Rate of Convergence to The Semicircle Law almost surely

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Abstract

The bounds for the L_p-norm, $p \geq 2$, for the Kolmogorov distance between spectral distribution function of random matrix and semicircle distribution function are obtained. It is shown that the rate of convergence almost surely has order $O(n^{-\frac{1}{2}+\eta})$, for any $\eta > 0$.

1 Introduction and results

In this paper we investigate the rate of convergence to the semicircle law of the empirical spectral distribution function. We consider bounds in the L_p-norm of the Kolmogorov distance between the empirical spectral distribution function and the semicircle distribution function. We obtain as well a rate of convergence almost surely for the empirical distribution function of random matrices as the dimension tends to infinity. In Götze, Tikhomirov ([5] and [6]) the rate of convergence the *expected* spectral distribution function to the limit distribution for the Wigner matrices and for the sample covariance matrices as well as the rate of convergence in probability has been studied.

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Let $X_{jk}, 1 \le j \le k < \infty$, be independent random variables with $\mathbf{E}\, X_{jk} = 0$ and $\mathbf{E}\, X_{jk}^2 = 1$ and let $X_{kj} = X_{jk}$, for $1 \le j < k < \infty$. For a fixed $n \ge 1$, denote by $\lambda_1 \le \ldots \le \lambda_n$ the eigenvalues of the symmetric $n \times n$ matrix

$$\mathbf{W} = \Big(W_{jk}\Big)_{j,k=1}^{n}, \quad W_{jk} = \frac{1}{\sqrt{n}} X_{jk}, \text{ for } 1 \le j \le k \le n,$$

and define their spectral distribution function

$$F_n(x) = \frac{1}{n} \sum_{j=1}^{n} I_{\{\lambda_j \le x\}},$$

where $I_{\{B\}}$ denote the indicator of an event B. We investigate the almost sure rate of convergence of the spectral distribution function $F_n(x)$ to the distribution function of Wigner's semicircle law.

Let $g(x)$ and $G(x)$ denote the density and the distribution function of the standard semicircle law, that is

$$g(x) = \frac{1}{2\pi} \sqrt{4 - x^2}\, I_{\{|x| \le 2\}}, \quad G(x) = \int_{-\infty}^{x} g(u) du.$$

Set

$$\Delta_n^* = \sup_x |F_n(x) - G(x)|, \qquad \Delta_n^{(p)} = \mathbf{E}^{\frac{1}{p}} \left(\Delta_n^*\right)^p,$$

$$\mathbf{R} := \mathbf{R}(z) := (\mathbf{W} - z\mathbf{I}_n)^{-1},$$

where $z = u + iv, v > 0$ and $\mathbf{I}_n$ denote the $n \times n$ identity matrix. Denote by $s(z)$ the Stieltjes transform of the semicircle law, that is

$$s(z) = -\frac{1}{2}\Big(z - \sqrt{z^2 - 4}\Big).$$

Let $M_q = \sup_{1 \le j,k \le n} \mathbb{E}|X_{jk}|^q$. Define

$$M_q^{(n)}(v) := \frac{M_q}{(nv)^{\frac{q}{4} - 1}}, \qquad \widetilde{M}_q^{(n)}(v) := nv M_q^{(n)}(v).$$

In what follows C denote a general positive constant and by "$\ll_p$" we denote an inequality which holds up to a constant factor exponentially depending on p.

Our main results are the following

Theorem 1.1. *Let $p \ge 1$. Then, for $v \ge C M_4^{\frac{1}{2}} p n^{-\frac{1}{2}}$, we have*

$$\mathbf{E}\left|\frac{1}{n} \operatorname{Tr} \mathbf{R} - \frac{1}{n} \mathbf{E} \operatorname{Tr} \mathbf{R}\right|^{2p} \le \frac{C^p p^{4p} \Big(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\Big)}{(nv)^{2p} |z + 2s(z)|^{2p}}. \qquad (1.1)$$

Theorem 1.2. *Let $p \geq 2$. Assume that*

$$M_{4p+4} := \sup_{1 \leq j \leq k \leq n} \mathbf{E}\,|X_{jk}|^{4p+4} \leq C_p < \infty.$$

Then

$$\Delta_n^{(p)} \leq Cp^2 (M_{2p+2} + \widetilde{M}_{4p+4}^{(n)}(v_0))^{\frac{1}{p}}\; n^{-\frac{1}{2}},$$

where $v_0 = CM_4^{\frac{1}{2}} pn^{-\frac{1}{2}}$.

Theorem 1.3. *Assume that for any integer $p \geq 1$,*

$$M_p = \sup_{1 \leq j \leq k \leq n} \mathbf{E}\,|X_{jk}|^p \leq C_p < \infty.$$

Then for any $\eta > 0$,

$$\Delta_n^* = \sup_x |F_n(x) - G(x)| = o(n^{-\frac{1}{2}+\eta}) \qquad a.s.,$$

where "$\xi_n = o(a_n)$ a.s." means that $a_n^{-1}\xi_n \to 0$ a.s.

Note that the results of Theorems 1.1 and 1.2 for $p = 1$ were obtained in Götze, Tikhomirov ([5]). In Bai et al. ([1]) it was shown that $\Delta_n^* = o(n^{-\frac{2}{5}+\eta})$. The rate of convergence of *expected* spectral distribution function to semicircle law of order $O(n^{-\frac{1}{2}})$ was shown independently in Bai et al. ([1]), Girko ([4]) and Götze, Tikhomirov ([5]). For more details see for example Götze, Tikhomirov ([5] and [6]) and references given there.

The paper is divided into six Sections. In Section 2 we collect results like the analog of Berry-Esseen's inequality for the Stieltjes transform proved in Götze, Tikhomirov ([5]). In Section 3 we formulated two main Lemmas which needed for the proof of Theorem 1.1. In Section 4 we prove Theorems 1.1–1.3. Section 5 contains some auxiliary Lemmas. In Section 6 and 7 we prove the main Lemmas.

2 Inequalities for the distance between distributions in terms of their Stieltjes transform

We define the Stieltjes transform $t(z)$ of a random variables ξ with the distribution function $F(x)$ (the Stieltjes transform $t(z)$ of the distribution function $F(x)$) via

$$t(z) := \mathbf{E}\,\frac{1}{\xi - z} = \int_{-\infty}^{\infty} \frac{1}{x - z} d\,F(x), \quad z = u + iv, \quad v > 0.$$

Given $\varepsilon > 0$ introduce the intervals $I_\varepsilon = [-2 + \varepsilon, 2 - \varepsilon]$ and $I'_\varepsilon = [-2 + \frac{1}{2}\varepsilon, 2 - \frac{1}{2}\varepsilon]$.

Lemma 2.1. *Let F be a distribution function and let G denote the semicircle distribution function. Denote their Stieltjes transforms by $t(z)$ and $s(z)$ respectively. Assume that $\int_{-\infty}^{\infty} |F(x) - G(x)|dx < \infty$. Let $v > 0$ and a and ε be positive numbers such that*

$$\gamma = \frac{1}{\pi} \int_{|u|\leq a} \frac{1}{u^2+1}\, du = \frac{3}{4},$$

and

$$\varepsilon > 2\, v\, a.$$

Then there exist some absolute constants C_1, C_2, C_3 such that

$$\sup_x |F(x) - G(x)| \leq C_1 \sup_{x \in I'_\varepsilon} |\mathrm{Im}\Big(\int_{-\infty}^{x} (t(z) - s(z))\, du\Big)| + C_2\, v + C_3\, \varepsilon^{3/2}.$$

A proof of Lemma 2.1 is given in Götze, Tikhomirov (2003) called GT03 henceforth.

To prove the Theorem 1.2 we shall use a modification of this smoothing inequality. Denote by $t_n(z)$ the Stieltjes transform of the spectral distribution function $F_n(x)$ and put $s_n(z) = \mathbf{E}\, t_n(z)$.

Lemma 2.2. *Let $x_0 \in I'_\varepsilon$. Then the following inequality holds*

$$\begin{aligned}
&\sup_x |F_n(x) - G(x)| \\
&\quad \leq C_1 \int_{-\infty}^{\infty} |(s_n(u+iV) - s(u+iV)\,|du + C_2\, v + C_3\, \varepsilon^{3/2} \\
&\quad + C_4 \sup_{x \in I'_\varepsilon} \left| \int_v^V (s_n(x+iu) - s(x+iu)du \right| \\
&\quad + C_1 \int_{-\infty}^{\infty} \left| \frac{1}{n} \mathrm{Tr}\, \mathbf{R}(u+iV) - \frac{1}{n} \mathbf{E}\, \mathrm{Tr}\, \mathbf{R}(u+iV)) \right| du \\
&\quad + C_1 \int_v^V \left| \frac{1}{n} \mathrm{Tr}\, \mathbf{R}(x_0+iu) - \frac{1}{n} \mathbf{E}\, \mathrm{Tr}\, \mathbf{R}(x_0+iu) \right| du \\
&\quad + C_2 \int_v^V \int_{x \in I_\varepsilon} \left| \frac{1}{n} \mathrm{Tr}\, \mathbf{R}^2(x+iu) - \mathbf{E}\, \frac{1}{n} \mathrm{Tr}\, \mathbf{R}^2(x+iu) \right| dx du, \qquad (2.1)
\end{aligned}$$

where $\mathbf{R} := (\mathbf{W} - z\mathbf{I}_n)^{-1}$.

3 The main Lemmas

Let $z = u + iv, \quad v > 0$. Recall that

$$s(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} d\, G(x) = -\frac{1}{2}(z - \sqrt{z^2-4}), \quad s_n(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} d\, \mathbf{E}\, F_n(x). \qquad (3.1)$$

By definition of $F_n(x)$, we may write

$$s_n(z) = \mathbf{E}\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{\lambda_j - z}\right) = \frac{1}{n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{R}(z) = \frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,R(j,j). \tag{3.2}$$

Let $\mathbf{W}(k)$ be the matrix obtained from $\mathbf{W}$ by deleting the kth row and kth column, and let $\mathbf{a}_k' = \big(X_{1k}, \ldots, X_{(k-1)k}, X_{(k+1)k}, \ldots, X_{nk}\big)$,
$\mathbf{R}_k := \mathbf{R}_k(z) = \left(\mathbf{W}(k) - z\mathbf{I}_{n-1}\right)^{-1} = \left(R_k(i,j)\right)_{\{i,j\}\in\{1,\ldots,k-1,k+1,\ldots,n\}}$. Set

$$\begin{aligned}\varepsilon_k &= \frac{1}{\sqrt{n}}X_{kk} - \frac{1}{n}\mathbf{a}_k'\left(\mathbf{W}(k) - z\mathbf{I}_{n-1}\right)^{-1}\mathbf{a}_k + s_n(z)\\ &= \frac{1}{\sqrt{n}}X_{kk} - \frac{1}{n}\left(\mathbf{a}_k'\mathbf{R}_k\mathbf{a}_k - \mathrm{Tr}\,\mathbf{R}_k\right) + \frac{1}{n}\left(\mathrm{Tr}\,\mathbf{R} - \mathrm{Tr}\,\mathbf{R}_k\right) - \frac{1}{n}\left(\mathrm{Tr}\,\mathbf{R} - \mathbf{E}\,\mathrm{Tr}\,\mathbf{R}\right).\end{aligned} \tag{3.3}$$

Introduce

$$\delta_n(z) = -\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\varepsilon_j\frac{1}{(z + s_n(z))(z + s_n(z) - \varepsilon_j)}. \tag{3.4}$$

By Lemma 3.1 GT03 and relations (3.2) and (3.3), we may write

$$\begin{aligned}R(j,j) &= -\frac{1}{z + s_n(z) - \varepsilon_j} = -\frac{1}{z + s_n(z)} - \frac{\varepsilon_j}{(z + s_n(z))(z + s_n(z) - \varepsilon_j)}\\ &= -\frac{1}{z + s_n(z)} + \frac{\varepsilon_j}{z + s_n(z)}R(j,j).\end{aligned} \tag{3.5}$$

This implies that

$$s_n(z) = \frac{1}{n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{R} = -\frac{1}{z + s_n(z)} + \delta_n(z). \tag{3.6}$$

In order to simplify the exposition we shall follow the notation of GT03:

$$\begin{aligned}&\Delta_E(\mathbf{R}) := \frac{1}{n}\Big(\mathrm{Tr}\,\mathbf{R} - \mathbf{E}\,\mathrm{Tr}\,\mathbf{R}\Big), && \Delta_E^{(j)}(\mathbf{R}) := \frac{1}{n}\Big(\mathrm{Tr}\,\mathbf{R}_j - \mathbf{E}\,\mathrm{Tr}\,\mathbf{R}_j\Big),\\ &\mathcal{Q}_j(\mathbf{R}) := \frac{1}{n}\Big(\mathbf{a}_j'\mathbf{R}_j\mathbf{a}_j - \mathrm{Tr}\,\mathbf{R}_j\Big), && \mathcal{Q}_j(\mathbf{R}^2) := \frac{1}{n}\Big(\mathbf{a}_j'\mathbf{R}_j^2\mathbf{a}_j - \mathrm{Tr}\,\mathbf{R}_j^2\Big),\\ &\mathcal{D}_j(\mathbf{R}) := \frac{1}{n}\Big(\mathrm{Tr}\,\mathbf{R} - \mathrm{Tr}\,\mathbf{R}_j\Big), && a_n(z) := (s_n(z) + z)^{-1},\ b_n(z) := (2s_n(z) + z)^{-1}.\end{aligned}$$

Here we may introduce

$$\varepsilon_j = \frac{1}{\sqrt{n}}X_{jj} - \mathcal{Q}_j(\mathbf{R}) + \mathcal{D}_j(\mathbf{R}) - \Delta_E(\mathbf{R}). \tag{3.7}$$

The proof of Theorem 1.1 is based on

Lemma 3.1. *For $p \ge 1$ and for $v \ge CM_4^{\frac{1}{2}} p n^{-\frac{1}{2}}$, the following inequality holds*

$$\mathbf{E}\,\big|\Delta_E(\mathbf{R})\big|^{2p} \ll_p \frac{p^{4p}}{(n^2v^3)^p}\Big(M_{2p}\sigma^{2p} + M_4^p + M_{4p}^{(n)}(v)\Big), \tag{3.8}$$

where

$$M_q^{(n)}(v) := \frac{M_q}{(nv)^{\frac{q}{4}-1}}.$$

Lemma 3.2. *Assuming the conditions of Theorem 1.1, the following inequalities hold*

$$\begin{aligned}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{4p} &\ll_p \frac{p^{4p}}{(nv)^{2p}}\big[M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\big], &(3.9)\\ \mathbf{E}\,|\Delta_E(\mathbf{R})|^{4p+2} &\ll_p \frac{p^{4p+2}}{(nv)^{2p+1}}\big[M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\big], &(3.10)\end{aligned}$$

where

$$\widetilde{M}_{8p+4}^{(n)}(v) = \frac{1}{(nv)^{2p-1}}M_{8p+4}.$$

Lemma 3.3. *Assuming the conditions of Theorem 1.1 the following inequality holds*

$$\mathbf{E}\,|\Delta_E(\mathbf{R})|^{4p-1} \ll_p \frac{p^{4p}}{|z+2s(z)|(nv)^{2p}}\big[M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\big]. \tag{3.11}$$

Lemma 3.4. *Let $1 \le \beta \le p$. Under the conditions of Theorem 1.1 assume that, for $1 \le k \le 2p - 2\beta + 2$,*

$$\mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2+k} \ll_p \frac{p^{4p}\Big(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\Big)}{(nv)^{2p}|z+2s(z)|^{2p-2\beta+2-k}}. \tag{3.12}$$

Then

$$\mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2} \ll_p \frac{p^{4p}\Big(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\Big)}{(nv)^{2p}|z+2s(z)|^{2p-2\beta+2}}. \tag{3.13}$$

4 The proofs of the Theorems

Proof of Theorem 1.1. According to Lemmas 3.2 and 3.3, we have, for $k = 0, 1$,

$$\mathbf{E}\,|\frac{1}{n}(\operatorname{Tr}\mathbf{R} - \mathbf{E}\operatorname{Tr}\mathbf{R})|^{4p-k} \le \frac{p^{4p}}{|z+2s(z)|^k(nv)^{2p}}\big[M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\big]. \tag{4.1}$$

Applying recurrently Lemma 3.4, for $k = 2, \ldots, 2p$, we get

$$\mathbf{E}\,|\frac{1}{n}(\operatorname{Tr}\mathbf{R} - \mathbf{E}\operatorname{Tr}\mathbf{R})|^{2p} \le \frac{p^{4p}}{|z+2s(z)|^{2p}(nv)^{2p}}\big[M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\big]. \tag{4.2}$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Applying the result of Lemma 2.1, we obtain

$$\begin{aligned}\mathbf{E}^{\frac{1}{p}}\sup_x|F_n(x)-G(x)|^p\le{}& Cv_0+\int_{-\infty}^{\infty}|s_n(u+iV)-s(u+iV)|du\\&+C\sup_{x\in I_\varepsilon}\int_{v_0}^{V}|s_n(x+iv)-s(x+iv)|dv\\&+C\mathbf{E}^{\frac{1}{p}}\Big(\int_{-\infty}^{\infty}|\operatorname{Tr}\mathbf{R}(u+iV)-\mathbf{E}\operatorname{Tr}\mathbf{R}(u+iV)|du\Big)^p\\&+C\mathbf{E}^{\frac{1}{p}}\Big(\int_{v_0}^{V}|\operatorname{Tr}\mathbf{R}(x_0+iv)-\mathbf{E}\operatorname{Tr}\mathbf{R}(x_0+iv)|dv\Big)^p\\&+C\mathbf{E}^{\frac{1}{p}}\Big(\int_{x\in I_\varepsilon}\int_{v_0}^{V}|\operatorname{Tr}\mathbf{R}^2(x+iv)-\mathbf{E}\operatorname{Tr}\mathbf{R}^2(x+iv)|dvdx\Big)^p.\end{aligned}\tag{4.3}$$

