$. The element g is called *hyperbolic* if $\dim A^0(g) = p - q$. Two hyperbolic elements g and h are called *transversal*, if $A^+(g) \cap A^-(h) = \{\,0\,\}$ and $A^+(h) \cap A^-(g) = \{\,0\,\}$. Let us denote the subgroup $\{x \in G_n \mid l(x) \in O(B)\}$ of G_n by G_B. We will say that $g \in G_B$ is *hyperbolic* if the linear part $l(x)$ is hyperbolic. Two affine transformations $g \in G_B$ and $h \in G_B$ are called *transversal* if their linear parts $l(g)$ and $l(h)$ are transversal. For an element $x \in G_B$, we write $A^+(x)$ (resp. $A^-(x)$, $A^0(x)$) instead of $A^+(l(x))$ (resp. $A^-(l(x))$, $A^0(l(x))$. Consider a hyperbolic element $g \in G_B$ without fixed points. Then there exists a g–invariant line L_g and the restriction of g to L_g is the translation by a non- zero vector t_g. Let us note that all such lines are parallel and vector t_g does not depend on the choice of L_g. We will assume that we fixed once and for ever some point in the affine space $\mathbb{R}^n$ as an origin point and the line L_g we define as a closest to the origin g –invariant line. We denote the restriction of B to $A^0(g)$ by B_g. The form B_g is positive definite. Therefore we can define $v^0(g) = t_g/B_g(t_g, t_g)^{1/2}$. It is clear that $B_g(v^0(g), v^0(g)) = 1$. Since, for every vector $v \in \mathbb{R}^n$, the vectors $v^0 \in \mathbb{R}^n$ and $l(g)v - v$ are orthogonal, $B(gx - x, v^0(g))$ does not depend on $x \in \mathbb{R}^n$ and

$$B(gx - x, v^0(g)) = B_g(t_g, t_g)^{1/2}$$

for any point $x \in \mathbb{R}^n$. Let Γ be a subgroup of G_B generated by $g_1, \dots, g_m$ and let $\omega : \Gamma \to \mathbb{Z}^+$ be the word metric on Γ with respect to this set of generators. Put

$$\chi(g) = \frac{t_g}{\omega(g)}.$$

It is clear that $\chi(g_i) = t_{g_i}$ for every $i = 1, \dots, m$.

Let us define the following affine subspaces: $E_g^+ = D^+(g) + L_g$, $E_g^- = D^-(g) + L_g$, $E_g^+ \cap E_g^- = C_g$. Let $\pi_g : \mathbb{R}^n \longrightarrow A^0(g)$ be the natural projection of $\mathbb{R}^n$ onto $A^0(g)$ along the subspace $A^+(g) \oplus A^-(g)$. Let $\tau_g : \mathbb{R}^n \longrightarrow C_g$ be the natural projection of $\mathbb{R}^n$ onto C_g along the subspace $A^+(g) \oplus A^-(g)$.

Let $g \in G_B$ be a hyperbolic transformation . We define $d_g(a, b)$ as the distance between $\tau_g(a)$ and $\tau_g(b)$ with respect to the (positive definite) form B_g that is

$$d_g(a, b) = B_g(v, v)^{1/2}$$

where $v = \tau_g(b) - \tau_g(a) = \pi_g(b - a)$. Put $d_g(a) = d_g(a, g(a))$. Define $\|v\|_g = B_g(\pi_g(v), \pi_g(v))^{1/2}$ where v is a vector in $\mathbb{R}^n$. Let us note, that $d_g(\ ,\ )$ (resp. $\|\ \|_g$) is a pseudometric (resp. a pseudonorm) and $d_g(a, g(a)) = \|v\|_g$, where $v = g(a) - a$.

Let $\|\,\mathbf{.}\,\|$ and d denote the norm and metric on $\mathbb{R}^n$ corresponding to the standard inner product on $\mathbb{R}^n$. For any two subsets A, B of $\mathbb{R}^n$ we put

$$d(A,B) = \max\left\{\max_{b\in B}\min_{a\in A} d(a,b), \max_{a\in A}\min_{b\in B} d(a,b)\right\}.$$

Let us note that on the set of compact subsets d is a metric. This metric induces the following metric $\widehat{d}$ on the set of all subspaces of $\mathbb{R}^n$, namely:

$$\widehat{d}(W_1, W_2) = \ d(S(W_1), S(W_2)),$$

where $W_1 \in \mathbb{R}^n,\ W_2 \in \mathbb{R}^n$ and $S(W) = \{w \in W \mid \|w\| = 1\}$ for $W \in \mathbb{R}^n$. A hyperbolic element $g \in O(B)$ is called *ε–hyperbolic* if $\widehat{d}(A^+(g), A^-(g)) \geq \varepsilon$. Two transversal elements g and h are *ε–transversal* if

$$\widehat{d}(A^+(g), A^-(h)\ ) \geq \varepsilon \ \textit{and} \quad \widehat{d}(A^-(g), A^+(h)\ ) \geq \varepsilon$$

Let Γ be a subgroup of the group $O(B)$ and let G be the Zariski closure of Γ. The group Γ is said to be *co–compactly dense in* $SO(B)$ (or *co–compactly dense* if it will cause no confusion) if the vector space $\mathbb{R}^n$ can be decomposed into a direct sum of two B–orthogonal, Γ–invariant subspaces V_1 and V_2 such that

(1) $G \supseteq SO(\widehat{B})$ where $\widehat{B} = B|_{V_1}$,

(2) the restriction of the form B to V_2 is positive definite .

It is obvious that in this case the connected component G^0 of G is a direct product $G^0 = SO(\widehat{B}) \times K$, where $K \leq O(B)$ is a connected compact subgroup in $GL(V_2)$. Observe that if Γ is dense in $SO(B)$ then Γ is co–compactly dense in $SO(B)$. It is clear, that if Γ is co–compactly dense in $SO(B)$ then the Zariski closure of Γ contains a normal subgroup $\widehat{G}$ such that the factor space $SO(B)/\widehat{G}$ is compact.

Let Γ be a subgroup of G_B. The group Γ is said to be a co–compactly dense in G_B if the linear part $l(\Gamma)$ is co–compactly dense in $SO(B)$. Suppose that Γ is co–compactly dense. Let G be the Zariski closure of $l(\Gamma)$. Let us consider the decomposition of an element $g = hk$ according to the direct product decomposition $G = \widehat{G} \times K$. We denote by l_1 (resp. l_2) the natural homomorphism $l_1 : G \to \widehat{G}$ (resp. $l_2 : G \to K$). It is easy to see, that g is hyperbolic (resp. ε–hyperbolic) if and only if $l_1(g)$ is hyperbolic (resp. ε–hyperbolic). Two hyperbolic elements g and h are transversal (resp. ε–transversal) if and only if $l_1(g)$ and $l_1(h)$ are transversal (resp. ε–transversal). Obviously $A^0 = D^+(g) \cap D^-(h) = (D^+(l_1(g)) \cap D^-(l_1(h))) \oplus V_2$ and $A^0(g) = A^0(l_1(g)) \oplus V_2$. To shorten notation we will write $\widehat{A}^0(g)$ (resp. $\widehat{D^{\pm}}(g)$) instead of $A^0(l_1(g))$ (resp. $D^+(l_1(g))$). We have the natural projection τ_1 of the affine space $\mathbb{R}^n$ onto to the affine space $A_1 = \mathbb{R}^n/V_2$ along V_2 and hence an induced homomorphism $\rho_1 : G \to \mathrm{Aff}\ A_1$ and similarly for $A_2 = \mathbb{R}^n/V_1$ we have the natural projection τ_2 and an induced homomorphism $\rho_2 : G \to \mathrm{Aff}\ A_2$. It is clear that $l(\rho_i(g)) = l_i(g)$ for $i = 1, 2$ and every element $g \in G$. Set $t_g^{(1)} = t_{\rho_1(g)}$, $t_g^{(2)} = t_{\rho_2(g)}$, $\chi_1(g) = \chi(\rho_1(g))$ and $\chi_2(g) = \chi(\rho_2(g))$ for every hyperbolic element $g \in G_B$.

Let $k \geq 1$ be a positive integer, and let f_1, f_2 be functions defined on a subset E of the direct product of k copies of the set of hyperbolic elements in G_B. We say that f_1 is *dominated* by f_2 and write $f_1 \ll f_2$ if for every $\varepsilon > 0$ there is a $c(\varepsilon) > 0$ such that $f_1(g_1, \ldots, g_k) \leq c(\varepsilon) f_2(g_1, \ldots, g_k)$ whenever $(g_1, \ldots, g_k) \in E$ and the elements $g_i, i = 1, \ldots, k$, are pairwise ε–transversal. If $f_1 \ll f_2$ and $f_2 \ll f_1$ then we write $f_1 \sim f_2$ and say that f_1 and f_2 are *equivalent.*

It is clear that for $a, b \in \mathbb{R}^n$

$$|d_g(a) - d_g(b)| \leq 2d_g(a, b) \ll d(a, b) \qquad (1.1).$$

We will quite often use the following inequality (see [AMS 3])

$$d(x, E_g^+) \ll s(g) d(x, g^{-1}x) \qquad (1.2).$$

Example. Let E be a subset of ε–hyperbolic elements in G_B and let f_1, f_2 (resp. f_3, f_4)

be the following functions from E to the set of real valued functions on $\mathbb{R}^n$ (resp. on $\mathbb{R}^n \times \mathbb{R}^n$) $f_1(g) = \|\cdot\|$, $f_2(g) = \|\cdot\|_g$ (resp. $f_3(g) = d(,)$, $f_4(g) = d_g(,)$). It is easy to see that $f_1 \sim f_3$ and $f_2 \sim f_4$.

1.2 Lemma. *There exists an $s(\varepsilon) < 1$ such that for any two ε–hyperbolic ε–transversal elements g and h with $s(g) < s(\varepsilon)$, $s(h) < s(\varepsilon)$ we have*

(1) the element gh is $\varepsilon/2$–hyperbolic and is $\varepsilon/2$–transversal to both g and h;

(2) $\widehat{d}(A^+(gh), A^+(g)) \ll s(g)$;

(3) $\widehat{d}(A^-(gh), A^-(h)) \ll s(h)$;

(4) $s(gh) \ll s(g)s(h)$.

1.3 Lemma. *Let $g_0, h_1, \ldots, h_m$ be ε–hyperbolic, pairwise ε–transversal elements, $s = \max\{s(g_0), s(h_1), \ldots, s(h_m)\}$ and $s_0 = s^{1/2}$. Let $g_\ell = g_0 \cdot h_{i_1}^{n_1} \cdot \cdots \cdot h_{i_\ell}^{n_\ell}$, $i_k \neq i_{k+1}$, $\ell \in \mathbb{Z}$, $\ell > 0$ and $M_\ell = |n_1| + \cdots + |n_\ell|$. Then there exists a constant $b(\varepsilon) < 1$ such that if $s < b(\varepsilon)$,*

(1) $s(g_\ell) \leq s_0^{M_\ell + 1}$;

(2) $\widehat{d}(A^+(g_{\ell-1}),\, A^+(g_\ell)) \leq \frac{\varepsilon}{2} s_0^{M_{\ell-1}}$;

(3) $\widehat{d}(A^+(g_0),\, A^+(g_\ell)) \leq \frac{\varepsilon}{2} s_0$;

(4) $\widehat{d}(A^-(g_\ell),\, A^-(h_{i_\ell})) \leq \frac{\varepsilon}{2} s_0^{i_\ell}$;

(5) $\widehat{d}(A^-(g_\ell),\, A^+(h_i) \cup A^-(h_i)) \geq \frac{\varepsilon}{2}$ for $i \neq i_\ell$;

(6) g_ℓ is $\varepsilon/2$–hyperbolic.

These lemmas, except the statement (5) in Lemma 1.3, were proved in [AMS 3, Corollary 5.9 and Lemma 5.10] for $p - q = 1$. The proof in the general case is almost verbatim repetition of the proofs given there. The proof of Lemma 1.3 (5) immediately follows from Lemma 1.2 (3).

We begin with results which give us an estimations of $d(q_0, C_g)$ and $d_g(q_0)$ where q_0 is a point in $\mathbb{R}^n$ and g is a product of hyperbolic transversal elements.

1.4 Lemma. *Let g and h be two ε–hyperbolic ε–transversal elements, such that gh is $\varepsilon/2$–hyperbolic and $\varepsilon/2$–transversal to both g and h. Let q_0 be a point in the affine space $\mathbb{R}^n$. Then there exists a constant $d(\varepsilon)$ such that*

(1) $d(q_0, E^+_{gh}) \leq d(q_0, E^+_g) + d(\varepsilon)s(g)\left[d_g(q_0) + d(q_0, C_g) + s(h)(d_h(q) + d(q_0, C_h))\right]$;

(2) $d(q_0, E^-_{gh}) \leq d(q_0, E^-_g) + d(\varepsilon)s(h)\left[d_h(q_0) + d(q_0, C_h) + s(g)(d_h(q_0) + d(q_0, C_g))\right]$;

(3) $d(q_0, C_{gh}) \leq d(\varepsilon)(s(h)+s(g))d(q_0, C_g)+d(\varepsilon)\left[d(q_0, E^+_g) + d(q_0, C_h)) + s(g)(d_g(q_0) + d_h(q_0)\right]$.

Proof. Put $E = E^-_g \cap E^+_h$. Let p_0 be a point in E such, that $d(q_0, E) = d(q_0, p_0)$. Denote $p_1 = h^{-1}p_0$, $p_2 = gp_0$. Let $q_1 = \pi_h(p_0)$, be the projection of p_0 onto C_h along the $A^+(h)$ and let $q_2 = \pi_g(p_0)$ be the projection of p_0 onto C_g along the $A^-(g)$. It is clear, that

$$d(q_0, E) \ll d(q_0, C_g) + d(q_0, C_h) \tag{1.3}$$

and

$$d(q_0, p_0) \ll d(q_0, C_g) + d(q_0, C_h) \tag{1.4}$$

Since, $d(q_0, \pi_h(q_0)) \ll d(q_0, C_h)$, $d(\pi_h(q_0), q_1) \ll d(\pi_h(q_0), q_0)+d(q_0, q_1)$, we have $d(\pi_h(q_0), q_1) \ll d(\pi_h(q_0), q_0) + d(q_0, p_0) + d(p_0, q_1)$.

It is easily seen that $d(p_0, q_1) \ll d(p_0, C_h)$ and $d(p_0, C_h) \ll d(q_0, C_h) + d(p_0, q_0)$. From (1.3) and (1.4) we conclude that

$$d(\pi_h(q_0), q_1) \ll d(q_0, C_g) + d(q_0, C_h) \tag{1.5}$$

By definition, $q_1 = \pi_h(p_0)$, then

$$d(h^{-1}q_1, q_1) \ll d(h^{-1}\pi_h(p_0), h^{-1}\pi_h(q_0)) + \ d(h^{-1}\pi_h(q_0), \pi_h(q_0)) + d(\pi_h(q_0), \pi_h(p_0)).$$

Since $d(\pi_h(p_0), \pi_h(q_0)) \ll d(p_0, q_0)$ and $d(h^{-1}\pi_h(q_0), \pi_h(q_0)) \ll d(h\pi_h(q_0), \pi_h(q_0))$ we have from (1.5)

$$d(h^{-1}q_1, q_1) \ll d_h(q_0) + d(q_0, C_g) + d(q_0, C_h) \tag{1.6}$$

Then from (1.1) we have $d(p_1, h^{-1}q_1) = d(h^{-1}p_0, h^{-1}q_1) \ll s(h)d(p_0, C_h)$.
Therefore $d(p_1, h^{-1}q_1) \ll s(h)(d(q_0, C_g) + d(q_0, C_h))$. Hence from (1.6) follows that

$$d(p_0, p_1) \ll s(h)(d(q_0, C_g) + d(q_0, C_h)) + d_h(q_0) + d(q_0, C_g) + d(q_0, C_h) \qquad (1.7)$$

By a similar argument, we have

$$d(g\ q_2,\ q_2) \ll d_g(q_0) + d(q_0, C_g) + d(q, C_h) \qquad (1.8)$$

and

$$d(p_0, p_2) \ll s(g)(d(q_0, C_g) + d(q_0, C_h)) + d_g(q_0) + d(q_0, C_g) + d(q_0, C_h) \qquad (1.9)$$

Therefore from (1.8), (1.9) we have $d(p_2, p_1) \ll (d_g(q_0) + d_h(q_0) + d(q_0, C_g) + d(q_0, C_h)$.
Since $d(p_2, p_1) = d(gp_0, h^{-1}p_0) = d(gh(h^{-1}p_0), h^{-1}p_0)$, by (1.1) we conclude that

$$d(p_1, E^-_{gh}) \ll s(gh)[d_g(q_0) + d_h(q_0) + d(q_0, C_g) + d(q_0, C_h)] \qquad (1.10).$$

It is an elementary geometric fact that

$$|d(q_0, E^-_{gh}) - d(q_0, E^-_h)| \ll sin\measuredangle(E^-_{gh}, E^-_h)\ d(q_0, p_1) + |d(p_1, E^-_{gh}) - d(p_1, E^-_h)| \qquad (1.11).$$

Then by (2) Lemma 1.2

$$sin\measuredangle(E^-_{gh}, E^-_h) \ll s(h) \qquad (1.12).$$

Therefore,

$$|d(q_0, E^-_{gh}) - d(q_0, E^-_h)| \ll s(h)(d(q_0, C_h) + d_h(q_0)) + s(gh)(d_g(q_0) + d_h(q_0) + d(q_0, C_g) + d(q_0, C_h)) \qquad (1.13)$$

By (3) Lemma 1.2, $s(gh) \ll s(g)s(h)$, therefore, there exists a constant $d(\varepsilon)$ such that

$$d(q_0, E^-_{gh}) \le d(q_0, E^-_h) + d(\varepsilon)s(h)[(d(q_0, C_h) + d_h(q_0)) + s(g)(d_g(q_0) + d(q_0, C_g))] \quad (1.14).$$

If x is a hyperbolic element then $E^-_x = E^+_{x^{-1}}$, $d_x(q_0) = d_{x^{-1}}(q_0)$ and $s(x) = s(x^{-1})$.
Therefore from (1.14) follows

$$d(q_0, E^+_{gh}) \le d(q_0, E^+_g) + d(\varepsilon)s(g)\,[(d(q_0, C_g) + d_g(q_0)) + s(h)(d_h(q_0) + d(q_0, C_h))] \quad (1.15).$$

Thus from (1.14), (1.15) and $C_{gh} = E^+_{gh} \cap E^-_{gh}$ we conclude that

$$d(q_0, C_{gh}) \le d(\varepsilon)(s(h)+s(g))d(q_0, C_g)+d(\varepsilon)\left[d(q_0, E^+_g) + d(q_0, C_h)) + s(g)(d_g(q_0) + d_h(q_0)\right] \quad (1.16).$$

□

1.5 Lemma. *Let g and h be two ε–hyperbolic ε–transversal elements, such that gh is $\varepsilon/2$–hyperbolic and $\varepsilon/2$–transversal to both g, h. Let q_0 be a point in the affine space $\mathbb{R}^n$. Then there is a constant $d(\varepsilon)$ such that*

$$d_{gh}(q_0) \le d_g(q_0) + d(\varepsilon)[d_h(q_0) + s(h)d_h(q_0) + s(g)d_g(q_0) + d(q_0, C_g) + d(q_0, C_h)]$$

Proof. Put $E = E^-_g \cap E^+_h$. Let p_0 be a point in E, such that $d(q_0, E) = d(q_0, p_0)$. Denote $q_1 = h^{-1}p_0$ and $q_2 = gp_0$. Let p_1 be the projection of p_0 onto C_h along subspace $A^+(h)$ and let p_2 be the projection of p_0 onto C_g along subspace $A^-(g)$. From (1.1)

$$|d_{gh}(q_0) - d_{gh}(q_1)| \ll d(q_0, q_1) \quad (1.17)$$

Put $v_1 = p_0 - q_1$ and $w_1 = p_1 - h^{-1}p_1$. Then $d(p_0, q_1) \ll d(p_0, C_h) + \|w_1\|_h = d(p_0, C_h) + d_h(p_0)$. Since $d(q_0, p_0) \ll d(q_0, C_h) + d(q_0, C_g)$ and $d_h(p_0) - d_h(q_0) \ll d(q_0, p_0)$, we have

$$d(q_0, q_1) \ll d_h(q_0) + d(q_0, C_h) + d(q_0, C_g) \quad (1.18).$$

Combining (1.17) with (1.18) we obtain

$$d_{gh}(q_0) - d_{gh}(q_1) \ll d_h(q_0) + d(q_0, C_h) + d(q_0, C_g) \quad (1.19).$$

By definition $d_{gh}(q_1) = d_{gh}(h^{-1}p_0, gh(h^{-1}p_0)) = d_{gh}(q_1, q_2)$. Then $d_{gh}(q_1) \le d_{gh}(q_1, p_0)+ d_{gh}(p_0, q_2)$. To estimate the first term $d_{gh}(q_1, p_0)$ we note that

$$|\, d_{gh}(q_1, p_0) - \|w_1\|_{gh}| \ll d(p_0, C_h) \quad (1.20).$$

Let u_1 (resp. u_2)be the projection of w_1 onto $D^+(gh)$ along $A^-(gh)$ (resp. $A^-(h)$). Then $\|u_1\|_{gh} = \|u_1\|_h = d_h(p_1)$, $\|u_2\|_{gh} = \|w_1\|_{gh}$. From Lemma 1.2 (3) we deduce $|\|u_1\|_{gh} - \|u_2\|_{gh}| \ll s(h)d_h(p_1)$. Therefore

$$| \, d_{gh}(q_1, p_0) - \|u_1\|_{gh}| \ll s(h)d_h(p_1) + d(p_0, C_h) \qquad (1.21)$$

