# MODIFIED LOG-SOBOLEV INEQUALITIES AND ISOPERIMETRY

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Abstract. We find sufficient conditions for a probability measure $\mu$ to satisfy an inequality of the type

$$\int_{\mathbb{R}^d} f^2 F\Big(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\Big)\, d\mu \le C \int_{\mathbb{R}^d} f^2 c^*\Big(\frac{|\nabla f|}{|f|}\Big)\, d\mu + A \int_{\mathbb{R}^d} f^2\, d\mu,$$

where $F$ is concave and $c$ (a cost function) is convex. In particular, for every convex $\mu$ satisfying $\int_{\mathbb{R}^d} e^{\varepsilon |x|^\alpha}\, d\mu < \infty$ for some $\varepsilon > 0$, $1 < \alpha \le 2$, we establish a family of tight inequalities interpolating between the $F$-Sobolev and modified log-Sobolev inequalities.

## 1. Introduction

The celebrated logarithmic Sobolev inequality

$$\text{Ent}_\mu f^2 := \int_{\mathbb{R}^d} f^2 \log\Big(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\Big)\, d\mu \le 2C \int_{\mathbb{R}^d} \big|\nabla f\big|^2\, d\mu, \tag{1}$$

where $\mu = e^{-V}\, dx$ is a probability measure, has numerous applications in probability theory, mathematical physics, and geometry. It appeared first in the work of Gross [12], where he established (1) for the standard Gaussian measure. Gross discovered that (1) implies the hypercontractivity of the semigroup $e^{tL}$ generated by $L = \Delta - \big\langle \nabla V, \nabla \big\rangle$.

The necessary and sufficient conditions for (1) have been intensively studied by many authors (see [1]). It is well-known that for every probability measure satisfying (1) there exists $\varepsilon > 0$ such that

$$e^{\varepsilon |x|^2} \in L^1(\mu). \tag{2}$$

It has been shown by Wang ([17]) that this assumption is sufficient provided $\mu$ is convex, i.e. has the form $\mu = e^{-V}\, dx$, where $V$ is a convex function. The Wang's proof is based on the properties of the associated diffusion semigroup. Bobkov [3] gave another proof of this result by using the Prékopa–Leindler theorem and isoperimetric inequalities. There exist non-convex measures satisfying (1). For example, there is a measure $\rho \cdot \mu$ such that $0 < a < \rho < b$ and $\mu$ satisfies (1).

Recall that (1) implies the well-known Poincaré inequality

$$\text{Var}_\mu f := \int_{\mathbb{R}^d} f^2\, d\mu - \Big(\int_{\mathbb{R}^d} f\, d\mu\Big)^2 \le C \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu. \tag{3}$$

The log-Sobolev inequality can be considered as a Poincaré-type inequality for the $L^2 \log L$-Orlicz norm. By using this observation and some classical results on Hardy's inequality with weights, Bobkov and Götze [4] established necessary and sufficient conditions for (1) on the real line. Namely, $\mu = \rho\, dx$ satisfies (1) if and only if

$$\sup_{x<m} F(x) \log\Big(\frac{1}{F(x)}\Big) \int_x^m \frac{dx}{\rho(x)} < \infty,$$

$$\sup_{x>m} (1 - F(x)) \log\Big(\frac{1}{1 - F(x)}\Big) \int_m^x \frac{dx}{\rho(x)} < \infty,$$

where $F(x) = \mu\Big((-\infty, x]\Big)$ and $m$ is the median of $\mu$.

It is well-known that (1) (as well as the classical Sobolev inequalities) is closely related to the isoperimetric inequalities. For $A \subset \mathbb{R}^d$ we denote by $\mu^+(A)$ the surface measure of the boundary $\partial A$:

$$\mu^+(A) = \underline{\lim}_{h\to 0} \frac{\mu(A^h) - \mu(A)}{h},$$

where $A^h = \{x : \mathrm{dist}(x, A) \le h\}$ is the $h$-neighborhood of $A$. It was proved by Ledoux [15] that the isoperimetric inequality of the Gaussian type

$$\mu^+(A) \ge c\varphi\big(\Phi^{-1}(\mu(A))\big)$$

implies (1). Here

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},\ \Phi(x) = \int_{-\infty}^{x} \varphi(s)\, ds.$$

Some sufficient conditions for (1) can be obtained by perturbation methods. For example, using the log-Sobolev inequality

$$\int_{\mathbb{R}^d} f^2 \log f^2\, dx \le \frac{1}{\pi e^2} \int_{\mathbb{R}^d} |\nabla f|^2\, dx,\ \int_{\mathbb{R}^d} f^2\, dx = 1$$

for Lebesgue measure, Carlen and Loss proved in [8] that $\mu = e^{-V}\, dx$ satisfies (1) provided that

$$\frac{1}{4}|\nabla V|^2 - \frac{1}{2}\Delta V - \pi e^2 V$$

is bounded from below and $\mu$ satisfies (3) (see also [2], [7]).

It follows from (2) that $\mu$ has a very fast decay. However, many distributions exhibit some weaker, yet useful properties. Below we consider the following generalizations of (1):

1) The defective log-Sobolev inequality

$$\mathrm{Ent}_\mu f^2 \le 2C \int_{\mathbb{R}^d} \big|\nabla f\big|^2\, d\mu + A \int_{\mathbb{R}^d} f^2\, d\mu.$$

2) The $F$-Sobolev inequalities

$$\int_{\mathbb{R}^d} f^2 F(f^2)\, d\mu \le 2C \int_{\mathbb{R}^d} \big|\nabla f\big|^2\, d\mu + A,$$

where $\int_{\mathbb{R}^d} f^2\, d\mu = 1$ and $F$ is a concave function.

3) The modified log-Sobolev inequality

$$\mathrm{Ent}_\mu f^2 \le C \int_{\mathbb{R}^d} f^2 c^*\Big(\frac{|\nabla f|}{|f|}\Big)\, d\mu \tag{4}$$

for some convex $c : \mathbb{R}^+ \to \mathbb{R}^+$. Here $c^*(x) = \sup_{y\in\mathbb{R}^+}\big(\langle x, y\rangle - c(y)\big)$.

Inequality of type 1) implies the hyperboundedness of the associated semigroups (see [10]). A basic example for 2) and 3) is given by the following measure on the real line:

$$\mu_\alpha = Z_\alpha e^{-|x|^\alpha}\, dx,$$

where $1 < \alpha \le 2$. It was proved in [11] that $\mu_\alpha$ satisfies (4) with

$$c(x) = c_{A,\alpha}(x) = \begin{cases} \frac{x^2}{2} & \text{if } |x| \le A \\ A^{2-\alpha}\frac{\frac{|x|^\alpha}{\alpha}}{2} + A^2\frac{\alpha-2}{2\alpha} & \text{if } |x| \ge A, \end{cases} \tag{5}$$

for every $A > 0$. By the tensorization argument the result holds also in the multi-dimensional case for the product measure $\prod_{i=1}^{d} \mu_\alpha(\,dx_i)$ and cost function $c_{d,A,\alpha}(x) = \sum_{i=1}^{d} c_{A,\alpha}(x_i)$. On the other hand, by a result from [2], $\mu_\alpha$ satisfies

$$\int f^2 \log^{2/\beta}(1+f^2)\,d\mu - \Big(\int f^2\,d\mu\Big)\log^{2/\beta}\Big(1+\int f^2\,d\mu\Big) \le C\int |\nabla f|^2\,d\mu,$$

where $\frac{1}{\alpha}+\frac{1}{\beta}=1$. One can easily verify that $c^*_{A,\alpha} = c_{A,\beta}$.

