LIMIT CORRELATION FUNCTIONS FOR FIXED TRACE RANDOM MATRIX ENSEMBLES

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Abstract. Universal limits for the eigenvalue correlation functions in the bulk of the spectrum are shown for a class of non-determinantal random matrices known as the fixed trace or the Hilbert-Schmidt ensemble. These universal limits have been proved before for determinantal Hermitian matrix ensembles and for some special classes of the Wigner random matrices.

1. Introduction and the statement of the result

Let $\mathcal{H}_N$ be the set of all $N \times N$ (complex) Hermitian matrices, and let $\operatorname{tr} A = \sum_{i=1}^{N} a_{ii}$ denotes the trace of a square matrix $A = (a_{ij})_{i,j=1}^{N}$. $\mathcal{H}_N$ is a real Hilbert space of dimension N^2 with respect to the symmetric bilinear form $(A, B) \mapsto \operatorname{tr} AB$. Let l_N denotes the unique Lebesgue measure on $\mathcal{H}_N$ which satisfies the relation $l_N(Q) = 1$ for every cube $Q \subset \mathcal{H}_N$ with edges of length 1. A Gaussian probability measure on $\mathcal{H}_N$ invariant with respect to all orthogonal linear transformations of $\mathcal{H}_N$ is uniquely defined up to a scaling transformation. Such measures form a one-parameter family $(\mu_N^s)_{s>0}$, where the measure μ_N^s is specified by its density

$$g_N^s(A) = \frac{1}{(\sqrt{s2\pi})^{N^2}} \exp\left(-\frac{1}{2s} \operatorname{tr} A^2\right) \tag{1.1}$$

with respect to l_N. Thus, for the random matrix X distributed according to μ_N^s we have

$$E_{\mu_N^s} \operatorname{tr} X^2 = sN^2. \tag{1.2}$$

The set $\mathcal{H}_N$ furnished with the measure μ_N^s is called the Gaussian Unitary Ensemble (GUE).

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Let X be a random $N \times N$ Hermitian matrix (that is a random variable taking values in $\mathcal{H}_N$). We consider the eigenvalues

$\lambda_1, \lambda_2, \dots, \lambda_N$ of the random matrix X as a finite sequence of *exchangeable* random variables. By definition, this means that their joint distribution P_N^X does not change under any permutation of these variables. Let for each $n, 1 \le n \le N$, $P_{n,N}^X$ denotes the joint distribution of some n of these N variables. Obviously, $P_{n,N}^X$ is a permutation invariant probability measure in $\mathbb{R}^n$. In particular, the measure $P_{1,N}^X$ describes the distribution of a single eigenvalue. By definition, the *n-point correlation measure* $\rho_{n,N}^X$ of a random matrix X is a non-normalized measure defined by the relation

$$\rho_{n,N}^X = \frac{N!}{(N-n)!} P_{n,N}^X. \tag{1.3}$$

For a measurable set $A \subset \mathbb{R}^n$ the amount $\rho_{n,N}^X(A)$ can be interpreted as the average number of $n-$tuples of eigenvalues getting into the set A. If the measure $\rho_{n,N}^X$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$, its Radon-Nikodym derivative $R_{n,N}^X$ is called the *n-point correlation function* of the random matrix X.

In particular, the measure $\rho_{1,N}^X$ has the total mass N. For a measurable set $E \subset \mathbb{R}^1$, the amount $\rho_{1,N}^X(E)$ expresses the expected number of the eigenvalues belonging to E. The corresponding density with respect to the Lebesgue measure in $\mathbb{R}^1$, if it exists, is called the *eigenvalue density* or the *density of states* (precaution: under the same names the normalized versions of the same measures are considered in the literature as well).

Let X_N be a random matrix with the distribution μ_N^s. For $n = 1, \dots, N$ we set $P_{n,N}^{\mathrm{GUE},s} = P_{n,N}^{X_N}$ and $\rho_{n,N}^{\mathrm{GUE},s} = \rho_{n,N}^{X_N}$. A classical result for the GUE says that we have

$$P_{1,N}^{\mathrm{GUE},1/N} \underset{N\to\infty}{\to} W, \tag{1.4}$$

where the measures converge in the weak sense and W is the standard Wigner measure on $[-2, 2]$ defined by the density

$$w(x) = (2\pi)^{-1}\sqrt{(4-x^2)_+},\ x \in \mathbb{R}. \tag{1.5}$$

In terms of correlation measures the same relation reads

$$\frac{1}{N}\rho_{1,N}^{\mathrm{GUE},1/N} \underset{N\to\infty}{\to} W. \tag{1.6}$$

For the $n-$point correlation measures we have a similar relation

$$\frac{1}{N^n}\rho_{n,N}^{\mathrm{GUE},1/N} \underset{N\to\infty}{\to} \underbrace{W \times \cdots \times W}_{n \text{ times}}, \tag{1.7}$$

which means that the eigenvalues become independent in the limit. However, for $n \geq 2$, the study of a finer asymptotics near a point on the principal diagonal of the cube $(-2,2)^n$ shows that for $R_{n,N}^{\mathrm{GUE},1/N}$ ([9, 10]): for every $u \in (-2,2)$ and $t_1, \dots, t_n \in \mathbb{R}^1$

(1.8)
$$\lim_{N\to\infty} \frac{1}{(Nw(u))^n} R_{n,N}^{\mathrm{GUE},1/N}\left(u + \frac{t_1}{Nw(u)}, \dots, u + \frac{t_n}{Nw(u)}\right) = \det\left(\frac{\sin\pi(t_i - t_j)}{\pi(t_i - t_j)}\right)_{i.j=1}^n .$$

This limit relation presents a pattern for many other results, in particular, for that of the present paper.

