

## ESTIMATES OF SOBOLEV NORMS OF TRIANGLE TRANSFORMATIONS

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### Abstract

We study increasing triangular transformations  $T$  of the  $n$ -dimensional cube  $\Omega = [0, 1]^n$  which transform a measure  $\mu$  into a measure  $\nu$ , where  $\mu$  and  $\nu$  are absolutely continuous Borel probability measures with densities  $\rho_\mu$  and  $\rho_\nu$ . It is shown that if there exist positive numbers  $\varepsilon$  and  $M$  such that  $\varepsilon < \rho_\mu < M$ ,  $\varepsilon < \rho_\nu < M$  and numbers  $\alpha, \beta > 1$  that such  $p = \alpha\beta(n-1)^{-1}(\alpha+\beta)^{-1} > 1$  and  $\varrho_\mu \in W^{\alpha,1}(\Omega)$ ,  $\varrho_\nu \in W^{\beta,1}(\Omega)$ , where  $W^{\alpha,1}$  denotes the Sobolev class, then the transformation  $T$  belongs to the class  $W^{p,1}(\Omega)$ .

The so called increasing triangular transformations have been investigated in work [1]. These are transformations of the form  $T = (T_1, \dots, T_n): \mathbb{R}^n \rightarrow \mathbb{R}^n$ , where  $T_1$  is a function of  $x_1$ ,  $T_2$  is a function of  $(x_1, x_2)$  and so on,  $T_i$  is a function of  $(x_1, \dots, x_i)$ , and  $T_i$  is increasing in  $x_i$ . The canonical version of  $T$  is described in work [1]. Since our statements do not depend on a Lebesgue version of  $T$  (and do not depend on a  $\mu$ -version as well because  $\mu$  is equivalent to Lebesgue measure), we may assume that the transformation  $T$  is canonical. The main result of the paper is the following theorem.

**Theorem.** *Let  $\mu$  and  $\nu$  be absolutely continuous Borel probability measures with densities  $\rho_\mu$  and  $\rho_\nu$  on  $\Omega = [0, 1]^n$ . Let  $T$  be an increasing triangular transformation that transforms the measure  $\mu$  into  $\nu$ . Let us assume that there exist*

- (1) *positive numbers  $\varepsilon$  and  $M$  such that  $\varepsilon < \rho_\mu < M$ ,  $\varepsilon < \rho_\nu < M$ ;*
- (2) *positive numbers  $\alpha, \beta > 1$  such that  $p_n = \alpha\beta(n-1)^{-1}(\alpha+\beta)^{-1} > 1$ ,  $\varrho_\mu \in W^{\alpha,1}(\Omega)$ , and  $\varrho_\nu \in W^{\beta,1}(\Omega)$ .*

*Then the transformation  $T$  belongs to the class  $W^{p_n,1}(\Omega)$ .*

*Proof.* We shall prove the statement in the case  $n = 2$ . We recall that the Sobolev class  $W^{p,r}(\Omega)$  (another notion is  $H^{p,r}(\Omega)$ ) is defined as the set of functions  $f \in L^p(\Omega)$  whose derivatives up to order  $r$  are elements of  $L^p(\Omega)$  (regarding Sobolev classes the reader is referred to [2]). In order to show that the transformation  $T$  belongs to the Sobolev class we shall express its derivatives as functions of the densities of the measures  $\mu$  and  $\nu$ . It is shown in work [1] that

$$T_1(x) = F_{\nu_1}^{-1}(F_{\mu_1}(x)), \quad T_2(x, y) = F_{\nu_{T_1(x)}}^{-1}(F_{\mu_x}(y)),$$

where  $\mu_x$  and  $\nu_x$  are conditional measures on  $\{x\} \times [0, 1]$  (on conditional measures see [3]). In our case the conditional measures determined by the densities

$$\rho_{\mu_x}(y) = \frac{\rho_\mu(x, y)}{\int_0^1 \rho_\mu(x, t) dt}, \quad \rho_{\nu_x}(y) = \frac{\rho_\nu(x, y)}{\int_0^1 \rho_\nu(x, t) dt}$$

with respect to Lebesgue measure. These densities are referred to as conditional densities. We shall denote the projections of the measures  $\mu$  and  $\nu$  to the interval  $\{(x, 0), x \in [0, 1]\}$  as  $\mu_1$  and  $\nu_1$ . We denote by  $F_\xi$  the distribution function of an absolutely continuous measure  $\xi$  with a positive density  $\varrho_\xi$  defined on the interval  $[0, 1]$ , i.e.,