The inequality $d(q_0, p_1) \leq d(q_0, p_0) + d(p_0, C_h) \ll d(q_0, C_h) + d(q_0, C_g)$ and (1.1) yield

$$d_h(p_1) - d_h(q_0) \ll d(q_0, C_h) + d(q_0, C_g) \qquad (1.22).$$

Thus we conclude from (1.21) and (1.22) that

$$| \, d_{gh}(q_1, p_0) - d_h(q_0) \, | \ll s(h)d_h(q_0) + d_h(q_0) + d(p_0, C_h) + d(q_0, C_g) \qquad (1.23).$$

Our next goal is to evaluate $d_{gh}(q_2, p_0)$. Put $v_2 = q_2 - p_0 = gp_0 - p_0$ and $w_2 = gp_2 - p_2$. Let u_3 (resp. u_4) be the projection of w_2 onto $D^-(gh)$ along $A^+(g)$ (resp. $A^+(gh)$). Then $\|u_3\|_{gh} = \|u_3\|_g = d_g(p_2)$ and $\|u_4\|_{gh} = \|w_2\|_{gh}$. From Lemma 1.2 (2) we deduce

$$| \, \|u_3\|_{gh} - \|u_4\|_{gh} \, | \ll s(g)d_g(p_2) \qquad (1.24)$$

. Since $d_{gh}(q_2, p_0) = \|v_2\|_{gh}$ and $| \, \|v_2\|_{gh} - \|w_2\|_{gh} \, | \ll d(p_0, C_g)$ we have

$$|d_{gh}(q_2, p_0) - \|w_2\|_{gh} \, | \ll d(p_0, C_g) \qquad (1.25)$$

Combining (1.24) with (1.25) we get

$$|d_{gh}(q_2, p_0) - d_g(p_2)| \leq |d_{gh}(q_2, p_0) - \|w_2\|_{gh} \, | + | \, \|u_3\|_{gh} - \|u_4\|_{gh} \, |$$

Then

$$|d_{gh}(q_2, p_0) - d_g(p_2)| \ll s(g)d_g(p_2) + d(p_0, C_g) \qquad (1.26).$$

From (1.1) we have $|d_g(q_0) - d_g(p_2)| \ll d(q_0, p_2)$. This together with $d(q_0, p_2) \leq d(q_0, p_0) + d(p_0, C_g)$ gives

$$|d_{gh}(q_2, p_0) - d_g(q_0)| \ll s(g)d_g(q_0) + d(q_0, C_h) + d(q_0, C_g) \qquad (1.27)$$

Recall that $d_{gh}(q_1) \leq d_{gh}(q_1,p_0) + d_{gh}(p_0,q_2)$. Therefore from (1.19), (1.23) and (1.27) there exists a constant $d(\varepsilon)$ such that

$$d_{gh}(q_0) \leq d_g(q_0) + d(\varepsilon)[d_h(q_0) + s(h)d_h(q_0) + s(g)d_g(q_0) + d(q_0,C_g) + d(q_0,C_h)]$$

which completes the proof. □

Let $g_0, h_1, \dots, h_m$ be ε–hyperbolic ε–transversal elements. Fix a point $q_0 \in \mathbb{R}^n$ and put $T = \max\{ d_{g_0}(q_0), d_{h_1}(q_0) \dots d_{h_m}(q_0) \}$, $C = \max\{d(q_0,C_{g_0}),\ d(q_0,C_{h_1}),\dots,d(q_0,C_{h_m})\}$ and $s = \max\{s(g_0), s(h_1),\dots,s(h_m)\}$. Let $g_\ell = g_0 h_{i_1}^{n_1}\dots h_{i_\ell}^{n_\ell}$, where $i_k \neq i_{k+1}$, $\ell \in \mathbb{Z}$, $\ell > 0$, $M_\ell = |n_1|+\dots+|n_\ell|$.According to Lemma 1.3 there exists a constant $b(\varepsilon) < 1$ such that if $s \leq b(\varepsilon)$, $i \neq n_\ell$ element $g_{\ell+1} = g_\ell h_i^{n_i}$ is $\varepsilon/2$–hyperbolic and $\varepsilon/2$–transversal to both g_ℓ and $h_i^{n_i}$. Put $a_\ell = d(q_0, E^+_{g_\ell})$, $a_{\ell+1} = d(q_0,E^+_{g_{\ell+1}})$, $b_\ell = d(q_0,C_{g_\ell})$, $b_{\ell+1} = d(q_0,C_{g_{\ell+1}})$, $d_\ell = d_{g_\ell}(q_0)$ and $d_{\ell+1} = d_{g_{\ell+1}}(q_0)$. We rewrite statements of Lemma 1.4 and Lemma 1.5 as follows:

$$a_{\ell+1} \leq a_\ell + d(\varepsilon)s(g_\ell)\left[d_\ell + b_\ell + s\left(h_{i_{\ell+1}}\right)^{|n_{i_{\ell+1}}|}\left(|n_{i_{\ell+1}}|T + C\right)\right].$$

$$b_{\ell+1} \leq d(\varepsilon)\left(s(h_{i_{\ell+1}})^{|n_{i_{\ell+1}}|} + s(g_\ell)\right)b_\ell + d(\varepsilon)(a_\ell + C + \ + s(g_\ell)d_\ell + s(h_{i_{\ell+1}})^{|n_{i_{\ell+1}}|}|n_{i_{\ell+1}}|T).$$

$$d_{\ell+1} \leq d_\ell + d(\varepsilon)[n_{i_{\ell+1}}T + s\left(h_{i_{\ell+1}}\right)^{|n_{i_{\ell+1}}|}|n_{i_{\ell+1}}|T + s(g_\ell)d_\ell + a_\ell + C].$$

Recall that $s = s_0^2$. There exists a constant $b_1(\varepsilon)$ such that if $s \leq b_1(\varepsilon)$ then $s_0 d(\varepsilon) < 1$ and $s_o^M M \leq \frac{s_0^{M/2}}{100}$ for all positive integers M and denote $Q = 2d(\varepsilon)(T+C)$. Then we can simplify inequalities above as follows

$$a_{\ell+1} \leq a_\ell + {s_0}^{M_\ell}(d_\ell + b_\ell + Q) \tag{1.28}$$

$$b_{\ell+1} \leq \frac{b_\ell}{10} + d(\varepsilon)a_\ell + {s_0}^{M_\ell}d_\ell + Q \tag{1.29}$$

$$d_{\ell+1} \leq d_\ell + |n_{\ell+1}|Q + {s_0}^{M_\ell}d_\ell + d(\varepsilon)a_\ell \tag{1.30}$$

1.6 Lemma. *There exists a positive constant $d_0(\varepsilon) < 1$ such that if $s \leq d_0(\varepsilon)$ then, for all ℓ, $a_\ell \ll 1, b_\ell \ll 1, d_\ell \ll M_\ell$*

Proof. Let $r_\ell = \max\{100d(\varepsilon)a_\ell,\ 100Q,\ 10b_\ell,\ 10d_\ell/M_\ell\}$. Then there exists $d_0(\varepsilon)$ such that for $s < d_0(\varepsilon)$ we have from (1.30)

$$d_{\ell+1} \le M_\ell r_\ell/10 + |n_{\ell+1}| r_\ell/100 + r_\ell/100 + r_\ell/100 \le M_{\ell+1} r_\ell/10 \tag{1.31}$$

and from (1.28), we have

$$b_{\ell+1} \le r_\ell/100 + r_\ell/100 + r_\ell/100 + r_\ell/100 \le r_\ell/10 \tag{1.32}$$

We can assume that the constant $d_0(\varepsilon)$ was chosen so that from (1.28) we have

$$100d(\varepsilon)a_{\ell+1} \le r_\ell \left(1 + 2M_\ell s_0{}^{\frac{M_\ell}{2}}\right) \tag{1.33}$$

Then

$$r_{\ell+1} \le r_\ell \left(1 + 2M_\ell s_0{}^{\frac{M_\ell}{2}}\right) \tag{1.34}$$

Put $Q_0(\varepsilon) = \prod_{i=0}^{\infty} \left(1 + 2M_\ell s_0{}^{\frac{M_\ell}{2}}\right)$. Then $Q_0 < \infty$. This proves the statement of lemma. $\square$

2. Orientation on maximal anisotropic spaces

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In this section we introduce orientations on maximal B–anisotropic subspaces. This is a generalization of orientations introduced on the one-dimensional B–anisotropic subspaces for the case $p - q = 1$ in [AMS 3].

Let us recall that our form B has signature (p, q), $p \le q$. Then there exists a basis $v_1, v_2, \ldots, v_p,\ w_1, w_2, \ldots, w_q$ of $\mathbb{R}^n$, such that for every vector $v = x_1 v_1 + \cdots + x_p v_p +$

$y_1w_1 + \cdots + y_qw_q$, we have

$$B(v,v) = x_1^2 + \cdots + x_p^2 - y_1^2 - \cdots - y_q^2$$

Consider the set Ψ of all maximal B–isotropic subspaces. Let X be the subspace spanned by $\{v_1, v_2, \ldots, v_p\}$ and Y be the subspace spanned by $\{w_1, w_2, \ldots, w_q\}$. It is clear, that $\mathbb{R}^n = X \oplus Y$. We have the two projections

$$\pi_X : \mathbb{R}^n \longrightarrow X \text{ and } \pi_{\mathrm{Y}} : \mathrm{\mathbb{R}^n} \longrightarrow \mathrm{Y}.$$

along Y and X respectively The restriction of π_Y to $V \in \Psi$ is a linear isomorphism $V \longrightarrow Y$. Hence if we fix an orientation on Y we have also fixed an orientation on each $V \in \Psi$. For $V \in \Psi$ let us denote the B–orthogonal complement of V by $V^\perp = \{z \in \mathbb{R}^n\ ;\ B(z,V) = 0\}$. We have $V \subset V^\perp$ since V is B–isotropic. We also have

$$\dim V^\perp = \dim V + (p-q) = p.$$

The restriction of π_X to $V^\perp$ is a linear isomorphism $V^\perp \longrightarrow X$. Hence if we fix an orientation on X we have also fixed an orientation on $V^\perp$ for each $V \in \Psi$. Thus we have orientations on both V and $V^\perp$ and we have naturally induced an orientation on any subspace W, such that $V^\perp = W \oplus V$. If $V_1 \in \Psi$ and $V_2 \in \Psi$ are transversal then $W = V_1^\perp \cap V_2^\perp$ is a subspace which is transversal to both V_1 and V_2, therefore $W \oplus V_1 = V_1^\perp$ and $W \oplus V_2 = V_2^\perp$. So there are two orientations ω_1 and ω_2 on W, where ω_i is defined if we consider W as a subspace in $V_i^\perp$. We have

2.1 Lemma. *The orientations defined above on W are the same if q is even and are opposite if q is odd.*

Proof. One can easily show that there exists a basis $\{e_1, \ldots, e_q, f_1, \ldots, f_q, e_1{}^*, \ldots e_{p-q}{}^*\}$ in $\mathbb{R}^n$ such that V_1 (resp. V_2) is the linear span of $(e_1, \ldots, e_q)$ (resp. of $(f_1, \ldots, f_q)$) and the form B with respect to this basis is given by the equation

$$B\left(x_1, \ldots, x_q, y_1, \ldots, y_q\, z_1, \ldots, z_{p-q}\right) = -x_1y_1 \cdots - x_qy_q + z_1^2 + \cdots + z_{p-q}^2.$$

Let us note that $V_1^\perp$ (resp. $V_2^\perp$) is the linear span of $\{e_1, \ldots, e_q, {e_1}^*, \ldots, {e_{p-q}}^*\}$ (resp. of $\{f_1, \ldots, f_q, {e_1}^*, \ldots, {e_{p-q}}^*\}$). Direct calculations show that, for every $j, 1 \leq j \leq q$, and every $t \in \mathbb{R}$, the transformation $h_j(t)$ defined by

$$e_i \to e_i, f_i \to f_i \quad if \;\; 1 \leq i \leq q, i \neq j$$

$$e_j \to \frac{e_j + f_j}{2} + \frac{\cos t \cdot (e_j - f_j)}{2} + \frac{\sin t \cdot e_1 *}{2}$$
$$f_j \to \frac{e_j + f_j}{2} - \frac{\cos t \cdot (e_j - f_j)}{2} - \frac{\sin t \cdot {e_1}^*}{2}$$
$${e_1}^* \to -\sin t \cdot (e_j - f_j) + \cos t \cdot {e_1}^*$$

belongs to $SO(B)$. Thus if we denote $h_1(\pi) \cdot \cdots \cdot h_q(\pi)$ by h_π, we have

$$h_\pi(e_i) = f_i, \;\; h_\pi(f_i) = e_i \quad and \quad h_\pi(e_1^*) = (-1)^q {e_1}^*.$$

Since $h_j(0) = Id$ and $h_j(t) \in SO(B)$ depends continuously on t we have

$$h_\pi \in S0(B).$$

The orientations on V and $V^\perp$ defined above depend continuously on $V \in \Psi$. Hence these orientations are invariant under the action of the group $SO(B)$. Now assuming that the basis $\{e_1, \ldots, e_q\}$ (resp. $\{e_1, \ldots, e_q, {e_1}^*, \ldots, {e_{p-q}}^*\}$) is positively oriented in V_1 (resp. in $V_1^\perp$) we get from (1) and (2) that the basis $\{f_1, \ldots, f_q\}$ (resp. $\{e_1, \ldots, e_q, (-1)^q {e_1}^*, \ldots, {e_{p-q}}^*\}$) is positively oriented in V_2 (resp. in $V_2^\perp$). This immediately implies the desired statement. □

Let $V_i, i = 1, \ldots, m$ be a sequence of pairwise transversal maximal B–isotropic subspaces in $\mathbb{R}^n$. Let W_i, $i = 1, \ldots, k$ be a sequence of pairwise transversal maximal B–isotropic subspaces in $\mathbb{R}^n$. Assume that V_1 and W_1 are transversal. Let V be a maximal

B–isotropic subspace in $\mathbb{R}^n$ transversal to both V_m and W_k. Set $V_i^\perp \cap V_{i+1}^\perp = V_i^0$ for $i = 1, \dots, m-1$ and $W_i^\perp \cap W_{i+1}^\perp = W_i^0$ for $i = 1, \dots, k-1$. Put $V^0 = V_1^\perp \cap W_1^\perp$. Fix a maximal B–anisotropic subspace V_0 in $V^\perp$. Set $U_1 = V^\perp \cap V_m$ and $U_2 = V^\perp \cap W_k$. Then for every $i = 2, \dots, m$ (resp. $i = 2, \dots, k$) we have the projection $\pi_i^+ : V_i^0 \to V_{i-1}^0$ (resp.$\pi_i^- : W_i^0 \to W_{i-1}^0$) along V_i (resp. W_i). Let π_1^+ (resp. π_1^-) be the projection of V_1^0 (resp. W_1^0) onto V^0 along V_1 (resp. W_1). Let τ_1 (resp. τ_2) be the projection of V_0 onto U_1 (resp. U_2) along V. Put $\tau^+ = \pi_1^+ \dots \pi_m^+ \tau_1$ and $\tau^- = \pi_1^- \dots \pi_k^- \tau_2$. Then the following is true.

2.2 Lemma. *If $p-q$ is odd and q is even, then there exists vector $v \in V^0$ such that $\tau^+(-v) \neq \tau^-(v)$.*

Proof. . Suppose contrary to our claim that $\tau^+(-v) = \tau^-(v)$. Let $e_1, \dots, p-q$ be a basis in V_0. It follows from lemma 2.1 that the basis $\tau^+(e_1), \dots, \tau^+(e_{p-q})$ of V^0 has the same orientation as the basis $\tau^-(e_1), \dots, \tau^-(e_{p-q})$. By our assumption $\tau^+(e_i) = -\tau^-(e_i)$ for all $i = 1, \dots, p-q$. Since $p-q$ is odd we have a contradiction. □

The statement of the lemma 2.2 is not true for example if $p = q+1$ and q is even [AMS3].

3. A composition of Euclidean parts of linear transformations

3.1 Notation. Let g and h be two hyperbolic transversal elements of $O(B)$. We will define an isometry $\pi_{g,h} : \ A^0(h) \longrightarrow A^0(g)$. The term isometry is used with respect to the positive definite form $B|W$, where W is a maximal B–anisotropic subspace. By definition, $A^-(g) \cap A^+(h) = \{0\}$ and $D^-(g) = (A^-(g))^\perp$ (resp. $D^+(h) = (A^+(h))^\perp$) therefore $A^-(g)$

(resp. $A^+(h)$) is the unique maximal isotropic subspace of $D^-(g)$ (resp. $D^+(h)$). Hence $D^+(h) \oplus A^-(g) = \mathbb{R}^n$ and $D^-(g) \oplus A^+(h) = \mathbb{R}^n$. Put $A^0_{h,g} = D(g)^- \cap D(h)^+$. We define isometries $\sigma_h^+ : A^0(h) \longrightarrow A^0_{h,g}$ and $\sigma_g^- : A^0_{h,g} \longrightarrow A^0(g)$, namely the projections along the isotropic subspaces $A^+(h)$ and $A^-(g)$, respectively . Let $\pi_{g,h} : A^0(h) \longrightarrow A^0(g)$ be the composition of these two isometries. Similarly, $D^-(h) \oplus A^+(g) = \mathbb{R}^n$ and $D^+(g) \oplus A^-(h) = \mathbb{R}^n$. As g and h are transversal we have $A^0_{g,h} \oplus A^+(g) = D^+(g)$ and $A^0_{g,h} \oplus A^-(h) = D^-(h)$. Therefore we also have an isometry $\tau_h^- : A^0_{g,h} \longrightarrow A^0(h)$ (resp. $\tau_g^+ : A^0(g) \longrightarrow A^0_{g,h}$) namely the projection along $A^+(g)$ (resp. along $A^-(h)$).

It is basic for our approach that any two hyperbolic transversal elements g and h of $O(B)$ yield an isometry $\pi_{g,h} : A^0(h) \longrightarrow A^0(g)$ of their Euclidean subspaces $A^0(h)$ and $A^0(g)$. These maps, the restrictions $h|^0_A(h)$ and $g|^0_A(g)$ and composites of such maps give a rich supply of orthogonal maps between the Euclidean subspaces $A^0(g)$. In the case that the linear part of Γ is Zariski dense in $SO(B)$ these isometries of $A^0(g)$ contains a dense subset of $SO(A^0(g))$. More precise statements are contains in 3.3 and 3.5. Here we use a result of Prasad and Rapinchuk [PR]. In the rest of the section it is described how to approximate certain maximal anisotropic subspace and their isometries by data of appropriate hyperbolic elements.

Next, consider a sequence $S = \{h_1, \ldots, h_k\}$ of hyperbolic elements of $O(B)$, such that each pair (h_i, h_{i+1}) is transversal. Recall that we denote by $O(B_g)$ the group of isometries of $A^0(g)$ if g is hyperbolic. The composition of the various isometries $\pi_{h_i,h_{i+1}}$, powers and inverses of $h_i|A^0(h_i) \in O(B_{h_i})$ gives many elements of $SO(B_{h_1})$. They actually form a dense subset of $SO(B_{h_1})$ for appropriate sequence S. In more detail, set $\pi_i = \pi_{h_1,h_2} \ldots \pi_{h_{i-1},h_i} : A^0(h_i) \longrightarrow A^0(h_1)$ for all $i > 1$. Define π_1 as the identity map. For $\widetilde{S} = \{g_1, \ldots, g_k\}$, where $g_i = h_i^{m_i}$ with positive integers m_i, we have $\pi_i(\widetilde{S}) = \pi_i(S)$ for all $i = 1, \ldots, k$. Therefore we will write π_i instead of $\pi_i\widetilde{(S)}$, if the set S is fixed. Put $o(g) = g \mid A^0(g)$ for every hyperbolic element $g \in O(B)$. Define $o_i(h_i) =$

$\pi_i o(h_i)\pi_i{}^{-1} \in O(B_{h_1})$ and $o(S) = o_1(h_1)o_2(h_2)\dots o_n(h_k) \in O(B_{h_1})$. If $\widetilde{S}$ is as above , then $o(\widetilde{S}) = o_1(g_1)o_2(g_2)\dots o_k(g_k) = o_1(h_1)^{m_1}o_2(h_2)^{m_2}\dots o_k(h_k)^{m_k} \in O(B_{h_1})$. Put $o_i = o_i(h_i) \in O(B_{h_1})$. For any k-tuple $M = (m_1, \dots, m_k)$ of positive integers m_i we put $o(M) = {o_1}^{m_1}{o_2}^{m_2}\dots {o_k}^{m_k}$.

Let Γ be a co–compactly dense subgroup of $SO(B)$. By definition, the connected component of the Zariski closure G of Γ is a direct product of a connected compact group K and $SO(\widehat{B})$. Recall that for a hyperbolic element g we denote by $\widehat{B}_g$ be the restriction of the quadratic form $\widehat{B}$ onto the subspace $\widehat{A}^0(g)$.