The case $\alpha \ge 2$ has been considered in [6]. The probability measure

$$\mu = Z_{\alpha,d}\prod_{i=1}^{d} e^{-|x_i|^\alpha}\,dx_i$$

on $\mathbb{R}^d$ satisfies

$$\mathrm{Ent}_\mu |f|^\beta \le C\int_{\mathbb{R}^d}\sum_i |\partial_{x_i} f|^\beta\,d\mu. \tag{6}$$

Among other generalizations of (1) let us mention an important result from [14] on a family of inequalities interpolating between log-Sobolev and Poincaré. If $1 < \alpha \le 2$, $1 \le p \le 2$, then for every smooth $f$

$$\int_{\mathbb{R}^d} f^2\,d\mu_\alpha - \Big(\int_{\mathbb{R}^d}|f|^p\,d\mu_\alpha\Big)^{2/p} \le C(2-p)^{2\left(1-\frac{1}{\alpha}\right)}\int_{\mathbb{R}^d}|\nabla f|^2\,d\mu_\alpha.$$

For further development and connections with the $F$-Sobolev inequality, see in [2], [18].

Inequality (4) is closely related to the Talagrand transportation inequality

$$W_c(\mu, f\cdot\mu) \le \mathrm{Ent}_\mu f, \tag{7}$$

where $f\cdot\mu$ is another probability measure and $W_c$ is the minimum of the Kantorovich functional for the cost function $c$ (see [16] for details). In fact, under broad assumptions on $c$, inequality (4) is stronger than (7). It was proved in [9] by the optimal transportation method that (4) holds for measures of the type $\mu = e^{-\Phi}\,dx$, where $\Phi$ satisfies

$$\Phi(b) - \Phi(a) \ge \big\langle \nabla\Phi(a), b-a\big\rangle + \alpha c(b-a)$$

for some $\alpha > 0$ and a proper choice of $c$.

In this paper we prove a sufficient condition for inequalities of the following type:

$$\int_{\mathbb{R}^d} f^2 F\Big(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\mu}\Big)\,d\mu \le C\int_{\mathbb{R}^d} f^2 c^*\Big(\frac{|\nabla f|}{|f|}\Big)\,d\mu + A\int_{\mathbb{R}^d} f^2\,d\mu, \tag{8}$$

where $F$ is concave and $c : \mathbb{R}^+ \to \mathbb{R}^+$ is convex (Theorem 2.1). This inequality unifies the defective modified log-Sobolev inequalities and the $F$-Sobolev inequalities. Obviously, the tight $F$-Sobolev inequality corresponds to the choice $c = |x|^2$, $A = 0$, and the modified Sobolev inequality corresponds to the choice $F = \log$, $A = 0$. Our estimate is based on the use of a special isoperimetric function

$$I_F(r) = \sup_{A\in\mathcal{M}_r}\frac{\mu(A)F\big(\frac{1}{\mu(A)}\big)}{\mu^+(A)}.$$

Here $\mathcal{M}_r = \{A : \mu(A) = \mu(\{x : |x| > r\})\}$. The main result can be roughly formulated in the following way: integrability of $\Phi(c(I_F))$, where $\Phi = (yF(y)-y)^*$, implies (8).

In Section 3 we prove sufficient conditions for the related tight inequalities. The case of the $F$-inequality follows immediately from the main result (Theorem 3.1). Then, under some additional assumptions, we prove a modification of (8), where $\int_{\mathbb{R}^d} f^2\mu$ is replaced by $\mathrm{Var}_\mu f$ (Theorem 3.3). In the proof we use some ideas from [11]. In Section 4 we apply our results to the case of convex measures. Following [11] we consider for every $1 < \alpha \le 2$ the corresponding family of cost functions $c_{A,\alpha}$ given by (5).

We also consider the generalized entropies defined by

$$f \to \int_{\mathbb{R}^d} f F_\tau\Bigl(\frac{f}{\mu(f)}\Bigr)\, d\mu,$$

where

$$F_\tau(x) = \begin{cases} \log x & \text{if } 0 < x \le e \\ \frac{1}{\tau}\bigl(\log^\tau x - 1\bigr) + 1 & \text{if } x \ge e, \end{cases}$$

and $\tau \le 1$.

**Theorem 1.1.** *Let $\mu$ be a convex measure on $\mathbb{R}^d$ such that*

$$\int_{\mathbb{R}^d} e^{\varepsilon |x|^\alpha}\, d\mu < \infty$$

*for some $1 < \alpha \le 2$ and $\varepsilon > 0$. Then for every $2\bigl(1 - \frac{1}{\alpha}\bigr) \le \tau \le 1$ there exists $C_\tau > 0$ such that for every smooth $f$ one has*

$$\int_{\mathbb{R}^d} f^2 F_\tau\Bigl(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\Bigr)\, d\mu \le C_\tau \int_{\mathbb{R}^d} f^2 c_{A,\frac{\tau\alpha}{\alpha-1}}\Bigl(\frac{|\nabla f|}{|f|}\Bigr)\, d\mu.$$

*In particular,*

$$\int_{\mathbb{R}^d} f^2 F_{2\left(1-\frac{1}{\alpha}\right)}\Bigl(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\Bigr)\, d\mu \le C_{2\left(1-\frac{1}{\alpha}\right)} \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu,$$

$$\mathrm{Ent}_\mu f^2 \le C_1 \int_{\mathbb{R}^d} f^2 c^*_{A,\alpha}\Bigl(\frac{|\nabla f|}{|f|}\Bigr)\, d\mu = C_1 \int_{\mathbb{R}^d} f^2 c_{A,\frac{\alpha}{\alpha-1}}\Bigl(\frac{|\nabla f|}{|f|}\Bigr)\, d\mu.$$

In particular, we generalize Wang's criterion for convex measures as well as a result from [11]. Note that unlike [11] we deal directly with multidimensional distributions and use a different cost function for $d \ge 2$. However, the multidimensional result of [11] follows from the one-dimensional ones by the standard tensorization argument. We also apply the method developed in Theorem 2.1 to establish a weaker version of (6) for the case of arbitrary convex measure (Theorem 4.3).

## 2. Main result

Consider a probability measure $\mu = \rho\, dx$ on $\mathbb{R}^d$. It will be assumed throughout the paper that $X := \mathrm{supp}(\mu)$ is convex. In addition, without loss of generality we assume that $0 \in X$. Set:

$$B_r = \{x : |x| \le r\}.$$

We denote by $R(X) \in (0, \infty]$ the smallest number such that $X \subset B_{R(X)}$. For every nonnegative function $f$ we denote by $\tilde{f}$ the corresponding spherical rearrangement, i.e. the function of the type $\tilde{f}(x) = g(|x|)$ such that $g$ is increasing and

$$\mu \circ f^{-1} = \mu \circ \tilde{f}^{-1}.$$

This can be rewritten as

$$\mu_f = \mu_r \circ g^{-1}$$

where $\mu_f = \mu \circ f^{-1}$ and $\mu_r$ is the image of $\mu$ under $x \to |x|$. For a probability measure $\nu$ on $\mathbb{R}^+$ let us define

$$F_\nu(t) = \nu([0, t))$$

and

$$G_\nu(u) = \{s : F_\nu(s) \geq u\}.$$

Then $g$ has the form

(9) $$g = G_{\mu_f} \circ F_{\mu_r}.$$

We denote by $B_r^c$ the complement of $B_r$ and by $R_t$ a number $R_t > 0$ such that

$$\mu(|x| \leq R_t) = t,\ R_1 = R(X).$$

Since $X$ is convex and $0 \in X$, $R_t$ is well-defined. Set:

$$t(r) = \mu(B_r^c) = 1 - \mu(B_r),\ r \geq 0.$$

For every $F : \mathbb{R}^+ \to \mathbb{R}$ we define the corresponding isoperimetric function

$$I_F(r) = \sup_{A \in \mathcal{M}_r} \frac{\mu(A) F\big(\frac{1}{\mu(A)}\big)}{\mu^+(A)},\ r > 0$$

where $\mathcal{M}_r = \big\{A : A \subset \mathbb{R}^d,\ \mu(A) = t(r)\big\}$. Equivalently,

$$I_F(r) = \frac{t(r) F\big(\frac{1}{t(r)}\big)}{\inf_{A \in \mathcal{M}_r} \mu^+(A)},\ r > 0.$$

We follow the agreement that $I_F(R(X)) = 0$.