$$F_\xi(x) = \int_0^x \rho_\xi(t) dt, \quad x \in [0, 1].$$

The function  $F_\xi$  has an inverse function because it is strictly increasing. Thus

$$\begin{aligned} \partial_y T_1(x) &= 0, & \partial_x T_1(x) &= \frac{F'_{\mu_1}(x)}{F'_{\nu_1}(T_1(x))} = \frac{\rho_{\mu_1}(x)}{\rho_{\nu_1}(T_1(x))}, \\ \partial_y T_2(x, y) &= \frac{F'_{\mu_x}(y)}{F'_{\nu_{T_1(x)}}(T_2(x, y))} = \frac{\rho_{\mu_x}(y)}{\rho_{\nu_{T_1(x)}}(T_2(x, y))}. \end{aligned} \quad (1)$$

All the three functions are bounded because the densities  $\rho_\mu$  and  $\rho_\nu$  are bounded and are separated from zero and thus their conditional densities and the densities of their projections are separated from zero too. Therefore, they are integrable on  $\Omega$  in any power. It only remains to prove that the function  $\partial_x T_2$  belongs to  $L^{p^2}(\Omega)$ . Let the density  $\rho_\nu$  be a smooth function. Then one has the equalities

$$\begin{aligned} \partial_x F_{\mu_x}(y) &= \partial_x \left( \frac{\int_0^y \rho_\mu(x, t) dt}{\int_0^1 \rho_\mu(x, t) dt} \right) \\ &= \frac{\int_0^y \partial_x \rho_\mu(x, t) dt \int_0^1 \rho_\mu(x, t) dt - \int_0^1 \partial_x \rho_\mu(x, t) dt \int_0^y \rho_\mu(x, t) dt}{\left( \int_0^1 \rho_\mu(x, t) dt \right)^2}, \end{aligned}$$

$$\begin{aligned} \partial_x F_{\nu_{T_1(x)}}(y) &= \partial_x \left( \frac{\int_0^y \rho_\nu(T_1(x), t) dt}{\int_0^1 \rho_\nu(T_1(x), t) dt} \right) \\ &= \frac{\int_0^y [\partial_x \rho_\nu](T_1(x), t) T_1'(x) dt \int_0^1 \rho_\nu(T_1(x), t) dt}{\left( \int_0^1 \rho_\nu(T_1(x), t) dt \right)^2} \\ &\quad - \frac{\int_0^1 [\partial_x \rho_\nu](T_1(x), t) T_1'(x) dt \int_0^y \rho_\nu(T_1(x), t) dt}{\left( \int_0^1 \rho_\nu(T_1(x), t) dt \right)^2}, \end{aligned}$$

$$F'_{\nu_{T_1(x)}}(y) = \partial_y \left( \frac{\int_0^y \rho_\nu(T_1(x), t) dt}{\int_0^1 \rho_\nu(T_1(x), t) dt} \right) = \frac{\rho_\nu(T_1(x), y)}{\int_0^1 \rho_\nu(T_1(x), t) dt}.$$

The following equality holds true as well:

$$\left[ \partial_x F_{\nu_{T_1(x)}}^{-1} \right] (y) = - \frac{\left[ \partial_x F_{\nu_{T_1(x)}} \right] (F_{\nu_{T_1(x)}}^{-1}(y))}{F'_{\nu_{T_1(x)}}(F_{\nu_{T_1(x)}}^{-1}(y))}. \quad (2)$$

Indeed, suppose that  $f(x, y) = F_{\nu_{T_1(x)}}(y)$  and  $\varphi(x, y) = F_{\nu_{T_1(x)}}^{-1}(y)$ . For any  $x$  and  $y$  we have the equality  $f(x, \varphi(x, y)) = y$ . Differentiation in  $x$  leads to the equality

$$\partial_x f(x, \varphi(x, y)) + \partial_y f(x, \varphi(x, y)) \partial_x \varphi(x, y) = 0.$$

We obtain the equality

$$\partial_x \varphi(x, y) = -\frac{\partial_x f(x, \varphi(x, y))}{\partial_y f(x, \varphi(x, y))}. \quad (3)$$

This leads to equality (2) in the case of smooth densities. Then we obtain the following chain of equalities:

$$\partial_x T_2(x, y) = \left[ \partial_x F_{\nu_{T_1(x)}}^{-1} \right] (F_{\mu_x}(y)) \partial_x F_{\mu_x}(y) = -\frac{\left[ \partial_x F_{\nu_{T_1(x)}} \right] (T_2(x, y))}{\left[ F'_{\nu_{T_1(x)}} \right] (T_2(x, y))} \partial_x F_{\mu_x}(y). \quad (4)$$