Let g and h be a two hyperbolic, transversal elements in G. The set $\{h_1, \dots, h_k\}$ of hyperbolic, pairwise transversal elements in $SO(B)$ is called a $D_{\mathbb{R}}(g,h)$–set, if (1) h_i is transversal to both g and h for $i = 1, \dots k$ and (2) the closure of the set $O(M)$ contains $SO(\widehat{B}_{h_1}) \times K$.

It is clear that if $\{h_1, \dots, h_k\}$ in $SO(B)$ is $D_{\mathbb{R}}(g,h)$–set, $g_0 \in SO(B)$ is hyperbolic element transversal to each $h_i, i = 1, \dots, k$ and to both g and h, then the sets $\{g_0, h_1, \dots, h_k\}$, $\{h_1, \dots, h_k, g_0\}$ and $\{g_0, h_1, \dots, h_k, g_0\}$ are $D_{\mathbb{R}}(g,h)$–sets. For example, if $\{h_1, \dots, h_k\}$ is $D_{\mathbb{R}}(g,h)$–set, then the sets $\{g, h_1, \dots, h_k\}$, $\{h_1, \dots, h_k, h\}$ and $\{g, h_1, \dots, h_k, h\}$ are $D_{\mathbb{R}}(g,h)$–sets.

3.2 Definition . *A hyperbolic element g in a semisimple connected algebraic group $\mathbb{G}$ is said to be $\mathbb{R}$– irreducible if the subgroup generated by $o(g)$ is dense in a maximal anisotropic torus in $\mathbb{G}$*

This definition was introduced by G.Prasad and A.Rapinchuk in [PR], where they have proved result about $\mathbb{R}$– irreducible elements of a Zariski dense semigroup in a semisimple connected group which is extremely important for us and used below.

3.3 Lemma. *Let Γ be a co–compactly dense subgroup in $SO(B)$, such that its Zariski closure G is a semisimple connected group. Let g and h be two hyperbolic transversal elements in Γ. Then there exists a $D_{\mathbb{R}}(g,h)$–set in Γ.*

Proof. It was shows for example in [AMS 4] that for any sequence of hyperbolic pairwise transversal elements $\{h_1, \ldots, h_t\}$ from Γ there exists a hyperbolic element h_{t+1} in Γ which is transversal to each $h_i, i = 1, \ldots, t$. Then there exists a sequence $S = \{h_1, h_2, \ldots, h_k\}$ of hyperbolic pairwise transversal elements of Γ such that

(i) h_i is transversal to both g and h for every $i = 1, \ldots, k$,

Let O_S be the closure (in the Euclidean topology) of the set $O = \{o(M)\}$. O_S is a product of tori and therefore O_S is a manifold. We assume that

(ii) O_S has maximal possible dimension.

We have to prove that $O_S = SO(\widehat{B}_{h_1}) \times K$. Suppose that $\dim(O_S) < \dim(O(\widehat{B}_{h_1}) \times K)$. The set O_S is constructible and therefore there exist Zariski closed subsets K_i and Zariski open subsets U_i of $O(\widehat{B}_{h_1}) \times K$,where $1 \leq i \leq r$, such that $O_S = \cup_{i=1}^{r} (K_i \cap U_i)$. Assume that K_i is a proper subset of $O(\widehat{B}_{h_1}) \times K$ for every i, $i = 1, \ldots, r$. The set of hyperbolic elements in $SO(\widehat{B}) \times K$, which are transversal to any its finite set of hyperbolic elements is Zariski open. It follows from [PR] that the set of $\mathbb{R}$–irreducible elements in the group Γ is Zariski dense in $G = SO(\widehat{B}) \times K$. Therefore there is an $\mathbb{R}$–irreducible element g of Γ which is transversal to each $h_i, i = 1, \ldots, r$ and to both g and h. Let R be the set $R = \{x \in G \mid y = xgx^{-1}$ is transversal to every h_i, $i = 0, 1, \ldots, r$ and $\pi(\langle o(y) \rangle) \subsetneq \widetilde{K}\}$, where $\widetilde{K} = \cup_{i=1}^{r} K_i$, and $\pi = \pi_r \pi_{h_r,y}$ and $\langle o(y) \rangle$ is the closure of the subgroup generated by $o(y)$ in $O(\widehat{B}_y) \times K$. It is clear that the set R is Zariski open. Let us show that this set is nonempty. Put $R_g = \{x \in G \mid x$ is hyperbolic, $A^+(x) = A^+(g), A^-(x) = A^-(g)$ and $\pi(\langle o(y) \rangle) \subsetneq K\}$, where $y = xgx^{-1}$ and $\pi = \pi_r \pi_{h_r,y}$. We clearly have $\langle o(y) \rangle = o(x) \langle o(g) \rangle\, o(x)^{-1}$ and $\pi = \pi_r \pi_{h_r,g}$. Since $\langle o(g) \rangle$ is a maximal torus of $O(\widehat{B}_g) \times K$, the set $\{o(x)\ \langle o(g) \rangle\ o(x)^{-1}, x \in R_g\}$ is an open subset of $O(B_g) \times K$. Therefore there exists an $x \in R_g$ such that $\pi(\langle o(y) \rangle) \subsetneq \widetilde{K}$. It is obvious that $R_g \subseteq R$, so R is nonempty. Thus $R \cap \Gamma \neq \varnothing$ since R is Zariski open and the subgroup Γ is Zariski dense. Let $x \in R \cap \Gamma$

and $h_{r+1} = xgx^{-1}$. Consider the new sequence $S = \{h_1, h_2, \ldots, h_r, h_{r+1}\}$, then $\dim O_S > d$. This contradicts our assumption. $\square$

Next we will define in the obvious way a topology on the set $\mathfrak{O}(B)$ of pair (W, g) where W is a maximal B–anisotropic vector subspace and $g \in O(B_W)$ (see definition 3.4). For every hyperbolic element $g \in G$ the pair $X_g = (A^0(g), o(g))$ is in $\mathfrak{O}(B)$. We thus have a map X from the set of all hyperbolic elements of Γ to $\mathfrak{O}(B)$, $X : \gamma \mapsto X_\gamma$, whose image we denote by $\mathfrak{O}(\Gamma)$. The main result of this section, lemma 3.6, says that for certain maximal B–anisotropic subspace W every element (W, o) is a limit point of $\mathfrak{O}(\Gamma)$ for every $o \in SO(B_W)$. This is the case for all W of the form $W = D^+(g) \cap D^-(h)$, where g and h are transversal hyperbolic elements of Γ. For the approximating element X_γ we can actually take an ε–hyperbolic element with ε bounded below by a positive bound.
Let $\mathfrak{O}(B)$ be the set of (W, o), where W is a maximal B-anisotropic subspace, B_W is the restriction of B to W and $o \in O(B_W)$. For any hyperbolic element g of $O(B)$ we have an element $X_g = (A^0(g), o(g)) \in \mathfrak{O}(B)$. Let $\mathfrak{O}(\Gamma) = \{X_g |\ g \in \Gamma,\ g \text{ is hyperbolic }\}$ and $\mathfrak{O}^\varepsilon(\Gamma) = \{X_g |\ g \in \Gamma,\ g \text{ is } \varepsilon\text{–hyperbolic }\}$.

We define a distance $\widetilde{d}$ between elements of $\mathfrak{O}(B)$. Recall the distance $\widehat{d}$ between subspaces of $\mathbb{R}^n$, see (1.1).

3.4 Definition . *Let $X = (W, g)$ and $Y = (U, h)$ be elements of $\mathfrak{O}(B)$. Put*

$$\widetilde{d}(X, Y) = \widehat{d}(W, U) + \max_{a \in S(W), b \in S(U)} \{|\, \|g(a) - h(b)\| - \|\, a - b\,\||\} \qquad (3.1).$$

Observe that $\widetilde{d}$ is a metric on $\mathfrak{O}^\varepsilon(\Gamma)$. Let $\{X_n\}_{n \in \mathbb{Z}}$ be a sequence of elements from $\mathfrak{O}(B)$. We write $X_n \rightrightarrows X$ if $\widetilde{d}\,(X_n, X) \to 0$ for $n \to \infty$. Denote by $\overline{\mathfrak{O}(\Gamma)}$ (resp. $\overline{\mathfrak{O}^\varepsilon(\Gamma)}$) the closure of $\mathfrak{O}(\Gamma)$ (resp. $\mathfrak{O}^\varepsilon(\Gamma)$) in $\mathfrak{O}(B)$.

Recall that for any two hyperbolic transversal elements $g \in O(B)$ and $h \in O(B)$ the intersection $A^0 = D(g)^+ \cap D(h)^-$ is a maximal B–anisotropic subspace.

3.5 Lemma *Let Γ be a Zariski dense subgroup in $SO(B)$. Let $g \in \Gamma$ and $h \in \Gamma$ be hyperbolic transversal elements, $A^0 = D^+(g) \cap D^-(h)$ and let B_{A^0} be the restriction of the form B to A^0. Then there exists an ε, such that $(A^0, \widetilde{o}) \in \overline{\mathfrak{O}^\varepsilon(\Gamma)}$ for every $\widetilde{o} \in SO(B_{A^0})$.*

Let us first prove the following corollary of this lemma.

3.6 Lemma *Let Γ be a Zariski dense subgroup in $SO(B)$.*

(1) If $X = (W, g_0) \in \mathfrak{O}(B)$ and $W = A^0(\gamma)$ for some ε–hyperbolic element $\gamma \in \Gamma$, then $(W, g) \in \overline{\mathfrak{O}^\varepsilon(\Gamma)}$ for every $g \in SO(B_W)$.

(2) If $X = (W, g_0) \in \overline{\mathfrak{O}^\varepsilon(\Gamma)}$, then $(W, g) \in \overline{\mathfrak{O}^\varepsilon(\Gamma)}$ for every $g \in SO(B_W)$.

Proof. . Let $g = h = \gamma$. Then, g and h are ε–transversal and $D^+(g) \cap D^-(h) = A^0(\gamma)$. Therefore $(W, x) \in \overline{\mathfrak{O}^\varepsilon(\Gamma)}$ for every $x \in SO(B_W)$, by Lemma 3.5. This proves (1).

Let us prove (2). It is clear that for every two maximal B–anisotropic subspaces W_1 and W_2, such that $\|W_1 - W_2\| < \varepsilon/2$ and any $x \in O(B_{W_1})$ there exists $y \in O(B_{W_2})$ such that $\|X - Y\| < \varepsilon$, where $X = (W_1, x)$ and $Y = (W_2, y)$. Hence (2) follows from (1) by using a diagonal process. □

The proof of lemma 3.5 will be divided into a sequence of lemmas.

Assume that g is an ε–hyperbolic element. Let W be a vector subspace in $\mathbb{R}^n$ such that $W \oplus D^-(g) = \mathbb{R}^n$ and $\widehat{d}(W, D^-(g)) \geq c$. Consider the ball $U(0, \delta) = \{t \in W \ |d(t, 0) \leq \delta\}$. Put $I_{\delta,c}(a) = a + U(0, \delta)$, where $a \in D^-(g)$.

Let us consider the decomposition of a vector $x = x^+ + x^0 + x^-$ according to the direct sum decomposition $A^+(g) \oplus A^0(g) \oplus A^-(g)$. Now suppose that $s(g) \to 0$. Then gx^+ expands exponentially to $+\infty$, gx^- contracts exponentially to 0 and $gx^0 = o(g)x^0$ remains in a sphere with respect to the positive definite form $B_{A^0(g)}$. Furthermore gW is close to $A^+(g)$ for every vector subspace W transversal to $D^-(g) = A^0(g) \oplus A^-(g)$. Hence for $I_{\delta,c}(a)$ the image $gI_{\delta,c}(a)$ intersects every subspace transversal to $A^+(g)$ in particular it will intersects $D^-(g^*)$ for every hyperbolic element g^* transversal to g. The image $gI_{\delta,c}(a)$ contains a ball of fixed radius in a vector space close to $A^+(g)$ with a center at the point of intersection. In the following lemmas we will give a quantitative version of this and generalize it in a straightforward way to sequences of elements.

3.7 Lemma . *Let $S^* = \{g_1, \ldots, g_{k-1}, g_k\}$ be a sequence of ε–hyperbolic elements such that g_i and g_{i+1} are ε–transversal for all $i = 1, \ldots, k$. Let $s = \max\{s(g_1), \ldots, s(g_{k-1}), s(g_k)\}$. Fix a vector $a_k \in D^-(g_k)$ and constants c, δ. Let $I_k = I_{\delta,c}(a_k)$. Then there exists a positive number $\beta = \beta(\varepsilon) < 1$, vectors $a_j \in D^-(g_j)$, $b_j \in A^0(g_j)$ and sets $I_j = I_{\delta,\varepsilon/2}(a_j, h_j)$, $j = 1, \ldots (k-1)$ such that for $s < \beta$ and $2 \leq j \leq k$ we have:*

(1) $g_j\,(I_j) \supseteq I_{j-1}$;

(2) $\pi_{g_j,g_{j-1}}(o(g_j)(b_j)) = b_{j-1}$

(3) $\|a_{j-1} - \tau^+_{g_j}(o(g_j)(b_j))\| \ll s \cdot \|a_k\|$

Proof. It is enough to prove the lemma for two ε–hyperbolic, ε–transversal elements g_1 and g_2. Let $I_{\delta,c}(a_2) = a_2 + U(0,\delta)$, where the vector a_2 belongs to $D^-(g_2)$ and let W be the vector subspace such that $U(0,\delta) \subseteq W$,and $\widehat{d}(W, D^-(g_2)) \geq c$. The projection of $U(0,\delta)$ onto $A^+(g_2)$ along $D^-(g_2)$ contains a ball $U(0,\delta^*)$ in $A^+(g_2)$, where $\delta \ll \delta^*$. It is clear that $g_2(U(0,\delta^*)) \supseteq U(0, s(g_2)^{-1}\delta^*)$. Therefore $g_2(U(0,\delta^*))$ contains a ball in $A^+(g_2)$ of a radius which tends to infinity for $s(g_2) \to 0$, uniformly in ε. Recall that δ is fixed. Similarly, $\widehat{d}\,(\,g_2W,\ A^+(g_2)\,) \ll s(g_2)c$. Then g_2W tends to $A^+(g_2)$ uniformly with ε when $s(g_2)$ tends to zero. Consequently there exists a positive number $\beta_0 = \beta_0(\varepsilon) < 1$ such that $g_2I_2 \cap D^-(g_1) \neq \varnothing$ for $s(g_2) < \beta_0$. There exists a positive integer $\beta_1 = \beta_1(\varepsilon)$ such that $\widehat{d}\,(\,g_2W,\ D^-(g_1)\,) \leq \varepsilon$ and the image of the projection of g_2I_2 onto $A^+(g_1)$ along $D^-(g_1)$

contains the ball $U(0,\delta)$ in $A^+(g_1)$ for $s(g_2) < \beta_1$. Put $\beta = \min\{\beta_0, \beta_1\}$. This completes the proof of (1).

Let $b_2 = \pi_{g_2}(a_2)$. Then there is a unique vector b_1 in $A^0(g_1)$ such that $b_1 = \pi_{g_1,g_2}(b_2)$. This proves (2).

There exists a point $x \in I_2$ such that $g_2(x) \in D^-(g_1)$. Put $a_1 = g_2(x)$. Then $x = x^+ + x_0 + x^-$, where $x^+ \in A^+(g_2)$, $x^+ \neq 0$, $x^- \in A^-(g_2)$, $x_0 \in A^0(g_2)$. It is clear, that $b_2 = x_0$ and $g_2x = g_2x^+ + o(g_2)x_0 + g_2x^-$. Since $\sigma^+_{g_2}(o(g_2)(b_2)) = g_2x^+ + o(g_2)x_0$, we have $\|a_1 - \sigma^+_{g_2}(o(g_2)(b_2)\| \ll s(g_2)\|a_2\|$. This finishes the proof. □

Lemma 3.8 . *Let the set S^* be as in Lemma 3.7. Assume that the elements g_1 and g_k are ε–transversal. Let a be a vector in $A^0_{g_1,g_k} = D^+(g_1) \cap D^-(g_k)$, $b = \tau^-_{g_k}(a)$, $c = \tau^+_{g_1}(o_1(g_1)o_2(g_2)\dots o_k(g_k)(\pi_k(b)))$ and let δ be a positive number. Then there exists a positive $\beta = \beta(\varepsilon)$ and a vector $c^* \in D^-(g_k)$ such that if $s < \beta$ we have:*

(1) $g_1g_2\dots g_kI_{\delta,\varepsilon}(a) \supseteq I_{\delta,\varepsilon/2}(c^*)$

(2) $\|c^* - c\| \ll s \cdot \|a\|$

Proof. Let us consider the set $\{h_1, h_2, \dots, h_{k+1}\}$ where $h_1 = g_k$ and $h_j = g_{j-1}$ for $2 \leq j \leq k+1$. Put $a_{k+1} = a$, $d = \varepsilon$. It is easy to verify that $\tau^+_{g_1}(o(g_1)(b_1)) = \tau^+_{g_1}o_1(g_1)\pi_{g_1,g_2}o(g_2)(b_2) =$
$\tau^+_{g_1}o_1(g_1)\pi_{g_1,g_2}o(g_2)\pi^{-1}_{g_1,g_2}\pi_{g_1,g_2}\pi_{g_2,g_3}o(g_3)(b_3) = \tau^+_{g_1}o_1(g_1)o_2(g_2)\pi_{g_2,g_3}o(g_3)(b_3) = \dots =$
$\tau^+_{g_1}(o_1(g_1)o_2(g_2)\dots o_{k+1}(g_{k+1})(\pi_{k+1}(b))) = c$, $b_{k+1} = b$ and $a_1 = c^*$. Then by lemma 3.7 (1), (3) we have precisely the assertion of the lemma. □

We now proceed to prove Lemma 3.5.

Proof. . Let $S = \{h_1, \dots, h_m\}$ be a $D_{\mathbb{R}}(g,h)$–set. As far as $\{g, h_1, \dots, h_m, h\}$ is also a $D_{\mathbb{R}}(g,h)$–set we will assume that $g = h_1, h = h_m$. Fix an ε such that all elements in S are 2ε–hyperbolic and 2ε–transversal.

Consider a positive m-tuple L, i.e. $L = (\ell_1, \ell_2, \ldots, \ell_m) \in \mathbb{Z}^m$ where $\ell_i > 0$, $i = 1, \ldots, m$. Put $|L| = \min_{1\le j\le m} \ell_j$. Recall that $A^0 = D^+(h_1) \cap D^-(h_m) = A^0_{h_1,h_m}$. Define $h(L) = h_1^{\ell_1} h_2^{\ell_2} \ldots h_m^{\ell_m}$. The idea is to find for every element $\widetilde{o} \in SO(B_{A^0})$ a sequence $\{L^{(r)}\}_{r\in\mathbb{N}}$ such that

(i) every element $h(L^{(r)}), r \in \mathbb{N}$, is ε-hyperbolic,

(ii) the sequence $X_r \rightrightarrows X$ when $r \to \infty$, where $X_r = \big(A^0(h(L^{(r)})), o(h(L^{(r)}))\big)$, $X = (A^0, \widetilde{o})$.

Recall that $\pi_m = \pi_{h_1,h_2} \ldots \pi_{h_{m-1},h_m}$, $\tau^-_{h_m} : A^0 \longrightarrow A^0(h_m)$ and $\tau^+_{h_1} : A^0(h_1) \longrightarrow A^0$. The composition $\tau^-_{h_m}\pi_m\tau^+_{h_1}$ is an element in $SO(B_{A^0})$as follows immediately from lemma 2.1 (3). Therefore, there exists an element o in $SO(B_{A^0})$ such that $\tau^+_{h_1} o\, \pi_m\, \tau^-_{h_m} = \widetilde{o}$. Recall that $o(L) = o_1^{\ell_1} \ldots o_m^{\ell_m}$. By lemma 3.3 (2), there exists a sequence $\{L^{(r)}\}_{r\in\mathbb{N}}$ such that $A^0(h(L^{(r)})) \to A^0$ and $o(L^{(r)}) \to o$ when $r \to \infty$. Hence $\tau^+_{h_1} o(L^{(r)})\, \pi_m\, \tau^-_{h_m} \to \widetilde{o}$ when $r \to \infty$. For every $i = 1, \ldots m$ there exists a sequence $\{k_i\}_{i\in\mathbb{N}}$ such that $k_i \to \infty$ and $o_i^{k_i} \to e$ since the group generated by o_i is compact. Thus we can and will assume that $o(L^{(r)}) \to o$ and $|L^{(r)}| \to \infty$ when $r \to \infty$.

Put $s_0^2 = s$. Fix $\beta(\varepsilon)$ as in lemma 1.3. It is clear that $s(g^n) = s(g)^{|n|}$ for any hyperbolic element g, $n \in \mathbb{Z}$. Then there exists a positive integer L_0 such that $s(h_i^{\ell_i}) < b(\varepsilon)$ for $|L^{(r)}| > L_0$. Therefore $h(L^{(r)})$ is ε–hyperbolic by lemma 1.3 (6), which proves (i).

Applying lemma 1.3 (3),(4) we obtain $\widehat{d}(A^+(h_1), A^+(h(L^{(r)}))) \le \varepsilon s_0^{2|L(r)|}$ and $\widehat{d}(A^-(h_m), A^-(h(L^{(r)}))) \le \varepsilon s_0^{2|L(r)|}$. Since $A^0(g) = D^+(g) \cap D^-(g)$, $D^+(g) = (A^+(g))^\perp$ and $D^-(g) = (A^-(g))^\perp$ for any hyperbolic element g, we conclude that

$$\widehat{d}(A^0(h(L^{(r)})), A^0) \ll s_0^{2|L(r)|} \qquad (3.2).$$

Fix a positive number δ. From (3.2) we have that there exists a positive integer L_0 such that $\widehat{d}(A^0(h(L^{(r)})), A^0) \le \delta$ for $|L^{(r)}| \ge L_0$.