In what follows we consider a convex cost function $c : \mathbb{R}^+ \to \mathbb{R}^+$. Let

$$c^*(x) = \sup_{y \in \mathbb{R}^+} \big(\langle x, y\rangle - c(y)\big).$$

We recall that $c$ is called superlinear if $\lim_{x\to\infty} \frac{c(|x|)}{|x|} = \infty$. In what follows for the sake of simplicity we set $\mu(f^2) = \int_{\mathbb{R}^d} f^2\, d\mu$.

**Theorem 2.1.** *Let $c : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex superlinear function such that $c(0) = 0$. Let $F$ be a function on $\mathbb{R}^+$ with the following properties:*

1) *$F$ is concave and increasing and $F(1) = 0$*
2) *$\lim_{y\to 0} yF(y) = 0$, $\lim_{y\to\infty} F(y) = \infty$.*

*Let $K > 1$. Assume that for $R = R_{\frac{K-1}{K}}$*

$$\int_{B_R^c} \Phi\Big(4c \circ I_F(|x|)\Big)\, d\mu < \infty,$$

*where*

$$\Phi(x) = \sup_{y \in \mathbb{R}^+} \big(\langle x, y\rangle - yF(y) + y\big) = \big(yF(y) - y\big)^*(x).$$

*Then for some sufficiently big $K > 1$ there exist $A > 0$, $C > 0$ such that for every smooth $f$ the following estimates hold:*

(10) $$\int_{\mathbb{R}^d} f^2 F\Big(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\Big)\, d\mu \leq 4 \int_{\{f^2 \geq K\mu(f^2)\}} f^2 c^*\Big(\frac{|\nabla f|}{|f|}\Big)\, d\mu + A \int_{\mathbb{R}^d} f^2\, d\mu$$

(11) $$\int_{\mathbb{R}^d} f^2 F\Big(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\Big)\, d\mu$$
$$\leq C \int_{\mathbb{R}^d} \big(f - \mu(f)\big)^2 c^*\Big(\frac{|\nabla f|}{\big|f - \mu(f)\big|}\Big)\, d\mu + A \cdot \mathrm{Var}_\mu f.$$

*Proof.* Let us fix some smooth Lipschitz function $f$. Without loss of generality we may assume that $f \ge \varepsilon > 0$. Set $\nu := g \cdot \mu$, where $g = F\big(\frac{f^2}{\int f^2 \, d\mu}\big)$. By a well-known result from measure theory one has

$$\int_{\mathbb{R}^d} f^2 F\Big(\frac{f^2}{\int_{\mathbb{R}^d} f^2 \, d\mu}\Big) = \int_{\mathbb{R}^d} f^2 g \, d\mu = \int_{\mathbb{R}^d} f^2 \, d\nu = \int_0^\infty \nu(f^2(x) > t) \, dt$$
$$= \int_0^\infty \Big(\int_{\{x: f^2(x) > t\}} g \, d\mu\Big) \, dt.$$

We split this integral in the following two parts:

$$I_1 = \int_0^{K\mu(f^2)} \Big(\int_{\{x: f^2(x) > t\}} g \, d\mu\Big) \, dt, \quad I_2 = \int_{K\mu(f^2)}^\infty \Big(\int_{\{x: f^2(x) > t\}} g \, d\mu\Big) \, dt.$$

The estimation of $I_1$ is quite elementary. By the concavity of $F$ one has

$$g \le F'(1)\Big(\frac{f^2}{\mu(f^2)} - 1\Big).$$

Hence

$$\frac{I_1}{F'(1)} \le \frac{1}{\mu(f^2)} \int_{\mathbb{R}^d} \Big(f^2 - \mu(f^2)\Big)\Big(\int_0^{K\mu(f^2)} I_{\{x: f^2(x) > t\}} \, dt\Big) \, d\mu.$$
$$= \frac{1}{\mu(f^2)} \int_{\mathbb{R}^d} \Big(f^2 - \mu(f^2)\Big) \min\Big(f^2, K\mu(f^2)\Big) \, d\mu$$
$$= \frac{1}{\mu(f^2)} \int_{\mathbb{R}^d} \Big(f^2 - \mu(f^2)\Big)\Big[\min\Big(f^2, K\mu(f^2)\Big) - \mu(f^2)\Big] \, d\mu.$$

The latter equals

$$\frac{1}{\mu(f^2)} \int_{\{f^2 \le K\mu(f^2)\}} \Big(f^2 - \mu(f^2)\Big)^2 d\mu + (K-1) \int_{\{f^2 \ge K\mu(f^2)\}} \Big(f^2 - \mu(f^2)\Big) \, d\mu.$$

The first term can be estimated in the following way:

$$\frac{1}{\mu(f^2)} \int_{\{f^2 \le K\mu(f^2)\}} \Big(f^2 - \mu(f^2)\Big)^2 d\mu$$
$$\le \frac{2}{\mu(f^2)} \int_{\{f^2 \le K\mu(f^2)\}} \Big(f^2 - \mu(f)^2\Big)^2 d\mu + \frac{2}{\mu(f^2)} \Big[\mathrm{Var}_\mu f\Big]^2$$
$$\le 4(K+1)^2 \int_{\mathbb{R}^d} \Big(f - \mu(f)\Big)^2 d\mu + 2\mathrm{Var}_\mu f = (4(K+1)^2 + 2)\mathrm{Var}_\mu f.$$

Further we get

$$\int_{\{f^2 \ge K\mu(f^2)\}} \Big(f^2 - \mu(f^2)\Big) \, d\mu \le \int_{\{f^2 \ge K\mu(f^2)\}} \Big(f^2 - \mu(f)^2\Big) \, d\mu.$$

One can easily check that

$$|f + \mu(f)| \le \frac{\sqrt{K} + 1}{\sqrt{K} - 1} |f - \mu(f)|$$

on $\{f^2 \ge K\mu(f^2)\}$. Hence

$$\int_{\{f^2 \ge K\mu(f^2)\}} \Big(f^2 - \mu(f^2)\Big) \, d\mu \le \frac{\sqrt{K} + 1}{\sqrt{K} - 1} \mathrm{Var}_\mu f.$$