We get

$$\left| \left[ \partial_x F_{\nu_{T_1(x)}} \right] (T_2(x, y)) \right| \leq 2 \frac{\int_0^1 |[\partial_x \rho_\nu](T_1(x), t)| T_1'(x) dt}{\int_0^1 \rho_\nu(T_1(x), t) dt}.$$

In addition, one has

$$|\partial_x F_{\mu_x}(y)| \leq 2 \frac{\int_0^1 |\partial_x \rho_\mu(x, t)| dt}{\int_0^1 \rho_\mu(x, t) dt}.$$

Due to the inequalities  $\rho_\nu \geq \varepsilon$  and  $\int_0^1 \rho_\mu(x, t) dt \geq \varepsilon$  for conditional density we have

$$\begin{aligned} |\partial_x T_2(x, y)| &\leq 4 \frac{\int_0^1 |[\partial_x \rho_\nu](T_1(x), t)| T_1'(x) dt \int_0^1 |\partial_x \rho_\mu(x, t)| dt}{\rho_\nu(T_1(x), T_2(x, y)) \int_0^1 \rho_\mu(x, t) dt} \\ &\leq \frac{4}{\varepsilon^2} \int_0^1 |[\partial_x \rho_\nu](T_1(x), t)| T_1'(x) dt \int_0^1 |\partial_x \rho_\mu(x, t)| dt. \quad (5) \end{aligned}$$

It is easy to see that, for any function  $f \in L^p(\Omega)$ , where  $p > 1$ , the function  $\int_0^1 f(x, t) dt$  belongs to  $L^p(\Omega)$  by Fubini's theorem. It follows by Hölder's inequality that if  $f \in L^\alpha(\Omega)$ ,  $g \in L^\beta(\Omega)$ , then  $fg \in L^p(\Omega)$  where  $p = \alpha\beta(\alpha + \beta)^{-1}$ . Thus to prove our statement in the case of smooth a density  $\rho_\nu$  it is enough to show that  $[\partial_x \rho_\nu](T_1(x), y) T_1'(x) \in L^\beta(\Omega)$ ,  $\partial_x \rho_\mu(x, y) \in L^\alpha(\Omega)$ . The hypotheses of the theorem imply that  $\partial_x \rho_\mu(x, y) \in L^\alpha(\Omega)$ . By the change of variables formula and the fact that  $T_1'(x) \leq M/\varepsilon$  we deduce that

$$\begin{aligned} \int_\Omega |[\partial_x \rho_\nu](T_1(x), y) T_1'(x)|^\beta dx dy &= \int_\Omega |\partial_x \rho_\nu(x, y)|^\beta (T_1'(T_1^{-1}(x)))^{\beta-1} dx dy \\ &\leq \frac{M^{\beta-1}}{\varepsilon^{\beta-1}} \int_\Omega |\partial_x \rho_\nu(x, y)|^\beta dx dy, \end{aligned}$$

where the existence of the right hand side of the equality implies the existence of the left hand side. Thus  $\partial_x T_2$  belongs to  $L^{p_2}(\Omega)$  and we obtain the following estimate:

$$\|\partial_x T_2\|_{L^{p_2}(\Omega)} \leq C \|\partial_x \rho_\nu\|_{L^\beta(\Omega)} \|\partial_x \rho_\mu\|_{L^\alpha(\Omega)}, \quad (6)$$

where  $C$  is a constant which depends only on  $\varepsilon$  and  $M$ .

