PURE POINT DIFFRACTION AND CUT AND PROJECT SCHEMES FOR MEASURES: THE SMOOTH CASE

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ABSTRACT. We present cut and project formalism based on measures and continuous weight functions of sufficiently fast decay. The emerging measures are strongly almost periodic. The corresponding dynamical systems are compact groups and homomorphic images of the underlying torus. In particular, they are strictly ergodic with pure point spectrum and continuous eigenfunctions. Their diffraction can be calculated explicitly. Our results cover and extend corresponding earlier results on dense Dirac combs and continuous weight functions with compact support. They also mark a clear difference in terms of factor maps between the case of continuous and non-continuous weight functions.

1. INTRODUCTION

This paper is concerned with the harmonic analysis behind certain models of aperiodic order. The latter is a specific form of order with long range correlations but no translation symmetry. It has attracted a lot of attention in the last two decades, compare the surveys and monographs [6, 23, 30, 35, 43].

This attention is partly due to the actual discovery of physical substances, later called quasicrystals, exhibiting such a form of order [44, 22]. Their key feature is a pure point diffraction spectrum combined with a non-periodic structure. (In a periodic structure pure point diffraction results easily from a Poisson summation type formula, see [10, 24] for further ideas in this direction.) This attention is also due to the conceptual mathematical relevance of aperiodic order as an intermediate form of (dis)order between periodicity and randomness. In fact, aperiodic order has highly distinctive and far from being understood geometric, combinatorial and Fourier analytic features.

The most prominent models of aperiodic order arise from so called cut and project schemes. They are called model sets or harmonious sets. They were introduced and first studied by Meyer in [29] for purely theoretical reasons. His investigations have later been generalised and extended in various directions (see [31, 32] for recent surveys and [38] for a recent inverse spectral type result). In the physics community cut and project models have been the objects of choice from the very beginning of theoretical investigation of quasicrystals [28].

In the study of aperiodic order and diffraction the use of dynamical systems has a long history going back to [12, 37] (see [20, 42, 45, 46] as well). Recently two further lines of research have proven fruitful: These are the systematic studies of notions of almost periodicity [7, 17, 34] and the replacement of sets by translation bounded measures [2, 3, 40, 25].

In line with these developments the basic aim of this paper is to extend the cut and project formalism to measures. More precisely, specific goals of this paper are

- to develop a cut and project scheme based on measures (instead of sets),
- to study the dynamical systems arising from these schemes,
- to investigate almost periodicity properties in this context.

Our results lead to a rather complete picture with quite strong properties being valid, provided the weight function is *smooth*, i.e., continuous and sufficiently fast decaying. In this case, almost periodicity is present in a rather strong form and (essentially) everything is determined by the underling torus dynamical system. More precisely, the arising measure dynamical systems are factors of the torus dynamical system. They carry a group structure and the factor map is a group homomorphism. Diffraction can be calculated explicitly.

In some sense our models are more regular than the "usual" cut and project schemes, where the weight function is the characteristic function of a Riemann integrable set. Mathematically, this is reflected in the almost periodicity properties of the underlying measures (as opposed to almost periodicity properties of averaged quantities like the autocorrelation). From the point of view of physics one may also argue in favour of our models: The strict cut off procedure in the usual model sets is highly idealised, whereas the cut off by continuous functions may be more realistic, at least in an averaged sense. Moreover, such models are used to analyse diffraction properties of random tilings [13, 18], whose vertex sets are derived from model sets, see also [8, 40].

For special cases some of these results are already known. Hof [20] presents results on continuous weight functions with compact support as a tool in his study of the usual model sets. Richard [40] has systematically investigated dense Dirac combs on $\mathbb{R}^d$. Our results cover and considerably extend the corresponding results of these authors, see Section 11 below.

We would like to emphasise that these results do no longer hold if the smoothness of the weight function is violated. More precisely, in the usual cut and project schemes, the arising dynamical system is neither a group nor a factor of the torus. On the contrary, the torus in that case is a factor of the dynamical system [1, 42], but not vice versa (see below Section 13). Thus, our results show in particular a change in the role of the torus system depending on the continuity of the weight function.

The paper is organised as follows: In Section 2 we recall background and notation. Section 3 presents the cut and project schemes for measures and gives our main results. The necessary investigation of almost-periodicity is carried out in Section 4. An abstract study of factors in our context is given in Section 5. Section 6 studies the dynamical systems arising from the measure cut and project schemes. After these preparations we discuss the proof of Theorem 3.1 in Section 7. A Weyl formula on uniform distribution is presented in Section 9. This is used to discuss the so-called Fourier Bohr coefficients and the proof of Theorem 3.3 in Section 10. Dense Dirac combs and other examples are studied in Section 11. Injectivity of the arising factor map is discussed in Section 12. Finally, in Section 13, we compare our results to those for usual model set dynamical systems.

2. Measure dynamical systems and diffraction

2.1. **Dynamical systems.** Whenever X is a σ-compact locally compact space (by which we mean to include the Hausdorff property), the space of continuous functions on X is denoted by $C(X)$, the subspace of continuous functions with compact support by $C_c(X)$ and the space of continuous bounded functions by $C_b(X)$. The latter two spaces are complete normed spaces when equipped with the supremum norm $\| \cdot \|_\infty$.

A topological space X carries the Borel σ-algebra generated by all closed subsets of X. By the Riesz-Markov representation theorem, the set $\mathcal{M}(X)$ of all complex regular Borel measures on X can then be identified with the dual space $C_c(X)^*$ of complex valued, linear functionals on $C_c(X)$ which are continuous with respect to a suitable topology, see [36, Ch.

6.5] for details. In particular, we write $\int_X f \, d\mu = \mu(f)$ for $f \in C_c(X)$. The space $\mathcal{M}(X)$ carries the vague topology, i.e., the weakest topology that makes all functionals $\mu \mapsto \mu(\varphi)$, $\varphi \in C_c(X)$, continuous. Alternatively, the vague topology arises by considering $\mathcal{M}(X)$ to be a subset of $\prod_{\varphi \in C_c(X)} \mathbb{C}$, which is equipped with the product topology, via

$$\mathcal{M}(X) \longrightarrow \prod_{\varphi \in C_c(X)} \mathbb{C}, \quad \nu \mapsto (\varphi \mapsto \nu(\varphi)).$$

The total variation of a measure $\mu \in \mathcal{M}(X)$ is denoted by $|\mu|$.

We will have to deal with various abelian groups. The group operation will be written additively as $+$ or $\dotplus$ if necessary to avoid misunderstandings. Now, let G be a σ-compact locally compact abelian (LCA) group. The Haar measure on G is denoted by m_G or dt. The dual group of G is denoted by $\widehat{G}$, and the pairing between a character $\hat{s} \in \widehat{G}$ and $t \in G$ is written as $(\hat{s}, t)$. As usual the Fourier transform $\widehat{f}$ of an integrable function f is defined by $\widehat{f}(\hat{s}) = \int_G \overline{(\hat{s}, t)} f(t) dt$.

Whenever G acts on the compact space Ω (which is then also Hausdorff by our convention) by a continuous action

$$\alpha \colon G \times \Omega \longrightarrow \Omega, \quad (t, \omega) \mapsto \alpha_t(\omega),$$

where $G \times \Omega$ carries the product topology, the pair (Ω, α) is called a *topological dynamical system* over G.

An α-invariant probability measure on Ω is then called *ergodic* if every measurable invariant subset of Ω has either measure zero or measure one. The dynamical system (Ω, α) is called *uniquely ergodic* if there exists a unique α-invariant probability measure on Ω, which then is ergodic by standard theory. (Ω, α) is called *minimal* if, for all $\omega \in \Omega$, the G-orbit $\{\alpha_t \omega : t \in G\}$ is dense in Ω. If (Ω, α) is both uniquely ergodic and minimal, it is called *strictly ergodic*.

Given an α-invariant probability measure m on Ω, we can form the Hilbert space $L^2(\Omega, m)$ of square integrable measurable functions on Ω. This space is equipped with the inner product

$$\langle f, g \rangle \;=\; \langle f, g \rangle_\Omega \;:=\; \int_\Omega \overline{f(\omega)} \, g(\omega) \, dm(\omega).$$

The action α gives rise to a unitary representation $T := T^\Omega := T^{(\Omega, \alpha, m)}$ of G on $L^2(\Omega, m)$ by

$$T_t \colon L^2(\Omega, m) \longrightarrow L^2(\Omega, m), \quad (T_t f)(\omega) \;:=\; f(\alpha_{-t} \omega),$$

for every $f \in L^2(\Omega, m)$ and arbitrary $t \in G$. An $f \in L^2(\Omega, m)$ is called an *eigenfunction* of T with *eigenvalue* $\hat{s} \in \widehat{G}$ if $T_t f = (\hat{s}, t) f$ for every $t \in G$. An eigenfunction (to $\hat{s}$, say) is called *continuous* if it has a continuous representative f with $f(\alpha_{-t} \omega) = (\hat{s}, t) f(\omega)$, for all $\omega \in \Omega$ and $t \in G$. The representation T is said to have *pure point spectrum* if the set of eigenfunctions is total in $L^2(\Omega, m)$. One then also says that the dynamical system (Ω, α) has *pure point dynamical spectrum*.

Finally, we will need the notion of factor of a dynamical system.

Definition 2.1. *Let two topological dynamical systems (Ω, α) and (Θ, β) under the action of G be given. Then, (Θ, β) is called a* factor *of (Ω, α), with factor map Φ, if $\Phi \colon \Omega \longrightarrow \Theta$ is a continuous surjection with $\Phi(\alpha_t(\omega)) = \beta_t(\Phi(\omega))$ for all $\omega \in \Omega$ and $t \in G$.*

2.2. **Measure dynamical systems.** We will be concerned with dynamical systems built from measures. These systems will be discussed next. They have been introduced in [2, 3], to which we refer for further details and proofs of the subsequent discussion.

A measure $\nu \in \mathcal{M}(G)$ is called *translation bounded* if there exist some $C > 0$ and an open non empty relatively compact set V in G so that

$$|\nu|(t + V) \leq C \tag{1}$$

for every $t \in G$, where $|\nu|$ is the total variation measure of ν. The set of all translation bounded measures satisfying (1) is denoted by $\mathcal{M}_{C,V}(G)$. The set of all translation bounded measures is denoted by $\mathcal{M}^\infty(G)$. As a subset of $\mathcal{M}(G)$, it carries the vague topology. $\mathcal{M}_{C,V}(G)$ is compact in this topology. There is an obvious action of G on $\mathcal{M}^\infty(G)$, again denoted by α, given by

$$\alpha \colon G \times \mathcal{M}^\infty(G) \longrightarrow \mathcal{M}^\infty(G), \quad (t, \nu) \mapsto \alpha_t \nu \quad \text{with} \quad (\alpha_t \nu)(\varphi) := \nu(\delta_{-t} * \varphi)$$

for $\varphi \in C_c(G)$. Here, δ_t denotes the unit point mass at $t \in G$ and the convolution $\omega * \varphi$ between $\varphi \in C_c(G)$ and $\omega \in \mathcal{M}^\infty(G)$ is defined by

$$\omega * \varphi(s) := \int \varphi(s - u) \, \mathrm{d}\omega(u).$$

It is not hard to see that α is continuous when restricted to a compact subset of $\mathcal{M}^\infty(G)$.

Definition 2.2. (Ω, α) *is called a dynamical system on the translation bounded measures on G (TMDS) if Ω is a compact α-invariant subset of $\mathcal{M}_{C,V}(G)$ for some open relatively compact V and $C > 0$.*

Every translation bounded measure ν gives rise to a (TMDS) $(\Omega(\nu), \alpha)$, where

$$\Omega(\nu) := \overline{\{\alpha_t \nu : t \in G\}}.$$

More precisely, if $\nu \in \mathcal{M}_{C,V}(G)$, then $\Omega(\nu) \subset \mathcal{M}_{C,V}(G)$.

As usual $\varphi \in C_b(G)$ is called *almost periodic* (in the sense of Bohr) if, for every $\epsilon > 0$, the set of $t \in G$ with $\|\delta_t * \varphi - \varphi\|_\infty \leq \epsilon$ is relatively dense in G. By standard reasoning this is equivalent to $\{\delta_t * \varphi : t \in G\}$ being relatively compact in in $C_b(G)$ (see e.g. [47]).

Definition 2.3. *A translation bounded measure ν is called strongly almost periodic if $\nu * \varphi$ is almost periodic (in the Bohr sense) for every $\varphi \in C_c(G)$.*

2.3. **Diffraction theory.** Having introduced our models, we can now discuss some key issues of diffraction theory, where we follow [2, 3, 7].