To estimate the second term in (3.1) note, that for every hyperbolic element g the projection of the vector space $\mathbb{R}^n$ onto $A^0(g)$ along $A^+(g)+A^-(g)$ is a g-equivariant map, i.e. $\pi_g(gv) = g\pi_g(v)$. for every vector $v \in \mathbb{R}^n$. Let us denote $h^{(r)} = h(L^{(r)})$ and $\|v\|_r = B(v,v)^{1/2}$ for any vector $v \in A^0(h^{(r)})$ and $\|v\|_0 = B(v,v)^{1/2}$ for any vector $v \in A^0$.

Let a (resp. b) be a vector in $S(A^0(h^{(r)})$ (resp. $\in S(A^0)$). Let b_1 be a vector in A^0 such that $\pi_{h^{(r)}}(b_1) = a$. Observe that $b_1 \in D^-(h_m)$. Consider a ball $U(0,\delta)$ in the subspace $A^+(h^{(r)})$. Then by lemma 1.3 (5) $b_1 + U(0,\delta) = I_{\delta,\varepsilon}(b_1)$. Since $U(0,\delta) \subseteq A^+(h^{(r)})$ then $\pi_{h^{(r)}}(I_{\delta,\varepsilon}(b_1)) = a$. It follows from lemma 1.3 (1) and lemma 3.9 (1) follows that there exists a positive integer L_1 and a vector $c^* \in D^-(h_m)$ such that $h^{(r)}I_{\delta,\varepsilon}(b_1) \supseteq I_{\delta,\varepsilon/2}(c^*)$ for $|L^{(r)}| \geq L_1$. Choose vector b_2 in $I_{\delta,\varepsilon}(b_1)$ such that $h^{(r)}(b_2) = c^*$. It is clear that $\|b_2 - b_1\| \leq \delta$. Let $L^{(r)} = (\ell_1^{(r)}, \ldots, \ell_m^{(r)})$. Set $c = \tau_{h_1}^+ o_1(h_1)^{\ell_1^{(r)}} \ldots o_m(h_m)^{\ell_m^{(r)}} \pi_m \tau_{h_m}^-(b_1)$. It follows from lemma 1.3 (1) and lemma 3.9 (2) that there exists a positive integer L_2 such that $\|c - c^*\| \leq \delta$ for $|L^{(r)}| \geq L_2$. On the other hand, $\tau_{h_1}^+ o_1(h_1)^{\ell_1^{(r)}} \ldots o_m(h_m)^{\ell_m^{(r)}} \pi_m \tau_{h_m}^- = o(L^{(r)})$. Hence $c = o(L^{(r)})(b_1)$. Note that there exists a positive integer L_3 such that $\|o(L^{(r)}(b_1) - \widetilde{o}(b_1)\| \leq \delta$. Obviously $a^* = \pi_{h^{(r)}}(c^*) = \pi_{h^{(r)}}(h^{(r)}(b_2)) = h^{(r)}(\pi_{h^{(r)}}(b_2)) = h^{(r)}(a)$. By definition $h^{(r)}(a) = o(h^{(r)})(a)$. Set $L = \max\{L_0, L_1, L_2, L_3\}$. Then for $|L^{(r)}| \geq L$ we have $|\,\|o(h^{(r)})(a) - \widetilde{o}(b)\| - \|a - b\|\,| \leq \|a^* - h^{(r)}(b_2)\| + |\|h^{(r)}(b_2) - \widetilde{o}(b)\| - \|b_2 - b\|| + \|a - b_2\| \leq \delta + \|c^* - c\| + \|c - \widetilde{o}(b)\| - \|b_2 - b\|| + \delta \leq 3\delta + \|o(L^{(r)})(b_1) - \widetilde{o}(b_1)\| + |\|\widetilde{o}(b_1) - \widetilde{o}(b)\| - \|b_1 - b\|| + \|b_1 - b_2\| \leq 5\delta$. Thus for $X_r = \left(A^0(h^{(r)}), o(h^{(r)})\right)$ and $X = (A^0, \widetilde{o})$ we have $X_r \rightrightarrows X$ when $r \to \infty$. $\square$

Note that we have actually proved more, namely

3.9 Lemma *Let Γ be a Zariski dense subgroup in $SO(B)$. Let $g \in \Gamma$ and $h \in \Gamma$ be hyperbolic transversal elements, $A^0 = D^+(g) \cap D^-(h)$ and let B_{A^0} be the restriction of the form B to A^0. Let $\{g_1, \ldots, g_k\}$ be a $D_{\mathbb{R}}(g,h)$–set. Consider a $D_{\mathbb{R}}(g,h)$–set $S = \{h_1, h_2, h_3, \ldots h_{m-1}, h_m\}$ where $m = k+2$, $h_1 = g$, $h_m = h$, $h_{i+1} = g_i$, $i = 1, \ldots, k$. Set*

$h(L^{(r)}) = h_1^{\ell_1^{(r)}} h_2^{\ell_2^{(r)}} \dots h_m^{\ell_m^{(r)}}$ *for a positive* m*-tuple* $L^{(r)} = \left(\ell_1^{(r)}, \ell_2^{(r)}, \dots, \ell_m^{(r)}\right) \in \mathbb{Z}^m$, $\ell_i^{(r)} > 0$, $i = 1, \dots, m$, *then*

(i) $\widehat{d}(A^+(h(L^{(r)})), A^+(h_1)) \to 0$ *for every sequence* $\{L^{(r)}\}_{r\in\mathbb{N}}$, *when* $|L^{(r)}| \to \infty$

(ii) $\widehat{d}(A^-(h(L^{(r)})), A^-(h_m)) \to 0$ *for every sequence* $\{L^{(r)}\}_{r\in\mathbb{N}}$, *when* $|L^{(r)}| \to \infty$.

(iii) *for every element* $o \in SO(B_{A^0})$ *there exists a sequence* $\{L^{(r)}\}_{r\in\mathbb{N}}$ *such that* $\left(A^0(h(L^{(r)})), o(h(L^{(r)}))\right) \rightrightarrows (A^0, o)$ *when* $|L^{(r)}| \to \infty$.

(iv) *there is a positive number* $s_0 < 1$ *such that* $s(h(L^{(r)})) \le s_0^{|L^{(r)}|}$

Let Γ be a co–compactly dense subgroup of $SO(B)$ such that its Zariski closure is a semisimple group. Then combining lemma 3.3, lemma 3.5 and lemma 3.9 we have

3.10 Lemma . *Let* $g \in \Gamma$ *and* $h \in \Gamma$ *be hyperbolic transversal elements,* $A^0 = D^+(g) \cap D^-(h)$. *Let* $\{g_1, \dots, g_k\}$ *be a* $D_{\mathbb{R}}(g,h)$*–set. Consider a* $D_{\mathbb{R}}(g,h)$*–set* $S = \{h_1, h_2, h_3, \dots h_{m-1}, h_m\}$ *where* $m = k+2$, $h_1 = g$, $h_m = h$, $h_{i+1} = g_i, i = 1, \dots, k$. *Set* $h(L^{(r)}) = h_1^{\ell_1^{(r)}} h_2^{\ell_2^{(r)}} \dots h_m^{\ell_m^{(r)}}$ *for a positive* m*-tuple* $L^{(r)} = \left(\ell_1^{(r)}, \ell_2^{(r)}, \dots, \ell_m^{(r)}\right) \in \mathbb{Z}^m$, $\ell_i^{(r)} > 0, i = 1, \dots, m$, *then*

(i) $\widehat{d}(A^+(h(L^{(r)})), A^+(h_1)) \to 0$ *for every sequence* $\{L^{(r)}\}_{r\in\mathbb{N}}$, *when* $|L^{(r)}| \to \infty$.

(ii) $\widehat{d}(A^-(h(L^{(r)})), A^-(h_m)) \to 0$ *for every sequence* $\{L^{(r)}\}_{r\in\mathbb{N}}$, *when* $|L^{(r)}| \to \infty$.

For every element $o \in SO(\widehat{B}_{A^0}) \times K$ *there exists a sequence* $\{L^{(r)}\}_{r\in\mathbb{N}}$ *such that*

(iii) $\left(A^0(h(L^{(r)})), o(h(L^{(r)}))\right) \rightrightarrows (A^0, o)$ *when* $|L^{(r)}| \to \infty$

(iv) *there is a positive number* $s_0 < 1$ *such that* $s(h(L^{(r)})) \le s_0^{|L^{(r)}|}$.

Recall that $V^0(g) = \{v \in \mathbb{R}^n \mid gv = v\}$ where g is a hyperbolic element. It is clear that $V^0(g) \le A^0(g)$.

3.11 Lemma . *Let* $X = (W, g)$ *be an element in* $\overline{\mathfrak{D}^{\varepsilon}(\Gamma)}$. *Set* $V^0 = \{v \in W \mid gv = v\}$. *Let* $\{X_r\}_{r\in\mathbb{N}}$ *be a sequence* $X_r = (A^0(g_r), o(g_r))$, $g_r \in \Gamma$, *such that* $X_r \rightrightarrows X$. *Set* $m_0 = \max_{N\to\infty}\{\max_{r\geq N} \dim V^0(g_r)\}$. *Then*

(i) $\dim V^0(g) \geq m_0$

(ii) for every δ *there exists* $N = N(\delta, \varepsilon)$ *such that* $\measuredangle(V^0(g_r), V^0(g)) \leq \delta$ *for* $r > N$.

(iii) if g_0 *is a regular element, then for every* δ *there exists an* $N = N(\delta, \varepsilon)$ *such that* $\widehat{d}(V^0(g_r), V^0(g)) \leq \delta$

Proof. (i). Assume that $\dim V^0(g) < m_0$. Without loss of generality we can assume that $\dim V^0(g_r) = m_0$ for all integers r. Then there exists a subspace W_0 in A^0 such that $\widehat{d}(V^0(g_r), W_0) \to 0$ when $r \to \infty$. Thus $\dim W_0 = m_o$ there is a vector $w_0 \in S(W_0)$ which is B–orthogonal to V_0. Consider a sequence of vectors $\{w_0^{(r)}\}_{r\in\mathbb{N}}$, $w_0^{(r)} \in S(V^0(g_r))$, such that $w_0^{(r)} \to w_0$ when $r \to \infty$. Since for every δ there exists a positive integer N_0 such that $\|g(w_0) - g_r(w_0^{(r)})\| \leq \|w_0 - w_0^{(r)}\| + \delta$ and $\|w_0 - w_0^{(r)}\| \leq \delta$, we have $\|g(w_0) - g_r(w_0^{(r)})\| \leq 2\delta$. Hence, $\|g(w_0) - w_0\| \leq \|g_r(w_0^{(r)}) - w_0^{(r)}\| + 3\delta$. By choice $\|g(w_0) - w_0\| \leq 3\delta$. Therefore $w_0 \in V^0$. This contradicts our assumption that $w_0 \in (A^0)^{\perp}$.

(ii). Assume without loss of generality that there exists a δ such that $\measuredangle(V^0(g_r), V^0(g)) \geq \delta$ and $\dim V^0(g_r)$ is the same for all positive integers r. Then there exists a subspace V^0 in A^0 such that $\widehat{d}(V^0(g_r), V^0) \to 0$ when $r \to \infty$. By assumption, there exists vector $v^0 \in S(V^0)$ such that $\min_{w\in S(V^0(g))} \measuredangle(v^0, w) \geq \delta$. Thus there is a constant c such that $\|g(v^0) - v^0\| > c$. Let $\{v_0^{(r)}\}_{r\in\mathbb{N}}$ be a sequence of vectors $v_0^{(r)} \in V^0(g_r)$ converging to v^0. Then by the same arguments as above for every δ_1 there is a positive integer N_1 such that $\|g(v^0) - v^0\| \leq \|g_r(v_0^{(r)}) - v_0^{(r)}\| + \delta_1$ for all $r \geq N_1$. Therefore $\|g(v^0) - v^0\| \leq \delta_1$ contrary to $\|g(v^0) - v^0\| > c$.

(iii) immediately follows from (ii). □

4. Euclidean parts of affine transformations

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In this section we take the translational parts of the elements of Γ into account. Remember that they are affine transformations. We show by using the orthogonal maps of the last section that they go in all directions in the sense that their convex hull in its interior. This will be crucial for going backwards by an element of Γ in the section 5, see in particular (5.1). The discussion is complicated by consider in the same time more general case that Γ is co-compactly dense.

Recall that G_B is the subgroup of the affine group $\mathrm{Aff}\mathbb{R}^n$ consisting of those elements g whose linear part belongs to $O(B)$, where B is a non degenerate quadratic from on $\mathbb{R}^n$ with signature (p, q), $p+q=n$. By definition, an element g in G_B is called hyperbolic (resp. ε–hyperbolic) if the linear part $\ell(g)$ of g is hyperbolic (resp. ε–hyperbolic). Two elements g and h in G_B are called transversal (resp. ε–transversal) if their linear parts $\ell(g)$ and $\ell(h)$ are transversal (resp. ε–transversal). Set $o(g) = o(\ell(g))$, where a g is a hyperbolic element in G_B (see (1.1)).

Unless otherwise stated we will assume that Γ *is a co–compactly dense subgroup of* G_B. Let $\mathbb{G}$ be the Zariski closure of the group Γ. Recall that by definition the vector space $\mathbb{R}^n$ is the direct sum of B–orthogonal, $\ell(\Gamma)$–invariant subspaces V_1 and V_2. There are two natural affine spaces $A_1 = \mathbb{R}^n/V_2$ and $A_2 = \mathbb{R}^n/V_1$, projections $\tau_1 : \mathbb{R}^n \to \mathbb{R}^n/V_2$ and $\tau_2 : \mathbb{R}^n \to \mathbb{R}^n/V_1$, induced homomorphisms $\rho_1 : \Gamma \to \mathrm{Aff}\ A_1$ $\rho_2 : \Gamma \to \mathrm{Aff}\ A_2$ and $\ell(\rho_1(g)) = \ell_1(g)$, $\ell(\rho_2(g)) = \ell_2(g)$. Recall that for a hyperbolic element g we defined $t_g^{(i)} = t_{\rho_i(g)}$, $i = 1, 2$.

Now we extend some definitions given in the previous chapter for linear transformations to affine transformations. Let $\mathfrak{O}_a(B)$ be the set of all (X, v) where $X = (W, \ell(g)) \in$

$\mathfrak{O}(B)$, $v \in W$, $\ell(g)(v) = v$ and $B(v,v) = 1$. We endow $\mathfrak{O}_a(B)$ with the topology induced from the product $\mathfrak{O}(B) \times W$. Namely, we say that the sequence $\{(X_n, v_n)\}_{n\in\mathbb{N}}$ in $\mathfrak{O}_a(B)$ converges to $(X,v) \in \mathfrak{O}_a(B)$ if $X_n \rightrightarrows X$ and $v_n \to v$. Put $\mathfrak{O}(\Gamma) = \mathfrak{O}(\ell(\Gamma))$ and $\mathfrak{O}^\varepsilon(\Gamma) = \mathfrak{O}^\varepsilon(\ell(\Gamma))$ for a subgroup Γ in G_B. Define $\mathfrak{O}_a(\Gamma) = \{\big(X_g, v_g^0\big) \mid X_g \in \mathfrak{O}(\Gamma),\ g$ a hyperbolic element of $\Gamma\}$ and $\mathfrak{O}^\varepsilon{}_a(\Gamma) = \{\big(X_g, v_g^0\big) \mid X_g \in \mathfrak{O}^\varepsilon(\Gamma),\ g$ an ε–hyperbolic element of $\Gamma\}$.

Assume that g and h are hyperbolic, transversal elements of Γ. Set $A^0 = A^0_{g,h} = D^+(g) \cap D^-(h)$. Let v_1 and v_2 be two non-zero vectors from A^0. We define the convex cone $Con_{\{v,w\}}$ in A^0 by $Con_{\{v_1,v_2\}} = \{v \in A^0 \,|\, B(v,v_1) > 0, B(v,v_2) < 0\}$. Let $\tau_h^- : A^0(h) \to A^0$ and $\tau_g^+ : A^0(g) \to A^0$ be the isometry defined in (3.1). Set $Con_{\{g,h\}} = Con_{\{\tau_g^+(v^0(g)),\tau_h^-(v^0(h))\}}$.

Let $g_0 \in \Gamma$ be a hyperbolic element transversal to both g and h. Consequently we have the following maximal B- anisotropic subspaces $A^0_{\{g,g_0^{-1}\}} = D^-(g) \cap D^+(g_0^{-1})$, $A^0_{\{g_0,h\}} = D^-(g_0) \cap D^+(h)$, $A^0 = D^+(g) \cap D^-(g)$. We conclude by repeating the arguments given before lemma 2.2 that there exist two isometries $\varsigma_{g_0}^+ : A^0(g_0) \to A^0$ and $\varsigma_{g_0}^- : A^0(g_0) \to A^0$. Namely, $\varsigma_{g_0}^+$ (resp. $\varsigma_{g_0}^-$) is the composition of the following projections $A^0(g_0) \to A^0_{\{g,g_0^{-1}\}}$ along $A^+(g_0^{-1})$, $A^0_{\{g,g_0^{-1}\}} \to A^0(g)$ along $A^-(g)$ and $A^0(g) \to A^0$ along $A^+(g)$ (resp. $A^0(g_0) \to A^0_{\{g_0,h\}}$ along $A^-(g_0)$, $A^0_{\{g_0,h\}} \to A^0(h)$ along $A^+(h)$ and $A^0(h) \to A^0$ along $A^-(h)$). Recall that $A^+(g_0^{-1}) = A^-(g_0)$ and $D^+(g_0^{-1}) = D^-(g_0)$. Define a convex cone $Con_{\{g,h\}}(g_0)$ by $Con_{\{g,h\}}(g_0) = Con_{\{\varsigma_{g_0}^+(v^0(g_0^{-1})),\varsigma_{g_0}^-(v^0(g_0))\}}$ and set $\widehat{Con}_{\{g,h\}}(g_0) = Con_{\{g,h\}}(g_0) \cap V_1$. It is easily seen that $\widehat{Con}_{\{g,h\}}(g_0) = Con_{\{\rho_1(g),\rho_1(h)\}}(\rho_1(g_0))$.

Recall that $a_b = bab^{-1}$.

4.1 Lemma. *(A) Let g and h be hyperbolic elements in Γ. Assume that Γ is Zariski dense in G_B and $p - q \geq 2$. Then the set $R_g(h) = \{x \in \mathbb{G} \,|\, h_x$ is transversal to g, $Con_{\{g,h_x\}} \neq \emptyset\}$ is a Zariski open non-empty subset of $\mathbb{G}$.*

(B) Assume that q is even and Γ is co–compactly dense in G_B. Let g and h be two hyperbolic transversal elements of Γ. Let $g_0 \in \Gamma$ be a hyperbolic element such that $t_{g_0}^{(1)} \neq 0$.

Then the set $R_{\{g,h\}}(g_0) = \{x \in \mathbb{G} \,|\, xg_0x^{-1}$ *is a hyperbolic element transversal to both* g *and* h, $\widehat{Con}_{\{g,h\}}(xg_0x^{-1}) \neq \emptyset\}$ *is a Zariski open non-empty subset of* $\mathbb{G}$.