In the sequel we shall use the following

Theorem 4.1. *Let $(\mathcal{X}_1,\mathcal{F}_1,\mu_1)$ and $(\mathcal{X}_2,\mathcal{F}_2,\mu_2)$ be positive measure spaces. Let K be a $\mu_1\times\mu_2$-integrable function on $\mathcal{F}_1\times\mathcal{F}_2$. Then for $q\ge1$*

$$\begin{aligned}&\Big(\int_{\mathcal{X}_2}\Big(\int_{\mathcal{X}_1}|K(x_1,x_2)|\mu(dx_2)\Big)^p\mu_2(dx_2)\Big)^{\frac{1}{p}}\\&\qquad\le\int_{\mathcal{X}_1}\Big(\int_{\mathcal{X}_2}|K(x_1,x_2)|^p\mu_2(dx_2)\Big)^{\frac{1}{p}}\mu_1(dx_1).\end{aligned}$$

See for example (Dunford, Schwartz [2], Part 1, v.6.11.13, p.580). Applying Theorem 4.1 to the r.h.s. of inequality (4.3), we get

$$\begin{aligned}\mathbf{E}^{\frac{1}{p}}\sup_x|F_n(x)-G(x)|^p\le{}& Cv_0+\int_{-\infty}^{\infty}|s_n(u+iV)-s(u+iV)|du\\&+C\sup_{x\in I_\varepsilon}\int_{v_0}^{V}|s_n(x+iv)-s(x+iv)|dv\\&+C\Big(\int_{-\infty}^{\infty}\mathbf{E}^{\frac{1}{p}}|\operatorname{Tr}\mathbf{R}(u+iV)-\mathbf{E}\operatorname{Tr}\mathbf{R}(u+iV)|^pdu\Big)\\&+C\Big(\int_{v_0}^{V}\mathbf{E}^{\frac{1}{p}}|\operatorname{Tr}\mathbf{R}(x_0+iv)-\mathbf{E}\operatorname{Tr}\mathbf{R}(x_0+iv)|^pdv\Big)\\&+C\Big(\int_{x\in I_\varepsilon}\int_{v_0}^{V}\mathbf{E}^{\frac{1}{p}}|\operatorname{Tr}\mathbf{R}^2(x+iv)-\mathbf{E}\operatorname{Tr}\mathbf{R}^2(x+iv)|^pdvdx\Big).\end{aligned}\tag{4.4}$$

Using inequalities (4.30), (5.64), (5.65) from GT03, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} &|s_n(u+iV)-s(u+iV)|du \\ &+C\sup_{x\in I_\varepsilon}\int_{v_0}^{V}|s_n(x+iv)-s(x+iv)|dv \le CM_4^{\frac{1}{2}}n^{-\frac{1}{2}}. \end{aligned}\tag{4.5}$$

Furthermore, similar to inequality (7.7) from GT03 we may show that

$$\int_{-\infty}^{\infty}\mathbf{E}^{\frac{1}{p}}|\frac{1}{n}(\operatorname{Tr}\mathbf{R}(u+iV)-\mathbf{E}\operatorname{Tr}\mathbf{R}(u+iV)|^p du \le Cn^{-\frac{1}{2}}M_p^{\frac{1}{2}}. \tag{4.6}$$

Applying Theorem 1.1, we get

$$\begin{aligned}\int_{x\in I_\varepsilon}\int_{v_0}^{V}&\mathbf{E}^{\frac{1}{p}}|\operatorname{Tr}\mathbf{R}^2(x+iv)-\mathbf{E}\operatorname{Tr}\mathbf{R}^2(x+iv)|^p dvdx \\ &\le Cp^2(M_{2p+1}+\widetilde{M}_{4p+2}^{(n)}(v))^{\frac{1}{p}}\int_{x\in I_\varepsilon}\int_{v_0}^{V}\frac{1}{|z+2s(z)|}\frac{1}{nv^2}dvdu \\ &\le Cn^{-\frac{1}{2}}p^2(M_{2p+1}+\widetilde{M}_{4p+4}^{(n)}(v_0))^{\frac{1}{p}}. \end{aligned}\tag{4.7}$$

Theorem 1.1 yields as well

$$\int_{v_0}^{V}\mathbf{E}^{\frac{1}{p}}|\operatorname{Tr}\mathbf{R}(x_0+iv)-\mathbf{E}\operatorname{Tr}\mathbf{R}(x_0+iv)|^p dv \le Cn^{-\frac{1}{2}}p^2(M_{2p+2}+\widetilde{M}_{4p+4}^{(n)}(v_0))^{\frac{1}{p}}. \tag{4.8}$$

Inequalities (4.3)–(4.8) together complete the proof. □

Proof of Theorem 1.3. It is enough to show that the series

$$\sum_{n=1}^{\infty}\mathbb{P}\{\Delta_n^* > Cn^{-\frac{1}{2}+\eta}\} < \infty \tag{4.9}$$

converges. Using Chebyshev's inequality and the result of Theorem 1.1, we get

$$\sum_{n=1}^{\infty}\mathbb{P}\{\Delta_n^* > Cn^{-\frac{1}{2}+\eta}\} \le \sum_{n=1}^{\infty} n^{-p\eta}p^2p(M_{2p+2}+n^{-\frac{p}{2}}M^{4p+4}). \tag{4.10}$$

Choosing $p=\frac{1}{\eta}+1$, we obtain the result. □

5 Auxiliary Lemmas

In this Section we prove some additional inequalities for resolvent matrices.

Lemma 5.1. *For $p \ge 1$ and for $v \ge C M_4^{\frac12} p n^{-\frac12}$, the following inequality holds*

$$\mathbf{E}\,\Big|\frac{1}{n}(\mathbf{a}_j'\mathbf{R}_j\mathbf{a}_j - \operatorname{Tr}\mathbf{R}_j)\Big|^{2p} \ll_p \frac{p^{2p}}{(nv)^p}\Big[M_4^p + M_{2p}\sigma^{2p} + M_{4p}^{(n)}(v)\Big]. \tag{5.1}$$

Proof. Using Rosenthal's inequality for quadratic forms (see the Appendix GT03), we obtain

$$\begin{aligned}
&\mathbf{E}\,\big|\mathbf{a}_j'\mathbf{R}_j\mathbf{a}_j - \operatorname{Tr}\mathbf{R}_j\big|^{2p} \\
&\ll_p \Bigg(\mathbf{E}\,\Big|\sum_{\substack{k=1\\k\ne j}}^{n}\mathbf{R}_j(k,k)(X_{jk}^2-1)\Big|^{2p} + \mathbf{E}\,\Big|\sum_{\substack{k,l=1\\k\ne l}}^{n-1}\mathbf{R}_j(k,l)X_{jk}X_{jl}\Big|^{2p}\Bigg) \\
&\ll_p p^{2p}\Bigg[M_4^p\mathbf{E}\,\Big(\sum_{k=1}^{n-1}|R_j(k,k)|^2\Big)^p + M_{2p}\sigma^{2p}\mathbf{E}\,\Bigg(\sum_{k\ne l}|R_j(k,l)|^2\Bigg)^p \\
&\qquad + M_{4p}\sum_{k=1}^{n-1}\mathbf{E}\,|R_j(k,k)|^{2p} + M_{2p}^2\sum_{k\ne l}\mathbf{E}\,|R_j(k,l)|^{2p}\Bigg].
\end{aligned} \tag{5.2}$$

Using the inequalities $|R_j(k,k)| \le v^{-1}$ and $\operatorname{Tr}|\mathbf{R}_j|^2 \le v^{-1}|\operatorname{Tr}\mathbf{R}_j|$, we get

$$\begin{aligned}
&\mathbf{E}\,\big|\mathbf{a}_j'\mathbf{R}_j\mathbf{a}_j - \operatorname{Tr}\mathbf{R}_j\big|^{2p} \\
&\ll_p p^{2p}\Bigg[(M_4^p + M_{2p}\sigma^{2p})\mathbf{E}\,\big(\operatorname{Tr}|\mathbf{R}_j|^2\big)^p + M_{4p}v^{-2p+2}\sum_{k=1}^{n-1}\mathbf{E}\,|R_j(k,k)|^2 \\
&\qquad + M_{2p}^2 v^{-2p+2}\sum_{k\ne l}\mathbf{E}\,|R_j(k,l)|^2\Bigg] \\
&\ll_p p^{2p}\Bigg[(M_4^p + M_{2p}\sigma^{2p})\mathbf{E}\,\Big(\frac{1}{v}|\operatorname{Tr}\mathbf{R}_j|\Big)^p + M_{4p}v^{-2p+2}\mathbf{E}\,\operatorname{Tr}|\mathbf{R}_j|^2 \\
&\qquad + M_{2p}^2 v^{-2p+2}\mathbf{E}\,\operatorname{Tr}|\mathbf{R}_j|^2\Bigg].
\end{aligned}$$

Furthermore, applying the inequalities $|\operatorname{Tr}\mathbf{R}_j| \le |\operatorname{Tr}\mathbf{R}| + v^{-1}$ and $|\operatorname{Tr}\mathbf{R}| \le n|\Delta_E(\mathbf{R})| +$

$n|\mathbf{E}\,\mathrm{Tr}\,\mathbf{R}|$, we obtain

$$\begin{aligned}
&\mathbf{E}\left|\mathbf{a}_j'\mathbf{R}_j\mathbf{a}_j - \mathrm{Tr}\,\mathbf{R}_j\right|^{2p}\\
&\ll_p p^{2p}\Bigg[\Big(M_4^p + M_{2p}\sigma^{2p}\Big)v^{-p}\mathbf{E}\,|\mathrm{Tr}\,\mathbf{R}|^p + \Big(M_4^p + M_{2p}\sigma^{2p}\Big)v^{-2p}\\
&\qquad\qquad + M_{4p}v^{-2p+1}|\mathbf{E}\,\mathrm{Tr}\,\mathbf{R}_j| + M_{4p}v^{-2p}\Bigg]\\
&\ll_p p^{2p}\Bigg(\Big(M_4^p + M_{2p}\sigma^{2p}\Big)v^{-p}n^p\mathbf{E}\,|\Delta_E(\mathbf{R})|^p\\
&+\Big(M_4^p + M_{2p}\sigma^{2p}\Big)v^{-p}n^p + M_{4p}v^{-2p+1}n + M_{4p}v^{-2p}\Bigg). \qquad (5.3)
\end{aligned}$$

Inequality (5.3) implies

$$\begin{aligned}
&\frac{1}{n^{2p}}\mathbf{E}\left|\mathbf{a}_j'\mathbf{R}_j\mathbf{a}_j - \mathrm{Tr}\,\mathbf{R}_j\right|^{2p}\\
&\ll_p p^{2p}\Bigg(\frac{M_4^p + M_{2p}\sigma^{2p}}{n^pv^p} + \frac{M_4^p + M_{2p}\sigma^{2p}}{(nv)^p}\mathbf{E}\,|\Delta_E(\mathbf{R})|^p + \frac{M_{4p}}{(nv)^{2p-1}}\Bigg). \qquad (5.4)
\end{aligned}$$

By Burkholder's inequality for martingales (see Hall and Heyde (1980), p. 24) we obtain

$$\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^p \ll_p \frac{p^p}{n^{\frac{p}{2}+1}}\sum_{j=1}^n \mathbf{E}\,|\gamma_j|^p,$$

where the martingale differences γ_j are defined as in the proof of Lemma 4.2 in GT03 (about the martingale differences method for the random matrices see Girko [3] as well). Since $|\gamma_j| \le \frac{2}{v}$, for $v \ge CM_4^{\frac{1}{2}}pn^{-\frac{1}{2}}$, we get

$$\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^p \ll_p \frac{p^p}{(\sqrt{n}v)^p} \le 1. \qquad (5.5)$$

Substituting (5.5) in (5.4), we obtain

$$\begin{aligned}
\frac{1}{n^{2p}}\mathbf{E}\left|\mathbf{a}_j'\mathbf{R}_j\mathbf{a}_j - \mathrm{Tr}\,\mathbf{R}_j\right|^{2p} &\ll_p \frac{p^{2p}(M_4^p + M_{2p}\sigma^{2p})}{(nv)^p} + \frac{p^{2p}M_{4p}}{(nv)^{2p-1}}\\
&\ll_p \frac{p^{2p}}{(nv)^p}\Big[M_4^p + M_{2p}\sigma^{2p} + M_{4p}^{(n)}(v)\Big]. \qquad (5.6)
\end{aligned}$$

□

Proof of Lemma 3.1. By Burkholder's inequality for martingales (see Hall and Heyde [7], p. 24) we obtain

$$\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{2p} \ll_p \frac{p^{2p}}{n^{p+1}}\sum_{j=1}^n \mathbf{E}\,|\gamma_j|^{2p}. \qquad (5.7)$$

Furthermore,

$$\mathbf{E}\,|\gamma_j|^{2p} \ll_p \Big(\mathbf{E}\,\Big|\frac{1}{n}(\mathbf{a}_j'\mathbf{R}_j^2\mathbf{a}_j - \operatorname{Tr}\mathbf{R}_j^2)\Big|^{2p} + \frac{1}{v^{2p}}\mathbf{E}\,|\varepsilon_j|^{2p}\Big), \tag{5.8}$$

where the martingale differences γ_j are defined in the proof of Lemma 4.2 in GT03.

Using definition (3.7) of ε_j and Lemma 3.3, we get

$$\begin{aligned}\mathbf{E}\,|\varepsilon_j|^{2p} &\ll_p \Big(\mathbf{E}\,\Big|\frac{1}{\sqrt{n}}X_{jj}\Big|^{2p} + \mathbf{E}\,|\mathcal{Q}_j(\mathbf{R})|^{2p} + \mathbf{E}\,|\mathcal{D}_j(\mathbf{R})|^{2p} + \mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p}\Big) \\ &\ll_p \Big(\frac{M_{2p}}{n^p} + \frac{p^{2p}}{(nv)^p}\Big[M_4^p + M_{2p}\sigma^{2p} + M_{4p}^{(n)}(v)\Big] + \frac{1}{(nv)^{2p}} + \mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p}\Big). \end{aligned} \tag{5.9}$$

Lemma 5.1 yields

$$\mathbf{E}\,\Big|\frac{1}{n}(\mathbf{a}_j'\mathbf{R}_j^2\mathbf{a}_j - \operatorname{Tr}\mathbf{R}_j^2)\Big|^{2p} \ll_p \frac{p^{2p}}{n^p v^{3p}}\Big[M_4^p + M_{4p}^{(n)}(v) + M_{2p}\sigma^{2p}\Big]. \tag{5.10}$$

Combining (5.1) – (5.10), we get

$$\begin{aligned}&\mathbf{E}\,\big|\Delta_E(\mathbf{R})\big|^{2p}\Big(1 - \frac{p^{2p}}{n^p v^{2p}}\Big) \\ &\qquad \ll_p p^{2p}\Big(\frac{M_{2p}}{(nv)^{2p}} + \frac{p^{2p}}{n^{2p}v^{3p}}\Big[M_4^p + M_{4p}^{(n)}(v) + M_{2p}\sigma^{2p}\Big] + \frac{1}{n^{3p}v^{4p}}\Big). \end{aligned} \tag{5.11}$$

The last inequality implies, for $v \ge CM_4^{\frac{1}{2}}pn^{-\frac{1}{2}}$,

$$\mathbf{E}\,\big|\Delta_E(\mathbf{R})\big|^{2p} \ll_p \frac{p^{4p}}{(n^2v^3)^p}\Big(M_4^p + M_{4p}^{(n)}(v) + M_{2p}\sigma^{2p}\Big).$$

□

Lemma 5.2. *For $v \ge CM_4^{\frac{1}{2}}pn^{-\frac{1}{2}}$, the following inequality holds*

$$\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,\big|R(j,j)\big|^{2p} \ll_p \Big(M_4^{2p} + M_{2p}\sigma^{2p} + M_{8p}^{(n)}(v)\Big).$$

Proof. We shall prove that, for $v \ge CM_4^{\frac{1}{2}}pn^{-\frac{1}{2}}$,

$$\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,\big|R(j,j)\big|^{2p} \le B\Big(\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,\big|R(j,j)\big|^{2p}\Big)^{\frac{1}{2}} + D, \tag{5.12}$$

where $B = B(n,v,p), D = D(n,v,p)$. Solving this inequality with respect to $\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,|R(j,j)|^{2p}$, we obtain, for $v \ge CM_4^{\frac{1}{2}}pn^{-\frac{1}{2}}$,

$$\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,\big|R(j,j)\big|^{2p} \le B^2 + 2D,$$

which proves Lemma 5.2.