Let (Ω, α) be a TMDS, equipped with an α-invariant measure m. Fix $\omega \in \Omega$ and let $\psi \in C_c(G)$ with $\int \psi(t) \, \mathrm{d}t = 1$ be given. Then, $\gamma_m : C_c(G) \longrightarrow \mathbb{C}$ defined by

$$\gamma_m(\varphi) := \int_\Omega \int_G \int_G \varphi(s + t) \psi(t) \, \mathrm{d}\omega(s) \, \mathrm{d}\widetilde{\omega}(t) \, \mathrm{d}m(\omega)$$

is a positive definite measure which does not depend on ψ (provided $\int \psi(t) \, \mathrm{d}t = 1$). Here, for $\nu \in \mathcal{M}(G)$, the measure $\widetilde{\nu}$ is defined by $\widetilde{\nu}(\varphi) := \overline{\nu(\overline{\varphi(-\cdot)})}$. The measure γ_m is called *autocorrelation measure*. Its Fourier transform exists and is called *diffraction measure* (see [14, 15] for definition and background on Fourier transforms on measures). This measure describes the outcome of actual diffraction experiments [11, 20]. If (Ω, α) is ergodic, γ_m can be calculated via a limiting procedure [2, 16, 20, 42]. More precisely, recall that the

convolution $\mu * \nu$ of two bounded measures μ and ν on G is defined to be the measure $\mu * \nu(\varphi) := \int \int \varphi(s+t) \, d\mu(s) \, d\nu(t)$. Now, if (Ω, α) is uniquely ergodic

$$(2) \qquad \gamma_m = \lim_{n \to \infty} \frac{1}{m_G(B_n)} \omega_{B_n} * \widetilde{\omega_{B_n}}$$

for every $\omega \in \Omega$ [2, Thm. 5]. Here, the limit is taken in the vague topology, ω_{B_n} denotes the restriction of ω to B_n and (B_n) is a van Hove sequence in G. This means [42] that for every compact $K \subset G$,

$$\lim_{n \to \infty} \frac{m_G(\partial^K B_n)}{m_G(B_n)} = 0,$$

where for arbitrary $A, K \subset G$ we set

$$(3) \qquad \partial^K A := ((K + A) \setminus A^\circ) \cup ((-K + \overline{G \setminus A}) \cap A),$$

where the bar denotes the closure of a set and the circle denotes the interior. If (Ω, α) is uniquely ergodic, we write γ instead of γ_m. We also recall the following result from [2].

Theorem 2.1. *Let (Ω, α) be a TMDS with invariant measure m. Then, the following assertions are equivalent.*

   (i) *The measure $\widehat{\gamma_m}$ is a pure point measure.*
   (ii) *T^Ω has pure point dynamical spectrum.*

In this case, the group generated by $\{\lambda \in \widehat{G} : \widehat{\gamma}(\{\lambda\}) > 0\}$ is the group of eigenvalues of T^Ω.

3. Cut and project schemes for measures: Main results

In this section, we introduce cut and project schemes and discuss our main results.

As usual a triple $(G, H, \widetilde{L})$ is called a cut-and-project scheme if G and H are locally compact σ-compact abelian groups and $\widetilde{L}$ is a *lattice* in $G \times H$ (i.e., a cocompact discrete subgroup) such that

   • the canonical projection $\pi : G \times H \longrightarrow G$ is one-to-one between $\widetilde{L}$ and $L := \pi(G)$ (in other words, $\widetilde{L} \cap (\{0\} \times H) = \{(0,0)\}$), and
   • the image $L^\star = \pi_{\mathrm{int}}(\widetilde{L})$ of the canonical projection $\pi_{\mathrm{int}} : G \times H \longrightarrow H$ is dense in H.

The group H is called the *internal* space. Given these properties of the projections π and π_{int}, one can define the $\star$-map as $(.)^\star : L \longrightarrow H$ via $x^\star := \left(\pi_{\mathrm{int}} \circ (\pi|_L)^{-1}\right)(x)$, where $(\pi|_L)^{-1}(x) = \pi^{-1}(x) \cap \widetilde{L}$, for all $x \in L$. This situation can be summarised in the following diagram.

$$
\begin{array}{ccccc}
G & \xleftarrow{\ \pi\ } & G \times H & \xrightarrow{\ \pi_{\mathrm{int}}\ } & H \\
\cup & & \cup & & \cup \text{ dense} \\
L & \xleftarrow{1-1} & \widetilde{L} & \longrightarrow & L^\star \\
\| & & & & \| \\
L & & \xrightarrow{\qquad \star \qquad} & & L^\star
\end{array}
$$

A cut and project scheme gives rise to a dynamical system in the following way: Define $\mathbb{T} := (G \times H)/\widetilde{L}$. By assumption on $\widetilde{L}$, $\mathbb{T}$ is a compact abelian group. Let

$$G \times H \longrightarrow \mathbb{T}, \quad (t, k) \mapsto [t, k],$$

be the canonical quotient map. There is a canonical continuous group homomorphism

$$\iota : G \longrightarrow \mathbb{T}, \ \ t \mapsto [t, 0].$$

The homomorphism ι has dense range as $L^\star$ is dense in H. It induces an action β of G on $\mathbb{T}$ via

$$\beta : G \times \mathbb{T} \longrightarrow \mathbb{T}, \ \ \beta_t([s, k]) := \iota(-t) + [s, k] = [s - t, k].$$

The dynamical system $(\mathbb{T}, \beta)$ will play a crucial role in our considerations. It is minimal and uniquely ergodic, as ι has dense range. Moreover, it has pure point spectrum. More precisely, the dual group $\widehat{\mathbb{T}}$ gives a set of eigenfunctions, which form a complete orthonormal basis by Peter-Weyl theorem (see [42] for further details). Later we will also meet the canonical injective group homomorphism

$$\kappa : H \longrightarrow \mathbb{T}, \ \ h \mapsto [0, h].$$

Definition 3.1. (a) *A quadruple $(G, H, \widetilde{L}, \rho)$ is called a* measure cut and project scheme *if $(G, H, \widetilde{L})$ is a cut and project scheme and ρ is an $\widetilde{L}$-invariant Borel measure on $G \times H$.*
(b) *Let $(G, H, \widetilde{L}, \rho)$ be a measure cut and project scheme. A function $f : H \longrightarrow \mathbb{C}$ is called* admissible *if it is measurable, locally bounded and for arbitrary $\varepsilon > 0$ and $\varphi \in C_c(G)$ there exists a compact $Q \subset H$ with*

$$\int_{G \times H} |\varphi(t + s)f(h + k)|(1 - 1_Q(h + k)) \, \mathrm{d}|\rho|(t, h) \leq \varepsilon$$

for every $(s, k) \in G \times H$, where 1_Q denotes the characteristic function of Q.

An example of a measure cut and project scheme is given by a cut and project scheme $(G, H, \widetilde{L})$ and $\rho := \delta_{\widetilde{L}} := \sum_{x \in \widetilde{L}} \delta_x$. It is not hard to see that then every Riemann integrable $f : H \longrightarrow \mathbb{C}$ is admissible. In this way all the "usual" cut and project schemes fall within measure cut and project schemes, see Section 11 and Section 13 for details.

Our focus here will be to investigate admissible functions which are continuous. However, some of our results will hold for arbitrary admissible functions.

Let a measure cut and project scheme $(G, H, \widetilde{L}, \rho)$ with an admissible f be given. As shown in Proposition 6.3 below, the map

$$(4) \qquad\qquad \nu_f : C_c(G) \longrightarrow \mathbb{C}, \ \ \varphi \mapsto \int_{G \times H} \varphi(t)f(h) \, \mathrm{d}\rho(t, h),$$

is a translation bounded measure. Thus, we can consider its hull

$$\Omega(\nu_f) := \overline{\{\alpha_t(\nu_f) : t \in G\}}.$$

By the discussion of the previous section, $(\Omega(\nu_f), \alpha)$ is then a TMDS.

Our main results are the following three. The first deals with the dynamical system side of the problem, the second and third deal with diffraction.

Theorem 3.1. *Let a measure cut and project scheme $(G, H, \widetilde{L}, \rho)$ with a continuous admissible f be given and ν_f be defined as in (4). Then, the following assertions hold.*

   (a) *ν_f is strongly almost periodic. In particular, $\Omega(\nu_f)$ has a unique abelian group structure such that $G \longrightarrow \Omega(\nu_f)$, $t \mapsto \alpha_t \nu_f$, is a continuous group homomorphism.*
   (b) *$(\Omega(\nu_f), \alpha)$ is a factor of $(\mathbb{T}, \beta)$ with factor map $\mu : \mathbb{T} \longrightarrow \Omega(\nu_f)$ given by $\mu([s, k])(\varphi) = \int f(h + k)\varphi(s + t) \, \mathrm{d}\rho(t, h)$. In fact, μ is a group homomorphism.*

(c) $(\Omega(\nu_f), \alpha)$ *is minimal, uniquely ergodic and has pure point spectrum with continuous eigenfunctions. The set of eigenvalues is contained in* $\{\lambda \circ \iota : \lambda \in \widehat{\mathbb{T}}\} \subset \widehat{G}$.

Before we can state the next results, we recall the following result on disintegration [26, Sec. 33]. Let $(G, H, \widetilde{L}, \rho)$ be a measure cut and project scheme. Let for $\xi = [s, h] \in \mathbb{T}$, the (well defined!) measure σ_ξ on $G \times H$ be given by $\sigma_\xi(g) = \sum_{(l,l^\star) \in \widetilde{L}} g(s + l, h + l^\star)$ for $g \in C_c(G \times H)$. Then, there exists a unique measure $\rho_\mathbb{T}$ on $\mathbb{T}$ with

$$(5) \qquad \int_{G \times H} g(s, h) \, d\rho(s, h) = \int_\mathbb{T} \sigma_\xi(g) \, d\rho_\mathbb{T}(\xi)$$

for all $g \in C_c(G \times H)$. In fact, (and this shows both existence and uniqueness) the measure $\rho_\mathbb{T}$ satisfies

$$\rho_\mathbb{T}(b) = \int_{G \times H} b([s, h]) \chi_Z(s, h) \, d\rho(s, h)$$

for $b \in C(\mathbb{T})$, whenever Z is a fundamental cell of $\widetilde{L}$ in $G \times H$ (i.e., Z is a measurable subset of $G \times H$ such that $Z \longrightarrow \mathbb{T}$, $(s, h) \mapsto [s, h]$, is bijective.)

Moreover, for a function $f : H \to \mathbb{C}$ define $\widetilde{f}(h) := \overline{f(-h)}$. For a measure ρ on $G \times H$, define $\overline{\rho}$ by $\overline{\rho}(g) = \overline{\rho(\overline{g})}$ for every $g \in C_c(G \times H)$.

Theorem 3.2. *Let a measure cut and project scheme* $(G, H, \widetilde{L}, \rho)$ *with a continuous admissible* f *be given. Then,* f *is integrable and in particular* $(f * \widetilde{f})(h)$ *exists for almost every h in H. For* $\varphi \in C_c(G)$ *and* $\xi = [s, h] \in \mathbb{T}$ *define*

$$\gamma_\xi(\varphi) := \frac{1}{(m_G \times m_H)(Z)} \sum_{(l,l^\star) \in \widetilde{L}} (f * \widetilde{f})(h - l^\star) \varphi(s - l)$$

whenever this exists. Note that this is well defined, i.e., $\gamma_\xi(\varphi)$ *does not depend on the chosen representative of* ξ. *Then, for every* $\varphi \in C_c(G)$ *and* $\rho \times \overline{\rho}$ *almost every* (ξ, η), $\gamma_{\xi - \eta}(\varphi)$ *exists and the autocorrelation* γ *of* ν_f *satisfies*

$$\gamma(\varphi) = \int \int \gamma_{\xi - \eta}(\varphi) \, d\rho_\mathbb{T}(\xi) \, d\overline{\rho_\mathbb{T}}(\eta).$$

If $\rho = \delta_{\widetilde{L}}$, *then* f *is square integrable,* $(f * \widetilde{f})(h)$ *exists for every* $h \in H$ *and* $\gamma = \gamma_0$.

Theorem 3.3. *Let a measure cut and project scheme* $(G, H, \widetilde{L}, \rho)$ *with a continuous admissible* f *be given and* ν_f *be defined as in* (4). *Let* $\widehat{\gamma}$ *be the associated diffraction measure. Then,* $\widehat{\gamma}$ *is a pure point measure supported on* $\{\lambda \circ \iota : \lambda \in \widehat{\mathbb{T}}\}$. *More precisely,* $\widehat{\gamma} = \sum_{\lambda \in \widehat{\mathbb{T}}} |c_\lambda|^2 \delta_{\lambda \circ \iota}$ *with*

$$c_\lambda := \frac{\rho_\mathbb{T}(\lambda)}{(m_G \times m_H)_\mathbb{T}(1)} \int_H f(h)(\lambda \circ \kappa)(h) dh = \lim_{n \to \infty} \frac{1}{m_G(B_n)} \nu_f(\chi_{B_n} \cdot \overline{(\lambda \circ \iota)})$$

for every van Hove sequence (B_n).

4. STRONGLY ALMOST PERIODIC MEASURES

In this section, we show that almost periodic measures on G give rise to topological groups, which are dynamical systems. Much of the material of this section can is at least implicitly contained in the literature, particularly in [15]. However, for the convenience of the reader,

and as our dynamical system perspective is not the usual approach to these results, we include proofs.