Finally we obtain

$$I_1 \le \Big[(4(K+1)^2+2)+(\sqrt{K}+1)^2\Big]F'(1)\mathrm{Var}_\mu f.$$

Let us estimate $I_2$. Let us set

$$A_t = \{x : f^2(x) > t\}.$$

By the concavity of $F$ one has

$$\begin{aligned}
I_2 &= \int_{K\mu(f^2)}^{\infty}\int_{\mathbb{R}^d} I_{A_t} F\Big(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\mu}\Big)\,d\mu\,dt \\
&\le \int_{K\mu(f^2)}^{\infty} \mu(A_t)\Big[F\Big(\int_{A_t}\frac{f^2}{\mu(A_t)\int_{\mathbb{R}^d} f^2\,d\mu}\Big)\,d\mu\Big]\,dt \\
&\le \int_{K\mu(f^2)}^{\infty} \mu(A_t)F\Big(\frac{1}{\mu(A_t)}\Big)\,dt \\
&= \int_{K\mu(f^2)}^{\infty} \mu(\{x : f^2(x) > t\})F\Big(\frac{1}{\mu(\{x : f^2(x) > t\})}\Big)\,dt.
\end{aligned}$$

Since $f$ is continuous and $X$ is convex, the function $t \mapsto \mu(A_t)$ is strictly increasing on

$$[\inf_{x\in\mathbb{R}^d} f^2(x), \sup_{x\in\mathbb{R}^d} f^2(x)].$$

Hence one can find a nondecreasing function $r(s)$ such that

$$\mu(A_s) = \mu(B^c_{r(s)})$$

and $r(0)=0$, $r(s)=R(X)$, if $s \ge \sup f^2$. Set

$$f_h(x) = \sup_{\{|x-y|\le h\}} f(y)$$

and

$$J(t) := \begin{cases} \int_{K\mu(f^2)}^t I_F(r(s))\,ds, & t \ge K\mu(f^2) \\ 0, & t \le K\mu(f^2) \end{cases}.$$

Then by the definition of $I_F$ we have

$$\begin{aligned}
I_2 &\le \int_{K\mu(f^2)}^{\infty} I_F\big(r(t)\big)\mu^+(A_t)\,dt \\
&\le \underline{\lim}_{h\to 0+}\int_{K\mu(f^2)}^{\infty} I_F(r(t))\frac{\mu(A_t^h)-\mu(A_t)}{h}\,dt,
\end{aligned}$$

where $\{x\in\mathbb{R}^d : f_h^2(x) > t\} = \{x\in\mathbb{R}^d : f^2(x) > t\}^h = A_t^h$. Applying the formula

$$\int \Phi(f^2)\,d\mu = \int_0^\infty \Phi'(t)\mu(A_t)\,dt,$$

which holds for every increasing $\Phi$ such that $\Phi(0)=0$, we get

$$\begin{aligned}
I_2 &\le \underline{\lim}_{h\to 0+}\int_{\mathbb{R}^d}\frac{J(f_h^2)-J(f^2)}{h}\,d\mu \\
&\le 2\int_{\{f^2\ge K\mu(f^2)\}} I_F\big(r(f^2)\big)|f||\nabla f|\,d\mu.
\end{aligned}$$

Let $\mathbb{R}_\delta = \{t : \mu \circ \left(f^2\right)^{-1}(t) > 0\}$ be the set of atoms of the measure $\mu \circ \left(f^2\right)^{-1}$. Note that $|\nabla f| = 0$ almost everywhere on $D = \{x : f^2(x) \in \mathbb{R}_\delta\}$. Hence by the Young inequality we find

$$
2\int_{\{f^2 \geq K\mu(f^2)\}} I_F(r(f^2))|f||\nabla f|\, d\mu \leq 2\int_{\{f^2 \geq K\mu(f^2)\}} f^2 c^*\Big(\frac{|\nabla f|}{|f|}\Big)\, d\mu + 2\int_{\{f^2 \geq K\mu(f^2)\} \cap D^c} f^2 \big[c \circ I_F\big(r(f^2)\big)\big]\, d\mu. \tag{12}
$$

Let $O_k = \{x : f^2(x) \geq K\mu(f^2)\} \cap D^c$. One has

$$
I_{O_k} = I_{\{f^2 \geq K\mu(f^2)\}} \cdot I_{\mathbb{R}^c_\delta}(f^2)
$$

and

$$
\begin{aligned}
&2\int_{O_k} f^2 c\big(I_F(r(f^2))\big)\, d\mu = 2\int_{\mathbb{R}^d} f^2 I_{O_k} c\big(I_F(r(f^2))\big)\, d\mu \\
&= \frac{1}{2}\mu(f^2)\int_{\mathbb{R}^d}\Big[\frac{f^2}{\mu(f^2)}\Big]\Big[4I_{O_k}c\big(I_F(|r(f^2)|)\big) - T\Big]\, d\mu + \frac{T}{2}\int_{\mathbb{R}^d} f^2\, d\mu \\
&\leq \frac{1}{2}\int_{\mathbb{R}^d} f^2\Big[F\Big(\frac{f^2}{\mu(f^2)}\Big) - 1\Big]\, d\mu + \frac{T}{2}\int_{\mathbb{R}^d} f^2\, d\mu \\
&+ \frac{1}{2}\mu(f^2)\int_{\mathbb{R}^d} \Phi\Big(4I_{O_k}c\big(I_F(r(f^2))\big) - T\Big)\, d\mu.
\end{aligned}
$$

Since $f$ and $\tilde{f}$ have the same laws considered as random variables on the probability space $(\mathbb{R}^d, \mu)$, one has

$$
\int_{\mathbb{R}^d} \Phi\Big(4I_{O_k}c\big(I_F(r(f^2))\big) - T\Big)\, d\mu = \int_{\mathbb{R}^d} \Phi\Big(4I_{\tilde{O}_k}c\big(I_F(r(\tilde{f}^2))\big) - T\Big)\, d\mu
$$

where $\tilde{O}_k = \{x : \tilde{f}^2(x) \geq K\mu(f^2)\} \cap \{x : \tilde{f}^2(x) \in \mathbb{R}^c_\delta\}$. By the definition of $r$ we have

$$
\mu\Big(B^c_{r(\tilde{f}^2)}\Big) = \mu\big(\{y : f^2(y) > \tilde{f}^2(x)\}\big) = \mu\big(\{y : \tilde{f}^2(y) > \tilde{f}^2(x)\}\big).
$$

We observe that the function $F_{\mu \circ (f^2)^{-1}}$ is continuous at every point $x \in \{x : \tilde{f}^2(x) \in \mathbb{R}^c_\delta\}$. Then if follows from representation (9) that

$$
\mu\Big(B^c_{r(\tilde{f}^2)}\Big) = \mu\big(\{y : \tilde{f}^2(y) > \tilde{f}^2(x)\}\big) = \mu(\{y : |y| > |x|\}).
$$

Hence $r(\tilde{f}^2)(x) = |x|$ on $\tilde{O}_k$. Moreover, if $x \in \{\tilde{f}^2 \geq K\mu(\tilde{f}^2)\}$, then by the Chebyshev inequality

$$
\mu\big(B^c_{|x|}\big) = \mu\Big(B^c_{r(\tilde{f}^2)}\Big) \leq \mu\big(\{\tilde{f}^2 \geq K\mu(\tilde{f}^2)\}\big) \leq 1/K.
$$

Hence $|x| = r(\tilde{f}^2(x)) \geq R_{(K-1)/K}$ if $x \in \{\tilde{f}^2 \geq K\mu(\tilde{f}^2)\}$. Thus

$$
\tilde{O}_k \subset \{x : |x| \geq R_{(K-1)/K}\}.
$$

Hence

$$
\int_{\mathbb{R}^d} \Phi\Big(4I_{\tilde{O}_k}c\big(I_F(r(\tilde{f}^2))\big) - T\Big)\, d\mu \leq \Phi(-T) + \int_{B^c_R} \Phi\Big(4c\big(I_F(|x|)\big) - T\Big)\, d\mu.
$$