Let us check now that inequality (5.12) holds. By equalities (3.7) and (3.6), we have

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|R(j,j)\big|^{2p} &\ll_p |a_n(z)|^{2p}\Big(1+\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\varepsilon_j|^{2p}|R(j,j)|^{2p}\Big)\\ &\ll_p \Big(1+\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\Big|\frac{1}{\sqrt{n}}X_{jj}\Big|^{2p}\big|R(j,j)\big|^{2p}+\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|\mathcal{Q}_j(\mathbf{R})\big|^{2p}\big|R(j,j)\big|^{2p}\\ &\quad+\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|\mathcal{D}_j(\mathbf{R})\big|^{2p}\big|R(j,j)\big|^{2p}+\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|\Delta_E(\mathbf{R})\big|^{2p}\big|R(j,j)\big|^{2p}\Big).\end{aligned}\tag{5.13}$$

Note that $|R(j,j)|\le v^{-1}$. It is obvious that

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\Big|\frac{1}{\sqrt{n}}X_{jj}\Big|^{2p}\big|R(j,j)\big|^{2p} &\le \frac{1}{n^{p+1}v^p}\sum_{j=1}^{n}\mathbf{E}\,|X_{jj}|^{2p}\big|R(j,j)\big|^{p}\\ &\le \frac{M_{4p}^{\frac12}}{(nv)^p}\Big(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|R(j,j)\big|^{2p}\Big)^{\frac12}.\end{aligned}\tag{5.14}$$

Using the inequality $|\operatorname{Tr}\mathbf{R}-\operatorname{Tr}\mathbf{R}_j|\le v^{-1}$, we get

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|\mathcal{D}_j(\mathbf{R})\big|^{2p}\big|R(j,j)\big|^{2p} &= \frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\Big|\frac{1}{n}(\operatorname{Tr}\mathbf{R}-\operatorname{Tr}\mathbf{R}_j)\Big|^{2p}\big|R(j,j)\big|^{2p}\\ &\le (n^{2p}\,v^{4p})^{-1}.\end{aligned}\tag{5.15}$$

Applying Lemma 3.1 and Cauchy's inequality, we obtain , for $v\ge CM_4^{\frac12}pn^{-\frac12}$,

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|\Delta_E(\mathbf{R})\big|^{2p}\big|R(j,j)\big|^{2p} &\le \frac{1}{v^p}\Big(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|\Delta_E(\mathbf{R})\big|^{4p}\Big)^{\frac12}\Big(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|R(j,j)\big|^{2p}\Big)^{\frac12}\\ &\ll_p \frac{p^{4p}}{n^{2p}v^{4p}}\Big(M_{4p}\sigma^{4p}+M_4^{2p}+M_{8p}^{(n)}(v)\Big)^{\frac12}\Big(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|R(j,j)\big|^{2p}\Big)^{\frac12}.\end{aligned}\tag{5.16}$$

Lemma 5.1 gives

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|\mathcal{Q}_j(\mathbf{R})\big|^{2p}\big|R(j,j)\big|^{2p} &\le \frac{1}{nv^p}\sum_{j=1}^{n}\mathbf{E}^{\frac12}\Big|\frac{1}{n}\big(\mathbf{a}_j'\mathbf{R}_j\mathbf{a}_j-\operatorname{Tr}\mathbf{R}_j\big)\Big|^{4p}\mathbf{E}^{\frac12}\big|R(j,j)\big|^{2p}\\ &\le \frac{1}{v^p}\Big(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\Big|\frac{1}{n}\big(\mathbf{a}_j'\mathbf{R}_j\mathbf{a}_j-\operatorname{Tr}\mathbf{R}_j\big)\Big|^{4p}\Big)^{\frac12}\Big(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|R(j,j)\big|^{2p}\Big)^{\frac12}\\ &\ll_p \frac{p^{2p}}{n^pv^{2p}}\Big[M_{4p}\sigma^{4p}+M_4^{2p}+M_{8p}^{(n)}(v)\Big]^{\frac12}\Big(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|R(j,j)\big|^{2p}\Big)^{\frac12}.\end{aligned}\tag{5.17}$$

From inequalities (5.13)–(5.17) we conclude that (5.12) holds with

$$\begin{aligned}B =&\frac{M_{4p}^{\frac12}}{(nv)^p}+\frac{p^{2p}}{n^pv^{2p}}\Big[M_4^{2p}+M_{4p}\sigma^{4p}+M_{8p}^{(n)}(v)\Big]^{\frac12},\\ D =&1+\frac{1}{n^{2p}v^{4p}}.\end{aligned}$$

This implies that, for $v\ge CM_4^{\frac12}pn^{-\frac12}$,

$$\begin{aligned}B^2+2D \le& C\Bigg(\frac{M_{4p}}{(nv)^{2p}}+\frac{p^{4p}}{n^{2p}v^{4p}}\Big[M_{4p}\sigma^{4p}+M_4^{2p}+M_{8p}^{(n)}(v)\Big]\Bigg)\\ \le& C\frac{p^{4p}}{(nv^2)^{2p}}\Big(M_{4p}\sigma^{4p}+M_4^{2p}+M_{8p}^{(n)}(v)\Big).\end{aligned}\tag{5.18}$$

The last inequality completes the proof. ☐

Proof of Lemma 3.2. According to Lemma 3.1, we have, for $v\ge CM_4^{\frac12}pn^{-\frac12}$,

$$\begin{aligned}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{4p}\ll_p&\ \frac{p^{8p}}{(n^2v^3)^{2p}}\Big(M_{4p}+M_4^{2p}+M_{8p}^{(n)}(v)\Big)\\ \ll_p&\ \frac{p^{4p}}{(nv)^{2p}}\Big(M_{4p}+M_4^{2p}+M_{8p}^{(n)}(v)\Big)\\ \ll_p&\ \frac{p^{4p}}{(nv)^{2p}}\Big(M_{4p}+M_{8p}^{(n)}(v)\Big)\\ \ll_p&\ \frac{p^{4p}}{(nv)^{2p}}\Big(M_{4p+2}+\frac{M_{8p+4}}{(nv)^{2p-1}}\Big).\end{aligned}\tag{5.19}$$

Analogously, we obtain

$$\begin{aligned}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{4p+2}\ll_p&\ \frac{p^{8p+4}}{(n^2v^3)^{2p+1}}\Big(M_{4p+2}+M_4^{2p+1}+M_{8p+4}^{(n)}(v)\Big)\\ \ll_p&\ \frac{p^{4p+2}}{(nv)^{2p+1}}\Big(M_{4p+2}+M_{8p+4}(nv)^{-2p}\Big)\\ \ll_p&\ \frac{p^{4p+2}}{(nv)^{2p+1}}\Big(M_{4p+2}+\frac{M_{8p+4}}{(nv)^{2p-1}}\Big).\end{aligned}\tag{5.20}$$

The Lemma is proved. ☐

6 Proof of main Lemma. I

Proof of Lemma 3.3. Using the representation (3.5), we get

$$\mathcal{A}_{4p-1}^{4p-1}:=\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{4p-1}=B_1+B_2+B_3+B_4+B_5,\tag{6.1}$$

where

$$
\begin{aligned}
B_1 &= -\delta_n(z)\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{4p-3}\overline{\Delta_E(\mathbf{R})},\\
B_2 &= a_n(z)\frac{1}{n\sqrt{n}}\sum_{j=1}^{n}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{4p-3}\overline{\Delta_E(\mathbf{R})}X_{jj}R(j,j),\\
B_3 &= -a_n(z)\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{4p-3}\overline{\Delta_E(\mathbf{R})}\mathcal{Q}_j(\mathbf{R})R(j,j),
\end{aligned}
\tag{6.2}
$$

$$
\begin{aligned}
B_4 &= a_n(z)\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{4p-3}\overline{\Delta_E(\mathbf{R})}\mathcal{D}_j(\mathbf{R})R(j,j),\\
B_5 &= -a_n(z)\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{4p-1}R(j,j).
\end{aligned}
\tag{6.3}
$$

At first we note that

$$B_5 = B_{51} + B_{52}, \tag{6.4}$$

with

$$B_{51} = -a_n(z)s_n(z)\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{4p-1}, \qquad B_{52} = a_n(z)\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{4p-1}\Delta_E(\mathbf{R}).$$

By Lemma 3.2, we have

$$|B_{52}| \ll_p \frac{p^{4p}}{(nv)^{2p}}\big[M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\big]. \tag{6.5}$$

Now we consider B_4. Using equality (3.5) again, we represent B_4 as

$$B_4 = -B_{41} + B_{42} - B_{43} + B_{44} - B_{45}, \tag{6.6}$$

with

$$\begin{aligned}
B_{41} &= (a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|\Delta_E(\mathbf{R})\big|^{4p-3}\overline{\Delta_E(\mathbf{R})}\mathcal{D}_j(\mathbf{R}),\\
B_{42} &= (a_n(z))^2\frac{1}{n\sqrt{n}}\sum_{j=1}^{n}\mathbf{E}\,\big|\Delta_E(\mathbf{R})\big|^{4p-3}\overline{\Delta_E(\mathbf{R})}\mathcal{D}_j(\mathbf{R})X_{jj}R(j,j),\\
B_{43} &= (a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|\Delta_E(\mathbf{R})\big|^{4p-3}\overline{\Delta_E(\mathbf{R})}\mathcal{D}_j(\mathbf{R})\mathcal{Q}_j(\mathbf{R})R(j,j),\\
B_{44} &= (a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|\Delta_E(\mathbf{R})\big|^{4p-3}\overline{\Delta_E(\mathbf{R})}\mathcal{D}_j^2(\mathbf{R})R(j,j),\\
B_{45} &= (a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|\Delta_E(\mathbf{R})\big|^{4p-1}\mathcal{D}_j(\mathbf{R})R(j,j).
\end{aligned}\tag{6.7}$$

It is easy to see that

$$|B_{41}| \le \frac{C}{nv}\mathcal{A}_{4p}^{4p-2} \le \frac{p^{4p-2}}{(nv)^{2p}}\big[M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v)\big]^{1-\frac{1}{2p}}. \tag{6.8}$$

We have the following bound for the term B_{42}

$$|B_{42}| \le \frac{C}{\sqrt{nv}}\frac{1}{nv}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\big|\Delta_E(\mathbf{R})\big|^{4p-2}|X_{jj}|. \tag{6.9}$$

Using the equality $\Delta_E(\mathbf{R}) = \Delta_E^{(j)}(\mathbf{R}) + \mathcal{D}_j(\mathbf{R}) - \mathbf{E}\,\mathcal{D}_j(\mathbf{R})$ and the bound $|\Delta_E^{(j)}(\mathbf{R}) + \mathcal{D}_j(\mathbf{R}) - \mathbf{E}\,\mathcal{D}_j(\mathbf{R})| \le |\Delta_E^{(j)}(\mathbf{R})| + \frac{2}{nv}$, we get

$$|B_{42}| \le \frac{C}{\sqrt{nv}}\frac{1}{nv}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-2}C_{4p-2}^{k}\mathbf{E}\,|\Delta_E^{(j)}(\mathbf{R})|^{4p-2-k}\mathbf{E}\,|X_{jj}|\left(\frac{2}{nv}\right)^{k}. \tag{6.10}$$

Note that for $1 \le q \le 4p-1$

$$\mathbf{E}\,|\Delta_E^{(j)}(\mathbf{R})|^{q} \le \sum_{k=0}^{q}C_q^k\mathbf{E}\,|\Delta_E(\mathbf{R})|^{q-k}\left(\frac{2}{nv}\right)^{k} \le \left(\mathcal{A}_{4p-1}+\frac{2}{nv}\right)^{q}.$$

The two last inequalities together imply that for $v \ge CM_4^{\frac{1}{2}}pn^{-\frac{1}{2}}$

$$|B_{42}| \le \frac{C}{nv}\left(\mathcal{A}_{4p-1}+\frac{4}{nv}\right)^{4p-2} \le \frac{C}{nv}\left(\mathcal{A}_{4p}+\frac{4}{nv}\right)^{4p-2}. \tag{6.11}$$

Using the result of Lemma 3.2, we obtain

$$|B_{42}| \le \frac{p^{4p}}{(nv)^{2p}}\left[M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v)\right]^{1-\frac{1}{2p}}. \tag{6.12}$$

Furthermore, we consider B_{43}.

$$|B_{43}| \le \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-2}C_{4p-2}^{k}\left(\frac{2}{nv}\right)^{k}\mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-2-k}|\mathcal{Q}_j(\mathbf{R})||R(j,j)|. \tag{6.13}$$

Conditioning with respect to σ-algebra F_j and applying Hölder's inequality, we get, for $1 \le q \le 4p-2$,

$$\begin{aligned}
\mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{q}|\mathcal{Q}_j(\mathbf{R})||R(j,j)| &\le \mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{q}\mathbf{E}^{\frac{1}{2}}\left(|\mathcal{Q}_j(\mathbf{R})|^2\big|F_j\right)\mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|^2\big|F_j\right)\\
&\le \frac{1}{\sqrt{nv}}\mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{q}\left(|s_{nj}|+\left|\Delta_E^{(j)}(\mathbf{R})\right|\right)^{\frac{1}{2}}\mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|\big|F_j\right)\\
&\le \frac{C}{\sqrt{nv}}\left(\mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-1}\right)^{\frac{q}{4p-1}}\left(\mathbf{E}\,|R(j,j)|^{2p}\right)^{\frac{1}{4p}}\\
&+\frac{C}{\sqrt{nv}}\left(\mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-1}\right)^{\frac{q+\frac{1}{2}}{4p-1}}\left(\mathbf{E}\,|R(j,j)|^{2p}\right)^{\frac{1}{4p}}.
\end{aligned} \tag{6.14}$$

Analogously to inequality (6.10) we obtain

$$\begin{aligned}
|B_{43}| &\le \frac{C}{nv}\frac{1}{\sqrt{nv}}(\mathcal{A}_{4p-1}+\frac{4}{nv})^{4p-2}(M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v))^{\frac{1}{4p}}\\
&+\frac{C}{nv}\frac{1}{\sqrt{nv}}(\mathcal{A}_{4p-1}+\frac{4}{nv})^{4p-\frac{3}{2}}(M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v))^{\frac{1}{4p}}\\
&\le \frac{C}{nv}\frac{1}{\sqrt{nv}}(\mathcal{A}_{4p}+\frac{4}{nv})^{4p-2}(M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v))^{\frac{1}{4p}}\\
&+\frac{C}{nv}\frac{1}{\sqrt{nv}}(\mathcal{A}_{4p}+\frac{4}{nv})^{4p-\frac{3}{2}}(M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v))^{\frac{1}{4p}}.
\end{aligned} \tag{6.15}$$

Using Lemma 3.2 again, we get

$$|B_{43}| \le \frac{p^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v)\right). \tag{6.16}$$

Furthermore, we have the obvious bound for the term B_{44}

$$|B_{44}| \le \frac{C}{n^2v^3}\mathcal{A}_{4p}^{4p-2} \le \frac{C}{nv}\mathcal{A}_{4p}^{4p-2} \le \frac{p^{4p-2}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v)\right). \tag{6.17}$$

Consider now the term B_{45}. We may write

$$\begin{aligned}|B_{45}| &\le \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{4p-1}|R_{j,j}| \\ &\le \frac{C}{nv}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{4p-1}\left(\frac{1}{n}\sum_{j=1}^{n}|R_{j,j}|^2\right)^{\frac{1}{2}} \\ &\le \frac{C}{nv\sqrt{v}}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{4p-1}\left|\frac{1}{n}\operatorname{Tr}\mathbf{R}\right|^{\frac{1}{2}} \\ &\le \frac{C}{nv\sqrt{v}}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{4p-1}\left(|s_n(z)|+|\Delta(\mathbf{R})|\right)^{\frac{1}{2}} \\ &\le \frac{C}{nv\sqrt{v}}\mathcal{A}_{4p-1}^{4p-1}+\frac{C}{nv\sqrt{v}}\mathcal{A}_{4p}^{4p-\frac{1}{2}}. \end{aligned} \tag{6.18}$$

Using Lemma 3.2, we get

$$\begin{aligned}|B_{45}| &\le \frac{C}{nv\sqrt{v}}\mathcal{A}_{4p-1}^{4p-1}+\frac{Cp^{4p}}{(nv)^{2p}}\left(M_4^{2p+1}+\widetilde{M}_{8p+4}^{(n)}(v)\right) \\ &\le \frac{C}{\sqrt{nv}}\mathcal{A}_{4p}^{4p-1}+\frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right) \\ &\le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right). \end{aligned} \tag{6.19}$$

Inequalities (6.8), (6.12), (6.16), (6.17), and (6.19) together imply

$$|B_4| \le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}_{8p+4}(v)\right). \tag{6.20}$$

Now we bound the term B_3. Using the representation (3.5) again, we obtain

$$B_3 = B_{31}+B_{32}+B_{33}+B_{34}+B_{35}, \tag{6.21}$$

where

$$\begin{aligned} B_{31} &= (a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\ |\Delta_E(\mathbf{R})|^{4p-3}\,\overline{\Delta_E(\mathbf{R})}\mathcal{Q}_j(\mathbf{R}), \\ B_{32} &= -(a_n(z))^2\frac{1}{n\sqrt{n}}\sum_{j=1}^{n}\mathbf{E}\ |\Delta_E(\mathbf{R})|^{4p-3}\,\overline{\Delta_E(\mathbf{R})}\mathcal{Q}_j(\mathbf{R})X_{jj}R(j,j), \\ B_{33} &= (a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\ |\Delta_E(\mathbf{R})|^{4p-3}\,\overline{\Delta_E(\mathbf{R})}\mathcal{Q}_j(\mathbf{R})^2R(j,j), \end{aligned}$$

$$B_{34} = -(a_n(z))^2 \frac{1}{n} \sum_{j=1}^{n} \mathbf{E}\, |\Delta_E(\mathbf{R})|^{4p-3}\, \overline{\Delta_E(\mathbf{R})} \mathcal{Q}_j(\mathbf{R}) \mathcal{D}_j(\mathbf{R}) R(j,j),$$

$$B_{35} = (a_n(z))^2 \frac{1}{n} \sum_{j=1}^{n} \mathbf{E}\, |\Delta_E(\mathbf{R})|^{4p-1}\, \mathcal{Q}_j(\mathbf{R}) R(j,j).$$

First note that $B_{34} = -B_{43}$. The bound for this term is given in (6.16). The term B_{35} is decomposed using again (3.5).

$$B_{35} = B_{351} + B_{352} + B_{353} + B_{354} + B_{355}, \tag{6.22}$$

where

$$\begin{aligned}
B_{351} &= -(a_n(z))^2 \frac{1}{n} \sum_{j=1}^{n} \mathbf{E}\, |\Delta_E(\mathbf{R})|^{4p-1}\, \mathcal{Q}_j(\mathbf{R}), \\
B_{352} &= (a_n(z))^2 \frac{1}{n\sqrt{n}} \sum_{j=1}^{n} \mathbf{E}\, |\Delta_E(\mathbf{R})|^{4p-1}\, \mathcal{Q}_j(\mathbf{R}) X_{jj} R(j,j), \\
B_{353} &= -(a_n(z))^2 \frac{1}{n} \sum_{j=1}^{n} \mathbf{E}\, |\Delta_E(\mathbf{R})|^{4p-1}\, \mathcal{Q}_j(\mathbf{R})^2 R(j,j), \\
B_{354} &= (a_n(z))^2 \frac{1}{n} \sum_{j=1}^{n} \mathbf{E}\, |\Delta_E(\mathbf{R})|^{4p-1}\, \mathcal{Q}_j(\mathbf{R}) \mathcal{D}_j(\mathbf{R}) R(j,j), \\
B_{355} &= -(a_n(z))^2 \frac{1}{n} \sum_{j=1}^{n} \mathbf{E}\, |\Delta_E(\mathbf{R})|^{4p-1}\, \Delta_E(\mathbf{R}) \mathcal{Q}_j(\mathbf{R}) R(j,j).
\end{aligned} \tag{6.23}$$