We start by introducing the relevant topology. For $\nu \in \mathcal{M}^\infty(G)$ and $\varphi \in C_c(G)$, the convolution $\nu * \varphi$ belongs to $C_b(G)$. Let $C_b(G)$ be equipped with the supremum norm. We then define the strong topology $\mathcal{T}_s$ to be the weakest topology on $\mathcal{M}^\infty(G)$ such that all maps

$$\mathcal{M}^\infty(G) \longrightarrow C_b(G), \;\; \nu \mapsto \nu * \varphi,$$

are continuous. Alternatively, we can describe this topology by considering $\mathcal{M}^\infty(G)$ as a subset of $\prod_{\varphi \in C_c(G)} C_b(G)$ which is equipped with the product topology via

$$i : \mathcal{M}^\infty(G) \longrightarrow \prod_{\varphi \in C_c(G)} C_b(G), \;\; i(\nu) := (\varphi \mapsto \nu * \varphi).$$

The projection on the ψ component

$$\prod_{\varphi \in C_c(G)} C_b(G) \longrightarrow C_b(G), \;\; x \mapsto x_\psi,$$

is denoted by p_ψ.

Proposition 4.1. $i(\mathcal{M}^\infty(G))$ is closed in $\prod_{\varphi \in C_c(G)} C_b(G)$.

Proof. Let (ν_n) be a net in $\mathcal{M}^\infty(G)$ such that $i(\nu_n)$ converges to $x \in \prod_{\varphi \in C_c(G)} C_b(G)$. Then, ν_n converge in particular in the vague topology to a measure ν and $\nu * \varphi$ belongs to $C_b(G)$ for every $\varphi \in C_c(G)$. Thus, ν is translation bounded and it is not hard to see that $i(\nu) = x$. $\square$

Lemma 4.2. Let $\nu \in \mathcal{M}^\infty(G)$ be given. The following assertions are equivalent:
  (i) *The measure ν is strongly almost periodic (i.e. $\nu * \varphi$ is Bohr almost periodic for every $\varphi \in C_c(G)$).*
  (ii) *$\{\alpha_t \nu : t \in G\}$ is relatively compact in $\mathcal{T}_s$.*
  (iii) *The topological space $\Omega(\nu)$ (the hull of ν in the vague topology) is a topological group with addition $\dot{+}$ satisfying $\alpha_s \nu \dot{+} \alpha_t \nu = \alpha_{s+t} \nu$ for all $s, t \in G$.*

Proof. (ii) $\Longrightarrow$ (i): Define $B := \{\alpha_t \nu : t \in G\}$ and $A_\varphi := \{\delta_t * (\nu * \varphi) : t \in G\}$ and $C_\varphi := \overline{A_\varphi}$. Then, $i(B)$ is relatively compact by (ii) and the definition of the strong topology. A direct calculation shows $A_\varphi = p_\varphi(i(B))$. As p_φ is continuous, compactness of C_φ follows now from $C_\varphi = \overline{A_\varphi} = \overline{p_\varphi(i(B))} = p_\varphi(\overline{i(B)})$.

(i) $\Longrightarrow$ (iii): For $R > 0$, let $K(R) \subset \mathbb{C}$ be the closed ball around the origin with radius R. As $\Omega(\nu)$ is compact, there exists for each $\varphi \in C_c(G)$ an $R_\varphi > 0$ with

$$\omega(\varphi) \in K(R_\varphi)$$

for every $\omega \in \Omega(\nu)$. We can and will therefore consider $\Omega(\nu)$ to be a compact subset of $\prod_{\varphi \in C_c(G)} K(R_\varphi)$. For $\varphi \in C_c(G)$ define $\check{\varphi} \in C_c(G)$ by $\check{\varphi}(t) := \varphi(-t)$. As ν is strongly almost periodic, for each $\varphi \in C_c(G)$, the function $v_\varphi := \nu * \check{\varphi}$ is almost periodic. Thus, the closure Ω_φ of $\{\delta_t * v_\varphi : t \in G\}$ with respect to the supremum norm $\|\cdot\|_\infty$ is a compact abelian group (see e.g. [19]) . Moreover, $j_\varphi : G \longrightarrow \Omega_\varphi, \; t \mapsto \delta_t * v_\varphi$, is a continuous group homomorphism. Then, $\prod_{\varphi \in C_c(G)} \Omega_\varphi$ is an abelian group, which is compact by Tychonov's theorem. Moreover,

$$j : G \longrightarrow \prod_{\varphi \in C_c(G)} \Omega_\varphi, \;\; t \mapsto (\varphi \mapsto j_\varphi(t)),$$

is a continuous group homomorphism. Obviously, $j(G)$ is a subgroup of $\prod_{\varphi \in C_c(G)} \Omega_\varphi$. In particular, its closure $\overline{j(G)}$ is a compact abelian group. We show that $\Omega(\nu)$ is homeomorphic to $\overline{j(G)}$:

Consider the evaluation at 0

$$\Delta : \prod_{\varphi \in C_c(G)} \Omega_\varphi \longrightarrow \prod_{\varphi \in C_c(G)} K(R_\varphi), \quad (\varphi \mapsto w_\varphi) \mapsto (\varphi \mapsto w_\varphi(0)).$$

Then, Δ is continuous, as each Ω_φ is equipped with the supremum norm. A short calculation shows

$$\Delta \circ j(t) = (\varphi \mapsto v_\varphi(-t)) = (\varphi \mapsto (\alpha_t \nu)(\varphi)) = \alpha_t \nu.$$

As Δ is continuous and $\overline{j(G)}$ compact, this gives

$$\Delta(\overline{j(G)}) = \overline{\Delta(j(G))} = \overline{\{\alpha_t \nu : t \in G\}} = \Omega(\nu).$$

Thus, Δ maps $\overline{j(G)}$ onto $\Omega(\nu)$.

Claim. Δ is one-to-one on $\overline{j(G)}$:

Proof of claim. As every $w = (\varphi \mapsto w_\varphi) \in j(G)$ satisfies $w_\varphi(t) = w_{\varphi(\cdot - t)}(0)$, the same holds for $w \in \overline{j(G)}$. Thus, the evaluations at 0 determine all coordinates and Δ is injective on $\overline{j(G)}$.

These considerations show that $\Delta : \overline{j(G)} \longrightarrow \Omega(\nu)$ is a homeomorphism. As $\overline{j(G)}$ is a compact abelian group, $\Omega(\nu)$ inherits the structure of a compact abelian group as well, and

$$\alpha_t \nu \dotplus \alpha_s \nu = \Delta(j(s)) \dotplus \Delta(j(t)) = \Delta(j(s) \dotplus j(t)) = \Delta(j(s+t)) = \alpha_{t+s} \nu.$$

This finishes the proof of this implication.

(iii) $\Longrightarrow$ (ii) : As in the proof of (ii)$\Longrightarrow$ (i), we define $B := \{\alpha_t \nu : t \in G\}$ and note that relative compactness of B is equivalent to relative compactness of $i(B)$.

Choose an arbitrary $\varphi \in C_c(G)$. We have to show that the closure Ω_φ of

$$\{\delta_s * \nu * \varphi : s \in G\}$$

in $(C_b(G), \| \cdot \|_\infty)$ is compact. Then, $\prod_{\varphi \in C_c(G)} \Omega_\varphi$ is compact by Tychonov's theorem, and relative compactness of $i(B) \subset \prod_{\varphi \in C_c(G)} \Omega_\varphi$ follows.

Obviously, the function $\alpha_\varphi : \Omega \longrightarrow \mathbb{C}, \omega \mapsto \omega * \varphi(0)$, is continuous. As Ω is a compact group, the map

$$\Omega \longrightarrow C(\Omega), \omega \mapsto \alpha_\varphi(\omega \dotplus \cdot)$$

is then continuous. Moreover,

$$C(\Omega) \longrightarrow C_b(G), b \mapsto (t \mapsto b(\alpha_t \nu)),$$

is continuous. Using $\alpha_t \nu \dotplus \omega = \alpha_t \omega$, we infer that

$$p_\varphi : \Omega \longrightarrow C_b(G), \omega \mapsto (t \mapsto \alpha_\varphi(\alpha_t \omega)),$$

is continuous as composition of continuous maps. In particular, $p_\varphi(\Omega)$ is compact. A direct calculation shows $p_\varphi(\alpha_s \nu) = \delta_s * \nu * \varphi$ yielding $\Omega_\varphi \subset p_\varphi(\Omega)$, and compactness of Ω_φ follows. $\square$

5. Factors

In this paper we study properties of $\Omega(\nu)$ for suitable translation bounded measures ν. Existence of a factor map

$$\mu : \mathbb{T} \longrightarrow \Omega(\nu), \tag{6}$$

with a suitable abelian compact group $\mathbb{T}$ will be of key importance. In this section, we provide abstract background to existence and use of such a factor map.

Factors inherit basic features of their underlying dynamical systems. The following statements summarises results proved in Section 3 of [3].

Proposition 5.1. [3, Sec. 3] *Let (Ω, α) be a topological dynamical system and let (Θ, β) be a factor with factor map $\Phi : \Omega \to \Theta$. For an invariant probability measure m on (Ω, α), we define the invariant probability measure $\Phi(m)$ on Θ by $\Phi(m)(g) := m(g \circ \Phi)$ for $g \in C_c(\Theta)$. The following assertions hold.*

(a) *If (Ω, α) is uniquely ergodic with invariant probability measure m, then (Θ, β) is uniquely ergodic with unique invariant probability measure $\Phi(m)$.*

(b) *If (Ω, α) has pure point dynamical spectrum when equipped with the invariant probability measure m, (Θ, β) has pure point dynamical spectrum when equipped with the measure $\Phi(m)$.*

(c) *If (Ω, α) is minimal, so is (Θ, β).*

(d) *If (Ω, α) is uniquely ergodic with pure point dynamical spectrum and all of its eigenfunctions continuous, the same holds for (Θ, β).*

We will be interested in special factors of compact groups. The relevant lemma is the following.

Lemma 5.2. *Let a compact abelian group $\mathbb{T}$ and a continuous group homomorphism $\iota : G \longrightarrow \mathbb{T}$ with dense range be given. Let β be the associated action of G on $\mathbb{T}$ i.e. $\beta_t(\xi) = \iota(t)\xi$ for $\xi \in \mathbb{T}$ and $t \in G$. If (Ω, α) is a factor of $(\mathbb{T}, \beta)$ with factor map μ, then there exists a unique topological group structure on Ω such that $\alpha_t\mu(0) \dot{+} \alpha_s\mu(0) = \alpha_{t+s}\mu(0)$. With respect to this group structure μ is a group homomorphism.*

Proof. By denseness, there can be at most one group structure with

$$\alpha_t\mu(0) \dot{+} \alpha_s\mu(0) = \alpha_{t+s}\mu(0).$$

Next, we show that

$$\mu(\eta) = \mu(\rho) \Longleftrightarrow \mu(\rho - \eta) = \mu(0). \tag{7}$$

Let $\eta = \lim \beta_{t_s}(0)$.

"$\Rightarrow$:" We have $-\eta = \lim \beta_{-t_s}(0)$ and

$$\mu(\rho - \eta) = \lim \mu(\beta_{-t_s}\rho) = \lim \alpha_{-t_s}\mu(\rho) = \lim \alpha_{-t_s}\mu(\eta) = \lim \mu(\beta_{-t_s}\eta) = \mu(0).$$

"$\Leftarrow$:" Set $\xi := \rho - \eta$. Thus, $\xi + \eta = \rho$ and

$$\mu(\rho) = \lim \mu(\xi + \iota(t_s)(0)) = \lim \mu(\beta_{t_s}\xi) = \lim \alpha_{t_s}\mu(\xi) = \lim \alpha_{t_s}\mu(0) = \lim \mu(\beta_{t_i}0) = \mu(\eta).$$

By continuity of μ, the set

$$U := \{\xi \in \mathbb{T} : \mu(\xi) = \mu(0)\}$$

is closed. Moreover, by (7), U is a subgroup. As G is abelian, U is normal and

$$\widetilde{\mu} : \mathbb{T}/U \longrightarrow \Omega, \; \rho + U \mapsto \mu(\rho),$$

is well defined, continuous and bijective. Thus, Ω is homeomorphic to the group $\mathbb{T}/U$ and inherits the desired group structure. $\square$

Theorem 5.1. *Let a compact abelian group $\mathbb{T}$ and a continuous group homomorphism $\iota : G \longrightarrow \mathbb{T}$ with dense range be given. Let β be the associated action of G on $\mathbb{T}$, i.e., $\beta_t(\xi) = \iota(t)\xi$. Let ν be a translation bounded measure on G. Then, the following assertions are equivalent:*

(i) *There exists a factor map $\mu : \mathbb{T} \longrightarrow \Omega(\nu)$.*
(ii) *The measure ν is strongly almost periodic, $\Omega(\nu)$ is a topological group satisfying $\alpha_s \nu + \alpha_t \nu = \alpha_{s+t} \nu$ and $\widehat{\Omega(\nu)} \subset \widehat{\mathbb{T}}$, where both groups are considered as subgroups of $\widehat{G}$.*

If there exists a cut and project scheme $(G, H, \widetilde{L})$ with $\mathbb{T} = (G \times H)/\widetilde{L}$, this is equivalent to

(iii) *There exists an $\widetilde{L}$-invariant measure σ on $G \times H$ together with a continuous disintegration $\sigma = \int_H \sigma_h \, dh$ (i.e. $H \longrightarrow \mathcal{M}(G)$, $h \mapsto \sigma_h$, is continuous and $\int_{G \times H} g(s, h) \, d\sigma(s, h) = \int_H \sigma_h(g(\cdot, h)) \, dh$ for every $g \in C_c(G \times H)$) and $\nu = \sigma_0$.*

Proof. (i)$\Longrightarrow$ (ii): As μ is a factor map, $\Omega(\nu)$ inherits by Lemma 5.2 the structure of an abelian group such that μ is a group homomorphism. As μ is onto, the inclusion $\widehat{\Omega(\nu)} \subset \widehat{\mathbb{T}}$ follows. As $\Omega(\nu)$ is a group, the measure ν is almost periodic by Lemma 4.2.