Proof. It is well known (see [AMS 3]) that if $\{g_0, g_1, \dots, g_m\} \subseteq O(B)$ is a set of hyperbolic elements and $G \leq O(B)$ is an irreducible subgroup then the set $\{x \in G \,|\, xg_0x^{-1}$ and g_i are transversal for $i = 1, \dots, m\}$ is a non-empty Zariski open subset of G.
(A). Let τ_x^- (resp. τ_x^+) be the projection $\tau_x^- : A^0(h_x) \to A_x^0$ along $A^-(h_x)$(resp. $\tau_x^+ : A^0(g) \to A_x^0$ along $A^+(g)$) where $A_x^0 = D^+(g) \cap D^-(h_x)$. Then $R_g(h) = \{x \in \mathbb{G} \,|\, \tau_x^+(v^0(g)) \neq \tau_x^-(v^0(h_x))$, h_x is transversal to $g\}$. Observe that this set is Zariski open. We claim that this set is non-empty . Indeed, since $p - q \geq 2$ the group $SO(B_h)$ acts transitively on $A^0(h)$ there exists $t \in O(B)$ such that $\tau_x^+(v^0(g)) \neq \tau_x^-(tv^0(h))$. Let $x \in G_B$ be a hyperbolic element such that $A^{\pm}(x) = A^{\pm}(h), \ell(x) = t$. Then $v^0(h_x) = tv^0(h)$, $\tau_x^+ = \tau_g^+$, $\tau_x^- = \tau_h^-$. Therefore x belongs to $R_g(h)$ and the proof of (A) is completed.
(B). Since $\widehat{Con}_{\{g,h\}}(g_0) = Con_{\{\rho_1(g), \rho_1(h)\}}(\rho_1(g_0))$ we can and will assume that $A_1 = \mathbb{R}^n$, $p + q = n$ and $\ell(\Gamma)$ is Zariski dense in $SO(B)$. Let g_0 be a hyperbolic element, transversal to both g and h. Since $\ell(g_o)$ has a fixed vector then $p - q$ is an odd number. From lemma 2.2 we conclude that there exists a vector $v \in A^0(g_0)$ such that $\varsigma_{g_0}^+(-v) \neq \varsigma_{g_0}^-(v)$. Therefore if $p - q = 1$ we have even more, namely, $\varsigma_{g_0}^+(-v^0(g_0)) = -\varsigma_{g_0}^-(v^0(g_0))$. Since $\varsigma_{g_0}^+(v^0(g_0^{-1})) = \varsigma_{g_0}^+(-v^0(g_0)$ we have $\varsigma_{g_0}^+(v^0(g_0^{-1})) \neq \varsigma_{g_0}^-(v^0(g_0))$. Consequently $e \in R_{\{g,h\}}(g_0)$ when $p - q = 1$. If $p - q \geq 2$ there exists an element $t \in SO(B_{g_0}) \cap G$ such that $tv^0(g_0) = v$. Then $tv^0(g_0^{-1}) = -v$. There is element $x \in \mathbb{G}$ such that $\ell(x) = t$, $A^{\pm}(x) = A^{\pm}(g_0)$. Write $g_x = xg_0x^{-1}$. Therefore $v^0(g_x) = v, v^0(g_x^{-1}) = -v$ and $\varsigma_{g_x}^+(v^0(g_x^{-1})) \neq \varsigma_{g_x}^-(v^0(g_x))$. Hence the element x belongs to the subset $R_{\{g,h\}}(g_0)$ Thus, the Zariski open subset $R_{\{g,h\}}(g_0)$ of $\mathbb{G}$ is non-empty set. □

Let g and h be two hyperbolic transversal elements. We will say that the set of affine transformations $\{h_1, \dots h_m\}$ is $D_{\mathbb{R}}(g, h)$–set if and only if the set of their linear parts

$\{\ell(h_1),\dots \ell(h_m)\}$ forms a $D_{\mathbb{R}}(\ell(g),\ell(h))$–set.

Let $h_1,\dots,h_m$ be a $D_{\mathbb{R}}(g,h)$–set. As usually we can and will assume that $h_1=g$, $h_m=h$. Recall that $h(L)=h_1^{\ell_1}\dots h_m^{\ell_m}$ for positive m-tuples $L=(\ell_1,\dots,\ell_m)\in\mathbb{Z}^m$. We claim that the following is true.

4.2 Lemma. *Assume that the linear part of Γ is Zariski dense in $SO(B)$. Let g and h be two hyperbolic transversal elements of Γ such that the convex cone $Con_{\{g,h\}}$ is non-empty. Let $v^0\in Con_{\{g,h\}}$, $B(v^0,\,v^0)=1$ and let o be a regular element of $SO(B_{A^0})$ such that $o(v^0)=v^0$. Put $X=(A^0,o)$. Then there exist sequences $\{L_+^{(r)}\}_{r\in\mathbb{N}}$ and $\{L_-^{(r)}\}_{r\in\mathbb{N}}$ such that*

(i) $|L_+^{(r)}|\to\infty$ *when* $r\to\infty$

(ii) $|L_-^{(r)}|\to\infty$ *when* $r\to\infty$.

(iii) $\left(X_{h(L_+^{(r)})},v^0_{h(L_+^{(r)})}\right)\rightrightarrows(X,v^0)$ *when* $r\to\infty$.

(iv) $\left(X_{h(L_-^{(r)})},v^0_{h(L_-^{(r)})}\right)\rightrightarrows(X,-v^0)$ *when* $r\to\infty$.

Proof. It follows from lemma 3.9 that there exists an infinite sequence of m–tuples $\{L^{(r)}\}_{r\in\mathbb{N}}$ such that $\left(A^0_{h(L^{(r)})},o(h(L^{(r)}))\right)\rightrightarrows(A^0,o)$ and $|L^{(r)}|\to\infty$ when $r\to\infty$. Recall that $L^{(r)}=\left(\ell_1^{(r)},\dots,\ell_m^{(r)}\right)$. Let us first show that for every $i=1,\dots,m$ there exists a subsequence $\{L_i^{(r)}\}_{r\in\mathbb{N}}$ of $\{L^{(r)}\}_{r\in\mathbb{N}}$ such that

(1) $|L_i^{(r)}|\to\infty$ when $r\to\infty$

(2) $\left(A^0_{h(L_i^{(r)})},o(h(L_i^{(r)}))\right)\rightrightarrows(A^0,o)$

(3) $\ell_k^{(r)}/\ell_i^{(r)}\to 0$ for $1\le k\le m, k\neq i$.

It suffices to show, that a sequence which fulfills the properties (1) and (2) can be changed in such a way that it will also have property (3). It is clear that for every $i = 1, \ldots, m$ there exists a sequence of positive m- tuples $\left(s_1^{(r)}, \ldots s_m^{(r)}\right)$ such that

(a) $o(h_k)^{s_k^{(r)}} \to e$ for $r \to \infty$ and $k = 1, \ldots, m$

(b) $s_k^{(r)} \to \infty$ for $r \to \infty$ and $k = 1, \ldots, m$

(c) $(s_k^{(r)} + \ell_k^{(r)})/(s_i^{(r)} + \ell_i^{(r)}) \to 0$ for $r \to \infty$ and $k \neq i, 1 \leq k \leq m$.

Put $\widetilde{\ell}_k^{(r)} = s_k^{(r)} + \ell_k^{(r)}$ and $L_i^{(r)} = \left(\widetilde{\ell}_1^{(r)}, \ldots, \widetilde{\ell}_m^{(r)}\right)$. It is easy to see that the sequence $\{L_i^{(r)}\}_{r\ in \mathbb{N}}$ fulfills the properties (1), (2) and (3).
The proof will be completed by showing that $v^0(h(L_1^{(r)})) \to v^0$ and $v^0(h(L_m^{(r)})) \to -v^0$ when $r \to \infty$.

Denote $L_1^{(r)} = L_+^{(r)}$ and $L_m^{(r)} = L_-^{(r)}$. Let $L_+^{(r)} = \left(\ell_1^{(r)}, \ldots, \ell_m^{(r)}\right)$. Set $g_1(L_+^{(r)}) = h_1^{\ell_1^{(r)}}$ and $g_2(L_+^{(r)}) = h_2^{\ell_2^{(r)}} \ldots h_m^{\ell_m^{(r)}}$. Clearly $h(L_+^{(r)}) = g_1(L_+^{(r)}) g_2(L_+^{(r)})$. Now for $L_-^{(r)} = \left(\ell_1^{(r)}, \ldots, \ell_m^{(r)}\right)$ set $g_1(L_-^{(r)}) = h_1^{\ell_1^{(r)}} \ldots h_{m-1}^{\ell_{m-1}^{(r)}}$ and $g_2(L_-^{(r)}) = h_m^{\ell_m^{(r)}}$ Clearly $h(L_-^{(r)}) = g_1(L_-^{(r)}) g_2(L_-^{(r)})$. We have $\dim V^0(h(L^{(r)})) \geq 1$, since $p - q$ is an odd integer. On the other hand $\dim V^0(o) = 1$, since o is a regular element in $SO(B_{A^0})$. Then $\widehat{d}\left(V^0(h(L^{(r)})), V^0(o)\right) \to 0$ when $r \to \infty$ by lemma 3.10 (ii). Therefore the sequence $\{v^0_{h(L^{(r)})}\}_{r \in \mathbb{N}}$ converges to the vector αv^0, where $|\alpha| = 1$. It thus remain to show that the sequence $\{v^0_{h(L_+^{(r)})}\}_{r \in \mathbb{N}}$ converges to a vector v^0 and the sequence $\{v^0_{h(L_-^{(r)})}\}_{r \in \mathbb{N}}$ converges to $-v^0$.

Let ε be a positive number such that the elements $h_i, i = 1, \ldots, m$, are 2ε–hyperbolic and the elements $h_i, h_j, 1 \leq i, j \leq m$ are 2ε–transversal. Fix a point q_0 in the affine space $\mathbb{R}^n$. Consider the sequence $\{L_+^{(r)}\}_{r \in \mathbb{N}}$. Let $C_r = E^-_{g_1(L_+^{(r)})} \cap E^+_{g_2(L_+^{(r)})}$ and let $p_0(r)$ be the point in C_r with $d(q_0, p_0(r)) = d(q_0, C_r)$. Let $g_1(L_+^{(r)}) p_0(r) = p_1(r)$ and $g_2(L_+^{(r)})^{-1} p_0(r) = p_2(r)$. Set $v(r) = p_1(r) - p_2(r)$ and $v_1(r) = p_1(r) - p_0(r)$ and $v_2(r) = p_0(r) - p_2(r)$. By lemma 1.3 (1), (5) and (6) there exists a positive integer N_1 such that for $|L^{(r)}| > N_1$

the elements $h_i^{\ell_i^{(r)}}$, $i = 1, \dots, m$, fulfill the hypotheses of the main lemma 1.6. Therefore, $d(p_0(r), C_{g_1(L_+^{(r)})}) \ll 1$, $d_{g_1(L_+^{(r)})}(p_0(r)) \sim \ell_1^{(r)}$, $d(p_0(r), C_{g_2(L_+^{(r)})}) \ll 1$ and $d_{g_2(L_+^{(r)})}(p_0(r)) \ll \ell_2^{(r)} + \cdots + \ell_m^{(r)}$ when $|L^{(r)}| > N_1$. Hence $\|v_1(r)\|/\|v(r)\| \to 1$ and $\|v_2(r)\|/\|v(r)\| \to 0$ when $r \to \infty$, by (3). It is immediate that $v(r)/\|v(r)\| \to v^0(h_1)$ when $r \to \infty$. Recall that $h_1 = g$. Thus

$$v(r)/\|v(r)\| \to v^0(g) \tag{4.1}$$

when $r \to \infty$. As $p - q$ is odd we have $\dim V^0(x) = 1$ for every regular element $x \in SO(B_{A^0})$. Therefore $\dim V^0(o) = 1$. Since by lemma 3.12, $\measuredangle V^0(o), V^0(h(L_+^{(r)})) \to 0$ when $r \to \infty$ we have $v^0(h(L_+^{(r)})) \to \alpha v^0$, where $\alpha = \pm 1$. From the definition of the vector $v^0(h(L_+^{(r)}))$ follows that $B(v^0(h(L_+^{(r)})), v(r)) > 0$. Hence from (4.1) we obtain $\alpha = 1$. This proves that $\left(X_{L_+^{(r)}}, v^0(h(L_+^{(r)}))\right) \rightrightarrows (X, v^0)$. By similar arguments, $\left(X_{L_-^{(r)}}, v^0(h(L_-^{(r)}))\right) \rightrightarrows (X, -v^0)$. □

Let $\widetilde{\Gamma}$ be the group generated by $h_1, \dots, h_m$ and let $\omega : \widetilde{\Gamma} \to \mathbb{Z}^+$ be the corresponding word metric. Since there exists an $N \in \mathbb{N}$ such that $h_1^N, \dots h_m^N$ freely generate a free group we have $\omega(h(L)) = \ell_1 + \dots \ell_m$ for $|L| > N$. It is possible to show that

$$\chi(h(L_+^{(r)})) \to B(\tau_g^+(t_g), v^0)v^0 \tag{4.2}$$

and

$$\chi(h(L_-^{(r)})) \to B(\tau_h^-(t_h), v^0)v^0 \tag{4.3}$$

when $r \to \infty$.

Let V be a vector space a subset of non-zero vectors $\{v_1, \dots, v_m\}$ of V is called $convex-full$ if

(i) $span\{v_1, \dots, v_m\} = V$

(ii) there exist positive numbers $\alpha_1, \dots, \alpha_m$ such that $\sum_1^m \alpha_i = 1$ and $\Sigma_1^m \alpha_i v_i = 0$

Let V be a Euclidean space. It is clear that a subset $\{v_1, \ldots, v_m\}$ of V is convex–full if and only if a subset $\{v_1/\|v_1\|, \ldots, v_m/\|v_m\|\}$ of V is convex–full. Recall that if g is a hyperbolic element then we consider $A^0(g)$ as a Euclidean space with respect to the positive definite quadratic form $B_g = B|_{A^0(g)}$.

Here is an elementary property of the concept of convex–full subset.

4.3 Lemma. *Let g be a hyperbolic element and $\{v_1, \ldots, v_m\}$ be a convex–full subset of $A^0(g)$. Then there is a constant $\vartheta = \vartheta(v_1, \ldots, v_m) < 1$ such that for every non-zero vector $v \in A^0(g)$ the following is true : there exists an index i_0 such that $\cos\measuredangle(v, v_{i_0}) < 0$ and $\sin\measuredangle(v, v_{i_0}) \leq \vartheta$.*

Proof. We can assume that $\|v_i\| = 1$ for all $i = 1, \ldots, m$. Put $\theta(v) = \sum_1^m B_g(v_i, v)^2$ where $v \in S(A^0(g))$. The number $\theta_1 = \inf_{v \in S(A^0(g))} \sum_1^m B_g(v_i, v)^2$ is positive since $S(A^0(g))$ is compact. Set $\vartheta = (1-\theta_1)^{1/2}$. Hence, if $\cos\measuredangle(v, v_i) \neq 0$ then $\sin\measuredangle(v, v_i) \leq \vartheta$. By definition there exist positive numbers $\alpha_1, \ldots, \alpha_m$ such that $\sum_1^m \alpha_i v_i = 0$. Hence $\sum_1^m \alpha_i B_g(v_i, v) = 0$. Thus there is an index i_0 such that $B_g(v_{i_0}, v) < 0$. Then $\cos\measuredangle(v, v_{i_0}) < 0$ and $\sin\measuredangle(v, v_{i_0}) \leq \vartheta$. □

We need the following

4.4 Definition. *Let g and h be two 2ε–hyperbolic, 2ε-transversal elements. We call a sequence $S = \{g_1, \ldots, g_k\}$ of ε–hyperbolic, pairwise ε-transversal elements a (θ, ε)–system for g and h if for every hyperbolic element g_0 we have*

(1) if g_0 is 2ε–transversal to both g and h, then g_0 is ε–transversal to every element g_i, $i = 1, \ldots, k$,

(2) if v is an arbitrary vector in $A^0(g_0)$ then there exists an index $i_0, 1 \leq i_0 \leq k$, such that $\cos\measuredangle(v^0_{g_{i_0}}, \pi_{g_{i_0}, g_0}(v)) \leq -\theta$.

When it will case no confusion we will say briefly $S = \{g_1, \ldots, g_k\}$ is a (θ, ε)–system.

4.5 Lemma *Let g and h be two 2ε–hyperbolic 2ε–transversal elements and $\{v_1, \ldots, v_m\}$ be a convex–full subset in $A^0 = D^+(g) \cap D^-(h)$. Let $\{h_i^{(r)}\}_{r\in\mathbb{N}}, i = 1, \ldots, m$ be sequences of hyperbolic elements such that when $r \to \infty$ for each $i = 1, \ldots, m$ we have*

(1) $\widehat{d}(A^+(h_i^{(r)}), A^+(g)) \to 0$

(2) $\widehat{d}(A^-(h_i^{(r)}), A^-(h)) \to 0$

(3) $v^0(h_i^{(r)}) \to v_i$.

Then there exist a positive $N = N(\varepsilon)$ and $\theta = \theta(g, h) < 1$ such that if $r \geq N$ the subset $\{h_1^{(r)}, \ldots, h_m^{(r)}\}$ is a (θ, ε)–system for g and h.

Proof. Let $\theta(v) = \sum_1^m B(v_i, \pi_{h,g_0}(v))^2$ where $v \in S(A^0(g))$. Then by the same arguments as in the proof of lemma 4.3 there exists a positive constant $\theta_1 = \theta_1 < 1$ such that

$$\inf_{v\in S(A^0(g_0))} \theta(v) \geq \theta_1 \qquad (4.4).$$

By (1) and (2), there exists a positive integer $R_1 = R_1(\varepsilon)$ such that for $r > R_1$ and all $i = 1, \ldots, m$,

$$\widehat{d}(A^\pm(h_i^{(r)}), A^\pm(g)) \leq \varepsilon/6 \qquad (4.5)$$

$$\widehat{d}(A^0, A^0(h_i^{(r)})) \leq \varepsilon/6 \qquad (4.6)$$

$$\|v_i - v^0(h_i^{(r)})\| \leq \varepsilon/6 \qquad (4.7)$$

Let g_0 be a hyperbolic element 2ε–transversal to both g and h and $\delta \leq \varepsilon$, then g_0 is ε–transversal to each $h_i^{(r)}, i = 1, \ldots, m$ and $r \geq R_1$. We also conclude from (1) and (2) that for $r \geq R_1$ the element $h_i^{(r)}$ is ε– hyperbolic for $i = 1, \ldots, m$. Therefore for all $i = 1, \ldots, m$ and all $r \geq R_1$ the isometries $\pi_{h_i^{(r)},g_0}$ and π_{h,g_0} are defined. It follows from (4.5) and (4.6) that for all $i = 1, \ldots, m$

$$\|\pi_{h_i^{(r)},g_0}(v) - \pi_{h,g_0}(v)\| \to 0 \qquad (4.8)$$

for every vector v in $S(A^0(g_0))$. Consider the functions

$$\theta^{(r)}(v) = \sum_1^m B(v^0(h_i^{(r)}), \pi_{h_i^{(r)},g_0}(v))^2$$

where $r \in \mathbb{N}$ and $v \in S(A^0(g_0))$. It follows easily from (4.8) that for every δ there exists a positive integer $R_2 = R_2(\varepsilon)$ such that for $r \geq R_2$

$$|\theta(v) - \theta^{(r)}(v)| \leq \delta \tag{4.9}$$

for all $v \in S(A^0(g_0))$. Combining (4.9) and (4.4) we conclude that there exists a number $R_3 = R_3(\varepsilon)$ such that for all $r \geq R_3$, we have

$$\inf_{v \in S(A^0(g_0))} \theta^{(r)}(v) \geq \theta_1/2 \tag{4.10}$$

Assume that there exists a subsequence $\{r_k\}_{k\in\mathbb{N}}$ such that $B(v^0(h_i^{(r_k)}), \pi_{h_i^{(r_k)},g_0}(v)) \geq 0$ for all $i = 1, \ldots, m$. Let $\alpha_1, \ldots \alpha_m$ be positive integers such that $\sum_1^m \alpha_i = 1$ and $\sum_1^m \alpha_i v_i = 0$. Let $\alpha_0 = \min\{\alpha_1, \ldots \alpha_m\}$. Then from (4.10) follows, that

$$\sum_1^m \alpha_i B(v^0(h_i^{(r_k)}), \pi_{h_i^{(r_k)},g_0}(v)) \geq \alpha_0\theta_1/2 \tag{4.11}$$

for all $v \in S(A^0(g_0))$ and $i = 1, \ldots, m$. On the other hand from (4.8) follows that there exists a positive integer $R_5 = R_5(\varepsilon)$ such that for $r \geq R_5$ and all $v \in S(A^0(g_0))$ we have

$$|\sum_1^m \alpha_i B(v^0(h_i^{(r)}), \pi_{h_i^{(r)},g_0}(v)) - \sum_1^m \alpha_i B(v_i^0, \pi_{h,g_0}(v))| \leq \alpha_0\theta_1/4 \tag{4.12}$$

Then combining (4.11) and (4.12) we have

$$0 = \sum_1^m \alpha_i B(v_i^0, \pi_{h,g_0}(v)) \geq \sum_1^m \alpha_i B(v^0(h_i^{(r)}), \pi_{h_i^{(r)},g_0}(v)) + \alpha_0\theta_1/4 > \alpha_0\theta_1/4 \tag{4.13}$$

a contradiction. Therefore, for every $r \geq max\{R_1, \ldots, R_5\}$ there exists an index i_0 such that $B(v^0(h_{i_0}^{(r)}), \pi_{h_{i_0}^{(r)},g_0}(v)) < 0$. It follows from (4.10) that $B(v^0(h_{i_0}^{(r)}), \pi_{h_{i_0}^{(r)},g_0}(v))^2 \geq \theta_1/2$. This completes the proof. $\square$

4.6 Lemma. *Assume that the linear part of* Γ *is Zariski dense in* $SO(B)$. *Let* g *and* h *be a* 2ε*–hyperbolic* 2ε*–transversal elements of* Γ *such that* $Con_{\{g,h\}} \neq \emptyset$. *Then there exists a positive number* $\theta = \theta(g,h) < 1$ *and a sequence of* ε*– hyperbolic, pairwise* ε*–transversal elements* $\{g_1, \dots, g_k\}$ *in the group* Γ *such that* $\{g_1, \dots, g_k\}$ *is a* (θ, ε)*–system for* g *and* h .

Proof. Let $\{h_1, \dots, h_m\}$ be a $D_{\mathbb{R}}(g,h)$–set. We can and will assume that $h_1 = g$ and $h_m = h$. Set $h(L) = h_1^{\ell_1} \dots h_m^{\ell_m}$ for positive m-tuples $L = (\ell_1, \dots, \ell_m) \in \mathbb{Z}^m$. Let $v_1, v_2, \dots, v_{p-q}$ be a basis of A^0. Assume that $B(v_i, v_i) = 1$ for all $i = 1, \dots, p-q$. It is obvious that $\{v_1, -v_1, \dots, v_{p-q}, -v_{p-q}\}$ is a convex–full subset of A^0.