The right hand side of this relation represents an example of the correlation function of a so-called determinantal (or fermionic) random point process [14]. In general the $n-$point correlation function R_n of such a process is given by the formula

$$R(u_1, \dots, u_n) = \det\big|K(u_i, u_j)|_{i,j=1}^n,$$

where K is the kernel of an integral operator on the line, which is trace class having been restricted to compact subsets of $\mathbb{R}$ and subject to some further conditions (see [14] for a detailed exposition). Moreover, in the asymptotic Hermitian random matrix theory $(K_N)_{N\geq 0}$ are the reproducing kernels of the subspaces of polynomials of degree $\leq N-1$ with respect to some weight on the line. In this case we call the corresponding matrix ensemble *determinantal.* The GUE gives an example of such an ensemble. To a large extent the asymptotic study of determinantal ensembles reduces to that of the respective kernels ([2, 4].

Outside the class of determinantal Hermitian random matrices only very few results on the asymptotics of the correlation function are known (see, for instance, the paper [9], where a mixture of determinantal measures is considered). In the present paper we investigate the following non-determinantal ensemble of random Hermitian matrices. Let

(1.9)
$$S_N^r = \{A \in \mathcal{H}_N : \mathrm{tr}\, A^2 = r^2\}$$

be the sphere in $\mathcal{H}_N$ of the radius $r > 0$ centered at the origin.Set $r = \sqrt{s}N$. The sphere $S_N^{\sqrt{s}N}$ carries a unique probability measure ν_N^s invariant with respect to all orthogonal linear transformations in the space $\mathcal{H}_N$. We call this measure the *fixed Hilbert-Schmidt norm ensemble* (or just HSE) to reserve the term "the fixed trace ensemble" for more general ones (see [1]). Let Y_N be a random matrix distributed according to ν_N^s. We set for $n = 1, \dots, N$ $P_{n,N}^{\mathrm{HSE},s} = P_{n,N}^{Y_N}$ and $\rho_{n,N}^{\mathrm{HSE},s} = \rho_{n,N}^{Y_N}$.

It is a well known result (see [10, 13]) that

$$P_{1,N}^{\mathrm{HSE},1/N} \underset{N\to\infty}{\to} W, \tag{1.10}$$

like in the case of GUE. In this note we prove that the correlation functions $R_{n,N}^{\mathrm{HSE},1/N}$ of arbitrary order n $(1 \le n \le N)$ have near every point $u \in (-2, 2)$, $u \neq 0$, the same determinantal limit with the kernel $\sin \pi(t_1 - t_2)/\pi(t_1 - t_2)$ as the GUE correlation functions (for $n = 1$ the limit equals 1).

More precisely, we establish in this paper the following result.

Theorem. *Let $R_{n,N}^{\nu,1/N}$ be the $n-$point correlation function of the eigenvalues for a random matrix uniformly distributed on the sphere $S_N^{\sqrt{N}}$. Then for every $u \in (-2, 0) \cup (0, 2)$ and $t_1, t_2, \dots, t_n \in \mathbb{R}^1$*

$$\frac{1}{(Nw(u))^n} R_{n,N}^{\nu,1/N}\left(u + \frac{t_1}{Nw(u)}, \dots, u + \frac{t_n}{N}\right) \to \det\left(\frac{\sin \pi(t_i - t_j)}{\pi(t_i - t_j)}\right)_{i,j=1}^n. \tag{1.11}$$

as $N \to \infty$. For every $\alpha > 0$ small enough and $K > 0$ the relation 1.11 holds uniformly in all $u \in (-2 + \alpha, 2 - \alpha)$ and $|t_1| < K|u|, \dots, |t_n| < K|u|$.

Corollary 1. *For every $\alpha > 0$, $L > 0$ the convergence in* (1.11) *is uniform with respect to $u \in (-2 + \alpha, -\alpha) \cup (\alpha, 2 - \alpha)$ and $|t_1| \le L, \dots, |t_n| \le L$.*

Excluding zero in the statement of Theorem is related to the techniques used here. We expect that the real behavior of the correlation functions near zero is the same as near any other point in $(-2, 2)$. In a separate note [8] by the present authors with A. Levina the zero case is treated by means of elementary methods which establishes the desired limiting relation in the topology of weak convergence of measures on compact sets. It is still open question how to extend our present result to this case.

Now we turn to discussing our result in the context of known facts about the GUE and sketching main steps of the proof. The guiding principle could be that results for the Hilbert-Schmidt ensembles are deducible from the corresponding results for GUE using the 'equivalence of ensembles' or the "concentration phenomenon". In our setup the primitive form of concentration is given by the law of large numbers for the squares of the Hilbert-Schmidt norms of the GUE random matrices. Supplemented by some estimates of the probabilities of large deviations, this is a main tool in [8]. However, our experience shows that usual deviation estimates are insufficient for proving local results for correlation functions of eigenvalues (with exception for eigenvalues near zero). This agrees with M.L. Mehta doubts ([10], Sect. 27.1, p.490) concerning the deducibility of the local results for correlation functions of the Hilbert-Schmidt ensembles from the corresponding results for GUE by using the equivalence of ensembles.

Solving this open problem in the present paper, we use a *local form* of concentration given by the local central limit theorem for the densities of the squared Hilbert-Schmidt norms.

First we represent the Hilbert-Schmidt measure as a conditional measure of the GUE, given the Hilbert-Schmidt norm of the GUE random matrix. Starting with the disintegration of the GUE according to the level sets of the Hilbert-Schmidt norm, we arrive to formula 2.10 which is the crucial ingredient of the proof. Here we have to extend the scaling parameter to the complex domain. The formal Fourier inversion applied to this formula gives an "heuristic proof" of the result. However, to make it rigorous we need asymptotic estimates in the complex domain for the the kernels related to the Hermite functions. This is done in Section 3, based on the results in [2, 3, 4]. Unfortunately, some of the results we need are contained in these papers not explicitly enough and should be extracted from the proofs rather than from the statements (see the proof of Lemma 2). With these estimates, we complete the proof. Note that the analytic part of the present paper may be viewed as a form of Tauberian theorem.

