A Stochastic Contraction Principle

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Abstract. We provide conditions for the existence of measurable solutions to the equation $\xi(T\omega) = f(\omega, \xi(\omega))$, where $T : \Omega \to \Omega$ is an automorphism of the probability space Ω and $f(\omega, \cdot)$ is a strictly (but not necessarily uniformly) contracting mapping.

Key words and Phrases: Random dynamical systems, Stochastic equations, Contraction mappings, Perron-Frobenius theory.

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Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $T : \Omega \to \Omega$ its automorphism, i.e., a one-to-one mapping such that T and T^{-1} are measurable and preserve the measure P. Let $(X, \mathcal{X})$ be a measurable space and $f(\omega, x)$ a jointly measurable mapping of $\Omega \times X$ into X. Our main goal in this paper is to provide conditions under which the equation

$$\xi(T\omega) = f(\omega, \xi(\omega)) \text{ (a.s.)} \tag{1}$$

has a solution in the class of measurable mappings $\xi : \Omega \to X$. We also will be interested in the uniqueness of this solution and properties of its stability. Equations of the type (1) arise in connection with various questions of the theory of random dynamical systems (Arnold 1998). Our work is motivated by the applications of results of this kind in the stochastic Perron-Frobenius theory (Evstigneev 1974, Arnold, Demetrius and Gundlach 1994, Kifer 1996).

Let Y be a measurable subset of X equipped with a metric ρ such that Y is separable with respect to this metric and the Borel measurable structure on Y coincides with the measurable structure induced from X. Define

$$f_k(\omega, x) := f(T^{k-1}\omega, x) \ (k = 0, \pm 1, \pm 2, ...), \tag{2}$$

$$f^{(k)}(\omega, x) := f_0(\omega) f_{-1}(\omega) ... f_{-k}(\omega)(x) \ (k = 0, 1, 2, ...).$$

We will assume that the space $(X, \mathcal{X})$ is standard[1] and the mapping f satisfies the following requirements.

(f.1) For each $\omega \in \Omega$, the mapping $f(\omega, x)$ transforms Y into itself and is continuous on Y with respect to the metric ρ.

(f.2) There is a sequence of $\mathcal{F}$-measurable sets $\Omega_0 \subseteq \Omega_1 \subseteq ... \Omega$ such that $P(\Omega_m) \to 1$ and, for each $m = 0, 1, 2, ...,$ and $\omega \in \Omega_m$, the following conditions hold:

(a) the set

$$X^{(m)}(\omega) := f^{(m)}(\omega, X)$$

is contained in Y and is compact with respect to the metric ρ;

(b) for all $x, y \in Y$ with $x \neq y$, we have

$$\rho(f^{(m)}(\omega, x), f^{(m)}(\omega, y)) < \rho(x, y) \tag{3}$$

(contraction property).

[1] A measurable space is called standard if it is isomorphic to a Borel subset of a complete separable metric space.

Since the sequence of sets Ω_m is increasing, there exists a measurable function $m(\omega)$ with non-negative integer values such that for each $\omega \in \Omega_1 \cup \Omega_2 \cup ...$ (and hence for almost all ω), we have $\omega \in \Omega_k$ for all $k \geq m(\omega)$.

The main result is as follows.

Theorem 1. *(i) There exists a measurable mapping* $\xi : \Omega \to Y$ *for which equation (1) holds and*

$$\lim_{m(\omega)\leq k\to\infty} \sup_{x\in X} \rho(\xi(\omega), f_0(\omega)...f_{-k}(\omega)(x)) = 0 \tag{4}$$

with probability one.

(ii) If $\eta : \Omega \to X$ *is any (not necessarily measurable) solution to (1), then* $\eta = \xi$ *a.s.*

(iii) Let $\mathcal{F}_0 \subseteq \mathcal{F}$ *be a* σ*-algebra such that the mappings* $f_{-k}(\omega, x)$, $k = 0, 1, ...$, *of the space* $\Omega \times X$ *into* X *are* $\mathcal{F}_0 \times \mathcal{X}$*-measurable and* $\Omega_m \in \mathcal{F}_0$ *for all* $m = 0, 1, ...$ *Then there exists an* $\mathcal{F}_0$*-measurable mapping* ξ *possessing the properties described in (i) and (ii).*

According to (4), the random sequence $f_0...f_{-k}(x)$ converges to $\xi(\omega)$ uniformly in x with probability one. Note that the distance ρ between $f_0...f_{-k}(x)$ and $\xi(\omega)$ involved in (4) is defined only if $f_0...f_{-k}(x) \in Y$. By virtue of (f.2), this inclusion holds for almost all ω and all $k \geq m(\omega)$, therefore the limit in (4) is taken over $k \geq m(\omega)$.