By Lemma 4.2 for every vector v_i, $i = 1, \dots, p-q$, there exist two sequences of positive m–tuples $\{L_i^{(r)}\}_{r \in \mathbb{N}}$ and $\{M_i^{(r)}\}_{r \in \mathbb{N}}$ such that $\left(X_{h(L_i^{(r)})}, v^0_{h(L_i^{(r)})}\right) \rightrightarrows (X, v_i)$ and $\left(X_{h(M_i^{(r)})}, v^0_{h(M_i^{(r)})}\right) \rightrightarrows (X, -v_i)$ when $r \to \infty$ where $X_{h(L_i^{(r)})} = \left(A^0(h(L_i^{(r)})), o(h(L_i^{(r)}))\right)$ and $X_{h(M_i^{(r)})} = \left(A^0(h(M_i^{(r)})), o(h(M_i^{(r)}))\right)$. It follows from lemma 4.5 that there exists a positive integer $N = N(\varepsilon)$ and a positive $\theta < 1$ such that sequence $\{h(L_1^{(r)}), h(M_1^{(r)}), \dots, h(L_{p-q}^{(r)}), h(M_{p-q}^{(r)})\}$ is a (θ, ε)–system for g and h when $r \geq N$. The lemma is proved. □

Our next goal is to show that under some restriction on the quadratic form B for any two 2ε–hyperbolic 2ε–transversal elements g and h of a co–compactly dense subgroup Γ of G_B there exists a (θ, ε)–system for g and h for some $\theta = \theta(g,h) < 1$. We will follow the same strategy we used in the proof of lemma 4.6. Namely, we will first construct sequences of hyperbolic elements $\{g_i(L^{(r)})\}_{r \in \mathbb{N}}$, $i = 1, \dots, m$ such that for $i = 1, \dots, m$ and $r \to \infty$, $A^0(g_i(L^{(r)})) \to A^0$ and $v^0(g_i(L^{(r)})) \to v_i$ where $\{v_1, \dots, v_m\}$ is a convex–full system in $A^0 = D^+(g) \cap D^-(h)$. Then the existence of a (θ, ε)–system will follow from lemma 4.5.

Let g, h and g_0 be 2ε–hyperbolic pairwise transversal elements of Γ such that

(1) $\widehat{Con}_{\{g,h\}}(g_0) \neq \varnothing$

(2) g_0 is $\mathbb{R}$–irreducible element.

Let $S = \{g_1, \dots, g_k\}$ be a $D_{\mathbb{R}}(g,h)$–set such that for every $i = 1, \dots, k$ the element g_i transversal to g_0. Hence we can add elements g, h, g_0 to the set S and will have a $D_{\mathbb{R}}(g,h)$–set $\{h_1, \dots, h_m\}$ where $h_1 = g,\ h_2 = g_0^{-1}, h_3 = g_1, \dots, h_{m-2} = h_k, h_{m-1} = g_0, h_m = h$. Set $h(L) = h_1^{\ell_1} \dots h_m^{\ell_m}$ for positive m-tuples $L = (\ell_1, \dots, \ell_m) \in \mathbb{Z}^m$. Let v_1^0 be a vector in $\widehat{Con}_{\{g,h\}}(g_o)$ such that $B(v_1^0,\, v_1^0) = 1$. There exists a regular element $o_1 \in SO(\widehat{B}_{A^0})$ such that $o_1(v_1^0) = v_1^0$. Let o_2 be the linear part of $\rho_2(g_0)$. Set $o = o_1 o_2$. Define

$$v_+ = |B(v_1^0, \varsigma_{g_0}^+(t_{g_0}^{(1)}))| v_1^0 - t_{g_0}^{(2)} \tag{4.14}$$

and

$$v_- = -|B(v_1^0, \varsigma_{g_0}^-(t_{g_0}^{(1)})| v_1^0 + t_{g_0}^{(2)} \qquad (4.15).$$

Set $v_+^{(1)} = v_+/\|v_+\|, v_-^{(1)} = v_-/\|v_-\|$. Obviously $v_+^{(1)},\ v_-^{(1)} \in A^0$.

Let us make a remark before we will start to prove the next lemma. To realize our objective we have to be able to calculate $\lim_{r\to\infty} v^0(h(L^{(r)}))$ for a certain sequence $L^{(r)}$). Since the group Γ is co–compactly dense its action on the affine space $\mathbb{R}^n$ splits into to two actions: one on the affine space A_1 and the other on A_2. Unfortunately, $v^0(h(L^{(r)})) \neq v^0(\rho_1(h(L^{(r)}))) + v^0(\rho_2(h(L^{(r)})))$. Fortunately since the two vectors $v^0(h(L^{(r)}))$ and $\chi(h(L^{(r)}))$ have the same direction it is enough to calculate $\lim_{r\to\infty} \chi(h(L^{(r)}))$. Let $\chi_1(h(L^{(r)})) = \chi(\rho_1(h(L^{(r)})))$ and $\chi_2(h(L^{(r)})) = \chi(\rho_2(h(L^{(r)})))$.Known that the elements $\{h_1^{\ell_1^{(r)}}, \dots, h_m^{\ell_m^{(r)}}\}$ freely generate free group Γ_r for r greater than some positive integer N. From this we will conclude that for $r \geq N$ the restriction of $\rho_i, i = 1, 2$ to Γ_r is an isometry with respect to the corresponding word metric. Therefore $\chi(h(L^{(r)})) = \chi_1(h(L^{(r)})) + \chi_2(h(L^{(r)}))$. This makes it possible to calculate $\lim_{r\to\infty} \chi(h(L^{(r)}))$.

4.7 Lemma. *Set $X = (A^0, o)$. Then there exist sequences $\{L_+^{(r)}\}_{r\in\mathbb{N}}$, $\{L_-^{(r)}\}_{r\in\mathbb{N}}$ such that:*

(i) $|L_+^{(r)}| \to \infty$ *when* $r \to \infty$

(ii) $|L_-^{(r)}| \to \infty$ *when* $r \to \infty$.

(iii) $\chi(h(L_+^{(r)})) \to v_+$

(iv) $\left(X_{h(L_+^{(r)})}, v^0_{h(L_+^{(r)})}\right) \rightrightarrows \left(X, v_+^{(1)}\right)$ *when* $r \to \infty$.

(v) $\chi(h(L_-^{(r)})) \to v_-$

(vi) $\left(X_{h(L_-^{(r)})}, v^0_{h(L_-^{(r)})}\right) \rightrightarrows \left(X, v_-^{(1)}\right)$ *when* $r \to \infty$.

Proof. Let $\{L^{(r)}\}_{r\in\mathbb{N}}$ be a sequence which fulfills the hypotheses of the lemma 3.10 such that $\left(A^0(h(L^{(r)})), o(h(L^{(r)}))\right) \rightrightarrows (A^0, o)$ when $r \to \infty$. It follows from lemma 3.11 that $v^0(g^{(r)}) \to \alpha v_1^0$ or $v^0(g^{(r)}) \to \beta v_1^0$ when $r \to \infty$, where $|\alpha| = |\beta| = 1$. The subset $\{h_3, \ldots, h_{m-2}\} = \{g_1, \ldots, g_k\}$ is a $D_{\mathbb{R}}(g,h)$–set. Therefore we can and will additionally assume that $o(h_i)^{\ell_i^{(r)}} \to e$ when $r \to \infty$ for $i = 1, 2, m-1, m$. We can now proceed analogously as in the proof of lemma 4.2 (a),(b) and (c) and conclude that there are two subsequences $\{L_+^{(r)}\}_{r\in\mathbb{N}}$ and $\{L_-^{(r)}\}_{r\in\mathbb{N}}$ of $\{L^{(r)}\}_{r\in\mathbb{N}}$ such that for $(\ell_1^{(r)}, \ldots, \ell_m^{(r)}) = L_+^{(r)}$

$$|L^{(r)}| \to \infty$$

we have when $r \to \infty$ and

$$\ell_1^{(r)} \geq \ell_i^{(r)} \quad \textit{for} \quad i \geq 3 \tag{4.16}$$

$$(\ell_1^{(r)})^2 \leq \ell_2^{(r)} \leq (\ell_1^{(r)})^3 \tag{4.17}$$

in particular $\ell_i^{(r)}/\ell_2^{(r)} \to 0$ for $i \neq 2$
and for the subsequence $(\ell_1^{(r)}, \ldots, \ell_m^{(r)}) = L_-^{(r)}$ we have

$$\ell_m^{(r)}/\ell_i^{(r)} \quad \textit{for} \quad i \leq m-2 \tag{4.18}$$

and

$$(\ell_m^{(r)})^2 \leq \ell_{m-1}^{(r)} \leq (\ell_m^{(r)})^3 \qquad (4.19)$$

in particulary $\ell_i^{(r)}/\ell_{m-1}^{(r)} \to 0$ for $i \neq m-1$. Since for every transformation γ we have $\chi(\gamma) = \|\chi(\gamma)\| v^0(\gamma)$ it is enough to prove (iii) and (v). The proof falls naturally into two parts.

Part 1. Put $g^{(r)} = \rho_1(h(L^{(r)}))$. Recall that for every hyperbolic element $\gamma \in \Gamma$ we have $A^{\pm}(\rho_1(\gamma)) = A^{\pm}(\gamma)$, $A^0(\gamma) = \widehat{A}^0(\gamma) \oplus V_2$, $\widehat{A}^0(\gamma) = A^0(\rho_1(\gamma))$ and $\widehat{A}^0 = D^+(\rho_1(g)) \cap D^-(\rho_1(h))$.

We claim that: (I) $\lim_{r\to\infty} \chi(g_+^{(r)}) = |B(v_1^0, \varsigma_{g_0}^+(t_{g_0}^{(1)}))|\, v_1^0$ (II) $\lim_{r\to\infty} \chi(g_-^{(r)}) = -|B(v_1^0, \varsigma_{g_0}^-(t_{g_0}^{(1)}))|\, v_1^0$.

Let us first show (I). Set $g_1^{(r)} = \rho_1(h_1^{\ell_1^{(r)}} h_2^{\ell_2^{(r)}})$ and $g_2^{(r)} = h_3^{\ell_3^{(r)}} \dots h_m^{\ell_m^{(r)}}$ where $(\ell_1^{(r)}, \dots, \ell_m^{(r)}) = L_+^{(r)}$. Clearly $g^{(r)} = g_1^{(r)} g_2^{(r)}$. Fix a point q_0 on the $\rho_1(g_0)$ invariant line $L_{\rho_1(g_0)}$. Let $C^{(r)} = E_{\rho_1(h_2)}^- \cap E_{g_2^{(r)}}^+$ and let q_1 be the projection of q_0 onto $C^{(r)}$ along $A^-(\rho_1(h_2))$. Let q_3 be the projection of q_1 onto $C_{g_2^{(r)}}$. It follows from lemma 1.6 that $d(q_0, q_1) \ll 1$ and $d(q_1, q_2) \ll 1$ for sufficiently large r. Let $p_0 = \rho_1(h_2^{\ell_2^{(r)}})q_0$. Set $E = E_{\rho_1(h_2)}^+ \cap E_{\rho_1(h_1)}^-$. Let $\widetilde{q}_0$ (resp. $\widetilde{p}_0$) be the projection of q_0 (resp. p_0) onto E along $A^+(\rho_1(h_2))$. Let $\widetilde{q}_1$ (resp. $\widetilde{p}_1$) be the projection of $\widetilde{q}_0$ (resp. $\widetilde{p}_0$) onto $C_{\rho_1(h_1)}$ along $A^-(\rho_1(h_1))$. Let $p^{(r)} = \rho_1(h_2^{\ell_2^{(r)}})q_1$ and let $\widetilde{p}^{(r)}$ be the projection of $p^{(r)}$ onto $E_{\rho_1(h_1)}^-$ along $A^+(\rho_1(h_2))$. It is clear that there is a point p_1 such that the vector $v = p_1 - q_1 \in A^+(\rho_1(h_2))$ and $\rho_1(h_2^{\ell_2^{(r)}})p_1 = \widetilde{p}^{(r)}$. Obviously

$$\|v\| \ll s(h_2)^{\ell_2^{(r)}} \qquad (4.20)$$

and $d(p_0, p^{(r)}) \ll s(h_2)^{\ell_2^{(r)}}$. Hence $d(\widetilde{p}_1, \widetilde{p}^{(r)}) \ll s(h_2)^{\ell_2^{(r)}}$. Since all projections considered above are isometries with respect to the quadratic form B, we have $d(\widetilde{p}_1, \widetilde{q}_1) \ll d(q_0, p_0) \ll \ell_2^{(r)}$. Therefore

$$d(\rho_1(h_1^{\ell_1^{(r)}})\widetilde{p}^{(r)}, \rho_1(h_1^{\ell_1^{(r)}})\widetilde{p}_1) \ll s(\rho_1(h_1))^{\ell_1^{(r)}} \ell_2^{(r)} \qquad (4.21)$$

Let $v_0^{(r)} = \widetilde{p}_1 - \widetilde{q}_1$. Then

$$\rho_1(h_1^{\ell_1^{(r)}})\widetilde{p}_1 = \rho_1(h_1^{\ell_1^{(r)}})\widetilde{q}_1 + \ell(\rho_1(h_1))^{\ell_1^{(r)}}(v_0^{(r)}) \tag{4.22}$$

It follows from $d(\widetilde{q}_1, q_0) \ll 1$ that $d(\rho_1(h_1^{\ell_1^{(r)}})\widetilde{q}_1, \widetilde{q}_1) \ll \ell_1^{(r)}$. Since $\|\ell(\rho_1(h_1))^{\ell_1^{(r)}} v_0^{(r)}\| \ll \|v_0^{(r)}\| \ll \|p_0 - q_0\| \ll \ell_2^{(r)}$ we deduce from (4.21) and (4.22) that

$$\|\rho_1(h_1^{\ell_1^{(r)}})\widetilde{p}_1 - \widetilde{q}_1\| \ll \ell_1^{(r)} + \ell_2^{(r)} \tag{4.23}$$

As $\rho_1(h_2^{\ell_2^{(r)}})p_1 = \widetilde{p}^{(r)}$ we have $g_1^{(r)}p_1 = \rho_1(h_1^{\ell_1^{(r)}})\widetilde{p}^{(r)}$. Combining this with $d(\widetilde{q}_1, p_1) \ll 1$ we conclude that

$$\|g_1^{(r)}p_1 - p_1\| \ll \ell_1^{(r)} + \ell_2^{(r)} \tag{4.24}$$

Applying (4.16) and (4.17) to (4.20), we see that $d((g_2^{(r)})^{-1}p_1, (g_2^{(r)})^{-1}q_1) \ll 1$. It follows from lemma 1.6 that $\|q_1 - (g_2^{(r)})^{-1}q_1\| \ll \sum_3^m \ell_i^{(r)}$. Therefore

$$\|p_1 - (g_2^{(r)})^{-1}p_1\| \ll \sum_3^m \ell_i^{(r)} \tag{4.25}$$

Set $v^{(r)} = g_1^{(r)}p_1 - (g_2^{(r)})^{-1}p_1/\omega(g^{(r)})$. Then $v^{(r)} = g^{(r)}x - x/\omega(g^{(r)})$, where $x = (g_2^{(r)})^{-1}p_1$. Combining (4.17) with (4.24) and (4.25) we conclude that the sequence $\{\|v^{(r)}\|\}_{r\in\mathbb{N}}$ is bounded. By definition $t_{g^{(r)}} = B(g^{(r)}x - x, v^0(g^{(r)}))v^0(g^{(r)})$ where x is an arbitrary point in the affine space A_1 and $\chi(g^{(r)}) = t_{g^{(r)}}/\omega(g^{(r)})$. Since the sequence $\{\|v^{(r)}\|\}_{r\in\mathbb{N}}$ is bounded and $v^0(g^{(r)}) \to \alpha v_1^0$ where $\alpha = \pm 1$, we assert that the sequences $\{B(v^{(r)}, v^0(g^{(r)}))v^0(g^{(r)})\}_{r\in\mathbb{N}}$ and $\{B(v^{(r)}, v_1^0)v_1^0\}_{r\in\mathbb{N}}$ converge to the same vector. It thus remains to show that the sequence $\{B(v^{(r)}, v_1^0)v_1^0\}_{r\in\mathbb{N}}$ converges to the vector $|B(v_1^0, \varsigma_{g_0}^+(t_{g_0}^{(1)}))|\, v_1^0$. Consequently we have to prove that $B(v^{(r)}, v_1^0) \to |B(v_1^0, \varsigma_{g_0}^+(t_{g_0}^{(1)}))|$ when $r \to \infty$. Let $v_1^{(r)} = p_1 - (g_2^{(r)})^{-1}p_1/\omega(g^{(r)})$. It follows immediately from (4.17) and (4.25) that $\|v_1^{(r)}\| \to 0$ when $r \to \infty$. Therefore it is remain to estimate $B(v_2^{(r)}, v_1^0)$, where $v_2^{(r)} = g^{(r)}p_1 - p_1/\omega(g^{(r)})$. Since $g_1^{(r)}p_1 = \rho_1(h_1^{\ell_1^{(r)}})\widetilde{p}^{(r)}$ it follows from (4.21) that

$$d(g_1^{(r)}p_1, \rho_1(h_1^{\ell_1^{(r)}})\widetilde{p}_1) \ll s(\rho_1(h_1))^{(\ell_1^{(r)})}\ell_2^{(r)} \tag{4.26}$$

Applying (4.17), (4.22) to (4.26) we assert that

$$\|(g^{(r)}p_1 - p_1) - (\rho_1(h_1^{\ell_1^{(r)}})\widetilde{q}_1 - \widetilde{q}_1) - (\widetilde{q}_1 - p_1) - \ell(\rho_1(h_1))^{\ell_1^{(r)}}(v_0^{(r)})\| \ll 1. \tag{4.27}$$

As $\|\rho_1(h_1^{\ell_1^{(r)}})\widetilde{q}_1 - \widetilde{q}_1\| \ll \ell_1^{(r)}$, $d(\widetilde{q}_1, p_1) \ll 1$ we have from (4.27) that $\|(g^{(r)}p_1 - p_1) - \ell(\rho_1(h_1))^{\ell_1^{(r)}}(v_0^{(r)})\| \ll \ell_1^{(r)}$. Since $\ell(\rho_1(h_1)) = o(\rho_1(h_1)$ we can rewrite the last inequality as follows

$$\|(g^{(r)}p_1 - p_1) - o(\rho_1(h_1))^{\ell_1^{(r)}}(v_0^{(r)})\| \ll \ell_1^{(r)}. \tag{4.29}$$

Let $w^{(r)} = o(\rho_1(h_1))^{\ell_1^{(r)}}(v_0^{(r)})/\omega(g^{(r)})$. It follows from (4.17) and (4.29) that the proof of (a) will be completed if we will show that $\lim_{r\to\infty} B(w^{(r)}, v_1^0) = |B(v_1^0, \varsigma_{g_0}^+(t_{g_0}^{(1)}))|$. Let w be a vector in $A^0(\rho_1(h_1))$ and let $\iota : A^0(\rho_1(h_1)) \to \widehat{A}^0$ be the projection onto $\widehat{A}^0$ along $A^+(\rho_1(h_1))$. Then $B(w, v_1^0) = B(\iota(w), v_1^0)$. It is easy to check that for $w_0^{(r)} = p_0 - q_0$ we have $\iota(v_0^{(r)}) = \varsigma_{g_0}^+(w_0^{(r)})$, since $g = h_1, g_0^{-1} = h_2$. Thus we have

$$B(w^{(r)}, v_1^0) = B(\iota(w^{(r)}), v_1^0) = B((\iota\, o(\rho_1(h_1))^{\ell_1^{(r)}}\iota^{-1})\varsigma_{g_0}^+(w_0^{(r)}), v_1^0)/\omega(g^{(r)}). \tag{4.30}$$

The point q_0 belongs to the $\rho_1(g_0)$–invariant line $L_{\rho_1(g_0)}$ and $g_0^{-1} = h_2$ therefore $w_0^{(r)} = \ell_2^{(r)} t_{g_0^{-1}}$. Since $o(\rho_1(h_1))^{\ell_1^{(r)}} \to e$ when $r \to \infty$ we conclude combining (4.30) with (4.17) that $\lim_{r\to\infty} B(w^{(r)}, v_1^0) = B(\tau_{g_0}^+(t_{g_0^{-1}}), v_1^0)$. The vector $t_{g_0^{-1}}$ has the same direction as the vector $v^0(g_0^{-1})$. By our assumption $v_1^0 \in \widehat{Con}_{g,h}(g_0)$. Hence $B(\tau_{g_0}^+(t_{g_0^{-1}}), v_1^0)$ is a positive number. Thus $B(\tau_{g_0}^+(t_{g_0^{-1}}), v_1^0) = |B(\tau_{g_0}^+(t_{g_0^{-1}}), v_1^0)| = |B(\tau_{g_0}^+(t_{g_0}), v_1^0)|$. This completes the proof of (I). To prove (II) consider $(\ell_1^{(r)}, \ldots, \ell_m^{(r)}) = L_-^{(r)}$ and set $g^{(r)} = \rho_1(h(L^{(r)})), g_1^{(r)} = \rho_1(h_1^{\ell_1^{(r)}} \ldots h_{m-2}^{\ell_{m-2}^{(r)}})$ and $g_2^{(r)} = \rho_1(h_{m-1}^{\ell_{m-1}^{(r)}} h_m^{\ell_m^{(r)}})$. Repeating the arguments given in the proof of (I) leads to a proof of (II).