2. Disintegration, a Fourier transform formula and the sketch of the proof

In this section we discuss a disintegration representation of the GUE in terms of the HSE, and derive a Fourier transform formula involving these matrix ensembles. We suppress in this section the "spectral" arguments of correlation functions and related quantities assuming that these arguments vary inside the domain described in Section 1.

For every $r > 0$ denote by S_N^r the sphere of radius r in $\mathcal{H}_N$ centered at the origin. Let for $s > 0$ X_N be a random matrix in $\mathcal{H}_N$ distributed according to $\mu_N^{\mathrm{GUE},s}$. Set $T_N = \mathrm{tr}(X_N^2/s)$ and $Y_N = NX_N/\sqrt{\mathrm{tr}\, X_N^2}$. The random variable T_N can be represented as a sum of N^2 squares of independent standard Gaussian random variables, hence it has the familiar $\chi^2_{N^2}$ distribution. Moreover, T_N and Y_N are independent, and Y_N is uniformly distributed on the sphere S_N^N in $\mathcal{H}_N$, that is Y_N is distributed according to ν_N^1 in the notation of the previous section. Then X_N can be represented as

$$X_N = \frac{Y_N}{N}\sqrt{sT_N} \tag{2.1}$$

with T_N and Y_N as above. Let γ_{N^2} denote the the probability density of T_N. Then it follows from (2.1) that

$$\mu_N^s = \int_0^\infty \nu_N^{u/N^2} \gamma_{N^2}(s^{-1}u)s^{-1}du. \tag{2.2}$$

As a consequence of (2.2), the correlation functions of GUE and HSE for $1 \le n \le N-1$ satisfy the relation

$$R_{n,N}^{\mathrm{GUE},s} = \int_0^\infty R_{n,N}^{\mathrm{HSE},u/N^2} \gamma_{N^2}(s^{-1}u)s^{-1}du. \tag{2.3}$$

Note that

$$ET_N = N^2,\ DT_N^2 = E(T_N - ET_N)^2 = 2N^2 \tag{2.4}$$

and, for every $m > 0$,

$$\gamma_m(u) = \begin{cases} (2^{m/2}\Gamma(m/2))^{-1} \mathrm{if}\, u \ge 0, \\ 0, \mathrm{if}\, u < 0. \end{cases} \tag{2.5}$$

Set for $s > 0$ $\gamma_{m,s}(\cdot) = s^{-1}\gamma_m(s^{-1}\cdot)$, so that $\gamma_{m,1} = \gamma_m$. Note that the same set of densities in a different parametrization appears in Lemma 4 as $(f_{a,p})_{a,p>0})$.

Observe now that the probability density of sT_N is given by $\gamma_{N^2,s}(\cdot)$. Then (2.3) can be rewritten as

$$R_{n,N}^{\mathrm{GUE},s} = \int_0^\infty R_{n,N}^{\mathrm{HSE},u/N^2} \gamma_{N^2,s}(u)du. \tag{2.6}$$

In particular, we have

$$R_{n,N}^{\mathrm{GUE},1/N} = \int_0^\infty R_{n,N}^{\mathrm{HSE},u/N^2} \gamma_{N^2,1/N}(u)du. \tag{2.7}$$

Our goal is to investigate the limiting behavior of $R_{n,N}^{\mathrm{HSE},1/N}$ when n is fixed, $N \to \infty$, and the "spectral" arguments of $R_{n,N}^{\mathrm{HSE},1/N}$ vary within a $\epsilon-$neighborhood of a part of the diagonal of $(-2,2)^n$. However, we prefer to consider the function $u \mapsto R_{n,N}^{\mathrm{HSE},u/N^2}$ rather than its value $R_{n,N}^{\mathrm{HSE},1/N}$ at N. More precisely, we shall deal with the product $u \mapsto R_{n,N}^{\mathrm{HSE},u/N^2}\gamma_{N^2,1/N}(u)$ which is a density likewise $\gamma_{N^2,1/N}(\cdot)$. Performing in (2.7) the change of variable $(u-N)/\sqrt{2}=v$ we obtain another relation

$$(2.8)\qquad \begin{aligned} R_{n,N}^{\mathrm{GUE},1/N} &= \int_{-\infty}^{\infty} R_{n,N}^{\mathrm{HSE},1/N+v\sqrt{2}/N^2}\sqrt{2}\gamma_{N^2,1/N}(N+v\sqrt{2})dv \\ &= \int_{-\infty}^{\infty} q_{N^2}(v)dv, \end{aligned}$$

where

$$(2.9)\qquad q_{N^2}(v) = R_{n,N}^{\mathrm{HSE},1/N+v\sqrt{2}/N^2}\sqrt{2}\gamma_{N^2,1/N}(N+v\sqrt{2}).$$

Notice that $v\mapsto\sqrt{2}\ \gamma_{N^2,1/N}(N+v\sqrt{2})$ is the probability density of the centered and normalized random variable $(T_{N^2}-N^2)/\sqrt{2N^2}$, and it tends to the standard normal density $\varphi: v\mapsto(1/\sqrt{2\pi})\exp(-v^2/2)$ as $N\to\infty$. Thus, the limit behavior of the density $\gamma_{N^2,1/N}(N+\cdot\sqrt{2})$ is well understood and we have to study $q_{N^2}(\cdot)$.