A key role in the proof of Theorem 1 is played by the contraction property (3). Note that this property is weaker than the one which is usually designated by this term. We do not assume that the left-hand side in (3) is not greater than $\kappa\rho(x, y)$, where κ is a constant less than one. It is known that the conventional Bahach contraction principle, dealing with mappings of complete metric spaces (see, e.g., Kolmogorov and Fomin 1957), is not valid for mappings with this weaker contraction property. It is valid, however, under assumptions of compactness. The deterministic version of our result which served as a prototype for Theorem 1 is as follows. Let Z be a space with metric r and $g : Z \to Z$ a mapping satisfying $r(g(u), g(v)) < r(u, v)$ $(u \neq v)$. If Z is compact, then g has a unique fixed point (cf. Eisenack and Fenske 1978). The proof of this fact is easy. Indeed, the mapping g and hence all its iterates g^k $(k = 1, 2, ...)$ are continuous. The set $Z^\infty := \cap_{k=1}^\infty g^k(Z)$ is compact and non-empty as an intersection of a nested family of compact sets $g^k(Z)$. We have $g(Z^\infty) = Z^\infty$, which implies, by virtue of the contraction property that the diameter of Z^∞ is zero, and so Z^∞ consists of a single point z satisfying $g(z) = z$. In the proof of Theorem 1 given below, we use basically similar considerations, adjusting them to the stochastic case.

Proof of Theorem 1. 1st step. Observe that $X^{(0)}(\omega) \supseteq X^{(1)}(\omega) \supseteq X^{(2)}(\omega) \supseteq$ ...and $X^{(k)}(\omega) \neq \emptyset$ for each k and ω. Consider the sets Ω_m

$(m=0,1,...)$ described in (f.2) and denote by $\bar{\Omega}$ their union. According to (f.2), $P(\bar{\Omega})=1$ and each $\omega\in\bar{\Omega}$ belongs to all Ω_k, $k\geq m(\omega)$. For $\omega\in\bar{\Omega}$, all the sets $X^{(k)}(\omega)$, $k\geq m(\omega)$, are contained in Y and compact, and so the set $X^\infty(\omega):=\cap_{k=0}^\infty X^{(k)}(\omega)\subseteq Y$ is non-empty and compact as an intersection of a nested sequence of non-empty compacta $X^{(k)}(\omega)$, $k\geq m(\omega)$.

2nd step. Define $\Omega^*=\cap_{k=-\infty}^{+\infty}(T^k\bar{\Omega})$. The set Ω^* is invariant and $P(\Omega^*)=1$. Let us show that

$$X^\infty(T\omega)=f(\omega,X^\infty(\omega)),\ \omega\in\Omega^*. \tag{5}$$

Equality (5) is equivalent to

$$X^\infty(\omega)=f(T^{-1}\omega,X^\infty(T^{-1}\omega)),\ \omega\in\Omega^*, \tag{6}$$

because $\omega\in\Omega^*$ if and only if $T^{-1}\omega\in\Omega^*$. To prove (6) let us observe that

$$f(T^{-1}\omega,\bigcap_{k=0}^{\infty}X^{(k)}(T^{-1}\omega))=\bigcap_{k=0}^{\infty}f(T^{-1}\omega,X^{(k)}(T^{-1}\omega)),\ \omega\in\Omega^*. \tag{7}$$

The inclusion "$\subseteq$" in (7) holds always. The opposite inclusion follows from the continuity of $f(T^{-1}\omega,\cdot)$ on Y and the fact that $X^{(k)}(\omega)$ are nested and compact in Y for all k large enough. By using (7), we obtain

$$f(T^{-1}\omega,X^\infty(T^{-1}\omega))=f(T^{-1}\omega,\bigcap_{k=0}^{\infty}X^{(k)}(T^{-1}\omega))=\bigcap_{k=0}^{\infty}f(T^{-1}\omega,X^{(k)}(T^{-1}\omega))=$$

$$\bigcap_{k=0}^{\infty}f_0(\omega,X^{(k)}(T^{-1}\omega))=\bigcap_{k=0}^{\infty}X^{(k+1)}(\omega)=X^\infty(\omega),\ \omega\in\Omega^*.$$