Conditioning given F_j and using Hölder's inequality, we get

$$\begin{aligned}
|B_{355}| \le \frac{C}{n} \sum_{j=1}^{n} \sum_{k=0}^{4p} C_{4p}^{k} \left(\frac{2}{nv}\right)^{k} \mathbf{E}\, \left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-k} \mathbf{E}^{\frac{1}{2}} \left(|\mathcal{Q}_j(\mathbf{R})|^2 \big| F_j\right) \\
\times \mathbf{E}^{\frac{1}{2}} \left(|R(j,j)|^2 \big| F_j\right).
\end{aligned}$$

Hence, we arrived

$$\begin{aligned}
|B_{355}| \le \frac{C}{n\sqrt{nv}} \sum_{j=1}^{n} \sum_{k=0}^{4p} C_{4p}^{k} \left(\frac{2}{nv}\right)^{k} \mathbf{E}\, \left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-k} \left|\frac{1}{n} \operatorname{Tr} \mathbf{R}_j\right|^{\frac{1}{2}} \\
\times \mathbf{E}^{\frac{1}{2}} \left(|R(j,j)|^2 \big| F_j\right).
\end{aligned}$$

Using $|\frac{1}{n}|\operatorname{Tr}\mathbf{R}_j| \le s_{nj}(z) + |\Delta_E^{(j)}(\mathbf{R})|$, we get

$$\begin{aligned}
|B_{355}| &\le \frac{C}{n\sqrt{nv}} \sum_{j=1}^{n}\sum_{k=0}^{4p} C_{4p}^{k} \left(\frac{2}{nv}\right)^{k} \mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-k} \left(|s_{nj}(z)| + \left|\Delta_E^{(j)}(\mathbf{R})\right|\right)^{\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|^2\middle|F_j\right) \\
&\le \frac{C}{n\sqrt{nv}} \sum_{j=1}^{n}\sum_{k=0}^{4p} C_{4p}^{k} \left(\frac{2}{nv}\right)^{k} \mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-k} \left(|s_{nj}(z)| + \left|\Delta_E^{(j)}(\mathbf{R})\right|\right)^{\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|\middle|F_j\right).
\end{aligned} \tag{6.24}$$

Applying Hölder's inequality again, we get

$$\begin{aligned}
|B_{355}| &\le \frac{C}{n\sqrt{nv}} \sum_{j=1}^{n}\sum_{k=0}^{4p} C_{4p}^{k} \left(\frac{2}{nv}\right)^{k} \mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-k} \left(|s_{nj}(z)| + \left|\Delta_E^{(j)}(\mathbf{R})\right|\right)^{\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|\middle|F_j\right) \\
&\le \frac{C}{\sqrt{nv}} \sum_{k=0}^{4p} C_{4p}^{k} \left(\frac{2}{nv}\right)^{k} \left(\frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p+1}\right)^{\frac{4p-k}{4p+1}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \left(\frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\,|R(j,j)|^{2p+1}\right)^{\frac{1}{4p+2}} \\
&\le C\left(\mathcal{A}_{4p+2} + \frac{4}{nv}\right)^{4p} \left(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\right)^{\frac{1}{4p+2}} \\
&\quad + C\left(\mathcal{A}_{4p+2} + \frac{4}{nv}\right)^{4p+\frac{1}{2}} \left(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\right)^{\frac{1}{4p+2}}.
\end{aligned} \tag{6.25}$$

The last inequality and Lemma 3.2 together imply

$$|B_{355}| \le \frac{Cp^{4p}}{(nv)^{2p}} \left(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\right). \tag{6.26}$$

Furthermore, note that

$$\begin{aligned}
|B_{354}| &\le \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-1} C_{4p-1}^{k} \left(\frac{2}{nv}\right)^{k} \mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-1-k} |\mathcal{Q}_j(\mathbf{R})|\,|R(j,j)| \\
&\le \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-1} C_{4p-1}^{k} \left(\frac{2}{nv}\right)^{k} \mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-1-k} \mathbf{E}^{\frac{1}{2}}\left(|\mathcal{Q}_j(\mathbf{R})|^2\middle|F_j\right) \mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|^2\middle|F_j\right).
\end{aligned}$$

Since

$$\mathbf{E}\,\left(|\mathcal{Q}_j(\mathbf{R})|^2\big|F_j\right) \le \frac{1}{n}\mathrm{Tr}\,|\mathbf{R}_j|^2 \le |s_{nj}(z)| + \left|\Delta_E^{(j)}(\mathbf{R})\right|, \tag{6.27}$$

we get

$$\begin{aligned}
|B_{354}| &\le \frac{C}{nv}\frac{1}{\sqrt{nv}}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-1} C_{4p-1}^{k}\left(\frac{2}{nv}\right)^{k}\mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-1-k}\left(|s_{nj}(z)| + \left|\Delta_E^{(j)}(\mathbf{R})\right|\right)^{\frac{1}{2}} \\
&\qquad\qquad \times \mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|\big|F_j\right) \\
&\le \frac{C}{nv}\left(\mathcal{A}_{4p+2} + \frac{4}{nv}\right)^{4p-1}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+1}\right)^{\frac{1}{4p+2}} \\
&+ \frac{C}{nv}\left(\mathcal{A}_{4p+2} + \frac{4}{nv}\right)^{4p-\frac{1}{2}}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+1}\right)^{\frac{1}{4p+2}}.
\end{aligned} \tag{6.28}$$

Applying Lemma 3.2, we obtain

$$|B_{354}| \le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\right). \tag{6.29}$$

Using that $|R(j,j)| \le v^{-1}$, we get

$$\begin{aligned}
|B_{352}| &\le \frac{C}{\sqrt{nv}}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{4p-1}\,|\mathcal{Q}_j(\mathbf{R})|\,|X_{jj}| \\
&\le \frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-1} C_{4p-1}^{k}\left(\frac{2}{nv}\right)^{k}\mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-1-k}|\mathcal{Q}_j(\mathbf{R})|.
\end{aligned}$$

Conditioning and applying (6.27), we conclude that

$$\begin{aligned}
|B_{352}| &\le \frac{1}{\sqrt{nv}}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-1} C_{4p-1}^{k}\left(\frac{2}{nv}\right)^{k}\mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-1}\left(|s_{nj}(z)| + \left|\Delta_E^{(j)}(\mathbf{R})\right|\right)^{\frac{1}{2}} \\
&\le \frac{C}{\sqrt{nv}}\left(\mathcal{A}_{4p} + \frac{4}{nv}\right)^{4p-1} + \frac{C}{\sqrt{nv}}\left(\mathcal{A}_{4p} + \frac{4}{nv}\right)^{4p-\frac{1}{2}}.
\end{aligned} \tag{6.30}$$

Analogously to inequality (6.29) we obtain

$$|B_{352}| \le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\right). \tag{6.31}$$

We investigate now the behavior of the term B_{351}. Since

$$|B_{351}| \le \frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{4p-1}\,|\mathcal{Q}_j(\mathbf{R})|, \tag{6.32}$$

we obtain the same bound as above

$$|B_{351}| \le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v)\right). \tag{6.33}$$

Finally, we bound B_{353}. Conditioning given F_j , we get

$$\begin{aligned}|B_{353}| &\le \frac{C}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-1}C^k_{4p-1}\left(\frac{2}{nv}\right)^k\mathbf{E}\,\left|\Delta^{(j)}_E(\mathbf{R})\right|^{4p-1-k}\mathbf{E}^{\frac12}\left(|\mathcal{Q}_j(\mathbf{R})|^4\middle|F_j\right)\\ &\qquad\times\mathbf{E}^{\frac12}\left(|R(j,j)|^2\middle|F_j\right)\\ &\le \frac{C}{nv\sqrt{v}}\frac{C}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-1}\left(\frac{2}{nv}\right)^k\mathbf{E}\,\left|\Delta^{(j)}_E(\mathbf{R})\right|^{4p-1-k}\left(|s_{nj}|+\left|\frac{1}{n}\mathrm{Tr}\,\mathbf{R}_j\right|\right)\\ &\qquad\times\mathbf{E}^{\frac12}\left(|R(j,j)|\middle|F_j\right)\\ &\le \frac{C}{\sqrt{nv}}\left(\mathcal{A}_{4p+2}+\frac{4}{nv}\right)^{4p-1}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+1}\right)^{\frac{1}{4p+2}}\\ &+\frac{C}{\sqrt{nv}}\left(\mathcal{A}_{4p+2}+\frac{4}{nv}\right)^{4p}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+1}\right)^{\frac{1}{4p+2}}.\end{aligned} \tag{6.34}$$

The last inequality yields

$$|B_{353}| \le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v)\right). \tag{6.35}$$

Inequalities (6.26), (6.29), (6.31), (6.33), (6.35) together imply

$$|B_{35}| \le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v)\right). \tag{6.36}$$

To bound B_{34} we note that

$$\begin{aligned}|B_{34}| &\le \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{4p-2}\,|\mathcal{Q}_j(\mathbf{R})|\,|R(j,j)|\\ &\le \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-2}C^k_{4p-2}\left(\frac{2}{nv}\right)^k\mathbf{E}\,\left|\Delta^{(j)}_E(\mathbf{R})\right|^{4p-2-k}\mathbf{E}^{\frac12}\left(|\mathcal{Q}_j(\mathbf{R})|^2\middle|F_j\right)\\ &\qquad\times\mathbf{E}^{\frac12}\left(|R(j,j)|^2\middle|\right).\end{aligned}$$

Using inequality $|R(j,j)|\le v^{-1}$ and (6.27), we get

$$\begin{aligned}|B_{34}| &\le \frac{C}{nv}\frac{1}{\sqrt{nv}}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-2}C_{4p-2}^{k}\left(\frac{2}{nv}\right)^{k}\mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-2-k}\\ &\qquad\qquad\qquad\times\left(|s_{nj}(z)|+\left|\Delta_E^{(j)}(\mathbf{R})\right|\right)^{\frac{1}{2}}\mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|\right)\\ &\le \frac{C}{nv}\left(\mathcal{A}_{4p}+\frac{4}{nv}\right)^{4p-2}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right)^{\frac{1}{4p+2}}\\ &\le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right).\end{aligned}\tag{6.37}$$

To bound B_{33} we use the representation (3.5) again. We may write

$$B_{33}=B_{331}+B_{332}+B_{333}+B_{334}+B_{335},\tag{6.38}$$

where

$$\begin{aligned}B_{331}&=-(a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{4p-3}\,\overline{\Delta_E(\mathbf{R})}\mathcal{Q}_j^2(\mathbf{R}),\\ B_{332}&=(a_n(z))^2\frac{1}{n\sqrt{n}}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{4p-3}\,\overline{\Delta_E(\mathbf{R})}\mathcal{Q}_j^2(\mathbf{R})X_{jj}R(j,j),\\ B_{333}&=-(a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{4p-3}\,\overline{\Delta_E(\mathbf{R})}\mathcal{Q}_j^3(\mathbf{R})R(j,j),\\ B_{334}&=(a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{4p-3}\,\overline{\Delta_E(\mathbf{R})}\mathcal{Q}_j^2(\mathbf{R})\mathcal{D}_j(\mathbf{R})R(j,j),\\ B_{335}&=-(a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{4p-1}\,\mathcal{Q}_j^2(\mathbf{R})R(j,j).\end{aligned}\tag{6.39}$$

Note that

$$\begin{aligned}|B_{332}| &\le \frac{C}{\sqrt{nv}}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-2}C_{4p-2}^{k}\left(\frac{2}{nv}\right)^{k}\mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-2-k}\mathbf{E}\left(|\mathcal{Q}_j(\mathbf{R})|^2\,\middle|F_j\right)\mathbf{E}\,|X_{jj}|\\ &\le \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-2}C_{4p-2}^{k}\left(\frac{2}{nv}\right)^{k}\mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-2-k}\left(|s_{nj}(z)|+\left|\Delta_E^{(j)}(\mathbf{R})\right|\right).\end{aligned}\tag{6.40}$$

Analogously

$$|B_{331}| \le \frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-2} C_{4p-2}^{k}\left(\frac{2}{nv}\right)^{k}\mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-2-k}\mathbf{E}\left(|\mathcal{Q}_j(\mathbf{R})|^2\left|F_j\right.\right)$$
$$\le \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-2} C_{4p-2}^{k}\left(\frac{2}{nv}\right)^{k}\mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-2-k}\left(|s_{nj}(z)|+\left|\Delta_E^{(j)}(\mathbf{R})\right|\right). \tag{6.41}$$

In addition,

$$|B_{334}| \le \frac{C}{nv^2}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{4p-2}\,|\mathcal{Q}_j(\mathbf{R})|^2. \tag{6.42}$$

Inequalities (6.40), (6.41) and (6.42) together imply

$$\max\{|B_{331}|,\ |B_{332}|\ |B_{334}|\} \le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right). \tag{6.43}$$

Consider now B_{333}. We have

$$|B_{333}| \le \frac{C}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-2} C_{4p-2}^{k}\left(\frac{2}{nv}\right)^{k}\mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-2-k}\mathbf{E}^{\frac{1}{2}}\left(|\mathcal{Q}_j(\mathbf{R})|^6\big|F_j\right)$$
$$\times\,\mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|^2\big|F_j\right).$$

This inequality and inequality (6.27) together imply

$$|B_{333}| \le \frac{C}{(nv)^{\frac{3}{2}}}\frac{1}{\sqrt{v}}\sum_{j=1}^{n}\sum_{k=0}^{4p-2} C_{4p-2}^{k}\left(\frac{2}{nv}\right)^{k}\mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-2-k}$$
$$\times\left(|s_{nj}(z)|+\left|\Delta_E^{(j)}(\mathbf{R})\right|\right)^{\frac{3}{2}}\mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|\big|F_j\right).$$

Applying Hölder inequality, Lemma 3.2 and the binomial formula, we obtain

$$|B_{333}| \le \frac{C}{(nv)}(\mathcal{A}_{4p}+\frac{4}{nv})^{4p-2}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right)^{\frac{1}{4p}}$$
$$+\frac{C}{(nv)}(\mathcal{A}_{4p+1}+\frac{4}{nv})^{4p-\frac{1}{2}}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right)^{\frac{1}{4p+2}}$$
$$\le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right). \tag{6.44}$$

Furthermore, consider B_{335}. We have

$$|B_{335}| \le \frac{C}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-1} C_{4p-1}^{k}\left(\frac{1}{nv}\right)^{k}\mathbf{E}\left|\Delta_{E}^{(j)}(\mathbf{R})\right|^{4p-1-k}\mathbf{E}^{\frac{1}{2}}\left(|\mathcal{Q}_j(\mathbf{R})|^4\left|F_j\right)\right. \\ \times\mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|^2\left|F_j\right)\right..$$

Rosenthal's inequality for quadratic forms (see for example Appendix of GT03) implies

$$\mathbf{E}\left(|\mathcal{Q}_j(\mathbf{R})|^4\left|F_j\right)\right. \le C\big(\operatorname{Tr}|\mathbf{R}_j|^2\big)^2 \le Cv^{-2}\big|\operatorname{Tr}\mathbf{R}_j\big|^2. \tag{6.45}$$

The last two inequalities together imply

$$\begin{aligned}|B_{335}| &\le \frac{C}{nv\sqrt{v}}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-1} C_{4p-1}^{k}\left(\frac{1}{nv}\right)^{k}\mathbf{E}\left|\Delta_{E}^{(j)}(\mathbf{R})\right|^{4p-1-k}\left(|s_{nj}(z)|+\left|\Delta_{E}^{(j)}(\mathbf{R})\right|\right) \\ &\qquad\times\mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|\left|F_j\right)\right. \\ &\le \frac{C}{\sqrt{nv}}\left(\mathcal{A}_{4p+2}+\frac{4}{nv}\right)^{4p-1}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right)^{\frac{1}{4p+2}} \\ &+\frac{C}{\sqrt{nv}}\left(\mathcal{A}_{4p+2}+\frac{4}{nv}\right)^{4p}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right)^{\frac{1}{4p+2}} \\ &\le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right).\end{aligned} \tag{6.46}$$

Inequalities (6.43), (6.44) and (6.46) together imply

$$|B_{33}| \le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right). \tag{6.47}$$

Now we bound B_{32}. Conditioning and applying Hölder's inequality, we get

$$|B_{32}| \le \frac{C}{\sqrt{nv}}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-2} C_{4p-2}^{k}\left(\frac{2}{nv}\right)^{k}\mathbf{E}\left|\Delta_{E}^{(j)}(\mathbf{R})\right|^{4p-2-k}\mathbf{E}^{\frac{1}{2}}\left(|\mathcal{Q}_j(\mathbf{R})|^2\middle|F_j\right) \\ \times\mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|\middle|F_j\right).$$

Using inequality (6.27), Hölder's inequality and the binomial formula, we conclude that

$$\begin{aligned}|B_{32}| &\le \frac{C}{nv}\left(\left(\mathcal{A}_{4p}+\frac{4}{nv}\right)^{4p-2}+\left(\mathcal{A}_{4p}+\frac{4}{nv}\right)^{4p-\frac{3}{2}}\right)\left(M_{4p+2}+\widetilde{M}_{8p+4}\right)^{\frac{1}{4p+1}} \\ &\le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right).\end{aligned} \tag{6.48}$$