Given u, due to the relation $\gamma_{N^2,s}(\cdot)=s^{-1}\gamma_{N^2}(s^{-1}\cdot)$ we can analytically extend $\gamma_{N^2,s}(u)$ to the the domain $\Re s>0$. Moreover, in the formula 2.6 the integral in the right hand side can be analytically continued in s in accordance with the above continuation of $\gamma_{N^2,s}(u)$. This leads to the corresponding continuation of $R_{n,N}^{\mathrm{GUE},s}$ so that 2.6 holds for s from the upper half-plane.

Now we shall evaluate the Fourier transform of the (nonprobabilistic) density q_{N^2} keeping in mind that by (2.8) it is a nonnegative integrable function.

Lemma 1.

$$(2.10)\qquad \int_{-\infty}^{\infty}\exp(ipv)q_{N^2}(v)dv = \phi_{N^2}(p)R_{n,N}^{GUE,1/((1-ip\sqrt{2}/N)N)},$$

where

$$(2.11)\qquad \phi_{N^2}(p)=\exp(-ipN/\sqrt{2})(1-ip\sqrt{2}/N)^{-(N^2/2)}$$

is the characteristic function of the random variable $(T_{N^2}-N^2)/\sqrt{2N^2}$.

Proof. Denoting by C_m the normalizing constant from the formula (2.5), we have

(2.12)

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \exp(ipv) q_{N^2}(v) dv \\
&= \int_{-\infty}^{\infty} \exp(ipv) R_{n,N}^{\mathrm{HSE},1/N+v\sqrt{2}/N^2} \sqrt{2}\gamma_{N^2,1/N}(N+v\sqrt{2}) dv \\
&= \int_{-\infty}^{\infty} \exp(ip(u-N)/\sqrt{2}) R_{n,N}^{\mathrm{HSE},u/N^2} \gamma_{N^2,1/N}(u) du \\
&= \exp(-ipN/\sqrt{2}) \int_{-\infty}^{\infty} \exp(ipu\sqrt{2}) R_{n,N}^{\mathrm{HSE},u/N^2} \gamma_{N^2,1/N}(u) du \\
&= \exp(-ipN/\sqrt{2}) \int_{-\infty}^{\infty} R_{n,N}^{\mathrm{HSE},u/N^2} C_{N^2} N (Nu)^{(N^2/2)-1} \\
&\quad \exp(-Nu(1-ip\sqrt{2}/N)/2) du \\
&= \exp(-ipN/\sqrt{2})(1-ip\sqrt{2}/N)^{-(N^2/2)+1} \int_{-\infty}^{\infty} R_{n,N}^{\mathrm{HSE},u/N^2} \\
&\quad C_{N^2} N (Nu(1-ip\sqrt{2}/N))^{(N^2/2)-1} \exp(-Nu(1-ip\sqrt{2}/N)/2) du \\
&= \exp(-ipN/\sqrt{2})(1-ip\sqrt{2}/N)^{-(N^2/2)} \\
&\quad \int_{-\infty}^{\infty} R_{n,N}^{\mathrm{HSE},u/N^2} \gamma_{N^2,1/((1-ip\sqrt{2}/N)N)}(u) du \\
&= \phi_{N^2}(p) R_{n,N}^{\mathrm{GUE},1/((1-ip\sqrt{2}/N)N)}.
\end{aligned}
$$

□

In the following we shall outline our approach. Write

$$
\frac{1}{(Nw(u))^n} q_{N^2}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{N^2}(p) \frac{1}{(Nw(u))^n} R_{n,N}^{\mathrm{GUE},1/((1-ip\sqrt{2}/N)N)} dp.
$$

Passing to the limit in the integral on the right hand side (uniformly with respect to the spectral variables), the right hand side has the same limit as

$$
\frac{1}{(Nw(u))^n} R_{n,N}^{\mathrm{GUE},1/N} \frac{1}{2\pi} \int_{\infty}^{\infty} \phi_{N^2}(p) dp
$$

or, in view of the local Central Limit Theorem (CLT) , as

$$
\frac{1}{\sqrt{2\pi}(Nw(u))^n} R_{n,N}^{\mathrm{GUE},1/N}.
$$

On the other hand, it follows from(2.9) that

$$\frac{1}{(Nw(u))^n} q_{N^2}(0) = \frac{1}{(Nw(u))^n} R_{n,N}^{\mathrm{HSE},1/N} \sqrt{2}\gamma_{N^2,1/N}(N). \tag{2.13}$$

Again, the local CLT implies that $\sqrt{2}\gamma_{N^2,1/N}(N) \to 1/\sqrt{2\pi}$. Therefore,

$$\frac{1}{(Nw(u))^n} R_{n,N}^{\mathrm{HSE},1/N}$$

tends to the same limit as

$$\frac{1}{(Nw(u))^n} R_{n,N}^{\mathrm{GUE},1/N},$$

and the conclusion follows.

In next Section these heuristic arguments will be made rigorous.

3. Proofs

For every $\alpha \in \mathbb{R}$, thorough the rest of this paper we will denote by $(\cdot)^\alpha$ the function

$$(\cdot)^\alpha : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C} : z \mapsto \exp \alpha \log z, \tag{3.1}$$

where log denotes the principal branch of the logarithm. We use $\sqrt{\cdot}$ as a notation for $(\cdot)^{1/2}$ extended to 0 by $\sqrt{0} = 0$.

First we state some known results on the asymptotics of Hermite polynomials and Hermite functions in the complex plane. These results will be employed in this section later. Asymptotic behavior for Hermite polynomials was first established in 1922 by Plancherel and Rotach. For a convenient form of these results we refer to the monograph [4] and the papers [3] (in particular, Appendix B)and[5] treating more general orthogonal polynomials.