(ii)$\Longrightarrow$(i): Dualising the inclusion $\widehat{\Omega(\nu)} \subset \widehat{\mathbb{T}}$ gives a map $\widehat{\widehat{\mathbb{T}}} \longrightarrow \widehat{\widehat{\Omega(\nu)}}$. This map is onto as $\widehat{\Omega(\nu)}$ is closed as a subgroup of the discrete group $\widehat{\mathbb{T}}$ (see e.g. Corollary 4.41 in [14]). By Pontryagin duality, we obtain a surjective map from $\mathbb{T}$ to $\Omega(\nu)$. It is easy to see that it is a factor map.

(i) $\Longrightarrow$ (iii): For $h \in H$ define $\sigma_h := \mu([0, h])$. Then, $\sigma_0 = \nu$ by definition and $h \mapsto \sigma_h$ is continuous as μ is continuous. Moreover, $\sigma = \int_H \sigma_h \, dh$ is $\widetilde{L}$-invariant as $\alpha_l(\sigma_{h+l^\star}) = \mu([l, h + l^\star]) = \mu([0, h]) = \sigma_h$ for every $(l, l^\star) \in \widetilde{L}$.

(iii) $\Longrightarrow$ (i): Define $\mu' : G \times H \longrightarrow \mathcal{M}(G)$ by $\mu'(t, h) := \alpha_t \sigma_h$. Then, μ' is continuous as $h \mapsto \sigma_h$ is and $\mu'(0, 0) = \nu$. Moreover, $\widetilde{L}$-invariance of σ gives

$$\int_H \sigma_h(g(\cdot, h)) \, dh = \sigma(g) = \sigma(g(\cdot - l, \cdot - l^\star)) = \int_H \alpha_l(\sigma_{h+l^\star})(g(\cdot, h)) \, dh$$

for every $g \in C_c(G \times H)$ and every $(l, l^\star) \in \widetilde{L}$. As $h \mapsto \sigma_h$ is continuous, this shows $\sigma_h = \alpha_l(\sigma_{h+l^\star})$ for every $(l, l^\star) \in \widetilde{L}$, and $\widetilde{L}$-invariance of μ' follows. $\square$

Example. We show that the previous result covers the "classic" quasiperiodic functions: Let a cut and project scheme $(G, H, \widetilde{L})$ be given with the associated canonical projection $p : G \times H \longrightarrow \mathbb{T}$. Let $A : \mathbb{T} \longrightarrow \mathbb{R}$ be continuous and set

$$\rho := A \circ p \, dt \times dh.$$

Thus, $a : G \longrightarrow \mathbb{R}$, $a(t) := A([t, 0])$ is quasiperiodic.

If we now define $\sigma_h := A([\cdot, h]) \, dt$, then $\sigma_0 = a \, dt$ and $h \mapsto \sigma_h$ is continuous with $\rho = \int_H \sigma_h \, dh$. Thus, the previous theorem applies to $\nu := a \, dt$, and we obtain a factor map and group homomorphism μ satisfying (6). In this sense, our results cover not only Dirac combs

but also quasiperiodic functions. Of course, for a quasiperiodic validity of (6) can also rather directly be shown from the definitions.

6. A study of admissibility

The aim of this section is to study admissibility.

Proposition 6.1. *Let* $(G, H, \widetilde{L}, \rho)$ *be a measure cut and project scheme. Then, the measure* ρ *is translation bounded.*

Proof. Let C be the closure of a nonempty open relatively compact set in $G \times H$. Let D be a compact subset of $G \times H$ containing a fundamental domain of $\widetilde{L}$. Then, by $\widetilde{L}$-invariance of ρ, for arbitrary $u \in G \times H$, there exists an $v \in D$ with $|\rho|(u + C) = |\rho|(v + C)$. As $D + C$ is compact and ρ is a Borel measure, this implies

$$|\rho|(u + C) \leq |\rho|(D + C) = const < \infty$$

for every $u \in G \times H$. The proposition follows. $\qquad\square$

Proposition 6.2. *Let* $(G, H, \widetilde{L}, \rho)$ *be a measure cut and project scheme and* $f : H \longrightarrow \mathbb{C}$ *be locally bounded and measurable. Then, the following assertions are equivalent:*

  (i) *The function* f *is admissible.*
  (ii) *For all* $\varepsilon > 0$ *and* $\varphi \in C_c(G)$ *there exists a* $\chi : H \longrightarrow [0, 1]$ *in* $C_c(H)$ *with*

$$\int_{G \times H} |\varphi(t + s) f(h + k)| (1 - \chi(h + k))| \, d|\rho|(t, h) \leq \varepsilon$$

  for every $s \in G$ *and* $k \in H$.
  (iii) *For all* $\varepsilon > 0$ *and* $\varphi \in C_c(G)$ *there exists a* $g \in C_c(H)$ *with*

$$\int_{G \times H} |\varphi(t + s)(f(h + k) - g(h + k))| \, d|\rho|(t, h) \leq \varepsilon$$

  for every $s \in G$ *and* $k \in H$.

Proof. (i)$\Longrightarrow$(ii): By Tietze's extension theorem, there exists a $\chi \in C_c(H)$ with $1 \geq \chi \geq 0$ and $\chi = 1$ on Q. Then $1 - 1_Q \geq 1 - \chi \geq 0$.
(ii)$\Longrightarrow$(iii): Set $g = \chi f$.
(iii) $\Longrightarrow$ (i): Set $Q := \text{supp}(g)$. $\qquad\square$

Proposition 6.3. *Let* $(G, H, \widetilde{L}, \rho)$ *be a measure cut and project scheme and* $f : H \longrightarrow \mathbb{C}$ *be admissible. For every* $k \in H$ *and* $\varphi \in C_c(G)$, *there exists a* $C_\varphi \geq 0$ *with*

$$\int |\varphi(s + t) f(h + k)| \, d|\rho|(t, h) \leq C_\varphi$$

for every $s \in G$. *In particular, for every* $(s, k) \in G \times H$ *the map*

$$\mu'(s, k) : C_c(G) \longrightarrow \mathbb{C}, \quad \varphi \mapsto \int_{G \times H} f(h + k) \varphi(t + s) \, d\rho(t, h)$$

is a translation bounded measure on G *and so is* $\nu_f = \mu'(0, 0)$.

Proof. Choose $k \in H$ and let φ in $C_c(G)$ be given. As f is admissible, we can find a continuous $\chi : H \longrightarrow [0,1]$ with compact support with $\int_{G \times H} |\varphi(s+t)f(k+h)|(1 - \chi(h+k)) \, \mathrm{d}|\rho|(t,h) \leq 1$ for every $s \in G$. Moreover, as ρ is translation bounded and $(t,h) \mapsto |\varphi(s+t)f(h+k)\chi(h+k)|$ is bounded with compact support, there exists a C' with

$$\int_{G \times H} |\varphi(s+t)f(h+k)\chi(h+k)| \, \mathrm{d}|\rho|(t,h) \leq C'.$$

The first statement follows with $C := 1 + C'$. Translation boundedness of the measures $\mu'(s,k)$ now follows easily by choosing nonnegative φ which are equal to 1 on the closure of an arbitrary open relatively compact V. □

Proposition 6.4. *Let $(G, H, \widetilde{L}, \rho)$ be a measure cut and project scheme and $f : H \longrightarrow \mathbb{C}$ be admissible. For every $K \subset H$ compact, $\varepsilon > 0$ and $\varphi \in C_c(G)$, there exists a compact Q_K with*

$$\int_{G \times H} |f(h+k)\varphi(t+s)(1 - 1_{Q_K}(h))| \, \mathrm{d}|\rho|(t,h) \leq \varepsilon$$

for every $s \in G$ and $k \in K$.

Proof. As f is admissible, we can find $\chi : H \longrightarrow [0,1]$ continuous with compact support with

$$\int_{G \times H} |\varphi(t+s)f(h+k)(1 - \chi(h+k))| \, \mathrm{d}|\rho|(t,h) \leq \varepsilon$$

for every $s \in G$ and $k \in H$. Then, $Q_K := \mathrm{supp}(\chi) - K$ has the desired properties. □

So, far our discussion of admissibility did not assume continuity of f. We will now come to a characterisation of admissibility for continuous f.

Proposition 6.5. *Let $(G, H, \widetilde{L}, \rho)$ be a measure cut and project scheme with metrisable H. Let $c > 0$ and $g : G \times H \longrightarrow [0, \infty)$ continuous be given with*

$$\int_{G \times H} g(t+s, h) \, \mathrm{d}|\rho|(t,h) \leq c$$

for every $s \in G$. Then,

$$\int_{G \times H} g(t+s, h+k) \, \mathrm{d}|\rho|(t,h) \leq c$$

for every $(s,k) \in G \times H$.

Proof. As ρ is $\widetilde{L}$-invariant, the assumption implies

$$\int_{G \times H} g(t+s+l, h+l^\star) \, \mathrm{d}|\rho|(t,h) \leq c$$

for every $s \in G$ and every $(l, l^\star) \in \widetilde{L}$. As s is arbitrary, this implies

$$\int_{G \times H} g(t+s, h+l^\star) \, \mathrm{d}|\rho|(t,h) \leq c$$

for every $s \in G$ and every $l^\star \in \widetilde{L}$. As $L^\star$ is dense in H by definition of a cut and project scheme and H is metrisable by assumption, we can find, for any $k \in H$, a sequence $(l_n^\star)$ in $L^\star$ with $l_n^\star \to k$. Now, the statement follows from Fatou's Lemma by continuity of g. □

Proposition 6.6. *Let $(G, H, \widetilde{L}, \rho)$ be a measure cut and project scheme with metrisable H. Let $f : H \longrightarrow \mathbb{C}$ be continuous. Then, f is admissible if and only if for arbitrary $\varepsilon > 0$ and $\varphi \in C_c(G)$ there exists a compact $Q \subset H$ with*

$$\int_{G \times H} |\varphi(t+s)f(h)| \, (1 - 1_Q(h)) \, \mathrm{d}|\rho|(t,h) \leq \varepsilon$$

for every $s \in G$.

Proof. The "only if" part is immediate from the definition of admissibility. The "if" part follows from the previous proposition. □

We finish this section by discussing restrictions on f imposed by the admissibility requirement.

Proposition 6.7. *Let $(G, H, \widetilde{L}, \rho)$ be a measure cut and project scheme and $f : H \longrightarrow \mathbb{C}$ admissible. Then, f is integrable with respect to the Haar measure on H.*

Proof. As ρ is $\widetilde{L}$-invariant, we can choose continuous $\varphi : G \longrightarrow [0, \infty)$ and $\psi : H \longrightarrow [0, \infty)$ with compact support such that

$$(8) \qquad\qquad \int \varphi(s+t)\psi(k-h) \, \mathrm{d}|\rho|(t,h) \geq 1$$

for all $(s, k) \in G \times H$.

Set $C := \int \psi \, \mathrm{d}h$. By admissibility, there exists a compact $Q \subset H$ such that

$$\int \varphi(t+s)|f(h+k)|(1 - \chi_Q(h+k)) \, \mathrm{d}|\rho|(t,h) \leq 1$$

for all $(s, k) \in G \times H$. This gives

$$\int_H \psi(k) \left(\int \varphi(t+s)|f(h+k)|(1 - \chi_Q(h+k)) \, \mathrm{d}|\rho|(t,h) \right) \mathrm{d}k \leq C.$$

Fubini's theorem and the translation invariance of the Haar measure then imply

$$\int_H |f(k)|(1 - \chi_Q(k)) \left(\int \varphi(s+t)\psi(k-h) \, \mathrm{d}|\rho|(t,h) \right) \mathrm{d}k \leq C.$$

Now, the lower bound (8) implies

$$\int_H |f(k)|(1 - \chi_Q(k)) \, \mathrm{d}k \leq C.$$

Thus, $f(k)(1 - \chi_Q(k))$ is integrable. As f is locally bounded, $f\chi_Q$ is also integrable, and integrability of f follows. □

Remark. The converse of this proposition does not hold, as can be seen by choosing $\rho := \sum_{x \in \widetilde{L}} \delta_x$ and and f continuous and integrable with "peaks". More precisely, the following holds.