Part 2. Our next objective is to show that $\lim_{r\to\infty} \chi_2(h(L_+^{(r)})) = -t_{g_0}^{(2)}$ and $\lim_{r\to\infty} \chi_2(h(L^{(r)}) = t_{g_0}^{(2)}$.

Since A_2 is a Euclidean space we have $d(a, \gamma_1\gamma_2 a) \leq d(a, \gamma_1 a) + d(a, \gamma_2 a)$ for any two Euclidean transformations γ_1 and γ_2 of Aff A_2 and any point $a \in A_2$. Hence there is constant $C = C(a, h_1, \dots, h_m)$ such that for every element γ in the group generated by $\rho_2(h_1), \dots, \rho_2(h_m)$ we have

$$d(a, \gamma a) \le C\omega(\gamma). \tag{4.31}$$

Let $V^0(\gamma) = \{v \,|\, \ell(\gamma)(v) = v\}$ be the subspace of γ–fixed vectors, where $\gamma \in \mathrm{Aff}\, A_2$. Let ς_γ be the projection $\varsigma_\gamma : V_2 \to V^0(\gamma)$ onto $V^0(\gamma)$ along the B–orthogonal complement to $V^0(\gamma)$. It is easy to check that for every $\gamma \in \mathrm{Aff}\, A_2$ we have

$$\varsigma_\gamma(\gamma a - a) = t_\gamma, \tag{4.32}$$

where a is an arbitrary point in A_2. Let $(\ell_1^{(r)}, \dots, \ell_m^{(r)}) = L_+^{(r)}$. Set $h^{(r)} = \rho_2(h(L_+^{(r)}))$, $h_1^{(r)} = \rho_2(h_1^{\ell_1^{(r)}} h_2^{\ell_2^{(r)}})$ and $h_2^{(r)} = \rho_2(h_3^{\ell_3^{(r)}} \dots h_m^{\ell_m^{(r)}})$.

Fix a point q_0 on a $\rho_2(g_0)$–invariant line $L_{\rho_2(g_0)}$. Recall that $h_2 = g_0^{-1}$. Then $v(r) = \rho_2(h_2^{\ell_2^{(r)}})q_0 - q_o = \ell_2^{(r)} t_{\rho_2(h_2)}$. Therefore

$$h_1^{(r)} q_0 - q_0 = (\rho_2(h_1^{\ell_1^{(r)}})q_0 - q_0) + \ell(\rho_2(h_1^{\ell_1^{(r)}})v(r). \tag{4.33}$$

Since $o(h_1^{\ell_1^{(r)}}) \to e$ when $r \to \infty$ applying (4.17), (4.31) to (4.33) we conclude that

$$(h_1^{(r)} q_0 - q_0)/\omega(h^{(r)}) \to t_{h_2}$$

when $r \to \infty$.

On the other hand from (4.31) follows that

$$(q_0 - (h_2^{(r)})^{-1} q_0)/\omega(h^{(r)}) \to 0$$

when $r \to \infty$. Set $a = (h_2^{(r)})^{-1} q_0$, then

$$(h^{(r)} a - a)/\omega(h^{(r)}) \to t_{h_2} \tag{4.34}$$

when $r \to \infty$. We choose the sequence $\{L^{(r)}\}$ such that the linear part of $h^{(r)}$ converges to the linear part of $\rho_2(g_0)$. Then $d(V^0(h^{(r)}), V^0(\rho_2(g_0))) \to 0$ when $r \to \infty$ by lemma

3.11 since g_0 is $\mathbb{R}$–irreducible. Thus $\chi(h^{(r)}) = \varsigma_{h^{(r)}}(h^{(r)}a - a/\omega(h^{(r)})) \to t_{h_2}$ when $r \to \infty$. This completes the proof of (iii) since $t_{h_2} = -t_{\rho_2(g_0)} = -t^{(2)}_{g_0}$. Let $(\ell^{(r)}_1, \ldots, \ell^{(r)}_m) = L^{(r)}_-$. Set $h^{(r)} = \rho_2(h(L^{(r)})), h^{(r)}_1 = \rho_2(h_1^{\ell_1^{(r)}} \ldots h_{m-2}^{\ell_{m-2}^{(r)}})$ and $h^{(r)}_2 = \rho_2(h_{m-1}^{\ell_{m-1}^{(r)}} h_m^{\ell_m^{(r)}})$ By verbatim repetition the above proof we come to the conclusion that $\chi(h^{(r)}) = \varsigma_{h^{(r)}}(h^{(r)}a - a/\omega(h^{(r)})) \to t_{h_{m-1}}$ when $r \to \infty$. This completes a proof of (v) since $t_{h_{m-1}} = t_{\rho_2(g_0)} = t^{(2)}_{g_0}$. □

In the next lemma we assume that Γ is co–compactly dense subgroup in G_B and there is a hyperbolic element $\gamma \in \Gamma$ such that .

4.8 Lemma. *Let g and h be 2ε–hyperbolic 2ε–transversal elements in Γ. Let $g_0 \in \Gamma$ be a 2ε–hyperbolic element 2ε–transversal to both g and h such that*

(1) $\widehat{Con}_{\{g,h\}}(g_0) \neq \varnothing$

(2) g_0 is $\mathbb{R}$–irreducible element

(3) the set $\{\ell(x)t^{(2)}_{g_0} \,|\, x \in \Gamma\}$ does not belong to any proper Γ– invariant subspace of V_2.

Then there exist positive constants $\theta = \theta(g,h) < 1$, $s < 1$ and a sequence of ε– hyperbolic, pairwise ε–transversal elements $\{g_1, \ldots, g_k\}$ in the group Γ such that $s(g_i) \leq s$ for all $i = 1, \ldots, k$ and $\{g_1, \ldots, g_k\}$ is a (θ, ε)–system for g and h .

Proof. Let v^0_1 and w^0_1 be two different vectors from the cone $\widehat{Con}_{\{g,h\}}(g_0)$ such that

$$|B(\varsigma^+_{g_0}(t_{g_0}), v^0_1)| = |B(\varsigma^-_{g_0}(t_{g_0}), w^0_1)| \qquad (4.35).$$

Then by simple geometrical arguments $|B(\varsigma^-_{g_0}(t_{g_0}), v^0_1)| = |B(\varsigma^+_{g_0}(t_{g_0}), w^0_1)|$. Set $\kappa_1 = |B(\varsigma^+_{g_0}(t_{g_0}), v^0_1)|$ and $\kappa_2 = |B(\varsigma^+_{g_0}(t_{g_0}), v^0_1)|$. Define $v_1 = \kappa_1 v^0_1 - t^{(2)}_{g_0}$, $v_2 = -\kappa_2 v^0_1 + t^{(2)}_{g_0}$, $v_3 = \kappa_1 w^0_1 - t^{(2)}_{g_0}$ and $v_4 = -\kappa_2 w^0_1 + t^{(2)}_{g_0}$. It follows from lemma 4.6 that there are sequences $L^{(r)}_i$ such that the following is true $\chi(h(L^{(r)}_i)) \to v_i$ for $1 \leq i \leq 4$. It is clear that $\kappa_2 v_1 + \kappa_1 v_2 + \kappa_1 w_1 + \kappa_2 w_2 = 0$. Observe that $\kappa_i > 0$ for $1 \leq i \leq 4$ and $span\{v_1, v_2, w_1, w_2\} \supseteq$

$\{v_1^0, w_1^0, t_{g_0}^{(2)}\}$. Consider another pair of vectors in the cone $\widehat{Con}_{\{g,h\}}(g_0)$ which fulfills (4.35). Repeating the above procedure enables us to construct sequences $L_i^{(r)}, i = 1, \dots, l$ such that if $v_i = \lim_{r\to\infty} \chi(h(L_i^{(r)}))$. Then the following is true:

(1) there are positive numbers α_i such that $\sum_1^l \alpha_i v_i = 0$

(2) the set $span\{v_1, \dots v_l\}$ contains a basis of the vector space $\widehat{A}^0$ and the vector $t_{g_0}^{(2)}$.

By our assumption (3) the set $\{\ell(x)t_{g_0}^{(2)} \,|\, x \in \Gamma\}$ contains a basis of V_2. Therefore for any proper space W of V_2 the set $\{x \in \Gamma \,|\, \ell(x)t_{g_0}^{(2)} \notin W\}$ is a non-empty Zariski open set. Combining this with lemma 4.1 we conclude that there are elements $x_i \in \Gamma$, $i = 1, \dots \widetilde{m}$ such that for the elements $g_i = x_i g_0 x_i^{-1}$ the following is true

(3) $span\{t_{g_0}^{(2)}, \dots t_{g_{\widetilde{m}}}^{(2)}\} = V_2$

(4) the cone $\widehat{Con}_{\{g,h\}}(g_i)$ is non-empty for every $i = 1, \dots, \widetilde{m}$.

Repeating the previous argument and replacing g_0 by g_i leads to sequences $\{\widetilde{h}_i^{(r)}\}_{r\in\mathbb{N}}$ such that for every i, $1 \le i \le \widetilde{m}$, $\chi(\widetilde{h}_i^{(r)}) \to v_i$ when $r \to \infty$ and

(5) there are positive number α_i such that $\sum_1^{\widetilde{m}} \alpha_i v_i = 0$

(6) $span\{v_1, \dots, v_{\widetilde{m}}\} \supseteq V_2$.

Collect these sequences together with sequences which fulfill properties (1) and (2). Then the set of their limit vectors will be a convex–full system in $\widehat{A}^0 \oplus V_2$. This together with lemma 4.5. completes the proof. $\square$

5. Main results

Fix a point q_0 in the affine space $\mathbb{R}^n$. Let Γ be a co–compactly dense subgroup in G_B and let S be a set of hyperbolic elements of Γ. Set $\Omega_{\varepsilon,d}(S) = \{ \gamma \in \Gamma \mid d(q_0, C_\gamma) \leq d\,,\ \gamma$ is $\varepsilon/2$–hyperbolic, $\varepsilon/2$–transversal to every element in $S\}$.

5.1 Lemma. *Let $S = \{g_1, \ldots, g_k\}$ be a (θ, ε)–system for some hyperbolic transversal elements. Then there are constants $\alpha = \alpha(\varepsilon, d) < 1$, $s^* = s^*(\varepsilon, d)$, $d^* = d^*(\varepsilon, d)$, $L_0 = L_0(\varepsilon, d)$ such that if*

(1) $s(g_i) \leq s^*$ for all $i = 1, \ldots, k$,

(2) $g \in \Omega_{\varepsilon,d}(g, h),\ s(g) \leq s^*$ and $d_g(q_0) > L_0$

then there exist an index i_0, $1 \leq i_0 \leq k$, and a positive integer $m \leq d_g(q_0)$ with the following property

$$d_{g_{i_0}^m g}(q_0) \leq \alpha d_g(q_0) + d^*. \tag{5.1}$$

Proof. Denote $s = \max\{s(g_1), \ldots, s(g_k), s(g)\}$ and $s_0 = s^{1/2}$. Take a positive constant $d_0(\varepsilon)$ as in lemma 1.6. Assume that $s \leq d_0(\varepsilon)$. Then the set $\{g_1, \ldots, g_k, g\}$ satisfies the hypotheses of the lemma 1.6 and lemma 1.3. It follows from lemma 1.6 that

$$d(q_0, C_{g_i^m g}) \ll 1 \tag{5.2}$$

and from lemma 1.3 that

$$\widehat{d}(A^+(g_i^m g),\ A^+(g_i)) \leq \frac{\varepsilon}{2} s_0^m \tag{5.3}$$

$$\widehat{d}(A^-(g_i^m g),\ A^-(g)) \leq \frac{\varepsilon}{2} s_0 \tag{5.4}$$

for all positive integers m and all $i = 1, \ldots, k$.

It then follows from (1.1) that it is sufficient to prove the lemma for a point p such that $d(p,q) \le d(\varepsilon)$. Let q_1 be a point in C_g^0 such that $d(q_0,q_1) = d(q_0, C_g^0)$. It is clear that $d(q_0,q_1) \le d(\varepsilon)$. Put $p_1 = gq_1$. By definition 4.4 there exists an index i_0 such that

$$cos(v^0_{g_{i_0}}, \pi_{g_{i_0},g_0}(v)) \le -\theta \tag{5.5}$$

where $v = p_1 - q_1$. Set $C^0 = E^-_{g_{i_0}} \cap E^+_g$, $C^0_{i_0} = E^+_{g_{i_0}} \cap E^-_g$. Recall that $A^0_{g,g_{i_0}} = D^-(g) \cap D^+(g_{i_0})$. Let q_2 be the projection of q_1 onto C^0 along the subspace $A^+(g)$. Let p_2 be the projection of p_1 onto C^0 along the subspace $A^+(g)$. For positive integers t set $p_3(t) = g^t_{i_0} p_2$. Let p_3 (resp. $q_3(t)$) be the projection of p_2 (resp. $p_3(t)$) onto $C^0_{g_{i_0}}$ along the subspace $A^-(g_{i_0})$. From (5.5) we conclude that there exists a positive integer $m \le \|v\|_g$ such that $|B(w,w)| \le \theta \|v\|_g + t_{g_{i_0}}$, where $w = q_3(m) - q_3$. Note that $\|v\|_g = d_g(q_1)$. Put $T = \max\{t_{g_1}, \dots, t_{g_k}\}$. Then

$$|B(w,w)| \le \theta d_g(q_1) + T \tag{5.6}$$

Recall that $\pi_{g^m_{i_0} g}$ is the projection of the vector space $\mathbb{R}^n$ onto $A^0(g_{i_0} g)$ along the subspace $A^+(g^m_{i_0} g) \oplus A^-(g^m_{i_0} g)$. Let w_1 be the projection of w onto $A^0_{g,g_{i_0}}$ along the subspace $A^+(g_{i_0})$. As this projection is an isometry, we have $B(w,w) = B(w_1,w_1)$. It follows from (5.3) and (5.6), that

$$\|\pi_{g^m_{i_0} g}(w) - w_1\| \ll s_0^m d_g(q_1) \tag{5.7}$$

It is clear that there is a constant $d_1 = d_1(\varepsilon, d)$ such that $d(q_3,q_1) \le d_1$. There is a constant $d_2 = d_2(\varepsilon, d)$ such that

$$|\,\|\pi_{g^m_{i_0} g}(w_2)\| - \|\pi_{g^m_{i_0} g}(w)\|\,| \le d_2 \tag{5.8}$$

for $w_2 = q_3(m) - q_1$. Define a vector $w_3 = p_3(m) - g^{-1}p_2$. It follows easily that $d(q_3(m), p_3(m)) \ll s_0^m d_g(q_1)$ and $d(p_2, g^{-1}p_2) \ll s_0 d_g(q_1)$. Therefore

$$\|w_2 - w_3\| \ll s_0^m d_g(q_1) + s_0 d_g(q_1) \tag{5.9}$$

By definition, $B(\pi_{g_{i_0}^m g}(w_3), \pi_{g_{i_0}^m g}(w_3))^{1/2} = d_{g_{i_0}^m g}(g^{-1}p_2)$. Hence from (1.1) follows that there is a constant $d_3 = d_3(\varepsilon, d)$ such that

$$|d_{g_{i_0}^m g}(q_1) - B(\pi_{g_{i_0}^m g}(w_2), \pi_{g_{i_0}^m g}(w_2))^{1/2}| \le s_0 d_g(q_1) + d_3 \qquad (5.10).$$

Combining (5.7) and (5.8) we conclude that there is a constant $d_4 = d_4(\varepsilon, d)$ such that

$$|B(\pi_{g_{i_0}^m g}(w_2), \pi_{g_{i_0}^m g}(w_2))^{1/2} - B(w_1, w_1)^{1/2}| \le d_4(s_0^m d_g(q_1) + d_2) \qquad (5.11).$$

Since $B(w, w) = B(w_1, w_1)$, combining (5.10), (5.11) and (5.6) we have that

$$d_{g_{i_0}^m g}(q_1) \le \theta d_g(q_1) + d_4(s_0^m d_g(q_1) + d_2) + T + s_0 d_g(q_1) + d_3.$$

We rewrite this inequality as

$$d_{g_{i_0}^m g}(q_1) \le (\theta + d_4 s_0^m d_g(q_1) + s_0) d_g(q_1) + d_2 d_4 + d_3 + T \qquad (5.12).$$

Let s_1 be a positive number, such that $\theta + d_4 s^{1/2} d_g(q_1) + s^{1/2} < 1$. Put $s^* = \min(s_1, d_0)$ and $d_5 = d_2 + d_3 + d_2 d_4 + T$. It follows from (5.12) that if $s \le s^*, s_0 = s^{1/2}$ and $\alpha = \theta + d_4 s_0^m d_g(q_1) + s_0$ then

$$d_{g_{i_0}^m g}(q_1) \le \alpha d_g(q_1) + d_4$$

and the proof is completed. □

5.2 Lemma. *Let $\{g_1, \ldots, g_k\}$ be a (θ, ε)–system. Assume that the constants ε, α, s^*, L_0 and d^* are as in lemma 5.1. Suppose that the element $g_0 \in \Omega_{\varepsilon/2,d}(S)$ and the $\{\theta, \varepsilon\}$–system $\{g_1, \ldots, g_k\}$ fulfill the hypotheses of lemma 5.1. Then there are $L_1 = L_1(\varepsilon, d)$ and $\beta < 1$, such that if $d_{g_0}(q_0) \ge L_1$ then there exists a positive integer $m \le d_{g_0}(q_0)$ and an index $i_0, 1 \le i_0 \le k$, such that*

$$d_{g_{i_0}^m g_0}(q_0) \le \beta d_{g_0}(q_0) \qquad (4.21)$$

Proof. It follows from lemma 5.1 that there exists a constant $L_0 = L_0(\varepsilon)$ such that if $d_g(q_0) \geq L_0$, then $d_{g_{i_0}^m g_0}(q_0) \leq \alpha d_{g_0}(q_0) + d^*$ for some index $i_0, 1 \leq i_0 \leq k$, and some positive integer $m \leq d_{g_0}(q_0)$. Let r be a positive integer such that $\alpha + \alpha^r < 1$. Put $L_1 = \max(L_o,\, \alpha^{-r} d^*)$ and $\beta = \alpha + \alpha^r$. Assume that $d_{g_0}(q_0) > L_1$. Then from (4.10) follows that $d_{g_{i_0}^m g_0}(q_0) \leq \alpha d_{g_0}(q_0) + \alpha^r d_{g_0}(q_0) \leq \beta d_{g_0}(q_0)$. □

Our proof of Theorem A and Theorem B is based on the following

5.3 Lemma. *Let Γ be a subgroup of G_B such that there exist 2ε–hyperbolic, pairwise 2ε–transversal elements g, h, g_0 and a (θ,ε)–system for g and h. Then Γ can not act properly discontinuously on the affine space $\mathbb{R}^n$.*

Proof. On the contrary suppose that Γ acts properly discontinuously. Let $S = \{g_1, \dots, g_k\}$ be a (θ,ε)–system for g and h. By definition 4.4 the elements g_0 and g_i are ε–transversal for every $i = 1, \dots, k$. Since for every positive integer N the set $\{g_1^N, \dots, g_k^N\}$ is (θ,ε)–system and g_0^N is ε–transversal to g_i^N for every $i = 1, \dots, k$ we can and will suppose that $s(g_i) \leq d_0(\varepsilon)$ for all $i = 0, 1, \dots, k$, where $d_0(\varepsilon)$ as in lemma 1.6. Then there is a positive integer M such that $\{g_0^M, g_1^M, \dots, g_k^M\}$ freely generate a free group (see [AMS 3]). Therefore we will assume that $\{g_0, g_1, \dots, g_k\}$ freely generate a free group and fulfill the hypotheses of lemma 1.3 and 1.6. Let $\Gamma(S)$ be the semigroup generated by $S = \{g_1, \dots, g_k\}$. For every positive integer l set $\Gamma_l = \{\, w\, g_0^l,\ w \in \Gamma(S)\,\}$. It follows from lemma 1.3 that every element $\gamma \in \Gamma_l$ is $\varepsilon/2$–hyperbolic and $\varepsilon/2$– transversal to g_i for every $i = 1, \dots, k$. From lemma 1.6 we conclude that there exists a positive number $b = b(\varepsilon)$ such that $\Gamma_l \subseteq \Omega_{\varepsilon/2,b}(S)$. Observe that since $\{\, g_0, g_1, \dots, g_k\,\}$ freely generate a free group we have $\Gamma_l \cap \Gamma_m = \emptyset$ for $l \neq m$. Let $\widetilde{\Gamma} = \cup_{l\in\mathbb{N}} \Gamma_l$. We claim that for every positive c the set $\widetilde{\Gamma}_c = \{\, \gamma \in \widetilde{\Gamma} \,|\, d_\gamma(q_0) \leq c\,\}$ is finite. Suppose contrary to our claim that this set is infinite for some c_0. Let q_γ be the point in C_γ such that $d(q_0, q_\gamma) = d(q_0, C_\gamma)$ where $\gamma \in \widetilde{\Gamma}_{c_0}$. By (1.1) for every $\varepsilon/2$–hyperbolic element γ we have $|d_\gamma(q_\gamma) - d_\gamma(q_0)| \ll d(q_o, q_\gamma)$.