Let $s > 0$, and let for every $N, N \geq 0$, $\tilde{p}_N(\cdot, s)$ be a polynomial of degree N with a positive leading coefficient satisfying the relations

$$\int_{\mathbb{R}} \tilde{p}_M(x,s)\tilde{p}_N(x,s) \exp\left(-x^2/(2s)\right) dx = \delta_{M,N},\ M, N = 0, 1, \ldots$$

The (monic) Hermite polynomials $(\tilde{H}_N(\cdot, s))_{N=0,1,\ldots}$ are defined by the expansion

$$\exp\left(\gamma x - \gamma^2 s/2\right) = \sum_{N=0}^{\infty} \tilde{H}_N(x,s) \frac{\gamma^N}{N!}$$

and for $M, N = 0, 1, \ldots$, satisfy the relations

$$\int_{\mathbb{R}} \tilde{H}_M(x,s)\tilde{H}_N(x,s) \exp\left(-x^2/(2s)\right) dx = \delta_{M,N}\sqrt{2\pi} s^{(M+N+1)/2} N!,$$

so that

$$\tilde{p}_N(\cdot, s) = \frac{1}{(2\pi)^{1/4} s^{(2N+1)/4} (N!)^{1/2}} \tilde{H}_N(\cdot, s), \;\; N = 0, 1, \ldots$$

Polynomials $\tilde{p}_N(\cdot, s)$ satisfy the difference equations

(3.2)
$$x\,\tilde{p}_N(x,s) = s^{1/2}\,\sqrt{N+1}\,\tilde{p}_{N+1}(x,s) + s^{1/2}\,\sqrt{N}\,\tilde{p}_{N-1}(x,s), \;\; N = 1, 2, \ldots$$

The Hermite functions $(\tilde{\varphi}_N(\cdot, s))_{N=0,1,\ldots}$ defined by

$$\tilde{\varphi}_N(x,s) \; = \tilde{p}_N(x,s) \exp\left(-x^2/(4s)\right), \; N = 0, 1, \ldots,$$

form an orthonormal sequence of functions in $L_2(\mathbb{R}, \lambda)$ where λ is the Lebesgue measure in $\mathbb{R}$.

Let us introduce standardized Hermite polynomials and functions (corresponding to the weight $\exp\left(-x^2/2\right)$) by the relations

$$H_N(\cdot) = \tilde{H}_N(\cdot, 1), \; p_N(\cdot) = \tilde{p}_N(\cdot, 1), \; \varphi_N(\cdot) = \tilde{\varphi}_N(\cdot, 1),$$

so that

$$\tilde{H}_N(x,s) = s^{N/2} H_N(xs^{-1/2}), \; \tilde{p}_N(x,s) = s^{-1/4} p_N(xs^{-1/2}),$$

$$\tilde{\varphi}_N(\cdot, s) = s^{-1/4} \varphi_N(xs^{-1/2}).$$

The above relations for the Hermite polynomials imply that the Hermite functions satisfy, for every $k \geq 1$, the following system of differential equations:

$$\begin{aligned} \varphi_k'(x) &= -\frac{x}{2}\varphi_k(x) + \sqrt{k}\varphi_{k-1}(x), \\ \varphi_{k-1}'(x) &= -\sqrt{k}\varphi_k(x) + \frac{x}{2}\varphi_{k-1}(x). \end{aligned} \tag{3.3}$$

The reproducing kernel $\tilde{K}_N(\cdot, \cdot, s)$ of the orthogonal projection in $L_2(\mathbb{R}, \lambda)$ (here λ is the Lebesgue measure) onto the linear span of $\tilde{\varphi}_0(\cdot, s), \ldots, \tilde{\varphi}_{N-1}(\cdot, s)$ is given by

$$\tilde{K}_N(x, y, s) = \sum_{k=0}^{N-1} \tilde{\varphi}_k(x,s)\tilde{\varphi}_k(y,s).$$

Setting

$$K_N(x,y) = \tilde{K}_N(x, y, 1),$$

we obtain

$$\tilde{K}_N(x, y, s) = s^{-1/2} K_N(xs^{-1/2}, ys^{-1/2}). \tag{3.4}$$

The following integral representation of the reproducing kernel is a version of formula (4.56) in [7]:

$$K_N(x,y) = \sqrt{N/2}\int_0^\infty (\varphi_N(x+\tau)\varphi_{N-1}(y+\tau) + \varphi_{N-1}(x+\tau)\varphi_N(y+\tau))d\tau, \tag{3.5}$$

so that

$$\begin{aligned} &\tilde{K}_N(x,y,s) \\ &= s^{-1/2}\sqrt{N/2}\int_0^\infty \big(\varphi_N(xs^{-1/2}+\tau)\varphi_{N-1}(ys^{-1/2}+\tau) \\ &+ \varphi_{N-1}(xs^{-1/2}+\tau)\varphi_N(ys^{-1/2}+\tau)\big)d\tau. \end{aligned} \tag{3.6}$$

The latter relations and our extension of the function $(\cdot)^\alpha$ to $\mathbb{C}\backslash(-\infty,0)$ allow us to continue $H_N(x,\cdot)$, $p_N(x,\cdot)$, $\varphi_N(x,\cdot)$ and $K_N(x,y,\cdot)$ to the same domain. Moreover, relations similar to (3.6) hold true for these continuation whenever the integrals are well-defined.

Let C_δ and D_δ be the disks in $\mathbb{C}$ of radius $\delta\in(0,1)$ with centers at 2 and -2, respectively. Let $\overline{\mathbb{C}}_+$ be the closed upper half-plane and $S_\delta=\{z\in\overline{\mathbb{C}}_+ : |\Re z|< 2-\delta/\sqrt{2}, \Im z\in[0,\delta/\sqrt{2})\}$. Define subsets A_δ and B_δ of $\overline{\mathbb{C}}_+$ by $A_\delta=\overline{\mathbb{C}}_+\backslash(S_\delta\cup C_\delta\cup D_\delta)$ and $B_\delta=S_\delta\backslash(C_\delta\cup D_\delta)$. Denote by w the analytic continuation of the standardized Wigner density (1.5) to the domain $\mathbb{C}\setminus\big((-\infty,2]\cup[2,\infty)\big)$.