Proposition 6.8. *Let $(G, H, \widetilde{L}, \rho)$ be a measure cut and project scheme with $\rho = \delta_{\widetilde{L}}$. Let f be continuous and admissible. Then, f is bounded and square integrable.*

Proof. Let a nonnegative $\varphi \in C_c(G)$ with $\varphi(0) = 1$ be given. As shown in Proposition 6.3, there exists $C_\varphi > 0$ with

$$\sum_{(l,l^\star) \in \widetilde{L}} \varphi(s+l)|f(l^\star)| \le C_\varphi$$

for every $s \in G$. As $L^\star$ is dense in H and f is continuous, we infer

$$\sup\{|f(h)| : h \in H\} = \sup\{|f(l^\star)| : l \in L\} \le C_\varphi.$$

This proves boundedness of f. As f is integrable by the previous proposition, square integrability follows from boundedness. $\square$

In some sense, admissibility with respect to $\delta_{\widetilde{L}}$ implies admissibility with respect to any other $\widetilde{L}$-invariant measure.

Proposition 6.9. *Let $(G, H, \widetilde{L}, \rho)$ be a measure cut and project scheme. Let f be continuous and admissible with respect to $(G, H, \widetilde{L}, \delta_{\widetilde{L}})$. Then, f is admissible with respect to $(G, H, \widetilde{L}, \rho)$.*

Proof. To simplify the notation, we write ζ for $\delta_{\widetilde{L}}$. As ρ is $\widetilde{L}$-invariant, there exists a finite measure ρ_0 supported on a fundamental domain Z of $\widetilde{L}$ with

$$\rho = \zeta * \rho_0.$$

As f is admissible with respect to ζ, for each $\varphi \in C_c(G)$ and $\varepsilon > 0$, there exists $Q \subset H$ compact with

$$\int_{G \times H} |\varphi(t+s)f(h+k)|(1 - 1_Q(h+k)) \, d|\zeta|(t,h) \le \varepsilon$$

for every $s \in G$ and $k \in H$. This gives

$$\int_{G \times H} |\varphi(t+s)f(h+k)|(1 - 1_Q(h+k)) \, d|\rho|(t,h) \le$$

$$\int_Z \int_{G \times H} |\varphi(t+s)f(h+k)|(1 - 1_Q(h+k)) \, d|\zeta|(t,h) \, d|\rho_0|(s,k) \le |\rho_0|(Z)\,\varepsilon$$

for every $s \in G$ and $k \in H$, and we obtain admissibility of f with respect to ρ. $\square$

7. Proof of Theorem 3.1

In this section we provide a proof of Theorem 3.1.

Lemma 7.1. *Let $(G, H, \widetilde{L}, \rho)$ be a measure cut and project scheme and $f : H \longrightarrow \mathbb{C}$ be admissible and continuous. The map $\mu' : G \times H \longrightarrow \mathcal{M}^\infty(G)$*

$$\mu'(s,k) : C_c(G) \longrightarrow \mathbb{C}, \quad \varphi \mapsto \int_{G \times H} f(h+k)\varphi(t+s) \, d\rho(t,h)$$

defined in Proposition 6.3 is continuous and $\widetilde{L}$-invariant, i.e., one has $\mu'(s+k, h+k^\star) = \mu'(s,h)$ for arbitrary $(s,h) \in G \times H$ and arbitrary $(k, k^\star) \in \widetilde{L}$.

Proof. Invariance is immediate from the definitions. Continuity follows easily from an $\varepsilon/3$-argument: Namely, let $\{(s_n, k_n)\}$ be a net for which $(s_n, k_n) \longrightarrow (s, k)$ in $G \times H$. We have to show

$$\mu'(s_n, k_n)(\varphi) \longrightarrow \mu'(s, k)(\varphi)$$

for every $\varphi \in C_c(G)$. Let $\varphi \in C_c(G)$ and $\epsilon > 0$ be given. As $\{k_n\}$ converges to k, there exists a compact neighbourhood K of k with $\{k_n : n \geq n_0\} \subset K$. Therefore, by Proposition 6.4, there exists a compact Q_K in H with

$$\int_{G \times H} |\varphi(t+s)f(h+k')(1 - 1_{Q_K}(h))| \, \mathrm{d}|\rho|(t,h) \leq \varepsilon/3$$

for every $s \in G$ and $k' \in K$. On the other hand, by continuity of f and φ and by compactness of Q_K, there obviously exists an n_1 with

$$\int_{G \times H} |\varphi(t+s_n)f(h+k_n) - \varphi(t+s)f(h+k)| \, 1_{Q_K}(h) \, \mathrm{d}|\rho|(t,h) \leq \varepsilon/3$$

for all $n \geq n_1$. Putting this together, we infer

$$|\mu'(s_n, k_n)(\varphi) - \mu'(s,k)(\varphi)| \leq \varepsilon$$

for all $n \geq \max\{n_0, n_1\}$. $\hspace{2cm}$ □

Lemma 7.2. *Let $(G, H, \widetilde{L}, \rho)$ be a measure cut and project scheme and $f : H \longrightarrow \mathbb{C}$ be admissible and continuous. Then, the dynamical system $(\Omega(\nu_f), \alpha)$ is a factor of $(\mathbb{T}, \beta)$ with factor map*

$$\mu : \mathbb{T} \longrightarrow \Omega(\nu_f), \ \ \mu([s,k]) := \mu'(s,k).$$

Proof. By Lemma 7.1 , $\mu' : G \times H \longrightarrow \mathcal{M}^\infty(G)$ is continuous and $\widetilde{L}$-invariant. Thus, $\mu : \mathbb{T} \longrightarrow \mathcal{M}^\infty(G)$ is well defined and continuous. By definition, $\alpha_t(\mu([s,k])) = \mu(\beta_t([s,k]))$. Thus, it remains to show that

$$\Omega(\nu_f) = \mu(\mathbb{T}).$$

As μ is a factor map, we have

(9) $$\mu(\beta_t[0,0]) = \alpha_t(\mu([0,0])) = \alpha_t(\mu'(0,0)) = \alpha_t(\nu_f).$$

By minimality of β we have

$$\mathbb{T} = \overline{\{\beta_t([0,0]) : t \in G\}}.$$

Now, continuity of μ, compactness of $\mathbb{T}$ and (9) imply

$$\mu(\mathbb{T}) = \mu(\overline{\{\beta_t([0,0]) : t \in G\}}) = \overline{\{\alpha_t(\nu_f) : t \in G\}} = \Omega(\nu_f).$$

This finishes the proof. $\hspace{2cm}$ □

After these preparations we are ready to prove our first main result.

Proof of Theorem 3.1. (a) / (b) By Lemma 7.2, $(\Omega(\nu_f), \alpha)$ is a factor of $(\mathbb{T}, \beta)$ with factor map μ. Thus, by Theorem 5.1, ν_f is strongly almost periodic and $(\Omega(\nu_f), \alpha)$ carries the desired group structure.

(c) As $(\Omega(\nu_f), \alpha)$ is a factor of $(\mathbb{T}, \beta)$, it inherits spectral properties according to Fact 5.1. Now, $(\mathbb{T}, \beta)$ is well known to be uniquely ergodic and minimal with pure point spectrum and continuous eigenfunctions (see e.g. [42]). As $(\Omega(\nu_f), \alpha)$ is a factor of $(\mathbb{T}, \beta)$, the eigenvalues of $(\Omega(\nu_f), \alpha)$ are eigenvalues of $(\mathbb{T}, \beta)$ as well. The eigenvalues of $(\mathbb{T}, \beta)$ can be determined easily [42]. Namely, each $\lambda \in \widehat{\mathbb{T}}$ is an continuous eigenfunction to the eigenvalue $\lambda \circ \iota$, and this is a complete set of eigenfunctions. Now, the statement on the eigenvalues follows. $\hspace{1cm}$ □

We finish this section with a proof of almost periodicity of ν_f for $f \in C_c(H)$. While this statement is clear from the main theorem and the abstract tools used above, it is instructive to give a direct proof.

Lemma 7.3. *Let $f \in C_c(H)$ be given. Then, ν_f is strongly almost periodic.*

Proof. Let $\varepsilon > 0$ and $\varphi \in C_c(G)$ be arbitrary. We have to show that the set P of all $p \in G$ with

$$\|\delta_p * (\nu_f * \varphi) - \nu_f * \varphi\|_\infty \leq \varepsilon$$

is relatively dense in G. As $f \in C_c(H)$ and ρ is translation bounded, there exists an open neighbourhood V of $0 \in H$ with

$$\left| \int_{G \times H} \varphi(t - s) f(h + k) \, \mathrm{d}\rho(s, h) - \int_{G \times H} \varphi(t - s) f(h) \, \mathrm{d}\rho(s, h) \right| \leq \varepsilon$$

for all $t \in G$ and $k \in V$. By $\widetilde{L}$-invariance of ρ, we have

$$(\nu_f * \varphi)(t - \ell) = \int_{G \times H} \varphi(t - \ell - s) f(h) \, \mathrm{d}\rho(s, h) = \int_{G \times H} \varphi(t - s) f(h + \ell^\star) \, \mathrm{d}\rho(s, h)$$

for all $(\ell, \ell^\star) \in \widetilde{L}$. Putting the last two equations together, we see that P contains all $\ell \in L$ with $\ell^\star \in V$. As V is open, this set is relatively dense. $\qquad \square$

8. Proof of Theorem 3.2

In this section we provide a proof of Theorem 3.2. We need a preparatory result.

Proposition 8.1. *Let (Ω, α) be a measure dynamical system with invariant measure m. For arbitrary $\varphi, \psi \in C_c(G)$*

$$\int_\Omega \int_G \int_G |\varphi(s + t)\psi(s)| \, \mathrm{d}|\omega|(s) \, \mathrm{d}|\widetilde{\omega}|(t) \, \mathrm{d}m(\omega) < \infty.$$

Proof. This follows easily from uniform translation boundedness of $\omega \in \Omega$. $\qquad \square$

Proof of Theorem 3.2. By Proposition 6.7, the function f is integrable. By Theorem 3.1, $(\Omega(\nu_f)), \alpha)$ is uniquely ergodic. Denote the unique invariant measure by m. Fix $\omega \in \Omega(\nu_f)$.

Recall now the definition of the measure $\gamma = \gamma_m$ as

$$\gamma(\varphi) := \int_\Omega \int_G \int_G \varphi(s + t)\psi(t) \, \mathrm{d}\omega(s) \, \mathrm{d}\widetilde{\omega}(t) \, \mathrm{d}m(\omega)$$

for $\varphi \in C_c(G)$, where $\psi \in C_c(G)$ is arbitrary with $\int \psi(t) \, \mathrm{d}t = 1$.

Define F on Ω by $F(\omega) := \int_G \int_G \varphi(s + t)\psi(t) \, \mathrm{d}\omega(s) \, \mathrm{d}\widetilde{\omega}(t)$. By Proposition 5.1 (a), we then have

$$\gamma(\varphi) = \int_\mathbb{T} F(\mu(\xi)) \, \mathrm{d}\xi.$$

Let Z be a fundamental cell of $\widetilde{L}$ in $G \times H$. In order to avoid a tedious factor $1/(m_G \times m_H)(Z)$ in the subsequent discussion, we will assume without loss of generality that $(m_G \times m_H)(Z) = 1$.

Recalling $\mu'(s, h) = \mu([s, h])$ and applying the discussion before Theorem 2, to $\sigma = m_G \times m_H$ instead of ρ, we obtain

$$\gamma(\varphi) = \int_Z F(\mu'(r, v)) \, \mathrm{d}(m_G \times m_H)(r, v).$$

Unwinding the definitions then gives

$$\gamma(\varphi) = \int_Z \int_{G \times H} \psi(-t - r) \left(\int_{G \times H} \varphi(s - t) f(h + v) \, \mathrm{d}\rho(s, h) \right) \overline{f}(k + v) \, \mathrm{d}\overline{\rho}(t, k) \, \mathrm{d}(m_G \times m_H)(r, v).$$

Note that the argument of φ does not include an r, as integration over ω contributes an r and integration over $\widetilde{\omega}$ contributes an $-r$ to the argument of φ. We now use the $\widetilde{L}$-invariance of $\overline{\rho}$ to obtain

$$\gamma(\varphi) = \int_Z \sum_{(l,l^\star) \in \widetilde{L}} \left(\int_Z \psi(-t - l - r)G(t + l, v)\overline{f}(k + v + l^\star)\,\mathrm{d}\overline{\rho}(t, k) \right) \,\mathrm{d}(m_G \times m_H)(r, v)$$

with $G(t, v) = \int_{G \times H} \varphi(s - t)f(h + v)\,\mathrm{d}\rho(s, h)$. By $\widetilde{L}$-invariance of ρ, we have $G(t + l, v) = G(t, v + l^\star)$, and we infer

$$\gamma(\varphi) = \int_{G \times H} \int_Z \psi(-t - r)G(t, v)\overline{f}(k + v)\,\mathrm{d}\overline{\rho}(t, k)\,\mathrm{d}(m_G \times m_H)(r, v).$$

By Proposition 8.1, we can now interchange the order of integration. Carrying out the integration over $m_G(r)$ gives $1 = \int_G \psi(-t - r)\,\mathrm{d}m_G(r)$, by assumption on ψ. The integration over $m_H(v)$ yields $(f * \widetilde{f})(h - k)$. Altogether, we end up with

$$\gamma(\varphi) = \int_{G \times H} \left(\int_Z (f * \widetilde{f})(h - k)\varphi(s - t)\,\mathrm{d}\rho(s, h) \right) \,\mathrm{d}\overline{\rho}(t, k).$$

We now split the integration over $G \times H$ into integrations on translates of the fundamental cell, yielding

$$\gamma(\varphi) = \int_Z \int_Z \left(\sum_{(l,l^\star) \in \widetilde{L}} (f * \widetilde{f})(h - k - l^\star)\varphi(s - t - l) \right) \,\mathrm{d}\rho(s, h)\,\mathrm{d}\overline{\rho}(t, k).$$

This is the desired formula.