Similarly, we obtain a bound for B_{31}. The equality $a^k - b^k = (a-b)\sum_{\nu=0}^{k-1} a^\nu b^{k-\nu}$ yields

$$|B_{31}| \le \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\,\left|(\Delta_E(\mathbf{R}))^{4p-2} - (\Delta_E^{(j)}(\mathbf{R}))^{4p-2}\right| |\mathcal{Q}_j(\mathbf{R})|$$
$$\le \frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-3} \mathbf{E}\,\left|\Delta_E(\mathbf{R}) - \Delta_E^{(j)}(\mathbf{R})\right| |\Delta(\mathbf{R})|^{4p-3-k} \left|\Delta_E^{(j)}(\mathbf{R})\right|^k |\mathcal{Q}_j(\mathbf{R})|.$$

Furthermore, the inequality $a^\nu b^{k-\nu} \le a^k + b^{k-\nu}$ implies

$$|B_{31}| \le \frac{Cp}{nv}\frac{1}{n}\sum_{j=1}^{n}\left(\mathbf{E}\,|\Delta(\mathbf{R})|^{4p-3}|\mathcal{Q}_j(\mathbf{R})| + \mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-3}|\mathcal{Q}_j(\mathbf{R})|\right).$$

Applying the inequality $|\Delta_E(\mathbf{R})| \le |\Delta_E^{(j)}| + \frac{1}{nv}$ and the binomial formula, we get

$$|B_{31}| \le \frac{Cp}{nv}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-3} C_{4p-3}^k \left(\frac{1}{nv}\right)^k \mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-3-k}|\mathcal{Q}_j(\mathbf{R})|.$$

Conditioning and using (6.27) and Hölder's inequality, we arrive

$$|B_{31}| \le \frac{Cp}{(nv)^{\frac{3}{2}}}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-3} C_{4p-3}^k \left(\frac{1}{nv}\right)^k \mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-3-k}\left(|s_{nj}(z)| + |\Delta_E^{(j)}(\mathbf{R})|\right)^{\frac{1}{2}}$$
$$\le \frac{Cp}{(nv)^{\frac{3}{2}}}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-3} C_{4p-3}^k \left(\frac{1}{nv}\right)^k \left(\mathcal{A}_{4p} + \frac{2}{nv}\right)^{4p-3-k}$$
$$+ \frac{Cp}{(nv)^{\frac{3}{2}}}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{4p-3} C_{4p-3}^k \left(\frac{1}{nv}\right)^k \left(\mathcal{A}_{4p} + \frac{2}{nv}\right)^{4p-\frac{5}{2}-k}$$
$$\le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_8^{2p+1} + \widetilde{M}_{8p+4}^{(n)}(v)\right). \tag{6.49}$$

Collecting inequalities (6.36), (6.37), (6.47), (6.48) and (6.49), we get

$$|B_3| \le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_8 + \widetilde{M}_{8p+4}^{(n)}(v)\right). \tag{6.50}$$

Now we bound B_2. Using the equality (3.5), we represent B_2 in the form

$$B_2 = B_{21} + B_{22} + B_{23} + B_{24} + B_{25}, \tag{6.51}$$

where

$$B_{21} = -\frac{a_n^2(z)}{n\sqrt{n}}\sum_{j=1}^{n}\mathbf{E}\,|\Delta(\mathbf{R})|^{4p-3}\overline{\Delta(\mathbf{R})}X_{jj},$$

$$B_{22} = \frac{a_n^2(z)}{n^2}\sum_{j=1}^{n}\mathbf{E}\,|\Delta(\mathbf{R})|^{4p-3}\overline{\Delta(\mathbf{R})}X_{jj}^2 R(j,j),$$

$$\begin{aligned}B_{23} &= -\frac{a_n^2(z)}{n\sqrt{n}}\sum_{j=1}^n \mathbf{E}\,|\Delta(\mathbf{R})|^{4p-3}\overline{\Delta(\mathbf{R})}X_{jj}\mathcal{Q}_j(\mathbf{R})R(j,j),\\
B_{24} &= \frac{a_n^2(z)}{n\sqrt{n}}\sum_{j=1}^n \mathbf{E}\,|\Delta(\mathbf{R})|^{4p-3}\overline{\Delta(\mathbf{R})}X_{jj}\mathcal{D}_j(\mathbf{R})R(j,j),\\
B_{25} &= -\frac{a_n^2(z)}{n\sqrt{n}}\sum_{j=1}^n \mathbf{E}\,|\Delta(\mathbf{R})|^{4p-1}X_{jj}R(j,j).\end{aligned}$$

It is easy to see that

$$\begin{aligned}|B_{21}| &\le \frac{C}{n}\mathbf{E}\,|\Delta(\mathbf{R})|^{4p-2}|\frac{1}{\sqrt{n}}\sum_{j=1}^n X_{jj}| \le \frac{C}{n}\mathcal{A}_{4p}^{4p-2}\mathbf{E}^{\frac{1}{2p}}\left|\frac{1}{\sqrt{n}}\sum_{j=1}^n X_{jj}\right|^{2p}\\
&\le \frac{Cp}{n}\mathcal{A}_{4p}^{4p-2}\left(1+n^{-p+1}M_{2p}\right)^{\frac{1}{2p}} \le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right).\end{aligned} \tag{6.52}$$

For the term B_{22} we set the following bound

$$|B_{22}| \le \frac{C^p}{nv}(\mathcal{A}_{4p}+\frac{4}{nv})^{4p-2}+\frac{C^p}{(nv)^{4p-1}} \le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_8+\widetilde{M}_{8p+4}^{(n)}(v)\right). \tag{6.53}$$

Furthermore, $B_{23}=B_{32}$ and thus inequality (6.48) implies

$$|B_{23}| \le \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right). \tag{6.54}$$

Analogously, $B_{24}=B_{42}$ and from (6.11) we have

$$|B_{24}| \ll_p \frac{Cp^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right). \tag{6.55}$$

Using the inequalities $|\Delta_E(\mathbf{R})| \le |\Delta_E^{(j)}(\mathbf{R})|+\frac{2}{nv}$ and $|R(j,j)| \le v^{-1}$, we get

$$\begin{aligned}|B_{25}| &\le \frac{C^p}{\sqrt{nv}}\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,|\Delta_E^{(j)}(\mathbf{R})|^{4p-1}\mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|\Big|F_j\right)+\frac{C^p}{(nv)^{4p-1}}\\
&\le \frac{C^p}{\sqrt{nv}}(\mathcal{A}_{4p}+\frac{4}{nv})^{4p-1}\left(\frac{1}{n}\sum_{jj=1}^n \mathbf{E}\,|R(j,j)|^{2p}\right)^{\frac{1}{2p}}+\frac{C^p}{(nv)^{4p-1}}\\
&\ll_p \frac{p^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right).\end{aligned} \tag{6.56}$$

Inequalities (6.52)–(6.56) together imply

$$|B_2| \ll_p \frac{p^{4p}}{(nv)^{2p}}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right). \tag{6.57}$$

The bound for the term B_1 follows from the bound for $\delta_n(z)$ in GT03, inequality (5.65). We have

$$|B_1| \le \frac{C}{nv}\mathcal{A}_{4p}^{4p-2} \ll_p \frac{p^{4p}}{(nv)^{2p}}\Big(M_{4p+2}+\widetilde{M}_{8p+4}\Big). \tag{6.58}$$

Finally, inequalities (6.58), (6.57), (6.50), (6.20), and (6.5) together imply

$$\mathcal{A}_{4p-1}^{4p-1} \ll_p \frac{p^{4p}}{(nv)^{2p}|z+2s(z)|}\Big(M_{4p+2}+\widetilde{M}_{8p+4}\Big). \tag{6.59}$$

□

7 Proof of the main Lemma. II

Proof of Lemma 3.4. Analogously to the proof of Lemma 3.3 we may write

$$\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-2} = \mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3+\mathcal{B}_4+\mathcal{B}_5, \tag{7.1}$$

where

$$\begin{aligned}
\mathcal{B}_1 &= -\delta_n(z)\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-4}\overline{\Delta(\mathbf{R})},\\
\mathcal{B}_2 &= a_n(z)\frac{1}{n\sqrt{n}}\sum_{j=1}^{n}\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-4}\overline{\Delta(\mathbf{R})}X_{jj}R(j,j),\\
\mathcal{B}_3 &= -a_n(z)\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-4}\overline{\Delta(\mathbf{R})}\mathcal{Q}_j(\mathbf{R})R(j,j),\\
\mathcal{B}_4 &= a_n(z)\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-4}\overline{\Delta(\mathbf{R})}\mathcal{D}_j(\mathbf{R})R(j,j),\\
\mathcal{B}_5 &= -a_n(z)\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-2}R(j,j).
\end{aligned}$$

At first we consider $\mathcal{B}_5$. We may write

$$\mathcal{B}_5 = \mathcal{B}_{51}+\mathcal{B}_{52},$$

where

$$\mathcal{B}_{51} = -a_n(z)s_n(z)\mathbf{E}\,\big|\Delta_E(\mathbf{R})\big|^{2p+2\beta-2}, \qquad \mathcal{B}_{52} = a_n(z)\mathbf{E}\,\big|\Delta_E(\mathbf{R})\big|^{2p+2\beta-2}\Delta_E(\mathbf{R}).$$

By the assumptions of Lemma 3.3, we have

$$|\mathcal{B}_{52}| \ll_p \frac{p^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\big[M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\big]. \tag{7.2}$$

Using the inequality $|z+2s_n(z)| \geq C|z+2s(z)|$ and solving equation (7.1) with respect to $\mathcal{A}_{2p+2\beta-2}^{2p+2\beta-2}$, we obtain

$$\begin{aligned}\mathcal{A}_{2p+2\beta-2}^{2p+2\beta-2} \ll_p & \frac{p^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+2}}\left[M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right] \\ & +\frac{C}{|z+2s(z)|}(\mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3+\mathcal{B}_4). \end{aligned} \tag{7.3}$$

Now we consider $\mathcal{B}_4$. Using the equality (3.5) again, we represent $\mathcal{B}_4$ as

$$\mathcal{B}_4 = -\mathcal{B}_{41}+\mathcal{B}_{42}-\mathcal{B}_{43}+\mathcal{B}_{44}-\mathcal{B}_{45}, \tag{7.4}$$

where

$$\begin{aligned}
\mathcal{B}_{41} &= (a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{2p+2\beta-4}\overline{\Delta_E(\mathbf{R})}\mathcal{D}_j(\mathbf{R}),\\
\mathcal{B}_{42} &= (a_n(z))^2\frac{1}{n\sqrt{n}}\sum_{j=1}^{n}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{2p+2\beta-4}\overline{\Delta_E(\mathbf{R})}\mathcal{D}_j(\mathbf{R})X_{jj}R(j,j),\\
\mathcal{B}_{43} &= (a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{2p+2\beta-4}\overline{\Delta_E(\mathbf{R})}\mathcal{D}_j(\mathbf{R})\mathcal{Q}_j(\mathbf{R})R(j,j),\\
\mathcal{B}_{44} &= (a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{2p+2\beta-4}\overline{\Delta_E(\mathbf{R})}\mathcal{D}_j^2(\mathbf{R})R(j,j),\\
\mathcal{B}_{45} &= (a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{2p+2\beta-2}\mathcal{D}_j(\mathbf{R})R(j,j).
\end{aligned}$$

It is easy to see that

$$|\mathcal{B}_{41}| \leq \frac{C}{nv}\mathcal{A}_{2p+2\beta-2}^{2p+2\beta-3}. \tag{7.5}$$

We have the following bound for the term $\mathcal{B}_{42}$

$$|\mathcal{B}_{42}| \leq \frac{C}{\sqrt{n}v}\frac{1}{nv}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left|\Delta_E(\mathbf{R})\right|^{2p+2\beta-3}|X_{jj}|.$$

Using the equality $\Delta(\mathbf{R}) = \Delta_E^{(j)}(\mathbf{R})+\mathcal{D}_j(\mathbf{R})-\mathbf{E}\,\mathcal{D}_j(\mathbf{R})$ and the bound $|\mathcal{D}_j(\mathbf{R})-\mathbf{E}\,\mathcal{D}_j(\mathbf{R})| \leq \frac{2}{nv}$, we get

$$|\mathcal{B}_{42}| \leq \frac{C}{\sqrt{n}v}\frac{1}{nv}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-3}C_{2p+2\beta-3}^{k}\mathbf{E}\,|\Delta_E^{(j)}(\mathbf{R})|^{2p+2\beta-3-k}\mathbf{E}\,|X_{jj}|\left(\frac{2}{nv}\right)^k.$$

Note that for $1 \le q \le 2p+2\beta-2$

$$\mathbf{E}\,|\Delta_E^{(j)}(\mathbf{R})|^q \le \sum_{k=0}^{q} C_q^k \mathbf{E}\,|\Delta_E^{(j)}(\mathbf{R})|^{q-k}\left(\frac{2}{nv}\right)^k \le \left(\mathcal{A}_{2p+2\beta-2}+\frac{2}{nv}\right)^q .$$

The two last inequalities together imply that for $v \ge CM_4^{\frac12}pn^{-\frac12}$

$$|\mathcal{B}_{42}| \le \frac{C}{nv}\left(\mathcal{A}_{2p+2\beta-2}+\frac{4}{nv}\right)^{2p+2\beta-3}. \tag{7.6}$$

Furthermore, we consider $\mathcal{B}_{43}$.

$$|\mathcal{B}_{43}| \le \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-3} C_{2p+2\beta-2}^k\left(\frac{2}{nv}\right)^k \mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^{2p+2\beta-3-k}|\mathcal{Q}_j(\mathbf{R})||R(j,j)|.$$

Conditioning with respect to the σ-algebra F_j and applying Hölder's inequality, we get, for $1 \le q \le 2p+2\beta-3$,

$$\begin{aligned}\mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^q|\mathcal{Q}_j(\mathbf{R})||R(j,j)| &\le \mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^q\mathbf{E}^{\frac12}\left(|\mathcal{Q}_j(\mathbf{R})|^2\middle|F_j\right)\mathbf{E}^{\frac12}\left(|R(j,j)|^2\middle|F_j\right)\\ &\le \frac{1}{\sqrt{nv}}\mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^q\left(|s_{nj}|+\left|\Delta_E^{(j)}(\mathbf{R})\right|\right)^{\frac12}\mathbf{E}^{\frac12}\left(|R(j,j)|\middle|F_j\right)\\ &\le \frac{C}{\sqrt{nv}}\left(\mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^{2p+2\beta-2}\right)^{\frac{q}{2p+2\beta-2}}\left(\mathbf{E}\,|R(j,j)|^{2p}\right)^{\frac{1}{4p}}\\ &+\frac{C}{\sqrt{nv}}\left(\mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^{2p+2\beta-2}\right)^{\frac{q+\frac12}{2p+2\beta-1}}\left(\mathbf{E}\,|R(j,j)|^{2p}\right)^{\frac{1}{4p}}.\end{aligned}$$

Analogously to inequality (7.6) we obtain

$$\begin{aligned}|\mathcal{B}_{43}| \le{}& \frac{C}{nv}\frac{1}{\sqrt{nv}}(\mathcal{A}_{2p+2\beta-2}+\frac{2}{nv})^{2p+2\beta-3}(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v))^{\frac{1}{4p}}\\ &+\frac{C}{nv}\frac{1}{\sqrt{nv}}(\mathcal{A}_{2p+2\beta-2}+\frac{2}{nv})^{2p+2\beta-\frac52}(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v))^{\frac{1}{4p}}.\end{aligned} \tag{7.7}$$

Furthermore, we have the following obvious bound for the term $\mathcal{B}_{44}$

$$|\mathcal{B}_{44}| \le \frac{C}{n^2v^3}\mathcal{A}_{2p+2\beta-2}^{2p+2\beta-3} \le \frac{C}{nv}\mathcal{A}_{2p+2\beta-2}^{2p+2\beta-3}. \tag{7.8}$$

Consider now the term $\mathcal{B}_{45}$. We may write

$$\begin{aligned}|\mathcal{B}_{45}| &\le \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\left|\Delta_E(\mathbf{R})\right|^{2p+2\beta-2}|R(j,j)| \le \frac{C}{nv}\mathbf{E}\,\left|\Delta_E(\mathbf{R})\right|^{2p+2\beta-2}\left(\frac1n\sum_{j=1}^{n}|R(j,j)|^2\right)^{\frac12}\\ &\le \frac{C}{nv\sqrt{v}}\mathbf{E}\,\left|\Delta_E(\mathbf{R})\right|^{2p+2\beta-2}\left|\frac1n\mathrm{Tr}\,\mathbf{R}\right|^{\frac12} \le \frac{C}{nv\sqrt{v}}\mathbf{E}\,\left|\Delta_E(\mathbf{R})\right|^{2p+2\beta-2}\left(|s_n(z)|+|\Delta_E(\mathbf{R})|\right)^{\frac12}\\ &\le \frac{C}{nv\sqrt{v}}\mathcal{A}_{2p+2\beta-2}^{2p+2\beta-2}+\frac{C}{nv\sqrt{v}}\mathcal{A}_{2p+2\beta-1}^{2p+2\beta-\frac32}.\end{aligned}$$

Simple calculations show that for $v \geq CM_4^{\frac{1}{2}} p n^{-\frac{1}{2}}$

$$\frac{1}{nv^{\frac{3}{2}}} \left(\frac{1}{(nv)^{2p}} \right)^{\frac{2p+2\beta-\frac{3}{2}}{2p+2\beta-1}} \leq \frac{C}{(nv)^{2p}}.$$

Using the assumptions of Lemma 3.4, we obtain

$$|\mathcal{B}_{45}| \leq \frac{C}{nv\sqrt{v}} \mathcal{A}_{2p+2\beta-2}^{2p+2\beta-2} + \frac{Cp^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}} \left(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v) \right). \tag{7.9}$$

Inequalities (7.5), (7.6), (7.7), (7.8), and (7.9) together imply

$$\begin{aligned} |\mathcal{B}_4| \leq \frac{1}{\sqrt{nv}} \Bigg(& \frac{Cp^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}} \left(M_{4p+2} + \widetilde{M}_{8p+4}(v) \right) \\ & + \frac{C}{nv} (\mathcal{A}_{2p+2\beta-2} + \frac{C}{nv})^{2p+2\beta-3} \left(M_{4p+2} + \widetilde{M}_{8p+4}(v) \right)^{\frac{1}{4p}} \\ & + \frac{C}{nv^{\frac{3}{2}}} (\mathcal{A}_{2p+2\beta-2} + \frac{C}{nv})^{2p+2\beta-2} \Bigg). \end{aligned} \tag{7.10}$$