Let us define for every $\alpha,\beta>0, \alpha+\beta\in(0,2)$, a set $S_{\alpha,\beta}$ by

$$S_{\alpha,\beta}=\{z\in\mathbb{C} : |\Re z|\le 2-\alpha, |\Im u|\le\beta\}. \tag{3.7}$$

Set for $H\in\mathbb{R}$ $d(H)=\sqrt{1+iH}$ and observe that

$$|d(H)|=(1+H^2)^{1/4}, \tag{3.8}$$

$$\Re d(H)=\sqrt{\frac{\sqrt{1+H^2}+1}{2}}, \tag{3.9}$$

$$\Im d(H)=\operatorname{sgn} H\sqrt{\frac{\sqrt{1+H^2}-1}{2}}, \tag{3.10}$$

and

$$\big(\Re d(H)\big)^2-\big(\Im d(H)\big)^2=1. \tag{3.11}$$

Note that for every $b \geq 0$ the equation

$$|\Im d(H)| = b$$

has a unique nonnegative solution

$$H_b = \sqrt{(1+2b^2)^2 - 1}. \tag{3.12}$$

It is clear that

$$\big|\Im\big(ud(H)\big)\big| \leq \beta. \tag{3.13}$$

if and only if $|H| \leq H_{\beta/|u|}$. Moreover, for $u \in [-2+\beta+a/2, 2-\beta-\alpha/2]$ and $|H| \leq H_{\beta/|u|}$ we also have

$$\big|\Re\big(ud(H)\big)\big| \leq 2 - \alpha/2 \tag{3.14}$$

because

$$\big|\Re\big(ud(H)\big)\big| = \sqrt{\big(\Im\big(d(H)\big)\big)^2 + u^2} \leq \sqrt{\beta^2 + (2-\alpha-\beta)^2} < \sqrt{(2-\alpha)^2}.$$

Thus, for any $u \in [-2+\beta+a/2, 2-\beta-\alpha/2]$ the relation

$$ud(H) \in S_{\alpha,\beta} \tag{3.15}$$

holds if and only if

$$|H| \leq H_{\beta/|u|} = \sqrt{(1+2(\beta/u)^2)^2 - 1}. \tag{3.16}$$

Lemma 2. *There exists a number $\alpha_0 \in (0,2)$ such that for every $\alpha \in (0,\alpha_0)$, $A > 0$ and a certain real $\beta = \beta(\alpha) > 0$, $\beta < \alpha/2$, the relation*

$$\begin{aligned} &\frac{1}{Nw(u)}\tilde{K}_N\left(\left(u + \frac{t_1}{Nw(u)}\right)d(H), \left(u + \frac{t_2}{Nw(u)}\right)d(H), N^{-1}\right) \\ &\underset{N\to\infty}{\longrightarrow} \frac{\sin\big(\pi(t_1-t_2)d(H)w(ud(H))/w(u)\big)}{\pi(t_1-t_2)d(H)}. \end{aligned} \tag{3.17}$$

holds uniformly respective to real numbers $u \in [-2+\alpha, 2-\alpha]$, $H \in [-H_{\beta/|u|}, H_{\beta/|u|}]$, and t_1, t_2 such that $|t_1| < A|u|$, $|t_2| < A|u|$.

Proof. Note that for every $\alpha > 0$ sufficiently small there exists $\beta = \beta(\alpha)$ such that for every $A > 0$

$$\frac{1}{Nw(v)}\tilde{K}_N\left(v + \frac{z_1}{Nw(v)}, v + \frac{z_2}{Nw(v)}, \frac{1}{N}\right) \underset{N\to\infty}{\longrightarrow} \frac{\sin\pi(z_1-z_2)}{\pi(z_1-z_2)} \tag{3.18}$$

uniformly with respect to all complex numbers $v \in S_{\alpha/2,\beta}$ z_1 and z_2, such that $|z_1| \leq A, |z_2| \leq A$. In fact this assertion (and much more general ones concerning some class of weights) is contained in papers [2] and [3] (see also the monograph [4]). Actually, Lemma 6.1 in [2] establishes the desirable result for real u, z_1 and z_2. The same reasoning applies to complex u, z_1 and z_2 satisfying the assumptions just made provided that α and β are sufficiently small. The boundedness property of a certain derivative involved is established in [2] and [3] for some complex neighborhood of a real point $u \in [-2+\alpha, 2-\alpha]$ (see relation (4.122) in [2]) which allows to bound it in a rectangular strip $S_{\alpha/2,\beta}$. Lowering β if necessary, in the rest of the proof we shall assume $\beta \leq \alpha/2$ and that the function $w(\cdot)$ does not vanish in $S_{\alpha/2,\beta}$ (recall that $w(\cdot)$ has no zeroes on $[-2+\alpha/2, 2-\alpha/2]$).

Setting in (3.18) $z_i = z_i' w(v)/w(v')$ $(i = 1, 2)$, we obtain

(3.19)
$$\frac{1}{Nw(v')}\tilde{K}_N\left(v+\frac{z_1'}{Nw(v')}, v+\frac{z_2'}{Nw(v')}, \frac{1}{N}\right) \underset{N\to\infty}{\to} \frac{\sin\bigl(\pi(z_1'-z_2')w(v)/w(v')\bigr)}{\pi(z_1'-z_2')}.$$

Note that for $v \in S_{\alpha/2,\beta}$ $\;C^{-1} \leq |w(v)| \leq C$ with some $C > 0$. Therefore, for every $A > 0$ (3.19) holds uniformly in $v \in S_{\alpha/2,\beta}$ and $|z_1'| \leq A, |z_2'| \leq A$.