If $\rho = \delta_{\widetilde{L}}$, then $\rho_{\mathbb{T}} = \delta_{[0,0]}$, and f is square integrable by Proposition 6.8. This yields the remaining statements. $\qquad\square$

9. Weyl theorem on uniform distribution

In this section we provide a proof of a Weyl result on uniform distribution, Theorem 9.1 and derive two corollaries. Such a result is of independent interest (and well known for usual model sets). Moreover, one of the corollaries will be used later when we calculate the Fourier Bohr coefficients by a limiting procedure.

Our proof is inspired by [21, 33], with one modification: we realize that the functional Λ defined below is translation invariant and hence must be a multiple of Haar measure. This allows us to get a grip on our rather abstract situation without more effort than the mentioned works.

Various boundary and non boundary type terms have to be estimated. To do so we use the following proposition (see [27] for similar results as well).

Proposition 9.1. *Let $V \subset G$ be a nonempty, open relatively compact set. Let $C \geq 0$ and $\nu \in \mathcal{M}_{C,V}(G)$ be arbitrary. Then,*

$$|\nu|(B) \leq \frac{m_G(B - V)}{m_G(V)}C$$

for every relatively compact $B \subset G$.

Proof. A direct calculation shows $\chi_B \leq \frac{1}{m_G(V)}\chi_{B-V} * \chi_V$. This gives

$$
\begin{aligned}
|\nu|(B) &\leq \frac{1}{m_G(V)} \int \int \chi_V(t-s)\,\chi_{B-V}(s)\,\mathrm{d}s\,\mathrm{d}|\nu|(t) \\
(Fubini) &= \frac{1}{m_G(V)} \int \chi_{B-V}(s)\left(\int \chi_V(t-s)\mathrm{d}|\nu|(t) \right)\mathrm{d}s \\
&\leq \frac{m_G(B-V)}{m_G(V)}C.
\end{aligned}
$$

This finishes the proof. $\qquad\square$

With this proposition we can easily derive the following "uniform" version of Lemma 1.1 of [42]. More precisely, Lemma 1.1 deals with a single translation bounded measure. Here, we consider all of $\mathcal{M}_{C,V}(G)$ simultaneously.

Lemma 9.2. *Let $V \subset G$ open, nonempty and relatively compact be given. Let (B_n) be a van Hove sequence in G and set*

$$
D := \sup_{n \in \mathbb{N}} \frac{m_G(B_n - V)}{m_G(B_n)} < \infty.
$$

(a) For every $C \geq 0$,

$$
\sup_{\nu \in \mathcal{M}_{C,V}(G), n \in \mathbb{N}} \frac{|\nu|(B_n)}{m_G(B_n)} < C\,\frac{D}{m_G(V)}.
$$

(b) For every $C \geq 0$ and $K \subset G$ compact

$$
\lim_{n \to \infty} \sup_{\nu \in \mathcal{M}_{C,V}(G)} \frac{|\nu|(\partial^K B_n)}{m_G(B_n)} = 0.
$$

(c) For every $C \geq 0$ and every $\varphi \in C_c(G)$

$$
\lim_{n \to \infty} \sup_{\nu \in \mathcal{M}_{C,V}(G)} \frac{1}{m_G(B_n)}\left| \int_G (\varphi * \nu)(t)\,\chi_{B_n}(t)\,\mathrm{d}t - \left(\int_G \varphi(t)\,\mathrm{d}t \right)\nu(B_n) \right| = 0.
$$

Proof. (a) This is immediate from the previous proposition.

(b) Set $\widetilde{K} := (K \cup \{0\}) - \overline{V}$. Then, $\widetilde{K}$ is compact and

$$
\partial^K B - V \subset \partial^{\widetilde{K}} B
$$

for every $B \subset G$. Here, the K- boundary of a set was defined in (3). Thus, the previous proposition gives

$$
|\nu|(\partial^K B) \leq m_G(\partial^{\widetilde{K}} B)\frac{C}{m_G(V)},
$$

and the statement follows as (B_n) is a van Hove sequence.

(c) A short calculation (see proof of Lemma 1.1 (c) in [42] as well) gives

$$
\left| \int_G (\varphi * \nu)(t)\,\chi_{B_n}(t)\,\mathrm{d}t - \left(\int_G \varphi(t)\,\mathrm{d}t \right)\nu(B_n) \right| \leq \int_G |\varphi(t)|\,\mathrm{d}t\,|\nu|(\partial^K B_n),
$$

and the statement follows from (b).

$\qquad\square$

Theorem 9.1 (Weyl Theorem). *Let a measure cut and project scheme $(G, H, \widetilde{L}, \rho)$ be given. Let $f \in C_c(H)$ be arbitrary. Then,*

$$\lim_{n \to \infty} \frac{1}{m_G(B_n)} \mu(\xi)(\chi_{B_n}) = \frac{\rho_{\mathbb{T}}(1)}{(m_G \times m_H)_{\mathbb{T}}(1)} \int_H f(h) \, \mathrm{d}h$$

uniformly in $\xi \in \mathbb{T}$, where $\rho_{\mathbb{T}}$ is defined in equation (5).

Proof. Note that the map $\mu : \mathbb{T} \longrightarrow \mathcal{M}^\infty(G)$ depends on f. As shown in Lemma 7.1, it is continuous. Thus, $\mu_\varphi : \mathbb{T} \longrightarrow \mathbb{C}$, $\xi \mapsto \mu(\xi)(\varphi)$, is continuous for every $\varphi \in C_c(G)$. Choose $\varphi \in C_c(G)$ with $\int_G \varphi(s) \, \mathrm{d}s = 1$. Define $\check{\varphi}$ by $\check{\varphi}(t) := \varphi(-t)$ and note $\int \check{\varphi} \, \mathrm{d}s = 1$.

As $(\mathbb{T}, \beta)$ is uniquely ergodic, and (B_n) is a van Hove sequence, the limit

$$\lim_{n \to \infty} \frac{1}{m_G(B_n)} \int_G \chi_{B_n}(t) \mu_\varphi(\beta_t \xi) \, \mathrm{d}t$$

exists uniformly in $\xi \in \mathbb{T}$. A short calculation shows $\mu_\varphi(\beta_t \xi) = (\check{\varphi} * \mu(\xi))(t)$, and we infer from part (c) of the previous lemma that the limit

$$\Lambda(f) := \lim_{n \to \infty} \frac{1}{m_G(B_n)} \mu(\xi)(\chi_{B_n})$$

exists uniformly in $\xi \in \mathbb{T}$. Apparently, $\Lambda : C_c(H) \longrightarrow \mathbb{C}$ is a linear functional.

We next show that Λ has a certain boundedness property: Choose $K \subset H$ compact. Let $V \subset G$ be open, nonempty and relatively compact. As ρ is translation bounded (see Proposition 6.1),

$$A := \sup \left\{ \int \chi_V(s + t) \chi_K(h + k) \, \mathrm{d}|\rho|(s, h) : (t, k) \in G \times H \right\} < \infty.$$

Thus, $\Omega(\nu_f) \subset \mathcal{M}_{A\|f\|_\infty, V}$ whenever $\mathrm{supp} f \subset K$. Therefore, part (a) of the previous lemma gives the existence of a constant C_K such that

$$(10) \qquad\qquad\qquad |\Lambda(f)| \leq C_K \|f\|_\infty$$

for every $f \in C_c(H)$ with $\mathrm{supp} f \subset K$. This shows that Λ is a measure on H.

It is not hard to see that $\Lambda(f(\cdot - l^\star)) = \Lambda(f)$ for every $f \in L^\star$. As $L^\star$ is dense in H, we infer from the continuity property (10) that $\Lambda(f(\cdot - h)) = \Lambda(f)$ for every $h \in H$.

To summarise, Λ is a translation invariant measure on $C_c(H)$. Thus, Λ is a multiple of the Haar measure and there exists $c_\rho \in \mathbb{C}$ with

$$\Lambda(f) = c_\rho \int_H f(h) \, \mathrm{d}h.$$

In order to determine c_ρ, we choose a van Hove sequence (C_m) in H. Let $\psi \in C_c(H)$ with $\int_H \psi \, \mathrm{d}h = 1$ be given and set $f_m := \chi_{C_m} * \psi$. Then, f_m is a smoothed version of χ_{C_m} and, in particular,

$$c_\rho = \lim_{m \to \infty} \frac{1}{m_H(C_m)} \Lambda(f_m).$$

Consider for fixed (and large) $m \in \mathbb{N}$

$$\frac{1}{m_H(C_m)} \Lambda(f_m) = \lim_{n \to \infty} \frac{1}{m_H(C_m) m_G(B_n)} \int_{G \times H} f_m(h) \chi_{B_n}(s) \, \mathrm{d}\rho(s, h).$$

For n and m large $\int_{G \times H} f_m(h)\chi_{B_n}(s)\,\mathrm{d}\rho(s,h)$ is, up to a boundary term, equal to

$$\rho_{\mathbb{T}}(1)\,\sharp\left\{\widetilde{L} \cap (B_n \times C_m)\right\},$$

and $m_H(C_m)m_G(B_n) = (m_G \times m_H)(B_n \times C_m)$ is, up to a boundary term, equal to

$$(m_G \times m_H)_{\mathbb{T}}(1)\,\sharp\left\{\widetilde{L} \cap (B_n \times C_m)\right\},$$

where $\sharp$ denotes the cardinality. This easily gives the desired value of c_ρ. □

Corollary 9.3. *Let a measure cut and project scheme $(G, H, \widetilde{L}, \rho)$ be given. Let $f : H \longrightarrow \mathbb{C}$ be continuous and admissible. Then,*

$$\lim_{n \to \infty} \frac{1}{m_G(B_n)}\mu(\xi)(\chi_{B_n}) = \frac{\rho_{\mathbb{T}}(1)}{(m_G \times m_H)_{\mathbb{T}}(1)}\int_H f(h)\,\mathrm{d}h$$

uniformly in $\xi \in \mathbb{T}$.

Proof. This follows by approximation. Let (B_n) be a van Hove sequence in G. Let $V \subset$ be open, nonempty and relatively compact, and choose $\varphi \in C_c(G)$ with $0 \leq \chi_V \leq \varphi$. For every $\varepsilon > 0$ we can then find by admissibility a $Q' \subset H$ compact such that

$$\int_{G \times H} |f(h+k)(1 - \chi_{Q'}(h+k))\varphi(s+t)|\,\mathrm{d}|\rho|(s,h) \leq \varepsilon$$

for all $(t,k) \in G \times H$. Thus, the measures $\varphi \mapsto \int_{G \times H} f(h)(1 - \chi_{Q'}(h))\varphi(s)\,\mathrm{d}\rho(s,h)$ belong to $\mathcal{M}_{C,V}(G)$ with arbitrarily small C, provided Q' is chosen large enough. By the explicit dependence on C in (a) of Lemma 9.2, we can then find $Q_1 \subset H$ compact with

$$(11) \qquad \frac{1}{m_G(B_n)}\int_{G \times H} |f(h)(1 - \chi_{Q_1}(h))|\chi_{B_n}(t)\,\mathrm{d}|\rho|(t,h) \leq \varepsilon$$

for all $n \in \mathbb{N}$. By Proposition 6.7, we can find $Q_2 \subset H$ compact

$$(12) \qquad \frac{\rho_{\mathbb{T}}(1)}{(m_G \times m_H)_{\mathbb{T}}(1)}\int_H |f(1 - \chi_{Q_2})|\,\mathrm{d}h \leq \varepsilon.$$

Set $Q := Q_1 \cup Q_2$ and choose $\chi \in C_c(H)$ with $\chi_Q \leq \chi$. We can now write $f = f\chi + f(1-\chi)$ and set $T_\xi^n(f) := \frac{1}{m_G(B_n)}\mu(\xi)(\chi_{B_n})$ and $T(f) := \frac{\rho_{\mathbb{T}}(1)}{(m_G \times m_H)_{\mathbb{T}}(1)}\int_H f(h)\,\mathrm{d}h$. Then

$$|T_\xi^n(f) - T(f)| \leq |T_\xi^n(f) - T_\xi^n(f\chi)| + |T_\xi^n(f\chi) - T(f\chi)| + |T(f\chi) - T(f)|.$$

Now, the first term is smaller than ε by (11) for all n, the second term goes to zero for $n \longrightarrow \infty$ by Weyl's theorem. The last term is smaller than ε by (12). As $\varepsilon > 0$ is arbitrary, this gives the desired convergence statement. □

Corollary 9.4. *Let a measure cut and project scheme $(G, H, \widetilde{L}, \rho)$ be given with $\rho \geq 0$. Let $f : H \longrightarrow \mathbb{R}$ be Riemann integrable. Then,*

$$\lim_{n \to \infty} \frac{1}{m_G(B_n)}\mu(\xi)(\chi_{B_n}) = \frac{\rho_{\mathbb{T}}(1)}{(m_G \times m_H)_{\mathbb{T}}(1)}\int_H f(h)\,\mathrm{d}h$$

uniformly in $\xi \in \mathbb{T}$.