Thus $d(q_0, \gamma q_\gamma) \le d(q_0, q_\gamma) + d(q_\gamma, \gamma q_\gamma) \ll 1$ for all $\gamma \in \widetilde{\Gamma}_{c_0}$ by (1.1). Since $\widetilde{\Gamma}_{c_0}$ is an infinite set this contradicts our assumption that the group Γ acts properly discontinuously. Let $c_l = \inf_{\gamma \in \Gamma_l} d_\gamma(q_0)$. The set $\{\gamma \in \Gamma_l \,|\, d_\gamma(q_0) \le c_l + 1\}$ is finite for every positive integer l. Therefore for every l there exists an element g_l from Γ_l such that $d_{g_l}(q_0) = \inf_{\gamma \in \Gamma_l} d_\gamma(q_0)$. Assume that the set $\{d_l\}_{l \in \mathbb{N}}$ is bounded and let c be a positive number such that $d_l \le c$ for all l. The set $\widetilde{S}_c$ is infinite since the sets Γ_l are disjoint a contradiction. Thus $d_l \to \infty$ when $l \to \infty$. Let L_1 be a constant as in lemma 5.2. Choose a positive integer l_0 such that $d_{l_0} \ge L_1 + 1$. Take $\gamma_{l_0} \in \Gamma_l$. Since $d_{\gamma_{l_0}}(q_0) \ge L_1 + 1$ it follows from lemma 5.2, that there exists an index i_0 such that $d_{g_{i_0}^m \gamma_{l_0}}(q_0) \le \beta d_{\gamma_{l_0}}(q_0)$, where $\beta < 1$. Then $d_{g_{i_0}^m \gamma_{l_0}}(q_0) < d_{\gamma_{l_0}}(q_0)$. Since $d_{\gamma_{l_0}}(q_0) = \inf_{\gamma \in \Gamma_{l_0}} d_\gamma(q_0)$ and $g_{i_0}^m \gamma_{l_0} \in \Gamma_{l_0}$ we have a contradiction and the lemma is proved. □

Theorem A. *If* $|p - q| \ge 2$ *and the Zariski closure of* $l(\Gamma)$ *contains* $SO(B)$, *then* Γ *cannot act properly discontinuously on* $\mathbb{R}^n$.

Proof. Let $\mathbb{G}$ be the Zariski closure of Γ. Suppose that $\ell(\Gamma)$ contains $SO(B)$. Since every subgroup of the group Γ acts properly discontinuously we may assume that $\ell(\Gamma)$ is Zariski dense in $SO(B)$ and Γ has no torsion. Suppose that a hyperbolic element $\gamma \in \Gamma$ has a fixed point q_0. Since Γ has no torsion, it does not act properly discontinuously. Hence the group Γ fulfills the hypotheses (i), (ii) chapter 4. It is well known (see [A]), that the set of hyperbolic pairwise transversal elements of Γ is infinite. Denote this set as $\Omega(\Gamma)$. Let g, h and g_0 be elements of $\Omega(\Gamma)$. We claim that we can assume that the cone $Con_{\{g,h\}} \neq \emptyset$. Indeed, let $g \in \Omega(\Gamma)$ and $h \in \Omega(\Gamma)$. Then by lemma 4.1 (A) the set $R_g(h) = \{x \in \mathbb{G} \,|\, xhx^{-1}$ is transversal to g and $Con_{\{g,h_x\}} \neq \emptyset\}$ is Zariski open and non-empty. The set $R = \{x \in \mathbb{G} \,|\, xhx^{-1}$ and g_0 are transversal $\}$ is Zariski open non-empty set. Therefore $\Gamma \cap R_g(h) \cap R \neq \emptyset$. Take $x \in \Gamma \cap R_g(h) \cap R$ and consider $xhx^{-1} \in \Omega(\Gamma)$ instead of h.

There exists an ε such that g, h and g_0 are 2ε–hyperbolic, pairwise 2ε–transversal elements. Since $Con_{\{g,h\}} \neq \emptyset$ it follows from lemma 4.6 that there exists a positive number $\theta = \theta(g,h) < 1$ and a sequence of ε– hyperbolic, pairwise ε–transversal elements $\{g_1, \dots, g_k\}$ in the group Γ such that $\{g_1, \dots, g_k\}$ is a (θ, ε)–system for g and h . Thus by lemma 5.3 the group Γ can not act properly discontinuously on the affine space $\mathbb{R}^n$. $\square$

5.4 Lemma *Let B be a quadratic form. (A) Assume that B is a positive definite quadratic form. Let $g_1, \dots, g_m$ be an elements of G_B such that the translation $t_{g_i} = 0$ for all $i = 1, \dots, m$. Then there exists a positive number $R = R(g_1, \dots, g_m)$ such that $g_N U(p_0, R) \cap U(p_0, R) \neq \emptyset$ for every m-tuple $N = \{n_1, \dots, n_m\}$, where $g_N = g_1^{n_1} \dots g_m^{n_m}$. (B) Assume that B is not positive definite. Let $g_1, \dots, g_m$ be hyperbolic pairwise transversal elements of G_B such that $t_{g_i} = 0$ for all $i = 1, \dots, m$. Then there exist constants $s_0 < 1$ and $R = R(g_1, \dots, g_m)$ such that if $s(g_i) \leq s_0, i = 1, \dots, m$ then $g_N U(p_0, R) \cap U(p_0, R) \neq \emptyset$ for every m-tuple $N = \{n_1, \dots, n_m\}$, where $g_N = g_1^{n_1} \dots g_m^{n_m}$ and $n_i \geq 0$ for $i = 1, \dots, m$.*

Proof. . (A) The proof is straightforward. (B). It follows from our assumption that every element $g_i, i = 1, \dots, m$, has a fixed point p_i. Let $d = \max_{1 \leq i,j \leq m}\{d(p_i, p_j)\}$. Define $C_i = E^+_{g_i} \cap E^-_{g_{i+1}}, i = 1, \dots, m-1$. Since for hyperbolic elements $g_1, \dots, g_m$ are pairwise transversal there exists a constant $d_1 = d_1(g_1, \dots, g_m)$ such that $d(C_i, C_j) \leq d_1$ for $1 \leq i, j \leq m$, we will assume that $d \leq d_1$. Our proof starts with the following observation. Let q_i be a point from C_{g_i}. Let $\widehat{p_i}$ be the projection of p_i onto C_i along $A^+(g_i)$ and let q_{i+1} be the projection of $\widehat{p_i}$ onto $C_{g_{i+1}}$ along $A^-(g_{i+1})$. Put $r_{i+1} = d(p_{i+1}, q_{i+1})$. Then there exists a positive number $s_i = s_i(d_1)$ such that if $\max\{s(g_i), s(g_{i+1}))\} \leq s_i$ then $g_{i+1} g_i U(p_i, \delta) \cap U(p_{i+1}, \delta + r_{i+1}) \neq \emptyset$.

Put $s_0 = \max\{s_i, i = 1, \dots m\}$. Assume that $s(g_i) \leq s_0$ for all $i = 1, \dots m$. Then if we will start with the ball $U(p_1, r_1)$, we will have $g_N U(p_1, r_1) \cap U(p_m, \widehat{R}) \neq \emptyset$ where $\widehat{R} = \sum_{i=1}^m r_i$. Thus if $R = \widehat{R} + d(p_0, p_1) + d(p_0 p_m)$ we have $g_N U(p_0, R) \cap U(p_0, R) \neq \emptyset$. $\square$

Now we will prove that the following is true for group Γ which is co–compactly dense in G_B

Theorem B. *If q is even, then Γ cannot act properly discontinuously on $\mathbb{R}^n$.*

Proof. The proof of theorem B will be divided into few steps.

Step 1. We will show here that for any hyperbolic element γ from Γ the element $\rho_1(\gamma)$ has no fixed point. Recall that $\rho_1(\gamma)$ is one part of the decomposition. To obtain a contradiction suppose that $\rho_1(\gamma)$ has a fixed point where γ is a hyperbolic element in Γ. It follows from [AMS 3] that there are elements $x_1, \ldots, x_{4n}$ of Γ and a positive integer N such that the elements $\gamma_i = x_i \gamma x_i^{-1}$ are pairwise transversal and freely generate a free group $\widehat{\Gamma}$. Observe that since $\rho_1(\gamma)$ has a fixed point then every element $\rho_1(\gamma_i)$, $i = 1, \ldots, 4n$, has a fixed point. Since for every positive integer M every element $\rho_1(\gamma_i^M)$, $i = 1, \ldots, 4n$ has a fixed point and these elements freely generate a free subgroup which still co–compactly dense in G_B we can and will assume, that $s(\rho_1(\gamma_i)) \leq s_0$ for every $i = 1, \ldots, 4n$, where s_0 as in lemma 5.4 and $\rho_1(\gamma_i)$, $i = 1, \ldots, 4n$ freely generate a free group. Hence $t^{(1)}_{\gamma_i} = t_{\rho_1(\gamma_i)} = 0$ for all $i = 1, \ldots, 4n$. Let $S_N = \{\gamma_1^{\ell_1} \ldots \gamma_{4n}^{\ell_{4n}}, \sum_1^{4n} \ell_i = N\, \ell_i \geq 0, i = 1, \ldots, 4n\}$. Set $\gamma_{L^{(r)}} = \gamma_1^{\ell_1^{(r)}} \ldots \gamma_{4n}^{\ell_{4n}^{(r)}}$ where $L^{(r)} = (\ell_1^{(r)}, \ldots, \ell_{4n}^{(r)})$. It is easily checked that $\sharp(S_N) = \binom{N}{4n} \geq (N/2)^{2n}$. Let q_0 be a point in $\mathbb{R}^n$ and put $q_1 = \tau_1(q_0)$, $q_2 = \tau_2(q_0)$. For the Euclidean transformations $\rho_2(\gamma_i)$, $i = 1, \ldots, 4n$, denote $T = \max\{\|t^{(2)}_{\gamma_1}\|, \ldots, \|t^{(2)}_{\gamma_{4n}}\|\}$ where $t^{(2)}_\gamma = t_{\rho_2(\gamma)}$. Then $\rho_2(\gamma_L) q_2 \in U(q_2, TN)$. Since for the volume of the ball $U(q_2, TN)$ we have $vol(U(q_2, TN)) \leq C(TN)^n$ where $C = C(\mathbb{R}^n)$ and $\sharp(S_N) \geq (N/2)^{2n}$ there exist two elements $\gamma_{L^{(1)}}$ and $\gamma_{L^{(2)}}$ of S_N such $d(\rho_2(\gamma_{L^{(1)}})q_2, \rho_2(\gamma_{L^{(2)}})q_2) \leq 1/N^2$. Then $d(\rho_2(\gamma_{L^{(2)}})^{-1} \rho_2(\gamma_{L^{(1)}}) q_2, q_2) \leq 1/N^2$. Let $\gamma_{L^{(1)}} = \widehat{g} h_1$ and $\gamma_{L^{(2)}} = \widehat{g} h_2$ where $h_1 = g_{i_1}^{\ell_{i_1}} \ldots g_{i_s}^{\ell_{i_s}}, s \leq 4n$, $h_2 = g_{j_1}^{\ell_{j_1}} \ldots g_{j_r}^{\ell_{j_r}}, r \leq 4n$ and $i_1 \neq j_1$. Let $h_N = h_2^{-1} h_1$. It follows from lemma 5.4 that there exists a constant $R = R(g_1, \ldots, g_{4n}, n)$ such that $\rho_1(h_N) U(q_1, R) \cap U(q_1, R) \neq \emptyset$. Let K_N be a compact set such that $q_0 \in K_N$, $U(q_1, R) = \tau_1(K_N)$ and

$U(q_2, 1/N^2) = \tau_2(K_N)$. Combining this with $d(\rho_2(h_N)q_2, q_2) \leq 1/N^2$ we have $h_N q_0 \in K_N$ for every N. Since $q_o \in K_N$ this contradicts our assumption that Γ acts properly discontinuously.

Step 2. Suppose that the assertion of the theorem is false and the dimension n of our affine space is minimal among the counterexamples to our theorem. Let us fist show that for every hyperbolic element $\gamma \in \Gamma$ the set $\{t^{(2)}_{x\gamma x^{-1}},\, x \in \Gamma\}$ contains a basis of the subspace V_2. Recall that $t^{(2)}_\gamma = t_{\rho_2(\gamma)}$. Denote by W the subspace defined by $W = span\,\{\, t^{(2)}_{x\gamma x^{-1}},\, x \in \Gamma\,\}$. Assume that $W \neq V_2$. Let $V_3 = W \oplus V_1$. We have a natural projection of the affine space $\mathbb{R}^n$ onto the affine space $A_3 = \mathbb{R}^n/V_3$ along V_3 and hence an induced homomorphism $\rho_3 : Aff(\mathbb{R}^n) \to Aff(A_3)$. Let $\widehat{\Gamma}$ be the group generated by the set $\{x\gamma x^{-1},\, x \in \Gamma\}$. The group $\rho_3(\gamma)$ is compact since $t_\gamma \in V_3$ for every $\gamma \in \widehat{\Gamma}$. Therefore $\rho_3(\Gamma)$ is a compact group. Hence there is a fixed point q_0 for $\rho_3(\Gamma)$ in A_3. Then the affine subspace $A = q_0 + V_3$ of dimension $\dim V_3 < n$ is $\widehat{\Gamma}$ invariant. Since $\widehat{\Gamma}$ is co–compactly dense in G_B and $\widehat{\Gamma}$ acts properly discontinuously on A this contradicts our assumption.

Let g and h be two hyperbolic transversal elements of Γ. Let g_0 be an $\mathbb{R}$–irreducible element of Γ. We claim that we may assume that g_0 is transversal to both g and h and $\widehat{Con}_{\{g,h\}}(g_0) \neq \emptyset$. Indeed, let $\mathbb{G}$ be the Zariski closure of Γ. Since $t^{(1)}_{g_0} \neq 0$ from lemma 4.1 (B) follows that the Zariski open set $R_{\{g,h\}}(g_0) = \{x \in \mathbb{G} \mid xg_0x^{-1}$ is transversal to both g and h and the set $\widehat{Con}_{\{g,h\}}(xg_0x^{-1}) \neq \emptyset\}$ is a non-empty subset of $\mathbb{G}$. Take $x \in R_{\{g,h\}}(g_0) \cap \Gamma$ and consider xg_0x^{-1} instead of g_0. Therefore we may assume that $\widehat{Con}_{\{g,h\}}(g_0) \neq \emptyset$. Fix an ε such that g, h and g_0 are 2ε–hyperbolic, pairwise 2ε–transversal elements. It follows from lemma 4.8 that there exist a $\theta < 1$ and (θ, ε)–system for g and h. Consequently the group Γ cannot act properly discontinuously on $\mathbb{R}^n$ by lemma 5.2 . □

Theorem C. *If* $q = 2$ *and* Γ *is a crystallographic group, then* Γ *is virtually solvable.*

We have to show that $\ell(\Gamma)$ is virtually solvable, since the kernel of ℓ is abelian, or equivalently that the Zariski closure of $\ell(\Gamma)$ is virtually solvable. The proof is done by contradiction, so we will assume from this point on that the Zariski closure of $\ell(\Gamma)$ is not solvable. Since a subgroup of finite index of crystallographic group is a crystallographic group, we can also assume that the Zariski closure G of $\ell(\Gamma)$ is connected.

5.5 Lemma *Assume that B is a quadratic form of a signature $(n-2,2)$ and G is a connected simple subgroup of $O(B)$ and $rank_{\mathbb{R}} G = 2$, then Γ is co–compactly dense in $SO(B)$.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of $O(B)$. We will use the following matrix realization of the Lie algebra $\mathfrak{g}$. Let J be the following matrix

$$J = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & I_{n-4} & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Then $\mathfrak{g}= \{A \in M_n(\mathbb{R}), AJ = JA^t\}$. There exists a maximal $\mathbb{R}$- split torus T in $O(B)$, whose Lie algebra $\mathfrak{t}$ is the following set $\mathfrak{t}=\{t \in M_n(\mathbb{R}), \varepsilon_1 \in \mathbb{R}, \varepsilon_2 \in \mathbb{R}\}$ where

$$t = \begin{pmatrix} \varepsilon_1 & 0 & \dots & 0 & 0 \\ 0 & \varepsilon_2 & \dots & 0 & 0 \\ \vdots & \vdots & 0_{n-4} & \vdots & \vdots \\ 0 & 0 & \dots & -\varepsilon_2 & 0 \\ 0 & 0 & \dots & 0 & -\varepsilon_1 \end{pmatrix}$$

Therefore all positive roots are the following: $\alpha = \varepsilon_1$, $\beta = \varepsilon_2$, $\alpha+\beta = \varepsilon_1+\varepsilon_2$, $\alpha-\beta = \varepsilon_1-\varepsilon_2$. The dimensions of the corresponding root spaces are as follows $dim V_\alpha = dim V_\beta = n-2$, $dim V_{\alpha+\beta} = dim V_{\alpha-\beta} = 1$.

We can assume that $T \leq G$. Let G_0 be the smallest connected simple subgroup of $O(B)$, containing T. Then $G_0 \leq G$. Let $\mathfrak{g_o}$ be the Lie algebra of G_0. Then the simple algebra $\mathfrak{g_o}$ contains $\mathfrak{t}$ and therefore all the following root spaces $V_{\alpha+\beta}, V_{-(\alpha+\beta)}, V_{\alpha-\beta}, V_{-(\alpha-\beta)}$. Let U^+ be the sum $U^+ = V_\alpha + V_\beta$ and $U^- = V_{-\alpha} + V_{-\beta}$. Let $\mathfrak{g}$ be the Lie algebra of G, then $\mathfrak{g} \cap U^+ \neq \{0\}$. This intersection is T–invariant, therefore $\mathfrak{g} \cap V_\alpha \neq \{0\}$ and $\mathfrak{g} \cap V_\beta \neq \{0\}$. There exists an element w_1 in the Weyl group of G_0 such that $w_1 V_\alpha {w_1}^{-1} = V_\beta$. Therefore, $w_1(V_\alpha \cap \mathfrak{g})\ {w_1}^{-1} = V_\beta \cap \mathfrak{g}$. Let K be the centralizer of T, $K = Z_{O(B)}(T)$. This group acts transitively on U^+ and U^-. There exists an element w_2 in the Weyl group of G_0 such that $w_2 U^+ {w_2}^{-1} = U^-$. Therefore one can find a connected co–compactly dense subgroup $\widehat{G}$ in $SO(B)$ such that if $\hat{\mathfrak{g}}$ is the Lie algebra of $\widehat{G}$ then $\hat{\mathfrak{g}} \cap U^+ = \mathfrak{g} \cap U^+$ and $\hat{\mathfrak{g}} \cap U^- = \mathfrak{g} \cap U^-$. Thus, the sets of unipotent elements in G and $\widehat{G}$ are the same. Hence $G = \widehat{G}$ since both of them are generated by their unipotent elements. □

Actually, using the same idea, one can prove that if the signature of a quadratic form B is $(n-k, k)$ and G is a connected simple subgroup in $O(B)$ of real rank k, then G is co–compactly dense in $SO(B)$.

5.6 Lemma *Assume G contains a simple group of real rank 2. Then the group G is a reductive group.*

Proof. Let U be the unipotent radical of G and let S be a semisimple part of G. There is one connected simple normal subgroup S_1 of S with real rank 2. From the previous lemma follows, that there are two S–invariant B-orthogonal subspaces V_1 and V_2, such that $S|_{V_1} = O(B|_{V_1})$, and the restriction $B|_{V_2}$ is a positive definite quadratic form. Let $V_0 = \{v \in V \mid tv = v \text{ for all t} \in \text{U}\}$ and let $V_0^\perp$ be the orthogonal complement to V_0. Put $W_0 = V_0^\perp \cap V_0$. It is obvious that the subspace W_0 is G–invariant and B- isotropic. Hence the projection of W_0 onto V_1 along V_2 is injective. Let w be a vector from W_0 and let the $w = w_1 + w_2$ be the decomposition of w according to the direct sum decomposition

$V = V_1 \oplus V_2$. We have $gv = v$ for every $g \in S_1$ and $v \in V_2$. Therefore $gw - w \in V_1$ for every element g from S_1. Thus $gw - w \in V_1 \cap W_0 = \{0\}$. Hence $gw = w$ for every $g \in S_1$ and $w \in W_0$. Combining $gw_1 = w_1$ and $gw = w$ we have $gw_2 = w_2$ for all $g \in S_1$. Therefore $w_2 = 0$ and $W_0 = \{0\}$. Then $V = V_0 \oplus V_0^{\perp}$. There exists a non-zero vector v from $V_0^{\perp}$ such that $tv - v \in V_0$ for every element t from the unipotent group U. Since $tv - v \in V_0^{\perp}$ we have $tv = v$ for all $t \in U$. Therefore $u \in V_0$. Thus $u \in V_0 \cap V_0^{\perp} = \{0\}$. A contradiction, hence $U = \{1\}$ □

Now we will prove the Theorem C.

Proof. Let G be the Zariski closure of $\ell(\Gamma)$. If the real rank of every semisimple subgroup of G is ≤ 1, then group Γ is virtually solvable (see [A]). Assume that there is a connected simple subgroup of G of a real rank 2. Then by the previous lemma group G is reductive. Obviously $[\,\ell(\Gamma), \ell(\Gamma)\,]$ is Zariski dense in $[\,G, G\,]$ and $[\,G, G\,]$ is a semisimple group. It follows from lemma 5.5 that the group $[\,\Gamma, \Gamma\,]$ is co–compactly dense in G_B and acts properly discontinuously on the affine space $\mathbb{R}^n$. From theorem B follows that this is impossible. This contradiction completes the proof . □

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