The fourth equality in this chain of relations holds because

$$X^{(k)}(T^{-1}\omega)=f_0(T^{-1}\omega)f_{-1}(T^{-1}\omega)...f_{-k}(T^{-1}\omega)(X)=$$

$$f_{-1}(\omega)f_{-2}(\omega)...f_{-k-1}(\omega)(X),$$

and so

$$f_0(\omega)X^{(k)}(T^{-1}\omega)=f_0(\omega)f_{-1}(\omega)...f_{-k-1}(\omega)(X)=X^{(k+1)}(\omega).$$

3rd step. For $\omega\in\Omega^*$, denote the diameter of the compact set $X^\infty(\omega)\subseteq Y$ by $\rho(\omega)$ and put $\rho(\omega)=+\infty$ if $\omega\in\Omega\setminus\Omega^*$. For $k=0,1,2,...$, put $\Omega_k^*:=\Omega^*\cap\Omega_k$ and for $\omega\in\Omega$ define

$$\rho^{(k)}(\omega)=\begin{cases}\operatorname{diam}X^{(k)}(\omega), & \text{if } \omega\in\Omega_k^*,\\ +\infty, & \text{otherwise.}\end{cases} \tag{8}$$

Recall that, for $\omega \in \Omega_k$ and hence for $\omega \in \Omega_k^*$, the set $X^{(k)}(\omega)$ is contained in Y and is compact, so that its diameter $\operatorname{diam} X^{(k)}(\omega)$ in the metric ρ is well-defined and finite.

Denote by $\mathcal{F}^P$ the completion of the σ-algebra $\mathcal{F}$ with respect to the measure P. We claim that $\rho^{(k)}(\omega)$ is an $\mathcal{F}^P$-measurable function of $\omega \in \Omega$. To prove this assertion we observe that for $\omega \in \Omega_k^*$, we have $\operatorname{diam} X^{(k)}(\omega) = \operatorname{diam} f^{(k)}(\omega, X)$, where $f^{(k)}(\omega, x) := f_0(\omega)f_{-1}(\omega)...f_{-k}(\omega)(x)$ is a measurable mapping from $\Omega \times X$ to X. Consequently, for each real a, the set Ω_k^a of $\omega \in \Omega_k^*$ satisfying $\operatorname{diam} f^{(k)}(\omega, X) > a$ is the projection on Ω_k^* of the set

$$\{(\omega, x, y) \in \Omega_k^* \times X \times X : \ \rho(f^{(k)}(\omega, x),\ f^{(k)}(\omega, y)) > a \ \} \tag{9}$$

measurable in $\Omega_k^* \times X \times X$. Since X (and hence $X \times X$) is standard, Ω_k^a is $\mathcal{F}^P$-measurable (see, e.g., Dellacherie and Meyer 1978, Theorem III.33). This implies that $\rho^{(k)}(\omega)$ is $\mathcal{F}^P$-measurable because $\rho^{(k)}(\omega) = +\infty$ outside Ω_k^*. Finally, $\rho(\omega)$ is $\mathcal{F}^P$-measurable because

$$\rho(\omega) = \lim_{k\to\infty} \rho^{(k)}(\omega) \text{ for } \omega \in \Omega^*, \tag{10}$$

which follows the fact that $X^{(k)}(\omega)$ are nested and compact in Y for all $\omega \in \Omega^*$ and $k \geq m(\omega)$.

4th step. Let us show that $\rho(\omega) = 0$ a.s.. Observe that equality (5) implies

$$X^\infty(\omega) = f(T^{-1}\omega, X^\infty(T^{-1}\omega)) = f(T^{-1}\omega)(X^\infty(T^{-1}\omega)) =$$

$$f(T^{-1}\omega)f(T^{-2}\omega)(X^\infty(T^{-2}\omega)) = ... = f(T^{-1}\omega)...f(T^{-m-1}\omega)(X^\infty(T^{-m-1}\omega)) =$$

$$f_0(\omega)...f_{-m}(\omega)(X^\infty(T^{-m-1}\omega)) = f^{(m)}(\omega, X^\infty(T^{-m-1}\omega)),\ \omega \in \Omega^*. \tag{11}$$

By virtue of (11) and (3), for $\omega \in \Omega_m^*$, we have

$$\rho(\omega) \leq \rho(T^{-m-1}\omega) \tag{12}$$

and

$$\text{if } \rho(\omega) > 0, \text{ then } \rho(\omega) < \rho(T^{-m-1}\omega). \tag{13}$$