Now we bound the term $\mathcal{B}_3$. Using the representation (3.5) again, we get

$$\mathcal{B}_3 = \mathcal{B}_{31} + \mathcal{B}_{32} + \mathcal{B}_{33} + \mathcal{B}_{34} + \mathcal{B}_{35}, \tag{7.11}$$

where

$$\begin{aligned} \mathcal{B}_{31} &= (a_n(z))^2 \frac{1}{n} \sum_{j=1}^{n} \mathbf{E}\, |\Delta_E(\mathbf{R})|^{2p+2\beta-4}\, \overline{\Delta_E(\mathbf{R})} \mathcal{Q}_j(\mathbf{R}), \\ \mathcal{B}_{32} &= -(a_n(z))^2 \frac{1}{n\sqrt{n}} \sum_{j=1}^{n} \mathbf{E}\, |\Delta_E(\mathbf{R})|^{2p+2\beta-4}\, \overline{\Delta_E(\mathbf{R})} \mathcal{Q}_j(\mathbf{R}) X_{jj} R(j,j), \\ \mathcal{B}_{33} &= (a_n(z))^2 \frac{1}{n} \sum_{j=1}^{n} \mathbf{E}\, |\Delta_E(\mathbf{R})|^{2p+2\beta-4}\, \overline{\Delta_E(\mathbf{R})} \mathcal{Q}_j(\mathbf{R})^2 R(j,j), \\ \mathcal{B}_{34} &= -(a_n(z))^2 \frac{1}{n} \sum_{j=1}^{n} \mathbf{E}\, |\Delta_E(\mathbf{R})|^{2p+2\beta-4}\, \overline{\Delta_E(\mathbf{R})} \mathcal{Q}_j(\mathbf{R}) \mathcal{D}_j(\mathbf{R}) R(j,j), \\ \mathcal{B}_{35} &= (a_n(z))^2 \frac{1}{n} \sum_{j=1}^{n} \mathbf{E}\, |\Delta_E(\mathbf{R})|^{2p+2\beta-2}\, \mathcal{Q}_j(\mathbf{R}) R(j,j). \end{aligned}$$

First at all note that $\mathcal{B}_{34} = \mathcal{B}_{43}$. The bound of this term is given in (6.15). The term $\mathcal{B}_{35}$ is decomposed using again (3.5). We may write

$$\mathcal{B}_{35} = \mathcal{B}_{351} + \mathcal{B}_{352} + \mathcal{B}_{353} + \mathcal{B}_{354} + \mathcal{B}_{355}, \tag{7.12}$$

where

$$\begin{aligned}
\mathcal{B}_{351} &= -(a_n(z))^2\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2}\,\mathcal{Q}_j(\mathbf{R}),\\
\mathcal{B}_{352} &= (a_n(z))^2\frac{1}{n\sqrt{n}}\sum_{j=1}^n \mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2}\,\mathcal{Q}_j(\mathbf{R})X_{jj}R(j,j),\\
\mathcal{B}_{353} &= -(a_n(z))^2\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2}\,\mathcal{Q}_j(\mathbf{R})^2R(j,j),\\
\mathcal{B}_{354} &= (a_n(z))^2\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2}\,\mathcal{Q}_j(\mathbf{R})\mathcal{D}_j(\mathbf{R})R(j,j),\\
\mathcal{B}_{355} &= -(a_n(z))^2\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2}\,\Delta_E(\mathbf{R})\mathcal{Q}_j(\mathbf{R})R(j,j).
\end{aligned} \tag{7.13}$$

To bound $\mathcal{B}_{355}$ we use the method of recursive inequalities. For $k \ge 1$, introduce the quantity

$$\mathcal{B}_{355}^{(k)} = (a_n(z))^2\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2}\,\Delta_E(\mathbf{R})^k\mathcal{Q}_j(\mathbf{R})R(j,j). \tag{7.14}$$

Conditioning given F_j and using Hölder's inequality, we get, for $k = 2p-2\beta+2$,

$$\begin{aligned}
|\mathcal{B}_{355}^{(2p-2\beta+2)}| \le \frac{C}{n}\sum_{j=1}^n\sum_{m=0}^{4p} C_{4p}^m\left(\frac{2}{nv}\right)^m \mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-m}\mathbf{E}^{\frac12}\left(|\mathcal{Q}_j(\mathbf{R})|^2\middle|F_j\right)&\\
\times\,\mathbf{E}^{\frac12}\left(|R(j,j)|^2\middle|F_j\right).&
\end{aligned}$$

Applying inequality (6.27), we get

$$\begin{aligned}
|\mathcal{B}_{355}^{(2p-2\beta+2)}| &\le \frac{C}{n\sqrt{nv}}\sum_{j=1}^n\sum_{m=0}^{4p} C_{4p}^m\left(\frac{2}{nv}\right)^m \mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-m}\left(|s_{nj}(z)|+\left|\Delta_E^{(j)}(\mathbf{R})\right|\right)^{\frac12}\\
&\qquad\times\,\mathbf{E}^{\frac12}\left(|R(j,j)|^2\middle|F_j\right)\\
&\le \frac{C}{n\sqrt{nv}}\sum_{j=1}^n\sum_{m=0}^{4p} C_{4p}^m\left(\frac{2}{nv}\right)^m \mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p-m}\left(|s_{nj}(z)|+\left|\Delta_E^{(j)}(\mathbf{R})\right|\right)^{\frac12}\\
&\qquad\times\,\mathbf{E}^{\frac12}\left(|R(j,j)|\middle|F_j\right).
\end{aligned}$$

Applying Hölder's inequality again, we arrived

$$\begin{aligned}
|\mathcal{B}_{355}^{(2p-2\beta+2)}| &\le \frac{C}{\sqrt{nv}} \sum_{m=0}^{4p} C_{4p}^{m} \left(\frac{2}{nv}\right)^{m} \Bigg[\left(\frac{1}{n} \sum_{j=1}^{n} \mathbf{E} \left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p+2} \right)^{\frac{4p-m}{4p+2}} \\
&+ \left(\frac{1}{n} \sum_{j=1}^{n} \mathbf{E} \left|\Delta_E^{(j)}(\mathbf{R})\right|^{4p+2} \right)^{\frac{4p-m+\frac{1}{2}}{4p+2}} \Bigg] \left(\frac{1}{n} \sum_{j=1}^{n} \mathbf{E}\,|R(j,j)|^{2p+1} \right)^{\frac{1}{4p+2}} \\
&\le \frac{C}{n\sqrt{nv}} \left(\mathcal{A}_{4p+2} + \frac{4}{nv} \right)^{4p} \left(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v) \right)^{\frac{1}{4p+2}} \\
&+ \frac{C}{n\sqrt{nv}} \left(\mathcal{A}_{4p+2} + \frac{4}{nv} \right)^{4p+\frac{1}{2}} \left(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v) \right)^{\frac{1}{4p+2}} .
\end{aligned} \tag{7.15}$$

The last inequality and Lemma 3.3 together imply

$$|\mathcal{B}_{355}^{(2p-2\beta+2)}| \le \frac{Cp^{4p}}{(nv)^{2p}} \left(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v) \right) \frac{C}{n\sqrt{nv}}. \tag{7.16}$$

Using the representation (3.5), we get

$$\mathcal{B}_{355}^{(k)} = \Gamma^{(k)} + \mathcal{B}_{355}^{(k+1)}, \tag{7.17}$$

where

$$\Gamma^{(k)} = \Gamma_1^{(k)} + \Gamma_2^{(k)} + \Gamma_3^{(k)} + \Gamma_4^{(k)},$$

with

$$\begin{aligned}
\Gamma_1^{(k)} &= (a_n(z))^3 \frac{1}{n\sqrt{n}} \sum_{j=1}^{n} \mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2}\, \Delta_E(\mathbf{R})^k \mathcal{Q}_j(\mathbf{R}), \\
\Gamma_2^{(k)} &= (a_n(z))^3 \frac{1}{n\sqrt{n}} \sum_{j=1}^{n} \mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2}\, \Delta_E(\mathbf{R})^k \mathcal{Q}_j(\mathbf{R}) X_{jj} R(j,j), \\
\Gamma_3^{(k)} &= (a_n(z))^3 \frac{1}{n\sqrt{n}} \sum_{j=1}^{n} \mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2}\, \Delta_E(\mathbf{R})^k \mathcal{Q}_j^2(\mathbf{R}) R(j,j), \\
\Gamma_4^{(k)} &= (a_n(z))^3 \frac{1}{n\sqrt{n}} \sum_{j=1}^{n} \mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2}\, \Delta_E(\mathbf{R})^k \mathcal{Q}_j(\mathbf{R}) \mathcal{D}_j(\mathbf{R}) R(j,j).
\end{aligned} \tag{7.18}$$

It is not difficult to obtain the following bound

$$|\Gamma_1^{(k)}| \ll_p \frac{1}{(nv)^{2p+2\beta+k-2}} + \frac{1}{n\sqrt{v}} \mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-2+k} + \frac{1}{n\sqrt{v}} \mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-\frac{3}{2}+k}. \tag{7.19}$$

Analogously, we can assert that

$$
\begin{aligned}
|\Gamma_2^{(k)}| &\ll_p \frac{1}{nv\sqrt{v}}\Big(\frac{1}{(nv)^{2p+2\beta-2+k}} + \mathbf{E}\,\Big|\Delta_E^{(j)}(\mathbf{R})\Big|^{2p+2\beta-2+k}|\mathcal{Q}_j(\mathbf{R})|\Big)\\
&\ll_p \frac{1}{nv\sqrt{v}}\Big(\frac{1}{(nv)^{2p+2\beta+k-2}} + \mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-2+k} + \mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-\frac{3}{2}+k}\Big). \qquad (7.20)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|\Gamma_3^{(k)}| &\le \frac{1}{v}\Big(\frac{1}{(nv)^{2p+2\beta-2+k}} + \mathbf{E}\,\Big|\Delta_E^{(j)}(\mathbf{R})\Big|^{2p+2\beta-2+k}|\mathcal{Q}_j(\mathbf{R})|^2\Big)\\
&\ll_p \frac{1}{nv^2}\Big(\frac{1}{(nv)^{2p+2\beta+k-2}} + \mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-2+k} + \mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-\frac{3}{2}+k}\Big). \qquad (7.21)
\end{aligned}
$$

Finally, we conclude that

$$
\begin{aligned}
|\Gamma_4^{(k)}| &\le \frac{1}{nv^2}\Big(\frac{1}{(nv)^{2p+2\beta-2+k}} + \mathbf{E}\,\Big|\Delta_E^{(j)}(\mathbf{R})\Big|^{2p+2\beta-2+k}|\mathcal{Q}_j(\mathbf{R})|^2\Big)\\
&\ll_p \frac{1}{nv^2}\Big(\frac{1}{(nv)^{2p+2\beta+k-2}} + \mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-2+k} + \mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-\frac{3}{2}+k}\Big). \qquad (7.22)
\end{aligned}
$$

By the assumption of Lemma 3.4, we have

$$
\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-2+k} \ll_p \frac{p^{4p}}{(nv)^{2p}|z+2s(z)|^{2p-2\beta+1}}\Big(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\Big). \qquad (7.23)
$$

Applying relations (7.17)–(7.23), for $k = 1, \dots, 2p-2\beta+2$, we get

$$
|\mathcal{B}_{355}| \ll_p (\frac{p}{\sqrt{nv}})^{\nu})\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-1} + |\mathcal{B}_{355}^{(2p-2\beta+2)}| \le \mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-1} + |\mathcal{B}_{355}^{(2p-2\beta+2)}|. \qquad (7.24)
$$

The last inequality, inequality (7.16) and the assumptions of Lemma 3.4 together imply

$$
|\mathcal{B}_{355}| \ll_p \frac{p^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\left(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v)\right). \qquad (7.25)
$$

Furthermore, note that

$$
\begin{aligned}
|\mathcal{B}_{354}| &\le \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2}\,|\mathcal{Q}_j(\mathbf{R})|\,|R(j,j)|\\
&\le \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2}\,\mathbf{E}^{\frac{1}{2}}\left(|\mathcal{Q}_j(\mathbf{R})|^2\big|F_j\right)\mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|^2\big|F_j\right).
\end{aligned}
$$

Applying inequality (6.27) and Hölder's inequality, we get

$$\begin{aligned}|\mathcal{B}_{354}| \le \frac{C}{nv}\left(\mathcal{A}_{2p+2\beta-1}+\frac{4}{nv}\right)^{2p+2\beta-2}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+1}\right)^{\frac{1}{4p+2}}\\ +\frac{C}{nv}\left(\mathcal{A}_{2p+2\beta-1}+\frac{4}{nv}\right)^{2p+2\beta-\frac{3}{2}}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+1}\right)^{\frac{1}{4p+2}}.\end{aligned} \tag{7.26}$$

The last inequality and Lemma 3.2 together yield

$$|\mathcal{B}_{354}| \le \frac{Cp^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\left(M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v)\right). \tag{7.27}$$

Using that

$$\begin{aligned}|\mathcal{B}_{352}| &\le \frac{C}{\sqrt{n}}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2}\,|\mathcal{Q}_j(\mathbf{R})|\,|X_{jj}||R(j,j)|\\ &\le \frac{1}{n\sqrt{n}}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-2}C^k_{2p+2\beta-2}\left(\frac{2}{nv}\right)^k\mathbf{E}\,\left|\Delta^{(j)}_E(\mathbf{R})\right|^{2p+2\beta-2-k}|\mathcal{Q}_j(\mathbf{R})|\,|X_{jj}||R(j,j)|,\end{aligned}$$

and applying Hölder inequality, we get

$$\begin{aligned}|\mathcal{B}_{352}| \le \frac{1}{nv}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-2}C^k_{2p+2\beta-2}\left(\frac{2}{nv}\right)^k\mathbf{E}\,\left|\Delta^{(j)}_E(\mathbf{R})\right|^{2p+2\beta-2-k}\left(|s_{nj}(z)|+\left|\Delta^{(j)}_E(\mathbf{R})\right|\right)^{\frac{1}{2}}\\ \times\mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|\big|F_j\right)\\ \le \frac{C}{nv}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+1}\right)^{\frac{1}{4p+2}}\\ \times\left(\left(\mathcal{A}_{2p+2\beta-1}+\frac{2}{nv}\right)^{2p+2\beta-2}+\frac{C}{\sqrt{nv}}\left(\mathcal{A}_{2p+2\beta-1}+\frac{2}{nv}\right)^{2p+2\beta-\frac{3}{2}}\right).\end{aligned} \tag{7.28}$$

Similar to inequality (7.27), we obtain

$$|\mathcal{B}_{352}| \le \frac{Cp^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\left(M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v)\right). \tag{7.29}$$

We investigate now the behavior of the term $\mathcal{B}_{351}$. We have

$$\begin{aligned}|\mathcal{B}_{351}| &\ll_p \frac{1}{(nv)^{2p+2\beta-2}}+\frac{1}{\sqrt{nv}}\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-2}+\frac{1}{\sqrt{nv}}(\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-1})^{\frac{2p+2\beta-\frac{3}{2}}{2p+2\beta-1}}\\ &\ll_p \frac{1}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\left(M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v)\right)+\frac{1}{\sqrt{nv}}\mathcal{A}^{2p+2\beta-2}_{2p+2\beta-2}.\end{aligned} \tag{7.30}$$

Finally, we bound $\mathcal{B}_{353}$. Conditioning given F_j, we get

$$|\mathcal{B}_{353}| \le \frac{C}{n}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-2} C_{2p+2\beta-2}^{k}\left(\frac{2}{nv}\right)^{k}\mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{2p+2\beta-2-k}\mathbf{E}^{\frac{1}{2}}\left(|\mathcal{Q}_j(\mathbf{R})|^4\big|F_j\right)$$
$$\times\,\mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|^2\big|F_j\right).$$

Using inequality (6.45), Hölder's inequality and the binomial formula, we get

$$|\mathcal{B}_{353}| \le \frac{C}{nv}\left(\mathcal{A}_{2p+2\beta-1}+\frac{4}{nv}\right)^{2p+2\beta-2}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+2\beta-1}\right)^{\frac{1}{2p+\beta-1}}$$
$$+\frac{C}{nv}\left(\mathcal{A}_{2p+2\beta}+\frac{4}{nv}\right)^{2p+2\beta-1}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+2\beta}\right)^{\frac{1}{2p+2\beta}}. \qquad (7.31)$$

By the assumptions of Lemma 3.4, we have

$$\frac{C}{nv}\left(\mathcal{A}_{2p+2\beta-1}+\frac{4}{nv}\right)^{2p+2\beta-2}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+2\beta-1}\right)^{\frac{1}{2p+\beta-1}}$$
$$\ll_p \frac{C}{nv}\left(\frac{p^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\right)^{1-\frac{1}{2p+2\beta-1}}\left(M_{4p+2}+M_{8p+4}^{(n)}(v)\right)^{1-\frac{1}{2p+2\beta-1}}$$
$$\times\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+2\beta-1}\right)^{\frac{1}{2p+\beta-1}}.$$

After simple calculations we get

$$\frac{C}{nv}\left(\mathcal{A}_{2p+2\beta-1}+\frac{4}{nv}\right)^{2p+2\beta-2}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+2\beta-1}\right)^{\frac{1}{2p+\beta-1}}$$
$$\ll_p \frac{1}{(nv)^{2p}}\left(\frac{Cp^{4p}}{|z+2s(z)|^{2p+2\beta+1}}\right)^{1-\frac{1}{2p+2\beta-1}}\left(M_{4p+2}+M_{8p+4}^{(n)}(v)\right)^{1-\frac{1}{2p+2\beta-1}}$$
$$\times\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+2\beta-1}\frac{1}{(nv)^{2\beta-1}}\right)^{\frac{1}{2p+\beta-1}}.$$