From now on we do not assume that the density  $\rho_\nu$  is smooth, but we suppose that the hypotheses of the theorem are fulfilled. There exists a sequence of smooth densities  $\rho_{\nu^{(m)}}$  convergent to  $\rho_\nu$  in the norm of  $W^{\beta,1}$ . In addition, we can choose it so that for  $\rho_{\nu^{(m)}}$  the hypotheses of the theorem are fulfilled with the same  $\varepsilon$ ,  $M$  and  $\beta$  for any  $m$ . Inequality (6) applied to the densities  $\rho_{\nu^{(m)}}$  and the corresponding triangular transformations  $T^{(m)}$  implies the boundedness of the sequence of functions  $\partial_x T_2^{(m)}$  in the class  $L^{p_2}(\Omega)$ . Now to prove the theorem it is enough to show that the sequence of functions  $T_2^{(m)}$  converges to  $T_2$  in  $L^{p_2}(\Omega)$ . Notice that because the absolute values of  $T_2^{(m)}$  and  $T_2$  do not exceed 1, it is enough to establish convergence in measure. It is proved in work [1] that if a sequence of absolutely continuous probability measures  $\nu_j$  defined on  $\mathbb{R}^n$  converges in variation to measure  $\nu$ , then sequence of canonical triangular transformations  $T_{\mu, \nu_j}$  converge in measure to  $T_{\mu, \nu}$  (in work [4], a generalization is obtained in the case where measure  $\mu$  also vary). Because convergence of densities in  $W^{\beta,1}(\Omega)$  implies convergence of measures in the variation norm, the sequence  $T_2^{(m)}$  converges to  $T_2$  in  $L^{p_2}(\Omega)$ . The statement in the case  $n = 2$  proved.

Now we apply induction on  $n$  and assume that the statement is proved if  $k < n$ . According to the construction of the canonical transformation  $T$  (see [1]), the first  $n - 1$  coordinates of the transformation  $T$  form the canonical transformation of the projections of the measures on the  $(n - 1)$ -dimensional cube in the hyperplane  $x_n = 0$ . We shall denote it by  $S$ , and the vector  $(x_1, \dots, x_{n-1})$  is denoted by  $x$ . Obviously, the hypotheses of the theorem are fulfilled for the projections of our measures. Indeed, the densities of the projections are positive, bounded and separated from zero, their derivatives  $\int_0^1 \partial_{x_i} \rho_\nu(x, x_n) dx_n$ ,  $\int_0^1 \partial_{x_i} \rho_\mu(x, x_n) dx_n$  are integrable in necessary powers. Therefore, the components  $T_i$ ,  $i = 1, \dots, n - 1$ , belong to the Sobolev class  $W^{p_{n-1}, 1}(\Omega)$ ,  $p_{n-1} > p_n$ . Thus it remains to prove the membership of  $\partial_{x_i} T_n(x, x_n)$  in  $L^{p_n}(\Omega)$ .

We shall use the following relation for  $T_n(x, x_n)$ :

$$T_n(x, x_n) = F_{\nu_{S(x)}}^{-1}(F_{\mu_x}(x_n)),$$

where  $\mu_x$  and  $\nu_x$  are conditional measures defined on the segments  $\{x\} \times [0, 1]$ . The derivative of  $T_n(x, x_n)$  in  $x_n$  has the same form as in (1), i.e.,

$$\partial_{x_n} T_n(x, x_n) = \frac{\rho_{\mu_x}(x_n)}{\rho_{\nu_{S(x)}}(T_n(x, x_n))}. \quad (7)$$

Hence it is integrable in any power. Suppose that the density  $\rho_\nu$  is a smooth function. Then the derivative in  $x_i$ ,  $i < n$ , has the same form as in (4), i.e.,

$$\partial_{x_i} T_n(x, x_n) = - \frac{\left[ \partial_{x_i} F_{\nu_{S(x)}} \right] (T_n(x, x_n))}{F'_{\nu_{S(x)}}(T_n(x, x_n))} \partial_{x_i} F_{\mu_x}(x_n).$$

Let us write out multipliers separately:

$$\left( F'_{\nu_{S(x)}}(T_n(x, x_n)) \right)^{-1} = \frac{\int_0^1 \rho_\nu(S(x), t) dt}{\rho_\nu(S(x), T_n(x, x_n))};$$

$$\begin{aligned} \left| \left[ \partial_{x_i} F_{\nu_{S(x)}} \right] (T_n(x, x_n)) \right| &= \left| \partial_{x_i} \left( \frac{\int_0^y \rho_\nu(S(x), t) dt}{\int_0^1 \rho_\nu(S(x), t) dt} \right) \Big|_{y=T_n(x, x_n)} \right| \\ &\leq \frac{2}{\int_0^1 \rho_\nu(S(x), t) dt} \int_0^1 \sum_{j=i}^{n-1} |[\partial_{x_j} \rho_\nu](S(x), t) \partial_{x_i} T_j(x_1, \dots, x_j)| dt, \\ |\partial_{x_i} F_{\mu_x}(x_n)| &= \left| \partial_{x_i} \left( \frac{\int_0^{x_n} \rho_\mu(x, t) dt}{\int_0^1 \rho_\mu(x, t) dt} \right) \right| \leq 2 \frac{\int_0^1 |\partial_{x_i} \rho_\mu(x, t)| dt}{\int_0^1 \rho_\mu(x, t) dt}. \end{aligned}$$