It follows from the properties of $F$ that $\lim_{x \to -\infty} \Phi(x) = 0$. Hence there exists $T > 0$ such that

$$
\int_{\mathbb{R}^d} \Phi\Big(4I_{\tilde{O}_k}c\big(I_F(r(\tilde{f}^2))\big) - T\Big)\, d\mu \leq 1
$$

and $T$ doesn't depend on $f$. Thus for a sufficiently big $T$ one has

$$\frac{1}{2}\int_{\mathbb{R}^d} f^2\Big[F\Big(\frac{f^2}{\mu(f^2)}\Big)-1\Big]\,d\mu+\frac{T}{2}\int_{\mathbb{R}^d} f^2\,d\mu$$
$$+\frac{1}{2}\mu(f^2)\int_{\mathbb{R}^d}\Phi\Big(4I_{O_k}c\big(I_F(r(f^2))\big)-T\Big)\,d\mu$$
$$\le\frac{1}{2}\int_{\mathbb{R}^d} f^2F\Big(\frac{f^2}{\mu(f^2)}\Big)\,d\mu+\frac{T}{2}\int_{\mathbb{R}^d} f^2\,d\mu.$$

and

$$I_2\le\frac{T}{2}\int_{\mathbb{R}^d} f^2\,d\mu+\frac{1}{2}\int_{\mathbb{R}^d} f^2F\Big(\frac{f^2}{\mu(f^2)}\Big)\,d\mu.$$

Combining all the inequalities obtained above, we get (10).

The proof of (11) is very similar and we just briefly describe the main difference. Instead of (12) we use

$$2\int_{\{f^2\ge K\mu(f^2)\}} I_F(r(f^2))|f||\nabla f|\,d\mu\le$$
$$C'\int_{\{f^2\ge K\mu(f^2)\}}\big(f-\mu(f)\big)^2c^*\Big(\frac{|\nabla f|}{|f-\mu(f)|}\Big)\,d\mu$$
$$+C'\int_{O_K}\big(f-\mu(f)\big)^2\big[c\circ I_F\big(r(f^2)\big)\big]\,d\mu.$$

This follows from the Young inequality and the observation that

$$f^2\le\frac{K}{(\sqrt{K}-1)^2}\big(f-\mu(f)\big)^2$$

on $\{f^2\ge K\mu(f^2)\}$. In the same way as above we estimate the second term by $\mathrm{Ent}_\mu\big(f-\mu(f)\big)^2$ and $\mathrm{Var}_\mu f$. Finally, we use (10) to estimate $\mathrm{Ent}_\mu\big(f-\mu(f)\big)^2$. The proof is complete. □

## 3. Tight estimates

In this section we establish some tight estimates, i.e. estimates whose right-hand sides vanish on constant functions. The following theorem is a direct corollary of (11).

**Theorem 3.1.** *Let $F$ and $\mu$ satisfy the assumptions of Theorem 2.1 for $c=\|x\|^2$. Then for every smooth $f$ one has*

$$\int_{\mathbb{R}^d} f^2F\Big(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\mu}\Big)\,d\mu\le C\int_{\mathbb{R}^d}|\nabla f|^2\,d\mu+A\cdot\mathrm{Var}f.$$

Unlike the $F$-Sobolev inequality, the case of tight modified log-Sobolev inequalities is more difficult. We use an idea from [11] and consider two cases: the case of large entropy and the case of small entropy. The large entropy case follows immediately from our main result. In the case of small entropy we reduce the problem to the $F$-inequality.

Recall the following Cheeger isoperimetric inequality:

(13) $$\min(\mu(A),1-\mu(A))\le K\mu^+(A).$$

This is equivalent to the following $L^1$-Poincaré-type inequality:

(14) $$\int_{\mathbb{R}^d}\Big|f-\int_{\mathbb{R}^d} f\,d\mu\Big|\,d\mu\le K\int_{\mathbb{R}^d}|\nabla f|\,d\mu.$$

In fact, (14) is stronger than the classical $L^2$-Poincaré inequality (see [5]). It is known that every convex measure satisfies (14) for some $K$ (see [13] and [5]).

**Lemma 3.2.** *Let $F$ satisfy the assumptions of Theorem 2.1. Suppose, in addition, that $yF(y)$ is convex and $\Phi \le \alpha + \beta\Phi'$. Then for all $f$ and $g$ one has*

$$\int_{\mathbb{R}^d} f^2 F\Big(\frac{g^2}{\int_{\mathbb{R}^d} g^2\, d\mu}\Big)\, d\mu \le \int_{\mathbb{R}^d} f^2 F\Big(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\Big)\, d\mu + (\alpha+\beta)\int_{\mathbb{R}^d} f^2\, d\mu.$$

*Proof.* Since $\Phi = (yF(y) - y)^*$, we get by the well-known relation for convex conjugated functions that

$$y = \Phi' \circ (yF(y) - y)' = \Phi'(F(y) + yF'(y) - 1). \tag{15}$$

Set

$$u = F\Big(\frac{g^2}{\mu(g^2)}\Big) + \frac{g^2}{\mu(g^2)} F'\Big(\frac{g^2}{\mu(g^2)}\Big) - 1$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^d} f^2 F\Big(\frac{g^2}{\mu(g^2)}\Big)\, d\mu \le \int_{\mathbb{R}^d} f^2 u\, d\mu + \int_{\mathbb{R}^d} f^2\, d\mu \\
&\le \int_{\mathbb{R}^d} f^2\, d\mu \Big(\int_{\mathbb{R}^d} \Big[\frac{f^2}{\mu(f^2)} F\Big(\frac{f^2}{\mu(f^2)}\Big) - \frac{f^2}{\mu(f^2)}\Big]\, d\mu\Big) \\
&+ \int_{\mathbb{R}^d} f^2\, d\mu \int_{\mathbb{R}^d} \Phi(u)\, d\mu + \int_{\mathbb{R}^d} f^2\, d\mu \\
&\le \int_{\mathbb{R}^d} f^2 F\Big(\frac{f^2}{\mu(f^2)}\Big)\, d\mu + \int_{\mathbb{R}^d} f^2\, d\mu\Big(\alpha + \beta\int_{\mathbb{R}^d} \Phi'(u)\, d\mu\Big).
\end{aligned}$$

It remains to observe that $\int_{\mathbb{R}^d} \Phi'(u)\, d\mu = 1$ by (15). □

Below we will use some additional assumptions on $F$. We say that $g : [0,\infty) \to \infty$ satisfies assumptions A1)–A2) if

A1) $g$ is concave and increasing

A2) $xg(x)$ is convex, $\lim_{x\to\infty} g(x) = \infty$, $g(1) = 0$ and

$$\lim_{x\to 0} xg(x) = 0.$$

**Theorem 3.3.** *Let $F$, $c$, $\mu$, and $K$ satisfy the assumptions of Theorem 2.1 for some $K \ge 2$. Assume that $v = c^*(\sqrt{x})$ is convex and superlinear. Assume that there exists $\Psi$ satisfying A1)–A2) such that*

1) *$\Psi$ has the form $\Psi = \psi(F(x))$ on $[K,\infty)$, where $\frac{x}{\psi(x)}$ is increasing on $[F(K),\infty)$,*

$$\lim_{x\to\infty} \frac{x}{\psi(x)} = \infty$$

*and for every $\varepsilon > 0$ there exist $\varepsilon' > 0$ such that*

$$v^*\Big(\varepsilon' \frac{x}{\psi(x)}\Big) < \varepsilon x \tag{16}$$

*for $x \ge K$*

2) *for some $\delta > 0$ and $R = R_{\frac{K-1}{K}}$*

$$\int_{B_R} \tilde{\Psi}\big(\delta |I_\Psi|^2\big)\, d\mu < \infty$$

*where*

$$\tilde{\Psi}(x) = \sup_{\{y>0\}} \big(\langle x,y\rangle - y\Psi(y) + y\big) = \big(y\Psi(y) - y\big)^*(x),$$

3) *there exist* $\alpha, \beta \geq 0$ *such that*

$$\tilde{\Psi} \leq \alpha + \beta \tilde{\Psi}'$$

*for* $x \geq K$.