Since $P(\Omega_m^*) = P(\Omega^* \cap \Omega_m) \to 1$, inequality (12) yields

$$\lim_{k\to\infty} P\{\rho(\omega) \leq \rho(T^{-k}\omega)\} \to 1. \tag{14}$$

We claim that (14) implies

$$\rho(\omega) = \rho(T^{-k}\omega) \text{ a.s. for all } k. \tag{15}$$

To deduce (15) from (14) we may assume that ρ is bounded by some constant C (we can always replace ρ by $\arctan \rho$). By setting $\Delta_k := \{\omega : \rho(\omega) \leq \rho(T^{-k}\omega)\}$, we write

$$E|\rho(\omega) - \rho(T^{-k}\omega)| \leq E(\rho(T^{-k}\omega) - \rho(\omega))\chi_{\Delta_k} + CP(\Omega \setminus \Delta_k),$$

where χ_{Δ_k} is the indicator function of Δ_k and

$$E(\rho(T^{-k}\omega) - \rho(\omega))\chi_{\Delta_k} = E(\rho(T^{-k}\omega) - \rho(\omega))\chi_{\Delta_k} - E(\rho(T^{-k}\omega) - \rho(\omega)) =$$

$$-E(\rho(T^{-k}\omega) - \rho(\omega))\chi_{\Omega\setminus\Delta_k} \leq CP(\Omega \setminus \Delta_k).$$

Consequently, $E|\rho(\omega) - \rho(T^{-k}\omega)| \to 0$, which implies (14).

Suppose $\rho(\omega) > 0$ with strictly positive probability. Then there exists a number m and a set $\Gamma \in \mathcal{F}$ contained in Ω_m^* such that $P(\Gamma) > 0$ and $\rho(\omega) > 0$ on Γ. By virtue of (13), we have $\rho(\omega) < \rho(T^{-m-1}\omega)$ for $\omega \in \Gamma$. On the other hand, we proved that $\rho(\omega) = \rho(T^{-m-1}\omega)$ for almost all ω. A contradiction.

5th step. Since the $\mathcal{F}^P$-measurable function $\rho(\omega)$ is zero a.s., there is a set $\tilde{\Omega} \in \mathcal{F}$ of full measure such that

$$\tilde{\Omega} \subseteq \Omega^* \text{ and } \rho(\omega) = 0 \text{ for each } \omega \in \tilde{\Omega}. \tag{16}$$

For such an ω, the set $X^\infty(\omega)$ consists of exactly one point, $\xi^\infty(\omega)$. Replacing $\tilde{\Omega}$ by $\cap_{k=-\infty}^{+\infty}(T^k\tilde{\Omega})$, we may assume that $\tilde{\Omega}$ is invariant. Fix any point $\tilde{y} \in Y$ and put $\xi(\omega) = \xi^\infty(\omega)$ for $\omega \in \tilde{\Omega}$ and $\xi(\omega) = \tilde{y}$ for $\omega \in \Omega \setminus \tilde{\Omega}$. For any $\omega \in \tilde{\Omega} \subseteq \Omega^*$ we have $T\omega \in \tilde{\Omega} \subseteq \Omega^*$, and so

$$\{\xi(T\omega)\} = \{\xi^\infty(T\omega)\} = X^\infty(T\omega) = f(\omega, X^\infty(\omega))$$

$$= f(\omega, \{\xi^\infty(\omega)\}) = f(\omega, \{\xi(\omega)\})$$

by virtue of (5). Consequently, $\xi(\omega)$ satisfies (1).

Consider the functions $\rho^{(k)}(\omega)$ defined by (8). For each $\omega \in \tilde{\Omega}$ and $k \geq m(\omega)$ we have $\omega \in \Omega_k$, and so

$$\sup_{x \in X} \rho(\xi(\omega), f_0 ... f_{-k}(x)) \leq \operatorname{diam} X^{(k)}(\omega) = \rho^{(k)}(\omega). \tag{17}$$

This implies (4) since $\lim \rho^{(k)}(\omega) = \rho(\omega) = 0$ on the set $\tilde{\Omega}$ of full measure.

6th step. To complete the proof of (i) it is sufficient to show that the mapping ξ constructed above coincides a.s. with some $\mathcal{F}$-measurable mapping ζ. Then ζ will be the sought-for solution to (1) possessing the properties listed in (i). To prove the existence such a mapping, we will consider the σ-algebra $\mathcal{F}_0$ described in (iii) and establish the existence of an $\mathcal{F}_0$-measurable mapping ζ such that $\zeta = \xi$ a.s..Thus we will prove (iii) and complete the proof of (i) (clearly (iii) can be applied to $\mathcal{F}_0 := \mathcal{F}$).