Proof. Again, this follows by approximation. As f is Riemann integrable, there exists for every $\varepsilon > 0$ $\varphi, \psi \in C_c(H)$ with

$$\varphi \le f \le \psi \quad \text{and} \quad \int (\psi - \varphi)\, dh \le \varepsilon.$$

Now, by Weyl's theorem Corollary 9.3, the desired convergence holds for both φ and ψ and the corollary follows easily. $\qquad\square$

10. Fourier-Bohr coefficients and the proof of Theorem 3.3

In this section we provide a proof of Theorem 3.3. Throughout, we will assume that a measure cut and project $(G, H, \widetilde{L}, \rho)$ and an admissible continuous $f : H \to \mathbb{C}$ is given.

Lemma 10.1. *Let* $\lambda \in \widehat{\widetilde{\mathbb{T}}}$ *and* $\xi \in \mathbb{T}$ *be given. Then, the limit*

$$c_\lambda(\xi) := \lim_{n \to \infty} \frac{1}{m_G(B_n)} \mu(\xi)(\chi_{B_n} \overline{\lambda \circ \iota})$$

exists and

$$c_\lambda(\xi) = \lambda(\xi) \frac{\rho_{\mathbb{T}}(\lambda)}{(m_G \times m_H)_{\mathbb{T}}(1)} \int f(h)\,(\lambda \circ \kappa)(h)\; dh.$$

Proof. This follows from a direct calculation using Weyl's Theorem. Choose $\xi = [s, k] \in \mathbb{T}$ arbitrary. Using $\lambda(\xi) = (\lambda \circ \iota)(s) \cdot (\lambda \circ \kappa)(k)$, we obtain after a short calculation

$$\mu(\xi)(\chi_{B_n} \overline{\lambda \circ \iota}) = \lambda(\xi) \int \chi_{B_n}(s + t)(\lambda \circ \kappa)(h + k) f(h + k) \overline{\lambda([t, h])}\; d\rho(t, h).$$

We can now appeal to Weyl's theorem with $f\,\lambda \circ \kappa$ instead of f and $\lambda([\cdot])\,\rho$ instead of ρ. $\quad\square$

Lemma 10.2. *Let* ν *be a translation bounded measure on* G *and* γ *the associated autocorrelation. Let* $\sigma \in \widehat{G}$ *be given. If*

$$c_\sigma = \lim_{n \to \infty} \frac{1}{m_G(B_n)} \nu(\chi_{B_n} \overline{\sigma})$$

exists for every van Hove sequence (B_n)*, then*

$$\widehat{\gamma}(\{\sigma\}) = |c_\sigma|^2.$$

Proof. For $G = \mathbb{R}^d$, this is proven by Hof in [20]. The proof can be adapted to our more general situation. More precisely, as discussed in section 2.3, we have $\gamma = \lim_{n \to \infty} \frac{1}{m_G(B_n)} \nu_{B_n} * \widetilde{\nu_{B_n}}$. Thus, Lemma 9.2 shows that there exists $C > 0$ and $V \subset G$ open and relatively compact, with

$$(13) \qquad \frac{1}{m_G(B_n)} \nu_{B_n} * \widetilde{\nu_{B_n}} \in \mathcal{M}_{C,V}(G), \text{ for all } n \in \mathbb{N} \text{ and } \gamma \in \mathcal{M}_{C,V}(G).$$

Moreover, by Theorem 11.3 of [15], we have

$$(14) \qquad \widehat{\gamma}(\{\sigma\}) = \lim_{n \to \infty} \frac{1}{m_G(B_n)} \gamma(\chi_{B_n} \overline{\sigma}).$$

Given (14) and (13), we can conclude as in the proof of Theorem 3.4 of [20]. $\qquad\square$

We can now come to the proof of Theorem 3.3.

Proof of Theorem 3.3. By Theorem 3.1, $(\Omega(\nu), \alpha)$ has pure point dynamical spectrum with eigenvalues contained in $\{\lambda \circ \iota : \lambda \in \widehat{\mathbb{T}}\}$. Thus, Theorem 2.1 gives that $\widehat{\gamma}$ is a pure point measure which can be written in the form

$$\widehat{\gamma} = \sum_{\lambda \in \widehat{\mathbb{T}}} w_\lambda \delta_{\lambda \circ \iota}$$

with suitable w_λ, $\lambda \in \widehat{\mathbb{T}}$. For these w_λ, we obtain from Lemma 10.1 and Lemma 10.2

$$w_\lambda = |c_\lambda(\xi)|^2,$$

where the right hand side does not depend on $\xi \in \mathbb{T}$. $\qquad\square$

Remark. In the situation discussed in this section, it is possible to show that

$$\lim_{n \to \infty} \frac{1}{m_G(B_n)} \mu(\xi)(\chi_{B_n} \hat{s}) = 0$$

for all $\xi \in \mathbb{T}$, whenever $\hat{s} \neq \lambda \circ \iota$ for a $\lambda \in \widehat{\mathbb{T}}$, see [27].

11. DENSE DIRAC COMBS

In this section, we restrict to the situation of $G = \mathbb{R}^d$, $H = \mathbb{R}^m$ and $\rho = \delta_{\widetilde{L}} = \sum_{x \in \widetilde{L}} \delta_x$. This setup (with $\widetilde{L} = \mathbb{Z}^{d+m}$) appears in [20] by regularising characteristic functions of model set windows. In a more general framework, the above setup is analysed in [40]. Both situations are subsumed by our theory, as we will now show. Note that [40] assumes a cut-and-project scheme with the additional assumption that L is dense in G and that the projection π_{int} is one-to-one between $\widetilde{L}$ and $L^\star$. We will not need these assumptions.

Our arguments rest on the following special case of Corollary 9.4, known as the density formula (see [41] as well). In order to formulate it in the variant discussed in [41, 40], we introduce the notation

$$\curlywedge(W) := \{x \in L : x^\star \in W\}$$

for $W \subset H$ relatively compact. Thus, the connection with the translation bounded measures discussed so far is given by the formula

$$\nu_{\chi_W} = \sum_{x \in \curlywedge(W)} \delta_x.$$

Corollary 11.1 (**Density formula**). *Let a cut and project scheme $(G, H, \widetilde{L})$ be given. Let $W \subset H$ be relatively compact such that χ_W is Riemann integrable. Then, for every van Hove sequence (B_n) in G,*

$$\lim_{n \to \infty} \frac{1}{m_G(B_n)} \left(\sum_{x \in \curlywedge(W+u) \cap (B_n+s)} 1 \right) = \frac{m_H(W)}{(m_G \times m_H)_{\mathbb{T}}(1)}$$

uniformly in $s \in G$ and in $u \in H$. $\qquad\square$

According to Proposition 6.6, a continuous function $f : \mathbb{R}^m \to \mathbb{C}$ is admissible if and only if for arbitrary $\epsilon > 0$ and $\varphi \in C_c(\mathbb{R}^d)$ there exists a compact $Q \subset \mathbb{R}^m$ with

$$\sum_{x \in L} |\varphi(x+s) f(x^\star)| \, (1 - 1_Q(x^\star)) \leq \epsilon$$

for all $s \in \mathbb{R}^d$, where 1_Q denotes the characteristic function of Q.

Theorem 11.1. *Assume a cut-and-project scheme* $(\mathbb{R}^d, \mathbb{R}^m, \widetilde{L})$. *Let the function* $f : \mathbb{R}^m \to \mathbb{C}$ *be continuous, with* $|x|^{m+\alpha}|f(x)| \leq C$ *for all* $x \in \mathbb{R}^m$, *for some constants* $C > 0$ *and* $\alpha > 0$. *Then* f *is admissible.*

Remark. The functions f considered in [40] satisfy the more restrictive condition $|x|^{m+1+\alpha}|f(x)| \leq C$ for some constants $C > 0$ and $\alpha > 0$.

Proof. Let (B_n) be a van Hove sequence in $\mathbb{R}^d$. For $l \in \mathbb{N}$, let $Q_l \subset \mathbb{R}^m$ denote the compact cube of sidelength l centred at the origin. The density formula yields for $n > n_0$ large enough the estimate

$$\left(\sum_{x \in \curlywedge (Q_1 + u) \cap (B_n + s)} 1 \right) \leq 2 \frac{m_G(B_n)}{(m_G \times m_H)_{\mathbb{T}}(1)}$$

uniformly in $s \in \mathbb{R}^d$ and in $u \in \mathbb{R}^m$, since $m_H(Q_1) = 1$. As $Q_{2(l+1)} \setminus Q_{2l}$ may be built from $(2l+2)^m - (2l)^m$ translated copies of Q_1, we obtain for $l \in \mathbb{N}$

$$\left(\sum_{\substack{x \in (B_n + s) \cap L \\ x^\star \in Q_{2(l+1)} \setminus Q_{2l}}} 1 \right) \leq 2 \frac{m_G(B_n)}{(m_G \times m_H)_{\mathbb{T}}(1)} \left((2l+2)^m - (2l)^m \right) \leq 2^{2m+1} \frac{m_G(B_n)}{(m_G \times m_H)_{\mathbb{T}}(1)} l^{m-1}$$

uniformly in $s \in \mathbb{R}^d$. The first estimate uses uniformity in u. To check admissibility of f, let $\epsilon > 0$ and $\varphi \in C_c(\mathbb{R}^d)$ be given. Fix $n > n_0$ such that $\mathrm{supp}(\varphi) \subset B_n$. We have for $l \in \mathbb{N}$ the estimate

$$\sum_{x \in L} |\varphi(x+s) f(x^\star)| \left(1 - 1_{Q_{2l}}(x^\star) \right) \leq ||\varphi||_\infty \sum_{\substack{x \in (B_n - s) \cap L \\ x^\star \notin Q_{2l}}} |f(x^\star)| = ||\varphi||_\infty \sum_{k=l}^\infty \sum_{\substack{x \in (B_n - s) \cap L \\ x^\star \in Q_{2(k+1)} \setminus Q_{2k}}} |f(x^\star)|$$

$$\leq ||\varphi||_\infty \sum_{k=l}^\infty \frac{C}{k^{m+\alpha}} 2^{2m+1} \frac{m_G(B_n)}{(m_G \times m_H)_{\mathbb{T}}(1)} k^{m-1} = \widetilde{C} \sum_{k=l}^\infty \frac{1}{k^{1+\alpha}},$$

where absolute convergence of the sum is used for reordering. The last expression is a bound independent of $s \in \mathbb{R}^d$. Now choose $l \in \mathbb{N}$ such that the bound does not exceed ϵ and set $Q := Q_{2l}$. By Proposition 6.6, we have shown that f is admissible. $\qquad \square$

According to Theorem 3.2, we obtain for the autocorrelation

$$\gamma = \frac{1}{(m_G \times m_H)_{\mathbb{T}}(1)} \sum_{l \in L} \eta(l)\, \delta_l, \qquad \eta(l) = \int_{\mathbb{R}^m} f(h) \overline{f(h - l^\star)}\, \mathrm{d}h,$$

compare [40, Thm. 9]. For the diffraction formula, consider the *dual lattice* $(\widetilde{L})^*$ of $\widetilde{L}$, given by

$$(\widetilde{L})^* = \left\{ (\gamma, \eta) \in \widehat{G} \times \widehat{H} \ : \ \gamma(l) \cdot \eta(l^\star) = 1 \text{ for all } (l, l^\star) \in \widetilde{L} \right\}.$$

As $\widetilde{L}$ is a closed subgroup of $G \times H$, we infer that $(\widetilde{L})^*$ is a closed subgroup of $\widehat{G} \times \widehat{H}$. By Pontryagin duality [14, Thm. 4.39], the group $(\widetilde{L})^*$ is, as a topological group, isomorphic to $\widehat{\mathbb{T}}$, an isomorphism from $\widehat{\mathbb{T}}$ to $(\widetilde{L})^*$ being given by $\lambda \mapsto (\lambda \circ \iota, \lambda \circ \kappa)$ for $\lambda \in \widehat{\mathbb{T}}$. Thus, $(\widetilde{L})^*$ is a discrete, cocompact subgroup of $\widehat{G} \times \widehat{H}$. Set now to $G \times H = \mathbb{R}^d \times \mathbb{R}^m$ with $\rho = \delta_{\widetilde{L}}$, and use the canonical identification $\widehat{\mathbb{R}^n} \simeq \mathbb{R}^n$, with $\eta(x) = e^{2\pi i \eta \cdot x}$ for $\eta \in \widehat{\mathbb{R}^n}$ and $x \in \mathbb{R}^n$. Then, Theorem 3.3 specialises to [40, Thm. 10].

12. Injectivity of the factor map

As discussed so far, a measure cut and project scheme $(G, H, \widetilde{L}, \rho)$ together with an admissible continuous function f yields a factor map $\mu : \mathbb{T} \longrightarrow \Omega(\nu_f)$. In this section, we discuss conditions to ensure that this factor map is one-to-one, i.e., an isomorphism.