*In addition, we assume that* $F$ *satisfies A1)-A2) and* $\mu$ *satisfies the Cheeger inequality. Then for some* $M > 1$ *there exist* $N, A, C$ *such that the following modified* $F$*-Sobolev inequality holds:*

$$(17) \quad \begin{aligned} &\int_{\mathbb{R}^d} f^2 F\Big(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\Big)\, d\mu \\ &\leq C \int_{\{f^2 \geq K \int_{\mathbb{R}^d} f^2\, d\mu\}} f^2 c^*\Big(M\frac{|\nabla f|}{|f|}\Big)\, d\mu + A \cdot \mathrm{Var}_\mu f. \end{aligned}$$

*Proof.* We follow the arguments from [11]. The case $\tau = 2\big(1 - \frac{1}{\alpha}\big)$ follows by Theorem 3.1. Let $\tau > 2\big(1 - \frac{1}{\alpha}\big)$. Consider a smooth function $f$. Without loss of generality one can assume that $f \geq \varepsilon > 0$. If $f$ satisfies $\int_{\mathbb{R}^d} f^2\, d\mu \leq \frac{1}{2A}\int_{\mathbb{R}^d} f^2 F\Big(\frac{f^2}{\mu(f^2)}\Big)\, d\mu$ ($A = A(K)$ is the same as in (10)), then (17) follows directly from Theorem 2.1. Hence one can assume that

$$(18) \quad \int_{\mathbb{R}^d} f^2 F\Big(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\Big)\, d\mu \leq 2A \int_{\mathbb{R}^d} f^2\, d\mu.$$

Note that if $\sup_{x\in\mathbb{R}^d} f^2 \leq K\mu(f^2)$, then by the concavity of $F$

$$\int_{\mathbb{R}^d} f^2 F\Big(\frac{f^2}{\int_{\mathbb{R}^d} f^2\, d\mu}\Big)\, d\mu \leq C(K)\mathrm{Var} f$$

(see the reasoning in Theorem 2.1). Hence without loss of generality one can assume that there exists $x_0$ such that $f(x_0) = \sqrt{K\mu(f^2)}$. Set

$$g(x) = f(x_0) + (f(x) - f(x_0))_+ P\Big(\frac{f^2}{\mu(f^2)}\Big)/P(K),$$

where

$$P(x) = \sqrt{\frac{F(x)}{\Psi(x)}} = \sqrt{\frac{F(x)}{\psi(F(x))}}.$$

Obviously, $g \geq f$ since $\frac{x}{\psi}$ is increasing. In addition, since $\Psi$ is increasing, we get by the Cauchy inequality

$$\int_{\mathbb{R}^d} g^2\, d\mu \leq C_1(K)\Big(\int_{\mathbb{R}^d} f^2\, d\mu + \int_{\{f^2 \geq K\mu(f^2)\}} f^2 F\Big(\frac{f^2}{\mu(f^2)}\Big)\Big).$$

In the same way as in Lemma 3.3(iii) in [11] the convexity of $xF(x)$ implies that

$$(19) \quad \int_{\{f^2 \geq K\mu(f^2)\}} f^2 F\Big(\frac{f^2}{\mu(f^2)}\Big)\, d\mu \leq C_2(K) \int_{\mathbb{R}^d} f^2 F\Big(\frac{f^2}{\mu(f^2)}\Big)\, d\mu.$$

Hence by (18) there exists $M = M(K)$ such that

$$\int_{\mathbb{R}^d} g^2\, d\mu \leq M \int_{\mathbb{R}^d} f^2\, d\mu.$$

Taking into account that $g \geq f$, one gets

$$\int_{\mathbb{R}^d} \big([g-g(x_0)]_+\big)^2 \Psi\Big(\frac{g^2}{\mu(g^2)}\Big)\, d\mu \geq \int_{\mathbb{R}^d} \big([g-g(x_0)]_+\big)^2 \Psi\Big(\frac{f^2}{M\mu(f^2)}\Big)\, d\mu$$

$$= \frac{1}{P^2(K)} \int_{\mathbb{R}^d} \big([f-f(x_0)]_+\big)^2 F\Big(\frac{f^2}{\mu(f^2)}\Big) \frac{\Psi\Big(\frac{f^2}{M\mu(f^2)}\Big)}{\Psi\Big(\frac{f^2}{\mu(f^2)}\Big)}\, d\mu.$$

By the concavity of $\Psi$ one has $\inf_{x \geq 2M} \frac{\Psi(x/M)}{\Psi(x)} = a > 0$. Hence

$$\int_{\mathbb{R}^d} \big([g-g(x_0)]_+\big)^2 \Psi\Big(\frac{g^2}{\mu(g^2)}\Big)\, d\mu \geq$$
$$\frac{a}{P^2(K)} \int_{\big\{f^2 \geq 2M\mu(f^2)\big\}} \big([f-f(x_0)]_+\big)^2 F\Big(\frac{f^2}{\mu(f^2)}\Big)\, d\mu$$
$$- \sup_{K \leq t \leq 2M} \Big| \frac{\Psi(t/M)}{\Psi(t)} F(t) \Big| \int_{\mathbb{R}^d} \big([f-f(x_0)]_+\big)^2\, d\mu.$$

Thus for some $A_1 = A_1(K) > 0$ one has

$$\int_{\mathbb{R}^d} \big([f-f(x_0)]_+\big)^2 F\Big(\frac{f^2}{\mu(f^2)}\Big)\, d\mu \leq$$
$$A_1 \int_{\mathbb{R}^d} \big([g-g(x_0)]_+\big)^2 \Psi\Big(\frac{g^2}{\mu(g^2)}\Big)\, d\mu + A_1 \int_{\mathbb{R}^d} \big([f-f(x_0)]_+\big)^2\, d\mu.$$

We observe that the second term on the right-hand side can be estimated by $Var_\mu f$, since $\big(f-f(x_0)\big)_+ \leq \big|f-\mu(f)\big|$. By Lemma 3.2 and Assumption 3) we obtain

$$\int_{\mathbb{R}^d} \big([g-g(x_0)]_+\big)^2 \Psi\Big(\frac{g^2}{\mu(g^2)}\Big)\, d\mu \leq$$
$$\int_{\mathbb{R}^d} \big([g-g(x_0)]_+\big)^2 \Psi\Big(\frac{([g-g(x_0)]_+)^2}{\mu([g-g(x_0)]_+)^2)}\Big)\, d\mu + (\alpha+\beta) \int_{\mathbb{R}^d} \big([g-g(x_0])_+\big)^2\, d\mu.$$

Since $\mu(x : g(x) > g(x_0)) = \mu(x : f(x) > f(x_0)) \leq \frac{1}{K} \leq \frac{1}{2}$, $g(x_0)$ is the median of $g$. According to [5], it follows from the Cheeger inequality that

$$\int_{\mathbb{R}^d} \big([g-g(x_0)]_+\big)^2\, d\mu \leq A_2 \int_{\mathbb{R}^d} \big|\nabla g\big|^2\, d\mu.$$