Fix some $x_0 \in X$, $y_0 \in Y$ and consider the mappings of Ω into Y ($k = 0, 1, ...$):

$$\zeta^k(\omega) := \begin{cases} f_0(\omega)...f_{-k}(\omega)(x_0) & \text{if } \omega \in \Omega_k, \\ y_0\,, & \text{otherwise.} \end{cases}$$

These mappings are $\mathcal{F}_0$-measurable by virtue of the assumptions on $f_{-k}(\omega, x)$ and Ω_k imposed in (iii). For each $\omega \in \tilde{\Omega}$ and $k \geq m(\omega)$, we have $\omega \in \Omega_k$ and

$$\rho(\xi(\omega), \zeta^k(\omega)) = \rho(\xi(\omega), f_0(\omega)...f_{-k}(\omega)(x_0) \leq \rho^{(k)}(\omega)$$

where $\rho^{(k)}(\omega) \to 0$ as $k \to \infty$ (see (10), (17) and (16)). Thus $\zeta^k(\omega) \to \xi(\omega)$ on a set $\tilde{\Omega} \in \mathcal{F}$, where $P(\tilde{\Omega}) = 1$.

Denote by Ω' the set of those $\omega \in \Omega$ for which the sequence $\zeta^k(\omega)$ converges in the metric ρ. Clearly, Ω' is the projection on Ω of the set

$$\{(\omega, x) \in \Omega \times Y : \lim \rho(x, \zeta^k(\omega)) = 0\}.$$

This set is measurable with respect to $\mathcal{F}_0 \times \mathcal{B}(Y) \subseteq \mathcal{F}_0 \times \mathcal{X}$ and, consequently, its projection Ω' on Ω is $\mathcal{F}_0^P$-measurable. We have $\tilde{\Omega} \subseteq \Omega'$, and so $P(\Omega') = 1$. Therefore there is an $\mathcal{F}_0$-measurable set $\hat{\Omega} \subseteq \Omega'$ with $P(\hat{\Omega}) = 1$. Define $\zeta(\omega)$ as $\lim \zeta^k(\omega)$ for $\omega \in \hat{\Omega}$ and as y_0 for $\omega \in \Omega \setminus \hat{\Omega}$. Then $\zeta(\omega)$ is $\mathcal{F}_0$-measurable and $\zeta(\omega) = \xi(\omega)$ for almost all ω (for $\omega \in \tilde{\Omega} \cap \hat{\Omega}$).

7th step. It remains to prove (ii). If $\eta : \Omega \to X$ is a mapping satisfying (1), then

$$\eta(\omega) = f(T^{-1}\omega, \eta(T^{-1}\omega)) = f(T^{-1}\omega)(\eta(T^{-1}\omega)) =$$

$$f(T^{-1}\omega)f(T^{-2}\omega)(\eta(T^{-2}\omega)) = ... = f(T^{-1}\omega)...f(T^{-k-1}\omega)(\eta(T^{-m-1}\omega)) \text{ (a.s.)},$$

which yields

$$\eta(\omega) = f_0(\omega)f_{-1}(\omega)...f_{-k}(\omega)(\eta(T^{-k-1}\omega)) \text{ (a.s.)}. \tag{18}$$

By combining (18) and (4), we get

$$\rho(\xi(\omega), \eta(\omega)) \leq \sup_{x \in X} \rho(\xi(\omega), f_0(\omega)...f_{-k}(\omega)(x)) \to 0 \ \ \text{(a.s.)},$$

and so $\xi(\omega) = \eta(\omega)$ a.s.

The proof is complete.

Remark 1. We note that part (b) of condition (f.2) can be replaced by a somewhat weaker assumption in which inequality (3) is supposed to hold for all distinct points x, y in $f^{(m)}(T^{-m-1}\omega, X) \cap Y$ (rather than in Y). This follows from the fact that the set $f^{(m)}(T^{-m-1}\omega, X) \cap Y$ contains $X^\infty(T^{-m-1}\omega)$—see Step 4 of the proof. In the deterministic case, this means that $f^{(m)}$ satisfies (3) for all $x \neq y$ in the compactum $f^{(m)}(X) \subseteq Y$.

Remark 2. We used the assumption of separability of the metric space Y to ensure that the metric $\rho(x, y)$ is a jointly measurable function of (x, y) (see (9)). Although we stated this assumption separately, it should be noted that it follows from the other conditions we postulated. Indeed, the space Y with its Borel σ-algebra $\mathcal{B}$ is standard, and so the cardinality of $\mathcal{B}$ cannot be greater than continuum. But a non-separable metric space always contains an uncountable discrete set, all subsets of which are closed and hence Borel, and then the cardinality of $\mathcal{B}$ is strictly greater than continuum.

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