First of all, note that injectivity of μ can be destroyed both by properties of ρ and of f:

Example. Let $(G, H, \widetilde{L}, \rho)$ be a measure cut and project scheme and $\rho := m_G \times m_H$. Let f be admissible and continuous. Then, $\mu(\xi) = (\int f \, \mathrm{d}h) \, m_G$ is independent of $\xi \in \mathbb{T}$, as can be seen by a direct calculation.

Example. Let $(G, H, \widetilde{L}, \rho)$ be a measure cut and project scheme. Let f be admissible and continuous with $f(\cdot - u) = f$ for some $u \neq 0$. Then, $\mu(\xi) = \mu(\xi + [0, u])$ for every $\xi \in \mathbb{T}$, and μ is not injective.

Our aim in this section is to prove the following theorem.

Theorem 12.1. *Let a cut and project scheme $(G, H, \widetilde{L})$ be given and f be admissible with respect to $\delta_{\widetilde{L}}$. Let ρ be an $\widetilde{L}$-invariant measure. If*

- $\rho_{\mathbb{T}}(\lambda) \neq 0$ *for every $\lambda \in \widehat{\widetilde{\mathbb{T}}}$,*
- f *does not have a nontrivial period, i.e., $f(\cdot - u) \neq f$ for every $u \neq 0$,*

then the map μ associated to $(G, H, \widetilde{L}, \rho)$ is one-to-one (and thus an isomorphism).

Remark. (a) Recall that admissibility of f with respect to $\delta_{\widetilde{L}}$ implies admissibility of f with respect to ρ, as shown in Lemma 6.9.
(b) Note that the first condition is satisfied for $\rho = \delta_{\widetilde{L}}$. In this case $\rho_{\mathbb{T}}(\lambda) = 1$ for every $\lambda \in \widehat{\widetilde{\mathbb{T}}}$.
(c) The condition that f has no periods also appears in the context of the usual cut and project schemes, where f is a characteristic function. There it is used to obtain a map from Ω to $\mathbb{T}$ (see [42, 5]). In some sense our aim is similar. We use this condition to prove injectivity of the map $\mathbb{T} \longrightarrow \Omega$, which then also implies existence of a map from Ω to $\mathbb{T}$.

The proof will be given at the end of this section after a series of intermediate results. We start by considering the case $\rho = \delta_{\widetilde{L}}$.

Lemma 12.1. *Let a measure cut and project scheme $(G, H, \widetilde{L}, \rho)$ be given with $\rho = \delta_{\widetilde{L}}$. If f is continuous and admissible and does not have a nontrivial period, then μ is injective.*

Proof. As μ is a group homomorphism by Theorem 3.1, it suffices to show that $\mu(0) = \mu(\eta)$ implies $\eta = 0$. Let $\eta = [(q, p)]$ be given with $\mu(0) = \mu(\eta)$. Thus,

$$(15) \qquad \sum_{(l, l^\star) \in \widetilde{L}} f(l^\star) \delta_l = \sum_{(l, l^\star) \in \widetilde{L}} f(l^\star + p) \delta_{l+q}.$$

As $L^\star$ is dense in H and f is continuous and does not vanish identically, there exists $l^\star \in L^\star$ with $f(l^\star) \neq 0$. Then, (15) implies that there exists $l' \in L$ with

$$l = l' + q,$$

and we infer that $q = l - l' \in L$. Then, (15) gives that

$$f(l^\star) = f(l^\star - q^\star + p)$$

for all $l \in L$. As $L^\star$ is dense in H and f is continuous, we obtain that $p - q^\star$ is a period of f. By assumption, all periods of f are trivial, and we obtain $p = q^\star$ and therefore

$$\eta = [(q,p)] = [(q,q^\star)] = 0.$$

This finishes the proof. $\qquad\square$

Proposition 12.2. *Let a measure cut and project scheme* $(G, H, \widetilde{L}, \rho)$ *and an admissible continuous* $f : H \longrightarrow \mathbb{C}$ *be given. Let* $\mu : \mathbb{T} \longrightarrow \Omega(\nu_f)$ *be the associated factor map. Let* $\varphi \in C_c(G)$ *be arbitrary and define* $\mu_\varphi : \mathbb{T} \longrightarrow \mathbb{C}$, $\xi \mapsto \mu(\xi)(\varphi)$. *Then*

$$\widehat{\mu_\varphi}(\lambda) = \widehat{\varphi}(\lambda \circ \iota)\, \widehat{f}(\lambda \circ \kappa)\, \widehat{\rho}(-\lambda).$$

Proof. Let $Z \subset G \times H$ be a fundamental cell of $\widetilde{L}$. Then $G \times H = \widetilde{L} + Z$. For $\lambda \in \widehat{\mathbb{T}}$ we can then calculate $\widehat{\mu_\varphi}(\lambda)$ as follows:

$$
\begin{aligned}
\widehat{\mu_\varphi}(\lambda) &= \int_{\mathbb{T}} \overline{(\lambda, \xi)} \mu_\varphi(\xi)\, \mathrm{d}\xi \\
&= \int_Z \overline{\lambda([s,k])} \left(\int_{G \times H} \varphi(s+t) f(k+h)\, \mathrm{d}\rho(t,h) \right) \mathrm{d}s\mathrm{d}k \\
&= \int_Z \overline{\lambda([s,k])} \left(\sum_{(l,l^\star) \in \widetilde{L}} \int_Z \varphi(s+l+t) f(k+l^\star+h)\, \mathrm{d}\rho(t,h) \right) \mathrm{d}s\mathrm{d}k \\
(Fubini) \quad &= \int_Z \lambda([t,h]) \left(\sum_{(l,l^\star) \in \widetilde{L}} \int_Z \varphi(s+l+t) f(k+l^\star+h) \overline{\lambda([s,k]) \lambda([t,h])}\, \mathrm{d}s\mathrm{d}k \right) \mathrm{d}\rho(t,h) \\
&= \int_Z \lambda([t,h]) \left(\int_{G \times H} \varphi(s+t) f(k+h) \overline{\lambda([s+t,h+k])} \mathrm{d}s\mathrm{d}k \right) \mathrm{d}\rho(t,h) \\
&= \int_Z \lambda([t,h]) \left(\int_{G \times H} \varphi(s) f(k) \overline{\lambda([s,k])}\, \mathrm{d}s\mathrm{d}k \right) \mathrm{d}\rho(t,h) \\
&= \int_Z \lambda([t,h]) \widehat{\varphi}(\lambda \circ \iota) \widehat{f}(\lambda \circ \kappa)\, \mathrm{d}\rho(t,h) \\
&= \widehat{\varphi}(\lambda \circ \iota)\, \widehat{f}(\lambda \circ \kappa)\, \widehat{\rho}(-\lambda).
\end{aligned}
$$

This finishes the proof. $\qquad\square$

We now come to the proof of injectivity.

Proof of Theorem 12.1. The proof of the general case will be reduced to the case treated in Lemma 12.1. We want to show injectivity of the group homomorphism μ associated to f and $(G, H, \widetilde{L}, \rho)$. We will need as well the group homomorphism μ^0 associated to f and $(G, H, \widetilde{L}, \delta_{\widetilde{L}})$. (Recall that f is admissible with respect to $\delta_{\widetilde{L}}$ by assumption.)

If μ is not injective, there exists by Theorem 3.1 an $\eta \neq 0$ with $\mu(\xi + \eta) = \mu(\xi)$ for all $\xi \in \mathbb{T}$. Thus, $\mu_\varphi : \mathbb{T} \longrightarrow \mathbb{C}$, $\xi \mapsto \mu(\xi)(\varphi)$ satisfies $\mu_\varphi(\xi + \eta) = \mu_\varphi(\xi)$ for all $\xi \in \mathbb{T}$ and $\varphi \in C_c(G)$. Taking Fourier transforms and using the previous proposition, we obtain

$$(\lambda, \eta) \widehat{\varphi}(\lambda \circ \iota)\, \widehat{f}(\lambda \circ \kappa)\, \widehat{\rho}(-\lambda) = \widehat{\varphi}(\lambda \circ \iota)\, \widehat{f}(\lambda \circ \kappa)\, \widehat{\rho}(-\lambda)$$

for all $\lambda \in \widehat{\mathbb{T}}$. As $\widehat{\rho}(-\lambda) \neq 0$ for all $\lambda \in \widehat{\mathbb{T}}$, this implies

$$(\lambda, \eta)\widehat{\varphi}(\lambda \circ \iota) \, \widehat{f}(\lambda \circ \kappa) = \widehat{\varphi}(\lambda \circ \iota) \, \widehat{f}(\lambda \circ \kappa)$$

for all $\lambda \in \widehat{\mathbb{T}}$. This, however, means that $(\eta, \lambda)\widehat{\mu^0_\varphi}(\lambda) = \widehat{\mu^0_\varphi}(\lambda)$ for all $\lambda \in \widehat{\mathbb{T}}$, with $\mu^0_\varphi : \mathbb{T} \longrightarrow \mathbb{C}$ given by $\mu^0_\varphi(\xi) = \mu^0(\xi)(\varphi)$. Taking the inverse Fourier transform, we obtain

$$\mu^0_\varphi(\xi + \eta) = \mu^0_\varphi(\xi)$$

for all $\xi \in \mathbb{T}$. As $\varphi \in C_c(G)$ is arbitrary, this gives

$$\mu^0(\xi + \eta) = \mu^0(\xi)$$

for all $\xi \in \mathbb{T}$. Now, μ^0 is just the map treated in Lemma 12.1, where its injectivity is shown. Thus, we obtain $\eta = 0$, and the proof is finished. $\qquad \square$

13. A COMPLEMENTARY RESULT

Let us shortly compare our results to the corresponding results for the "usual" model sets. In our notation this case can be described as follows (see Section 11 as well): Let $(G, H, \widetilde{L})$ be a cut and project scheme and consider the measure

$$\rho := \delta_{\widetilde{L}} = \sum_{x \in \widetilde{L}} \delta_x$$

on $G \times H$. Now, let χ_W be the characteristic function of a compact set $W \subset H$, which is the closure of its interior and whose boundary has Haar measure 0. Then, we can form ν_{χ_W} exactly as above. This measure has the form $\sum_{x \in \curlywedge(W)} \delta_x$ with the uniformly discrete set $\curlywedge(W) := \{x \in L : x^* \in W\} \subset G$. Identifying the measure ν_{χ_W} with the uniformly discrete set $\curlywedge(W)$ in G, we can apply results of Schlottmann [42] to obtain a factor map

$$\Phi : \Omega(\nu_{\chi_W}) \longrightarrow \mathbb{T}$$

This map is $1 : 1$-almost everywhere, i.e. there exists a set $\mathbb{T}_0$ of Haar measure zero in $\mathbb{T}$ such that Φ is one-to-one on $\Phi^{-1}(\mathbb{T} \setminus \mathbb{T}_0)$. It turns out that Φ is indeed not injective if ν_{χ_W} is not periodic. More precisely, the following is proved in [5].

Theorem. *Let ν_{χ_W} and Φ be given as in the preceding paragraph. Then, Φ is injective if and only if ν_{χ_W} is crystallographic, i.e. the set $\{t \in G : \alpha_t \nu_{\chi_W} = \nu_{\chi_W}\}$ is a cocompact discrete subgroup of G.*

This theorem has the following consequence.

Corollary 13.1. *Let the notation be as in the preceding theorem. If ν_{χ_W} is not crystallographic, then $(\Omega(\nu_{\chi_W}), \alpha)$ is not a factor of $(\mathbb{T}, \beta)$.*

Proof. Assume the contrary. Then, there exists a factor map

$$\psi : \mathbb{T} \longrightarrow \Omega(\nu_{\chi_W}).$$

Thus, $\Phi \circ \psi : \mathbb{T} \longrightarrow \mathbb{T}$ is a factor map as well. With the canonical homomorphism $\iota : G \longrightarrow \mathbb{T}$, $\iota(t) = [t, 0]$ and the definition of β we therefore obtain

$$(\Phi \circ \psi)(\iota(-t)) = (\Phi \circ \psi)(\beta_t[0, 0]) = \beta_t(\Phi \circ \psi)([0, 0]) = \iota(-t) \dotplus (\Phi \circ \psi)([0, 0]) = (\Phi \circ \psi)([0, 0]) \dotplus \iota(-t),$$

where we write $\dotplus$ to denote the product in $\mathbb{T}$. As $\iota(G)$ is dense in $\mathbb{T}$ and $(\Phi \circ \psi)$ is continuous, we infer

$$(\Phi \circ \psi)(\xi) = (\Phi \circ \psi)([0,0]) \dotplus \xi$$

for every $\xi \in \mathbb{T}$. In particular, $(\Phi \circ \psi)$ is injective as it is just translation by $(\Phi \circ \psi)([0,0])$.

On the other hand, by the previous Theorem, Φ is not injective as ν_{χ_W} is not crystallographic. As ψ is a factor map, it is onto. Therefore, non-injectivity of Φ leads to non-injectivity of $\Phi \circ \psi$. This contradiction shows that $(\Omega(\nu_{\chi_W}), \alpha)$ is not a factor of $(\mathbb{T}, \beta)$. $\square$

The corollary shows that our main result does indeed crucially depend on the smoothness of the weight function f, as it becomes false for characteristic functions of compact sets. In some sense, the dynamical systems associated with continuous weight functions are closer to periodic systems than to model set systems.

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