By Assumption 2) and Theorem 3.1 the measure $\mu$ satisfies the $\Psi$-Sobolev inequality

$$\int_{\mathbb{R}^d} \big([g-g(x_0)]_+\big)^2 \Psi\Big(\frac{([g-g(x_0)]_+)^2}{\mu([g-g(x_0)]_+)^2}\Big)\, d\mu \leq A_3 \int_{\mathbb{R}^d} |\nabla g|^2\, d\mu.$$

Combining the estimates obtained above, we get

$$\int_{\mathbb{R}^d} \big([f-f(x_0)]_+\big)^2 F\Big(\frac{f^2}{\mu(f^2)}\Big)\, d\mu \leq C' \Big(\int_{\mathbb{R}^d} |\nabla g|^2\, d\mu + \mathrm{V}ar_\mu f\Big).$$

Let us estimate $\nabla g$. Set $h = \frac{f^2}{\mu(f^2)}$. We recall that $\Psi = \psi(F)$. One has

$$\nabla g = \Big[\frac{\big(f-f(x_0)\big)_+}{P(h)\cdot P(K)} \Big(\frac{F'}{\psi(F)} - \frac{F\psi'(F)F'}{\psi^2(F)}\Big)(h) \frac{f}{\mu(f^2)}\Big] \nabla f$$
$$+ \Big[I_{\{f \geq f(x_0)\}} \frac{P(h)}{P(K)}\Big] \nabla f.$$

Let us show that for some $B > 0$ one has

$$|\nabla g|^2 \le B P^2(h) |\nabla f|^2.$$

It sufficient to verify that

$$\frac{\bigl(f - f(x_0)\bigr)_+}{P^2(h)} \Bigl(\frac{F'}{\psi(F)} - \frac{F\psi'(F)F'}{\psi^2(F)}\Bigr)(h) \frac{f}{\mu(f^2)}$$

is bounded. Since $\frac{f\bigl(f-f(x_0)\bigr)_+}{\mu(f^2)} \le h$ and $P^2 = F/\psi(F)$, we have to estimate

$$\frac{x\psi(F)}{F}\Bigl(\frac{F'}{\psi(F)} - \frac{F\psi'(F)F'}{\psi^2(F)}\Bigr) = \frac{xF'}{F} - \frac{x\psi'(F)F'}{\psi(F)} = \frac{xF'}{F}\Bigl(1 - \frac{F\psi'(F)}{\psi(F)}\Bigr).$$

This quantity is uniformly bounded. Indeed, since $\frac{x}{\psi(x)}$ and $\psi$ are increasing, if follows from the inequality $\Bigl(\frac{x}{\psi(x)}\Bigr)' \ge 0$ that $0 \le \frac{x\psi'(x)}{\psi(x)} \le 1$. The boundedness of $\frac{xF'}{F}$ follows by the concavity of $F$. Finally, we obtain

$$\int_{\mathbb{R}^d} \bigl([f - f(x_0)]_+\bigr)^2 F\bigl(h\bigr)\, d\mu \le C \int_{\{f^2 \ge K\mu(f^2)\}} |\nabla f|^2 \frac{F(h)}{\Psi(h)}\, d\mu.$$

The right-hand side can be estimated by

$$C \int_{\{f^2 \ge K\mu(f^2)\}} f^2 v\Bigl(N^2 \frac{|\nabla f|^2}{f^2}\Bigr)\, d\mu + C \int_{\{f^2 \ge K\mu(f^2)\}} f^2 v^* \Bigl[\frac{1}{N^2}\frac{F(h)}{\psi(F(h))}\Bigr]\, d\mu$$

for arbitrary $N$. Hence by (16) for arbitrary $\varepsilon > 0$ and sufficiently big $N$ the latter doesn't exceed

$$C \int_{\{f^2 \ge K\mu(f^2)\}} f^2 v\Bigl(N^2 \frac{|\nabla f|^2}{f^2}\Bigr)\, d\mu + C\varepsilon \int_{\{f^2 \ge K\mu(f^2)\}} f^2 F\bigl(h\bigr)\, d\mu.$$

Hence by (19) one has

$$\begin{aligned} &\int_{\mathbb{R}^d} \bigl(f - f(x_0)\bigr)_+^2 F\Bigl(\frac{f^2}{\mu(f^2)}\Bigr)\, d\mu \\ &\le C \int_{\{f^2 \ge K\mu(f^2)\}} f^2 c^*\Bigl(N\frac{|\nabla f|}{|f|}\Bigr)\, d\mu + \varepsilon' \int_{\mathbb{R}^d} f^2 F\Bigl(\frac{f^2}{\mu(f^2)}\Bigr)\, d\mu, \end{aligned}$$

where $\varepsilon'$ can be chosen arbitrarily small. It remains to prove that

$$\begin{aligned} &\int_{\mathbb{R}^d} f^2 F\Bigl(\frac{f^2}{\mu(f^2)}\Bigr)\, d\mu \le A \cdot \mathrm{V}ar_\mu f \\ &+ B \int_{\mathbb{R}^d} \bigl(f - \sqrt{K\mu(f^2)}\bigr)_+^2 F\Bigl(\frac{f^2}{\mu(f^2)}\Bigr)\, d\mu. \end{aligned}$$

This was proved in [11] (see (20), (30) and the proof of Proposition 3.6) for the partial case $\mu(f^2) = 1$, $K = 2$ and a more special choice of $F$. However, the estimate can be easily generalized to our case, which completes the proof. □

## 4. Application to convex measures

**Lemma 4.1.** *Let $\mu$ be a convex measure. Then* $\sup_{r \ge R_{1/2}} \frac{I_{\log}(r)}{r} < \infty$.

*Proof.* We apply the following estimate from [3]:

$$2r\mu^+(A) \ge \mu(A) \log \frac{1}{\mu(A)} + (1-\mu(A)) \log \frac{1}{1-\mu(A)} + \log \mu\{|x-x_0| \le r\}, \tag{20}$$

which holds for every convex measure $\mu$, every set $A$, every point $x_0$, and any $r > 0$. Let $\mu(A) \le 1/2 - \varepsilon$, where $\varepsilon > 0$. Choose $r$ in such a way that $\mu(A) = \mu(B_r^c)$. Then

$$(1-\mu(A)) \log \frac{1}{1-\mu(A)} + \log \mu(B_r) = \mu(B_r^c) \log \mu(B_r). \tag{21}$$

Pick $\delta = \delta(\varepsilon)$ such that

$$\Bigl(\frac{1}{2} + \varepsilon\Bigr)^{\frac{1}{1-\delta}} \ge \frac{1}{2} - \varepsilon.$$

Then

$$\mu(B_r) \ge \Bigl(\frac{1}{2} + \varepsilon\Bigr) \ge \Bigl(\frac{1}{2} - \varepsilon\Bigr)^{1-\delta} \ge \mu^{1-\delta}(B_r^c).$$

Therefore,

$$(1-\delta)\mu(A) \log \frac{1}{\mu(A)} + \mu(B_r^c) \log \mu(B_r) = \mu(A)\Bigl(\log \frac{\mu(B_r)}{\mu^{1-\delta}(B_r^c)}\Bigr) \ge 0.$$