Under the assumptions of the lemma we have for every $u \in [-2+\beta(\alpha)+\alpha/2, 2-\beta(\alpha)-\alpha/2]$ and every $H \in [-H_{\beta/|u|}, H_{\beta/|u|}]$, by equivalence of (3.15) and (3.16),

$$ud(H) \in S_{\alpha,\beta}.$$

Setting $v' = u \in [-2+\alpha, 2-\alpha]$, $v = ud(H) \in S_{\alpha,\beta(\alpha)}$, $z_i' = t_i d(H)$, $(i = 1, 2)$ we obtain (3.17). □

Corollary 2. *Under the conditions of Lemma 2 for every $\alpha \in (0, \alpha_0)$ and $A > 0$ we have, with some constant $C(\alpha, A)$,*

(3.20)
$$\left|\frac{1}{N}\tilde{K}_N\left(\left(u+\frac{t_1}{Nw(u)}\right)d(H), \left(u+\frac{t_2}{Nw(u)}\right)d(H), \frac{1}{N}\right)\right| \leq C(\alpha, A)$$

for every real numbers u, t_1, t_2 such that $u \in [-2+\alpha, 2-\alpha]$, $|t_1| \leq |u|A$, $|t_2| \leq |u|A$, and every $H \in [-H_{\beta/|u|}, H_{\beta/|u|}]$ and natural N.

We now derive estimates in $C \setminus S_{\alpha,\beta}$.

Lemma 3. *For every $\beta > 0$ there exist such constants $C(\beta)$ and $M(\beta)$ that the inequalities*

$$\bigl|p_N(z\sqrt{N})\bigr| \leq C(\beta)N^{-1/4}M^N(\beta)|z|^N. \tag{3.21}$$

and

$$\bigl|p_{N-1}\bigl(z\sqrt{N}\bigr)\bigr| \leq C(\beta)(N)^{-1/4}M^{N-1}(\beta)|z|^{N-1}. \tag{3.22}$$

hold for every N and every z with $\Im z \geq \beta$.

Proof. According to known results about the Plancherel-Rotach asymptotics in the complex plane for the Hermite polynomials (Theorem

7.185 in [4], see also [17] and the references therein), we have in our notation, uniformly in z from every compact set contained in $(\mathbb{C}\cup\{\infty\})\setminus[-\sqrt{2},\sqrt{2}]$,

$$\begin{aligned}\tilde{H}_N(z,(2N)^{-1})\\ =\frac{1}{2}\Big[\Big(\frac{z-\sqrt{2}}{z+\sqrt{2}}\Big)^{1/4}+\Big(\frac{z+\sqrt{2}}{z-\sqrt{2}}\Big)^{1/4}\Big]\\ \times\frac{\exp\big(N(z-\sqrt{z^2-2})^2/4\big)}{(z-\sqrt{z^2-2})^N}\Big(1+O\Big(\frac{1}{N}\Big)\Big).\end{aligned}\tag{3.23}$$

For the monic orthogonal Hermite polynomials (with respect to the weight $\exp(-x^2/2)$) we obtain

$$\begin{aligned}H_N(z\sqrt{N})=(2N)^{N/2}\tilde{H}_N(z/\sqrt{2},(2N)^{-1})\\ =\frac{1}{2}\Big[\Big(\frac{z-2}{z+2}\Big)^{1/4}+\Big(\frac{z+2}{z-2}\Big)^{1/4}\Big]\\ \times 2^N N^{N/2}\frac{\exp\big(N(z-\sqrt{z^2-4})^2/8\big)}{(z-\sqrt{z^2-4})^N}\Big(1+O\Big(\frac{1}{N}\Big)\Big)\end{aligned}\tag{3.24}$$

uniformly in z for every compact from $(\mathbb{C}\cup\{\infty\})\setminus[-2,2]$. Passing to normalized polynomials we see that

$$\begin{aligned}p_N(z\sqrt{N})=\frac{1}{(2\pi)^{1/4}(N!)^{1/2}}H_N(z\sqrt{N})\\ =\frac{2^N N^{N/2}}{2(2\pi)^{1/4}(N!)^{1/2}}\Big[\Big(\frac{z-2}{z+2}\Big)^{1/4}+\Big(\frac{z+2}{z-2}\Big)^{1/4}\Big]\\ \times\frac{\exp\big(N(z-\sqrt{z^2-4})^2/8\big)}{(z-\sqrt{z^2-4})^N}\Big(1+O\Big(\frac{1}{N}\Big)\Big),\end{aligned}\tag{3.25}$$

and, in view of the Stirling formula $N!=\sqrt{2\pi}N^{N+\frac{1}{2}}e^{-N}(1+O(1/N))$, we obtain

$$\begin{aligned}p_N(z\sqrt{N})=\frac{1}{2(2\pi)^{1/2}(N)^{1/4}}\Big[\Big(\frac{z-2}{z+2}\Big)^{1/4}+\Big(\frac{z+2}{z-2}\Big)^{1/4}\Big]\\ \times\frac{\exp\big(N(\frac{1}{2}+\log 2+(z-\sqrt{z^2-4})^2/8)\big)}{(z-\sqrt{z^2-4})^N}\Big(1+O\Big(\frac{1}{N}\Big)\Big),\end{aligned}\tag{3.26}$$