Similarly to inequality (5) we obtain the estimate

$$\partial_{x_i} T_n(x, x_n) \leq \frac{4}{\varepsilon^2} \cdot \int_0^1 \sum_{j=i}^{n-1} |[\partial_{x_j} \rho_\nu](S(x), t) \partial_{x_i} T_j(x_1, \dots, x_j)| dt \int_0^1 |\partial_{x_i} \rho_\mu(x, t)| dt. \quad (8)$$

By the inductive assumption for  $j = 1, \dots, n-1$  the function  $\partial_{x_i} T_j(x_1, \dots, x_j)$  belongs to  $L^{p_j}(\Omega)$ . In particular, for any  $j$  this expression belongs to  $L^{p_{n-1}}(\Omega)$ . The function  $\int_0^1 |\partial_{x_i} \rho_\mu(x, t)| dt$  belongs to  $L^\alpha(\Omega)$  and one has  $\partial_{x_j} \rho_\nu(x, t) \in L^\beta(\Omega)$  for any  $j$ . Then by the change of variable formula (see [1, p. 7]) we obtain

$$\int_\Omega |[\partial_{x_j} \rho_\nu](S(x), t)|^\beta dx dt = \int_\Omega |\partial_{x_j} \rho_\nu(x, t)|^\beta \prod_{k=1}^{n-1} \partial_{x_k} T_k(x_1, \dots, x_k) dx dt,$$

where  $\varepsilon/M \leq \partial_{x_k} T_k(x_1, \dots, x_k) \leq M/\varepsilon$  according to (7). Thus  $[\partial_{x_j} \rho_\nu](S(x), t)$  belongs to  $L^\beta(\Omega)$ . By using Hölder's inequality we obtain that the right hand side of inequality (8) and therefore the left hand side belongs to  $L^q(\Omega)$  where  $1/q = 1/\alpha + 1/\beta + 1/p_{n-1}$ , i.e.,  $q = p_n$ . In addition, the following chain of equalities holds true:

$$\begin{aligned} \|\partial_{x_i} T_n\|_{L^{p_n}(\Omega)} &\leq C \max_{i \leq j \leq n-1} \|\partial_{x_i} T_j\|_{L^{p_{n-1}}(\Omega)} \max_{i \leq j \leq n-1} \|\partial_{x_j} \rho_\nu\|_{L^\beta(\Omega)} \|\partial_{x_i} \rho_\mu\|_{L^\alpha(\Omega)} \\ &\leq C \max_{i \leq j \leq n-1} \|\partial_{x_i} T_j\|_{L^{p_j}(\Omega)} \|\rho_\nu\|_{W^{\beta,1}(\Omega)} \|\rho_\mu\|_{W^{\alpha,1}(\Omega)}, \end{aligned}$$

where  $C$  is a constant depending only on  $\varepsilon$  and  $M$ . Then by induction we can obtain the estimate

$$\|\partial_{x_i} T_n\|_{L^{p_n}(\Omega)} \leq C_1 \|\rho_\nu\|_{W^{\beta,1}(\Omega)}^{n-1} \|\rho_\mu\|_{W^{\alpha,1}(\Omega)}^{n-1}, \quad (9)$$

where  $C_1$  is a constant number depending only on  $\varepsilon$  and  $M$ .

From now on we do not assume that the density  $\rho_\nu$  is smooth. As in the case  $n = 2$  let us find a sequence of smooth densities  $\rho_{\nu^{(m)}}$  for which the hypotheses of the theorem are fulfilled with the same  $\varepsilon$ ,  $M$  and  $\beta$  for any  $m$ , and the sequence  $\rho_{\nu^{(m)}}$  converges to  $\rho_\nu$  in  $W^{\beta,1}(\Omega)$ . By inequality (9) applied to the densities  $\rho_{\nu^{(m)}}$  and the corresponding triangular transformations  $T^{(m)}$ , it is easy to show the boundedness of the sequence of functions  $\partial_{x_i} T_n^{(m)}$  in the class  $L^{p_n}(\Omega)$ . The functions  $T_n^{(m)}$  converge to  $T_n$  in  $L^{p_n}(\Omega)$ . Hence  $T_n$  is a limit of the sequence of functions  $T_n^{(m)}$  in  $W^{p_n,1}(\Omega)$ . Theorem is completely proved.

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## References

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