Using $|R(j,j)| \le v^{-1}$, we may rewrite the last inequality as follows

$$\begin{aligned}
&\frac{C}{nv}\left(\mathcal{A}_{2p+2\beta-1}+\frac{4}{nv}\right)^{2p+2\beta-2}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+2\beta-1}\right)^{\frac{1}{2p+\beta-1}}\\
&\quad\ll_p \frac{1}{(nv)^{2p}}\left(\frac{Cp^{4p}}{|z+2s(z)|^{2p+2\beta+1}}\right)^{1-\frac{1}{2p+2\beta-1}}\left(M_{4p+2}+M_{8p+4}^{(n)}(v)\right)^{1-\frac{1}{2p+2\beta-1}}\\
&\quad\times\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p}\frac{1}{(nv^2)^{2\beta-1}}\right)^{\frac{1}{2p+\beta-1}}.
\end{aligned}$$

Combining this fact with the inequality $nv^2 \ge C$, we get

$$\begin{aligned}
&\frac{C}{nv}\left(\mathcal{A}_{2p+2\beta-1}+\frac{4}{nv}\right)^{2p+2\beta-2}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+2\beta-1}\right)^{\frac{1}{2p+\beta-1}}\\
&\quad\ll_p \frac{1}{(nv)^{2p}}\left(\frac{Cp^{4p}}{|z+2s(z)|^{2p+2\beta+1}}\right)^{1-\frac{1}{2p+2\beta-1}}\left(M_{4p+2}+M_{8p+4}^{(n)}(v)\right)^{1-\frac{1}{2p+2\beta-1}}\\
&\quad\times\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p}\right)^{\frac{1}{2p+\beta-1}}\\
&\quad\ll_p \frac{1}{(nv)^{2p}}\left(\frac{Cp^{4p}}{|z+2s(z)|^{2p+2\beta+1}}\right)\left(M_{4p+2}+M_{8p+4}^{(n)}(v)\right). \qquad (7.32)
\end{aligned}$$

Analogously we bound the second term in the r.h.s. of (7.31). This implies that

$$|\mathcal{B}_{353}| \le \frac{Cp^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right). \qquad (7.33)$$

Inequalities (7.25), (7.27), (7.29), (7.30), (7.33) together yield

$$|\mathcal{B}_{35}| \le \frac{Cp^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right)+\frac{1}{\sqrt{nv}}\mathcal{A}_{2p+2\beta-2}^{2p+2\beta-2}. \qquad (7.34)$$

To bound $\mathcal{B}_{34}$ we note that

$$|\mathcal{B}_{34}| \le \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-3}\,|\mathcal{Q}_j(\mathbf{R})|\,|R(j,j)|.$$

Conditioning given F_j and applying inequality (6.27), we get

$$|\mathcal{B}_{34}| \le \frac{C}{nv}\frac{1}{\sqrt{n}v}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-3} C_{2p+2\beta-3}^{k}\left(\frac{2}{nv}\right)^{k}\mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^{2p+2\beta-3-k}$$
$$\times\left(|s_{nj}(z)|+\left|\Delta_E^{(j)}(\mathbf{R})\right|\right)^{\frac{1}{2}}\mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|\right). \tag{7.35}$$

This implies that

$$|\mathcal{B}_{34}| \le \frac{C}{nv}\left(\mathcal{A}_{2p+2\beta-2}+\frac{4}{nv}\right)^{2p+2\beta-3}\left(M_{4p+2}+\widetilde{M}_{8p+4}^{(n)}(v)\right)^{\frac{1}{4p+2}}. \tag{7.36}$$

To bound $\mathcal{B}_{33}$ we use the representation (3.5) again. We may write

$$\mathcal{B}_{33} = \mathcal{B}_{331}+\mathcal{B}_{332}+\mathcal{B}_{333}+\mathcal{B}_{334}+\mathcal{B}_{335}, \tag{7.37}$$

where

$$\begin{aligned}
\mathcal{B}_{331} &= -(a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-4}\,\overline{\Delta_E(\mathbf{R})}\mathcal{Q}_j^2(\mathbf{R}),\\
\mathcal{B}_{332} &= (a_n(z))^2\frac{1}{n\sqrt{n}}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-4}\,\overline{\Delta_E(\mathbf{R})}\mathcal{Q}_j^2(\mathbf{R})X_{jj}R(j,j),\\
\mathcal{B}_{333} &= -(a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-4}\,\overline{\Delta_E(\mathbf{R})}\mathcal{Q}_j^3(\mathbf{R})R(j,j),\\
\mathcal{B}_{334} &= (a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-4}\,\overline{\Delta_E(\mathbf{R})}\mathcal{Q}_j^2(\mathbf{R})\mathcal{D}_j(\mathbf{R})R(j,j),\\
\mathcal{B}_{335} &= -(a_n(z))^2\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-2}\,\mathcal{Q}_j^2(\mathbf{R})R(j,j).
\end{aligned} \tag{7.38}$$

Note that

$$|\mathcal{B}_{332}| \le \frac{C}{\sqrt{n}v}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-3} C_{2p+2\beta-3}^{k}\left(\frac{2}{nv}\right)^{k}\mathbf{E}\,\left|\Delta_E^{(j)}(\mathbf{R})\right|^{2p+2\beta-3-k}$$
$$\times\mathbf{E}\left(|\mathcal{Q}_j(\mathbf{R})|^2\,\big|F_j\right)\mathbf{E}\,|X_{jj}|.$$

Using inequality (6.27) and Hölder's inequality, we get

$$\begin{aligned}
|\mathcal{B}_{332}| \le{}& \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-3} C^k_{2p+2\beta-2}\left(\frac{2}{nv}\right)^k \mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{2p+2\beta-3-k} \\
&\times\left(|s_{nj}(z)|+\left|\Delta_E^{(j)}(\mathbf{R})\right|\right) \\
\le{}& \frac{C}{nv}(\mathcal{A}_{2p+2\beta-2}+\frac{2}{nv})^{2p+2\beta-3}+\frac{C}{nv}\mathcal{A}^{2p+2\beta-2}_{2p+2\beta-2}+\frac{C}{(nv)^{2p}}.
\end{aligned}\tag{7.39}$$

Analogously,

$$\begin{aligned}
|\mathcal{B}_{331}| \le{}& \frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-3} C^k_{2p+2\beta-3}\left(\frac{2}{nv}\right)^k \mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{2p+2\beta-3-k}\mathbf{E}\left(|\mathcal{Q}_j(\mathbf{R})|^2\left|F_j\right.\right) \\
\le{}& \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-3} C^k_{2p+2\beta-3}\left(\frac{2}{nv}\right)^k \mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{2p+2\beta-3-k}\left(|s_{nj}(z)|+\left|\Delta_E^{(j)}(\mathbf{R})\right|\right) \\
\le{}& \frac{C}{nv}(\mathcal{A}_{2p+2\beta-2}+\frac{2}{nv})^{2p+2\beta-3}+\frac{C}{nv}\mathcal{A}^{2p+2\beta-2}_{2p+2\beta-2}+\frac{C}{(nv)^{2p}}.
\end{aligned}\tag{7.40}$$

In addition,

$$\begin{aligned}
|\mathcal{B}_{334}| \le{}& \frac{C}{nv^2}\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\Delta_E(\mathbf{R})|^{2p+2\beta-3}\,|\mathcal{Q}_j(\mathbf{R})|^2 \\
\le{}& \frac{C}{nv}(\mathcal{A}_{2p+2\beta-2}+\frac{2}{nv})^{2p+2\beta-3}+Cnv\mathcal{A}^{2p+2\beta-2}_{2p+2\beta-2}+\frac{C}{(nv)^{2p}}.
\end{aligned}\tag{7.41}$$

Inequalities (6.40), (7.40) and (7.41) together imply

$$\max\{|\mathcal{B}_{331}|,\ |\mathcal{B}_{332}|\ |\mathcal{B}_{334}|\} \le \frac{C}{nv}(\mathcal{A}_{2p+2\beta-2}+\frac{2}{nv})^{2p+2\beta-3}+Cnv\mathcal{A}^{2p+2\beta-2}_{2p+2\beta-2}+\frac{C}{(nv)^{2p}}.\tag{7.42}$$

Now consider $\mathcal{B}_{333}$. We have

$$\begin{aligned}
|\mathcal{B}_{333}| \le{}& \frac{C}{n}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-3} C^k_{2p+2\beta-3}\left(\frac{2}{nv}\right)^k \mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{2p+2\beta-3-k}\mathbf{E}^{\frac{1}{2}}\left(|\mathcal{Q}_j(\mathbf{R})|^6\middle|F_j\right) \\
&\times\mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|^2\middle|F_j\right).
\end{aligned}$$

This inequality and inequality (6.27) together imply

$$\begin{aligned}
|\mathcal{B}_{333}| \le{}& \frac{C}{(nv)^{\frac{3}{2}}}\frac{1}{\sqrt{v}}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-3} C^k_{2p+2\beta-3}\left(\frac{2}{nv}\right)^k \mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{2p+2\beta-3-k} \\
&\times\left(|s_{nj}(z)|+\left|\Delta_E^{(j)}(\mathbf{R})\right|\right)^{\frac{3}{2}}\mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|\middle|F_j\right).
\end{aligned}$$

Applying Hölder's inequality and $\sqrt{nv} \geq C$, we get

$$\begin{aligned} |\mathcal{B}_{333}| &\leq \frac{C}{(nv)}(\mathcal{A}_{2p+2\beta-2} + \frac{4}{nv})^{2p+2\beta-3} \left(M_{4p+2} + \widetilde{M}^{(n)}_{8p+4}(v)\right)^{\frac{1}{4p}} \\ &+ \frac{C}{(nv)}(\mathcal{A}_{2p+2\beta-1} + \frac{4}{nv})^{2p+2\beta-\frac{3}{2}} \left(M_{4p+1} + \widetilde{M}^{(n)}_{8p+4}(v)\right)^{\frac{1}{4p+2}} \\ &\leq \frac{Cp^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}} \left(M_{4p+2} + \widetilde{M}^{(n)}_{8p+4}(v)\right) \\ &+ \frac{C}{(nv)}(\mathcal{A}_{2p+2\beta-2} + \frac{4}{nv})^{2p+2\beta-3} \left(M_{4p+2} + \widetilde{M}^{(n)}_{8p+4}(v)\right)^{\frac{1}{4p}}. \qquad (7.43) \end{aligned}$$

Furthermore, consider $\mathcal{B}_{335}$. We have

$$\begin{aligned} |\mathcal{B}_{335}| \leq \frac{C}{n}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-2} C^k_{2p+2\beta-2}\left(\frac{1}{nv}\right)^k \mathbf{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{2p+2\beta-2-k} \mathbf{E}^{\frac{1}{2}}\left(|\mathcal{Q}_j(\mathbf{R})|^4 \,|F_j\right) \\ \times \mathbf{E}^{\frac{1}{2}}\left(|R(j,j)|^2 \,|F_j\right). \end{aligned}$$

The last inequality and inequality (6.45) together yield

$$\begin{aligned} |\mathcal{B}_{335}| &\leq \frac{C}{\sqrt{nv}}\left(\mathcal{A}_{2p+2\beta-1} + \frac{4}{nv}\right)^{2p+2\beta-2}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+2\beta-1}\right)^{\frac{1}{2p+2\beta-1}} \\ &+ \frac{C}{\sqrt{nv}}\left(\mathcal{A}_{2p+2\beta-1} + \frac{4}{nv}\right)^{2p+2\beta-1}\left(M_{4p+2} + \widetilde{M}^{(n)}_{8p+4}(v)\right)^{\frac{1}{4p+2}}. \end{aligned}$$

By the assumption of Lemma 3.4

$$\begin{aligned} |\mathcal{B}_{335}| &\leq \frac{Cp^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\left(M_{4p+2} + \widetilde{M}^{(n)}_{8p+4}(v)\right) \\ &+ \frac{C}{(nv)^{2p}}\frac{1}{(nv)^{\frac{2\beta-1}{2p+2\beta-1}}}\left(M_{4p+2} + \widetilde{M}^{(n)}_{8p+4}(v)\right)^{1-\frac{1}{2p+2\beta-1}}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|R(j,j)|^{2p+2\beta-1}\right)^{\frac{1}{2p+2\beta-1}} \\ &\leq \frac{Cp^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\left(M_{4p+2} + \widetilde{M}^{(n)}_{8p+4}(v)\right). \qquad (7.44) \end{aligned}$$

Inequalities (7.42), (7.43) and (7.44) together imply

$$\begin{aligned} |\mathcal{B}_{33}| &\leq \frac{Cp^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\left(M_{4p+2} + \widetilde{M}^{(n)}_{8p+4}(v)\right) \\ &+ \frac{C}{(nv)}(\mathcal{A}_{2p+2\beta-2} + \frac{4}{nv})^{2p+2\beta-3}\left(M_{4p+2} + \widetilde{M}^{(n)}_{8p+4}(v)\right)^{\frac{1}{4p}}. \qquad (7.45) \end{aligned}$$

Now we bound $\mathcal{B}_{32}$. Applying the binomial formula and Hölder's inequality, we get

$$
\begin{aligned}
|\mathcal{B}_{32}| \le \frac{C}{\sqrt{nv}}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-3} C_{2p+2\beta-3}^{k}\left(\frac{1}{nv}\right)^{k}\mathbb{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{2p+2\beta-3-k} \\
\times\, \mathbb{E}^{\frac{1}{2}}\left(|\mathcal{Q}_j(\mathbf{R})|^2\Big|F_j\right)\mathbb{E}^{\frac{1}{2}}\left(|R(j,j)|\Big|F_j\right).
\end{aligned}
$$

Combining (6.27) with Hölder's inequality gives

$$
\begin{aligned}
|\mathcal{B}_{32}| &\le \frac{C}{nv}\frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-3} C_{2p+2\beta-3}^{k}\left(\frac{1}{nv}\right)^{k}\mathbb{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{2p+2\beta-3-k} \\
&\qquad\qquad \times (|s_{nj}(z)| + |\Delta_E^{(j)}(\mathbf{R})|)^{\frac{1}{2}}\mathbb{E}^{\frac{1}{2}}\left(|R(j,j)|\Big|F_j\right) \\
&\le \frac{C}{nv}\left(\left(\mathcal{A}_{2p+2\beta-2}+\frac{4}{nv}\right)^{2p+2\beta-3} + \left(\mathcal{A}_{2p+2\beta-1}+\frac{4}{nv}\right)^{4p-\frac{3}{2}}\right) \\
&\qquad\qquad \times \left(M_{4p+2}+\widetilde{M}_{8p+4}\right)^{\frac{1}{4p+1}}.
\end{aligned}
$$

By the assumptions of Lemma 3.4,

$$
\begin{aligned}
|\mathcal{B}_{32}| &\le \frac{Cp^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right) \\
&\quad + \frac{C}{nv}\left(\mathcal{A}_{2p+2\beta-2}+\frac{4}{nv}\right)^{2p+2\beta-3}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right)^{\frac{1}{4p+1}}. \qquad (7.46)
\end{aligned}
$$

To bound $\mathcal{B}_{31}$ we need some preliminary bounds. Note that

$$
\begin{aligned}
\Delta(\mathbf{R}) - \Delta_E^{(j)}(\mathbf{R}) &= \frac{1}{n}(\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}_j) - \frac{1}{n}\mathbf{E}\,(\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}_j) \\
&= \frac{1}{n^2}(\alpha_j'\mathbf{R}_j^2\alpha_j - \operatorname{Tr}\mathbf{R}_j^2) + \frac{1}{n^2}(\operatorname{Tr}\mathbf{R}_j^2 - \mathbf{E}\operatorname{Tr}\mathbf{R}_j^2) \\
&\quad + \frac{1}{n}(\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}_j)\varepsilon_j - \frac{1}{n}\mathbf{E}\,(\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}_j)\varepsilon_j. \qquad (7.47)
\end{aligned}
$$

Applying this equality, we get

$$
\begin{aligned}
|\mathcal{B}_{31}| &\le \frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\left|(\Delta(\mathbf{R}))^{2p+2\beta-3} - (\Delta_E^{(j)}(\mathbf{R}))^{2p+2\beta-3}\right||\mathcal{Q}_j(\mathbf{R})| \\
&\le \frac{1}{n}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-4}\mathbf{E}\left|\Delta(\mathbf{R}) - \Delta_E^{(j)}(\mathbf{R})\right||\Delta(\mathbf{R})|^{2p+2\beta-4-k}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{k}|\mathcal{Q}_j(\mathbf{R})|.
\end{aligned}
$$

The last inequality yields

$$
\begin{aligned}
|\mathcal{B}_{31}| &\le \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-4}\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-4-k}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{k}|\mathcal{Q}_j(\mathbf{R})|\left|\mathcal{Q}_j(\mathbf{R}^2)\right| \qquad (=\Sigma_1)\\
&+\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-4}\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-4-k}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{k}|\mathcal{Q}_j(\mathbf{R})|\left|\Delta_E^{(j)}(\mathbf{R}^2)\right| \qquad (=\Sigma_2)\\
&+\frac{1}{n^2v}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-4}\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-4-k}\left|\Delta_E^{(j)}(\mathbf{R})\right|^{k}|\mathcal{Q}_j(\mathbf{R})|\,|\varepsilon_j|. \qquad (=\Sigma_3)
\end{aligned}
\tag{7.48}
$$

Furthermore, by the binomial formula,

$$
\begin{aligned}
\Sigma_1 \le \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-4}\sum_{\nu=0}^{2p+2\beta-4-k}&C^{\nu}_{2p+2\beta-4-k}\left(\frac{1}{nv}\right)^{\nu}\\
&\times\mathbf{E}\,|\Delta_E^{(j)}(\mathbf{R})|^{2p+2\beta-4-\nu}|\mathcal{Q}_j(\mathbf{R})||\mathcal{Q}_j(\mathbf{R}^2)|.
\end{aligned}
$$