Hence by (21) we obtain

$$(1-\delta)\mu(A) \log \frac{1}{\mu(A)} + (1-\mu(A)) \log \frac{1}{1-\mu(A)} + \log \mu\{|x-x_0| \le r\} \ge 0$$

and $\frac{2r}{\delta}\mu^+(A) \ge \mu(A) \log \frac{1}{\mu(A)}$. It remains to show that

$$\sup_{R_{1/2} \le r \le R_{1/2+\varepsilon}} \frac{I_{\log}(r)}{r} < \infty.$$

But this follows easily from (20). One has to pick a sufficiently big number $\tilde{R}$ such that

$$\inf_{0 \le \tau \le \varepsilon} (1/2 + \tau) \log \frac{1}{1/2 + \tau} + \log \mu(\{x : |x| \le \tilde{R}\}) \ge 0.$$

Then $I_{\log}(r) \le \tilde{R}$. The proof is complete. ∎

**Corollary 4.2.** *For every convex measure and $0 < \beta \le 1$ there exist $R_0 > 0$ and $C > 0$ such that*

$$I_{\log^\beta}(r) \le C r \log^{\beta-1}\Bigl(\frac{1}{t(r)}\Bigr).$$

*on $[R_0, \infty)$. In particular, if $e^{\varepsilon|x|^\alpha} \in L^1(\mu)$, $\varepsilon > 0$, the Chebyshev inequality implies*

$$I_{\log^\beta}(r) \le C r^{1-\alpha(1-\beta)}.$$

Now we are ready to prove Theorem 1.1. The functions $F_\tau$ and $c_{A,\alpha}$ are defined in the introduction. It is easy to check that

$$e^{my^{\frac{1}{\tau}}} \le \bigl(xF_\tau(x) - x\bigr)^*(y) \le e^{My^{\frac{1}{\tau}}} \tag{22}$$

for sufficiently big $y$ and some $M \ge m > 0$.

**Proof of Theorem 1.1:** The result follows by Theorem 3.3, Theorem 3.1, Corollary 4.2, and (22). We set $F = F_\tau$ and $\Psi = F_{\frac{2}{\beta}}$, where $\beta = \frac{\alpha}{\alpha-1}$. Thus $\psi$ is an increasing function such that $F_{\frac{2}{\beta}} = \psi(F_\tau)$. One can easily compute that

$$\psi(x) = \begin{cases} x, & x \le 1 \\ \frac{\beta}{2}\big([1+\tau(x-1)]^{\frac{2}{\tau\beta}} - 1\big) + 1, & x \ge 1 \end{cases}$$

and

$$c = \delta \cdot c_{A,\beta_\tau}$$

for some $\delta > 0$, where $\beta_\tau = \left(\frac{\tau\alpha}{\alpha-1}\right)^*$ is the Hölder dual to $\frac{\tau\alpha}{\alpha-1}$. We have to check that hypothese of Theorem 3.3 are fulfilled for a sufficiently big $K$ and small $\delta$. The details of the proof are quite obvious and we omit them here. The key estimate establishes that the growth of $\Phi(4c(I_F))$ is dominated by $e^{\varepsilon'|x|^\alpha}$ and follows directly from Corollary 4.2 and (22). Recall that every convex measure satisfies the Cheeger inequality according to [13] and [5]. □

Finally, we prove an inequality of the type (6).

**Theorem 4.3.** *Let $\mu$ be a convex measure such that $\int_{\mathbb{R}^d} e^{\varepsilon|x|^\alpha}\,d\mu < \infty$ for some $1 < \alpha$ and $\varepsilon > 0$. Then the following inequality holds:*

$$\mathrm{Ent}_\mu|f|^\beta \le C\Big[\int_{\mathbb{R}^d} |\nabla f|^\beta\,d\mu + \mathrm{Var}_\mu|f|^{\frac{\beta}{2}}\Big].$$

*Proof.* Set: $g^2 = |f|^\beta$. Apply Theorem 2.1 to $g^2$ in place of $f^2$. Following the proof we get

$$\mathrm{Ent}_\mu|f|^\beta \le C\mathrm{Var}|f|^\beta + C\int_{|f|^\beta \ge K\mu(|f|^\beta)} I_{\log}\big(r(f^\beta)\big)|f|^{\beta-1}|\nabla f|\,d\mu$$

for some $K > 1$. By the Hölder inequality for every $\delta > 0$ there exists $N(C,\delta) > 0$ such that

$$C\int_{|f|^\beta \ge K\mu(|f|^\beta)} I_{\log}\big(r(f^\beta)\big)|f|^{\beta-1}|\nabla f|\,d\mu \le N\int_{\mathbb{R}^d} |\nabla f|^\beta\,d\mu +$$
$$\delta\int_{|f|^\beta \ge K\mu(|f|^\beta)} I_{\log}^{\frac{\beta}{\beta-1}}\big(r(f^\beta)\big)|f|^\beta\,d\mu.$$

Since $|f| \le C(K,\beta)|f-\mu(f)|$ on $\{|f|^\beta \ge K\mu(|f|^\beta)\}$, we get by the same arguments as in Theorem 2.1

$$\begin{aligned}
&\delta\int_{|f|^\beta \ge K\mu(|f|^\beta)} I_{\log}^{\frac{\beta}{\beta-1}}\big(r(f^\beta)\big)|f|^\beta\,d\mu \\
&\le \delta C(K,\beta)\int_{\mathbb{R}^d} I_{\log}^{\frac{\beta}{\beta-1}}\big(r(f^\beta)\big)|f-\mu(f)|^\beta\,d\mu \\
&\le C\int_{\mathbb{R}^d} |f-\mu(f)|^\beta\,d\mu + \frac{1}{2}\mathrm{Ent}_\mu|f-\mu(f)|^\beta,
\end{aligned}$$

where $C < \infty$ whenever

$$\int_{B_R^c} \exp\Big(\delta I_{\log}^{\frac{\beta}{\beta-1}}\big(|x|\big)\Big)\,d\mu < \infty$$

for $R = R_{\frac{K-1}{K}}$. By Corollary 4.2 one has $I_{\log}^{\frac{\beta}{\beta-1}}\bigl(|x|\bigr) \le C'|r|^{\frac{\beta}{\beta-1}} = C'r^{\alpha}$. Hence, choosing a sufficiently small $\delta$, we obtain

$$\int_{B_R^c} \exp\Bigl(\delta I_{\log}^{\frac{\beta}{\beta-1}}\bigl(|x|\bigr)\Bigr)\, d\mu < \infty.$$

Since $\mu$ satisfies the Cheeger inequality, one has

$$\int_{\mathbb{R}^d} |f - \mu(f)|^{\beta}\, d\mu \le C(\beta) \int_{\mathbb{R}^d} |\nabla f|^{\beta}\, d\mu$$

(see [5] for the proof). Finally, we arrive at the estimate

$$\mathrm{Ent}_{\mu}|f|^{\beta} \le C\mathrm{Var}|f|^{\frac{\beta}{2}} + N' \int_{\mathbb{R}^d} |\nabla f|^{\beta}\, d\mu + \frac{1}{2}\mathrm{Ent}_{\mu}|f - \mu(f)|^{\beta}. \tag{23}$$

In particular, applying this estimate to $f - \mu(f)$, we get:

$$\mathrm{Ent}_{\mu}|f - \mu(f)|^{\beta} \le 2C \int_{\mathbb{R}^d} |f - \mu(f)|^{\beta} + 2N' \int_{\mathbb{R}^d} |\nabla f|^{\beta}\, d\mu \le$$
$$\bigl(2CC(\beta) + 2N'\bigr) \int_{\mathbb{R}^d} |\nabla f|^{\beta}\, d\mu$$

and we get the claim by (23). □

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