Conditioning given F_j and applying inequality (6.27), we obtain

$$
\begin{aligned}
\Sigma_1 &\le \frac{1}{n^3v^2}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-4}\sum_{\nu=0}^{2p+2\beta-4-k}C^{\nu}_{2p+2\beta-4-k}\left(\frac{1}{nv}\right)^{\nu}\\
&\qquad\times\mathbf{E}\,|\Delta_E^{(j)}(\mathbf{R})|^{2p+2\beta-4-\nu}(|s_{nj}(z)|+|\Delta_E^{(j)}(\mathbf{R})|)\\
&\le \frac{1}{n^2v^2}(\mathcal{A}_{2p+2\beta-2}+\frac{4}{nv})^{2p+2\beta-4}.
\end{aligned}
\tag{7.49}
$$

The bound for Σ_2 is similar. We have

$$
\begin{aligned}
\Sigma_2 \le \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-4}\sum_{\nu=0}^{2p+2\beta-4-k}&C^{\nu}_{2p+2\beta-4-k}\left(\frac{1}{nv}\right)^{\nu}\\
&\times\mathbf{E}\,|\Delta_E^{(j)}(\mathbf{R})|^{2p+2\beta-4-\nu}|\mathcal{Q}_j(\mathbf{R})||\Delta_E^{(j)}(\mathbf{R}^2)|.
\end{aligned}
$$

The last inequality and inequality (6.27) yield

$$
\begin{aligned}
\Sigma_2 \le \frac{1}{n^{\frac52}v^{\frac12}}\sum_{j=1}^{n}\sum_{k=0}^{2p+2\beta-4}\sum_{\nu=0}^{2p+2\beta-4-k}&C^{\nu}_{2p+2\beta-4-k}\left(\frac{1}{nv}\right)^{\nu}\\
&\times\mathbf{E}\,|\Delta_E^{(j)}(\mathbf{R})|^{2p+2\beta-4-\nu}(|s_{nj}(z)|+|\Delta_E^{(j)}(\mathbf{R})|)^{\frac12}|\Delta_E^{(j)}(\mathbf{R}^2)|.
\end{aligned}
$$

Using the binomial formula, we obtain

$$\begin{aligned}\Sigma_2 &\le \frac{C}{n^{\frac32}\sqrt{v}}\left(\mathcal{A}_{2p+2\beta-2}+\frac{4}{nv}\right)^{2p+2\beta-4}\mathbf{E}^{\frac{1}{2p+2\beta-2}}|\Delta(\mathbf{R}^2)|^{2p+2\beta-2}\\ &+\frac{C}{n^{\frac32}\sqrt{v}}\left(\mathcal{A}_{2p+2\beta-2}+\frac{4}{nv}\right)^{2p+2\beta-\frac72}\mathbf{E}^{\frac{1}{2p+2\beta-2}}|\Delta(\mathbf{R}^2)|^{2p+2\beta-2}\\ &\le \frac{C}{n^{\frac32}\sqrt{v}}\left(\mathcal{A}_{2p+2\beta-2}+\frac{4}{nv}\right)^{2p+2\beta-4}\mathbf{E}^{\frac{1}{2p+2\beta-2}}|\Delta(\mathbf{R}^2)|^{2p+2\beta-2}.\end{aligned}\tag{7.50}$$

By Lemma 3.1 and Cauchy's integral formula we have, for $v \ge CM_4^{\frac12}n^{-\frac12}$,

$$\mathbf{E}^{\frac{1}{2p+2\beta-3}}|\Delta(\mathbf{R}^2)|^{2p+2\beta-3} \le \frac{Cp^2}{nv^{\frac52}}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right)^{\frac{1}{4p+1}}.\tag{7.51}$$

The last inequality implies that

$$\Sigma_2 \ll_p \frac{p}{(nv)^2}(\mathcal{A}_{2p+2\beta-2}+\frac{4}{nv})^{2p+2\beta-4}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right)^{\frac{1}{4p+1}}.\tag{7.52}$$

Analogously we obtain the bound for Σ_3. Collecting inequalities (7.34), (7.35), (7.45), (7.46) and (7.48), we get

$$\begin{aligned}|\mathcal{B}_3| &\le \frac{Cp^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\left(M_8+\widetilde{M}^{(n)}_{8p+4}(v)\right)\\ &+\frac{p}{(nv)^2}(\mathcal{A}_{2p+2\beta-2}+\frac{4}{nv})^{2p+2\beta-4}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right)^{\frac{1}{4p+1}}.\end{aligned}\tag{7.53}$$

Now we bound $\mathcal{B}_2$. Using equality (3.5), we represent $\mathcal{B}_2$ in the form

$$\mathcal{B}_2=\mathcal{B}_{21}+\mathcal{B}_{22}+\mathcal{B}_{23}+\mathcal{B}_{24}+\mathcal{B}_{25},\tag{7.54}$$

where

$$\begin{aligned}\mathcal{B}_{21}&=-\frac{a_n^2(z)}{n\sqrt{n}}\sum_{j=1}^n\mathbb{E}|\Delta(\mathbf{R})|^{2p+2\beta-4}\overline{\Delta(\mathbf{R})}X_{jj},\\ \mathcal{B}_{22}&=\frac{a_n^2(z)}{n^2}\sum_{j=1}^n\mathbb{E}|\Delta(\mathbf{R})|^{2p+2\beta-4}\overline{\Delta(\mathbf{R})}X_{jj}^2R(j,j),\\ \mathcal{B}_{23}&=-\frac{a_n^2(z)}{n\sqrt{n}}\sum_{j=1}^n\mathbb{E}|\Delta(\mathbf{R})|^{2p+2\beta-4}\overline{\Delta(\mathbf{R})}X_{jj}\mathcal{Q}_j(\mathbf{R})R(j,j),\\ \mathcal{B}_{24}&=\frac{a_n^2(z)}{n\sqrt{n}}\sum_{j=1}^n\mathbb{E}|\Delta(\mathbf{R})|^{2p+2\beta-4}\overline{\Delta(\mathbf{R})}X_{jj}\mathcal{D}_j(\mathbf{R})R(j,j),\\ \mathcal{B}_{25}&=-\frac{a_n^2(z)}{n\sqrt{n}}\sum_{j=1}^n\mathbb{E}|\Delta(\mathbf{R})|^{2p+2\beta-2}X_{jj}R(j,j).\end{aligned}$$

It is easy to see that

$$\begin{aligned}
|\mathcal{B}_{21}| &\le \frac{C}{n}\mathbb{E}|\Delta(\mathbf{R})|^{2p+2\beta-3}|\frac{1}{\sqrt{n}}\sum_{j=1}^{n}X_{jj}| \\
&\le \frac{C}{n}\mathcal{A}_{2p+2\beta-2}^{2p+2\beta-3}\,\mathbb{E}^{\frac{1}{2p+2\beta-2}}\left|\frac{1}{\sqrt{n}}\sum_{j=1}^{n}X_{jj}\right|^{2p+2\beta-2} \\
&\le \frac{Cp}{n}\mathcal{A}_{2p+2\beta-2}^{2p+2\beta-3}\left(1+n^{-p+1}M_{2p+2\beta-2}\right)^{\frac{1}{2p+2\beta-2}} \\
&\le \frac{C}{(nv)}\mathcal{A}_{2p+2\beta-2}^{2p+2\beta-3}\left(M_8+\widetilde{M}_{8p+4}^{(n)}(v)\right)^{\frac{1}{2p+2\beta-2}}. \qquad (7.55)
\end{aligned}$$

For the term $\mathcal{B}_{22}$ we have the following bound

$$\begin{aligned}
|\mathcal{B}_{22}| &\le \frac{1}{nv}\sum_{k=0}^{2p+2\beta-3}C_{2p+2\beta-3}^{k}\left(\frac{1}{nv}\right)^{k}\mathbf{E}\,|\Delta_E^{(j)}(\mathbf{R})|^{2p+2\beta-3-k} \\
&\le \frac{1}{nv}\left(\mathcal{A}_{2p+2\beta-2}+\frac{4}{nv}\right)^{2p+2\beta-3}. \qquad (7.56)
\end{aligned}$$

Furthermore, $\mathcal{B}_{23}=\mathcal{B}_{32}$. Thus inequality (7.46) yields

$$\begin{aligned}
|\mathcal{B}_{23}| \le{}& \frac{Cp^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right) \\
&+\frac{C}{nv}\left(\mathcal{A}_{2p+2\beta-2}+\frac{4}{nv}\right)^{2p+2\beta-3}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right)^{\frac{1}{2p+2\beta-2}}. \qquad (7.57)
\end{aligned}$$

Analogously, $\mathcal{B}_{24}=\mathcal{B}_{42}$ and (6.11) gives

$$|\mathcal{B}_{24}| \ll_p \frac{C}{nv}\left(\mathcal{A}_{2p+2\beta-2}+\frac{4}{nv}\right)^{2p+2\beta-3}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right)^{\frac{1}{2p+2\beta-2}}. \qquad (7.58)$$

To bound B_{25} we use the representation (3.5) again and obtain

$$\mathcal{B}_{25}=\mathcal{B}_{251}+\mathcal{B}_{252}+\mathcal{B}_{253}+\mathcal{B}_{254}+\mathcal{B}_{255}, \qquad (7.59)$$

where

$$\begin{aligned}
\mathcal{B}_{251} &= \frac{a_n^3(z)}{n\sqrt{n}}\sum_{j=1}^{n}\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-2}X_{jj}, \\
\mathcal{B}_{252} &= -\frac{a_n^3(z)}{n^2}\sum_{j=1}^{n}\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-2}X_{jj}^2R(j,j), \\
\mathcal{B}_{253} &= \frac{a_n^3(z)}{n\sqrt{n}}\sum_{j=1}^{n}\mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-2}X_{jj}\mathcal{Q}_j(\mathbf{R})R(j,j),
\end{aligned}$$

$$\mathcal{B}_{254} = -\frac{a_n^3(z)}{n\sqrt{n}} \sum_{j=1}^{n} \mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-2} X_{jj} \mathcal{D}_j(\mathbf{R}) R(j,j),$$

$$\mathcal{B}_{255} = \frac{a_n^3(z)}{n\sqrt{n}} \sum_{j=1}^{n} \mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-1} X_{jj} R(j,j). \tag{7.60}$$

We have the following bounds.

$$|B_{251}| \le \frac{C}{n} \mathcal{A}_{2p+2\beta-1}^{2p+2\beta-2} \mathbf{E}^{\frac{1}{2p+2\beta-1}} \left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j j \right)^{\frac{1}{2p+2\beta-1}}$$

$$\ll_p \frac{p^{4p}}{(nv)^{2p} |z+2s(z)|^{2p+2\beta+1}} \left(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v) \right). \tag{7.61}$$

Furthermore,

$$|B_{252}| \le \frac{C}{nv} (\mathcal{A}_{2p+2\beta-1} + \frac{4}{nv})^{2p+2\beta-2}$$

$$\ll_p \frac{p^{4p}}{(nv)^{2p} |z+2s(z)|^{2p+2\beta+1}} \left(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v) \right), \tag{7.62}$$

and

$$|\mathcal{B}_{253}| \le \frac{C}{nv} \left(\mathcal{A}_{2p+2\beta-1} + \frac{4}{nv} \right)^{2p+2\beta-2} \left(\frac{1}{n} \sum_{j=1}^{n} \mathbf{E}\,|R(j,j)|^{2p+1} \right)^{\frac{1}{4p+2}}$$

$$\ll_p \frac{p^{4p}}{(nv)^{2p} |z+2s(z)|^{2p+2\beta+1}} \left(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v) \right). \tag{7.63}$$

It is easy to see that

$$|\mathcal{B}_{254}| \le \frac{C}{nv} \frac{1}{n\sqrt{nv}} \sum_{j=1}^{n} \mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-2} |X_{jj}|$$

$$\le \frac{C}{nv} \left(\mathcal{A}_{2p+2\beta-1} + \frac{4}{nv} \right)^{2p+2\beta-2} \tag{7.64}$$

$$\ll_p \frac{p^{4p}}{(nv)^{2p} |z+2s(z)|^{2p+2\beta+1}} \left(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v) \right). \tag{7.65}$$

Finally, we conclude that

$$|\mathcal{B}_{255}| \le \frac{1}{n\sqrt{nv}} \sum_{j=1}^{n} \mathbf{E}\,|\Delta(\mathbf{R})|^{2p+2\beta-1} |X_{jj}|$$

$$\le \frac{C}{\sqrt{nv}} \left(\mathcal{A}_{2p+2\beta-1} + \frac{4}{nv} \right)^{2p+2\beta-1} \tag{7.66}$$

$$\ll_p \frac{p^{4p}}{(nv)^{2p} |z+2s(z)|^{2p+2\beta+1}} \left(M_{4p+2} + \widetilde{M}_{8p+4}^{(n)}(v) \right). \tag{7.67}$$

Inequalities (7.61)–(7.66) together imply that

$$|\mathcal{B}_{25}| \ll_p \frac{p^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\left(M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v)\right) \tag{7.68}$$

Combining inequalities (7.55)–(7.68), we get

$$\begin{aligned}|\mathcal{B}_2| \ll_p &\frac{p^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+1}}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right)\\ &+\frac{C}{nv}\left(\mathcal{A}_{2p+2\beta-2}+\frac{4}{nv}\right)^{2p+2\beta-3}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right)^{\frac{1}{2p+2\beta-2}}.\end{aligned} \tag{7.69}$$

The bound for the term $\mathcal{B}_1$ follows from the bound for $\delta_n(z)$ from the paper GT03. We have

$$|\mathcal{B}_1| \le \frac{C}{nv}\mathcal{A}^{2p+2\beta-3}_{2p+2\beta-2}. \tag{7.70}$$

Collecting the bounds (7.70), (7.69), (7.53), (7.10) and using (7.3), we obtain

$$\begin{aligned}\mathcal{A}^{2p+2\beta-2}_{2p+2\beta-2} \le &\frac{Cp^{4p}}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+2}}\left(M_{4p+2}+\widetilde{M}^{(n)}_{8p+4}(v)\right)\\ &+\frac{C}{nv|z+2s(z)|}(\mathcal{A}_{2p+2\beta-2}+\frac{C}{nv})^{2p+2\beta-3}\left(M_{4p+2}+\widetilde{M}_{8p+4}(v)\right)^{\frac{1}{4p}}\\ &+\frac{C}{nv^{\frac{3}{2}}|z+2s(z)|}(\mathcal{A}_{2p+2\beta-2}+\frac{C}{nv})^{2p+2\beta-2}\\ &+\frac{C}{nv|z+2s(z)|}\mathcal{A}^{2p+2\beta-3}_{2p+2\beta-2}\\ &+\frac{p}{(nv)^2|z+2s(z)|}(\mathcal{A}_{2p+2\beta-2}+\frac{4}{nv})^{2p+2\beta-4}\left(M_{4p+2}+\widetilde{M}_{8p+4}\right)^{\frac{1}{4p+1}}.\end{aligned} \tag{7.71}$$

In order to find a solution to this recursive inequality we use the following

Lemma 7.1. *Let $x \ge 0$, $\alpha > 0$, $\rho > 0$, $B > 0$, $0 < \varepsilon < \frac{1}{4}$ and $q \ge 2$ such that*

$$x^q \le \varepsilon(x+\alpha)^q + C_1\rho(x+\alpha)^{q-1} + C_2\rho^2(x+\alpha)^{q-2} + B. \tag{7.72}$$

Then there exists some constant C such that

$$x^q \le \max\{(B^{\frac{1}{q}}+q\alpha)^q, (B^{\frac{1}{q}}+C\rho)^q\}. \tag{7.73}$$

Proof. Assume that

$$x^q \ge (B^{\frac{1}{q}}+q\alpha)^q. \tag{7.74}$$

This assumption and inequality (7.73) together imply that

$$\begin{aligned}x^q &\le \varepsilon x^q(1+\frac{1}{Cq})^q + C_1\rho x^{q-1}(1+\frac{1}{q})^{q-1} + C_2\rho^2 x^{q-2}(1+\frac{1}{q})^{q-2} + B\\ &\le \frac{3}{4}x^q + 3C_1\rho x^{q-1} + 3C_2\rho^2 x^{q-2} + B.\end{aligned} \tag{7.75}$$

Dividing by x^{q-2} and solving with respect to x^q, we get

$$x^2 \leq 12C_1\rho x + 12C_2\rho^2 + B^{\frac{2}{q}}. \tag{7.76}$$

Solving the last inequality, we obtain

$$x \leq (C\rho + B^{\frac{1}{q}}). \tag{7.77}$$

This implies that

$$x^q \leq (C\rho + B^{\frac{1}{q}})^q. \tag{7.78}$$

The assumption (7.74) and inequality (7.78) together complete the proof. ☐

Now apply Lemma 7.1 to inequality (7.71). We choose

$$\begin{aligned} x &= \mathcal{A}_{2p+2\beta-2}, \quad \varepsilon = \frac{C}{nv^{\frac{3}{2}}|z+2s(z)|} \\ \rho &= \frac{p^{\frac{1}{2}}(M_{4p+2} + \widetilde{M}^{(n)}_{8p+4}(v))^{\frac{1}{4p}}}{nv|z+2s(z)|^{\frac{1}{2}}}, \quad \alpha = \frac{C}{nv}, \quad q = 2p+2\beta-2. \\ B &= \frac{p^{4p}(M_{4p+2} + \widetilde{M}^{(n)}_{8p+4}(v))}{(nv)^{2p}|z+2s(z)|^{2p+2\beta+2}}. \end{aligned} \tag{7.79}$$

Note that $|z+2s(z)| \geq \sqrt{v}$. This implies that for $v \geq v_0$

$$\varepsilon \leq \frac{C}{nv^2} \leq \frac{1}{4}. \tag{7.80}$$

Furthermore, it is not difficult to show that

$$B^{\frac{1}{2p+2\beta-2}} \geq Cp\rho. \tag{7.81}$$

Applying Lemma 7.1, we get

$$\mathcal{A}^{2p+2\beta-2}_{2p+2\beta-2} \leq B(1 + \frac{C}{2p+2\beta-2})^{2p+2\beta-2} \leq CB. \tag{7.82}$$

This bound concludes the proof of Lemma 3.4. ☐

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