so that we have

$$(3.27)\quad \begin{aligned} &p_N(z\sqrt{N})\\ &==\frac{1}{2(2\pi)^{1/2}(N)^{1/4}}\left[\left(\frac{z-2}{z+2}\right)^{1/4}+\left(\frac{z+2}{z-2}\right)^{1/4}\right]\\ &\quad\times\frac{\exp\left(\frac{N}{2}(1+((z-\sqrt{z^2-4})/2)^2)\right)}{\big((z-\sqrt{z^2-4})/2\big)^N}\left(1+O\Big(\frac{1}{N}\Big)\right)\end{aligned}$$

uniformly in z from any compact subset of $(\mathbb{C}\cup\{\infty\})\setminus[-2,2]$. Let $z\in\mathbb{C}\setminus[-2,2]$ and $x=x(z)$ be the root of the equation $x+x^{-1}=z$ satisfying $|x|<1$ so that

$$x=x(z)=\big(z-\sqrt{z^2-4}\big)/2$$

where $z\mapsto\kappa(z)=\sqrt{z^2-4}$ is, by definition, a univalent analytic function in $\mathbb{C}\setminus[-2,2]$ satisfying $\kappa^2(z)=z^2-4$ and $\kappa(t)>0$ for $t>2$.

For every $r\in(0,1)$ the inequality $|x(z)|\le r$ defines the exterior domain E_r of an ellipse with the focal points $-2,2$. Thus for $z\in E_r$ we obviously have

$$|x(z)|=\big|\big(z-\sqrt{z^2-4}\big)/2\big|\le r<1,$$

and

$$\Re\Big(\big(z-\sqrt{z^2-4}/2\big)^2\Big)\le r^2/4<1/4.$$

Note that for every $r\in(0,1)$

$$\min_{|x(z)|=r}|z|=r^{-1}-r$$

which for $z\in E_r$ implies

$$(3.28)\quad \begin{aligned} &\frac{1}{\big|\big(z-\sqrt{z^2-4}\big)/2\big|}\\ &=\frac{1}{|x(z)|}=|z-x(z)|\le|z|+1\le(1+|z|^{-1})|z|\\ &\qquad\le\left(1+\frac{1}{r^{-1}-r}\right)|z|.\end{aligned}$$

Further, for $z\in E_r(0<r<1)$ we can write

$$(3.29)\quad \begin{aligned} &|z-2|\\ &=\big|\big(x(z)-1\big)+\big(x^{-1}(z)-1\big)\big|\ge 2|x(z)-1||x^{-1}(z)-1|\\ &=2|x(z)|^{-1}|1-x(z)|^2\ge 2r^{-1}|1-x(z)|^2\ge 2r^{-1}|1-r|^2\\ &=2(r^{-1/2}-r^{1/2})^2,\end{aligned}$$

and analogously

$$|z+2| \geq 2(r^{-1/2} - r^{1/2})^2.$$

This implies that the function

$$\left|\left(\frac{z-2}{z+2}\right)^{1/4} + \left(\frac{z+2}{z-2}\right)^{1/4}\right|$$

is bounded above on E_r by a constant depending on $r \in (0,1)$. Combining these estimates with (3.27) we arrive, for a fixed $r \in (0,1)$ and every $z \in E_r$, to

$$\left|p_N(z\sqrt{N})\right| \leq C_1(r) N^{-1/4} L^N(r) |z|^N. \tag{3.30}$$

We set $z_n = z\sqrt{N/(N-1)}$ and note that, by convexity of $\mathbb{C} \setminus E_r$, $z_N \in E_r$ if so does z. Hence, applying (3.30) to p_{N-1}, we see that for every $z \in E_r$

$$\begin{aligned} &\left|p_{N-1}\big(z\sqrt{N}\big)\right| \\ &= \left|p_{N-1}\big(z_N\sqrt{N-1}\big)\right| \leq C_1(r)(N-1)^{-1/4} L^{N-1}(r)|z_N|^{N-1} \\ &= C_1(r)\big(1+1/(N-1)\big)^{(2N-3)/4} N^{-1/4} L^{N-1}(r)|z|^{N-1}. \end{aligned}$$

Since $\big(1+1/(N-1)\big)^{(2N-3)/4} \to e^{1/2}$ as $N \to \infty$ we conclude that

$$\left|p_{N-1}\big(z\sqrt{N}\big)\right| \leq C_2(r)(N)^{-1/4} L^{N-1}(r)|z|^{N-1}. \tag{3.31}$$

For every $r \in (0,1)$ the inequality

$$\Im z \geq r^{-1} - r$$

implies

$$|x(z)| \leq r.$$

Indeed, if $x(z) = |x(z)|e^{ib}$ with some real b then

$$\Im z = \Im\big(x^{-1}(z) + x(z)\big) = (|x(z)|^{-1} - |x(z)|)\sin b \geq r^{-1} - r,$$

proving our claim (we used that $|x(z)| < 1$, $\Im z > 0$ and, as a consequence, $\sin b > 0$). Now the assertion of the lemma follows from (3.30),(3.31). □

Lemma 4. *For every $\beta > 0$ and a certain constant $C(\beta), M(\beta)$ the inequality*

$$|N^{-1}\tilde{K}_N\big(z_1, z_2, N^{-1}\big)| \leq C(\beta) N^{-3/2} M^{2N-1}(\beta) 2^N \Gamma(N) \Im(z_1))^N (\Im(z_2))^N, \tag{3.32}$$

holds for every z_1, z_2 with $\Im z_i \geq \beta$ and $\Re z_i \geq \Im z_i$ $(i = 1,2)$.

Proof. Let

$$f_{a,p}(x) = \frac{1}{\Gamma(p)} a^p x^{p-1} e^{-ax}, p > 0, x > 0,$$

be the *gamma density* [6] with parameters $a > 0$, $p > 0$, Furthermore, let $\Re z \geq \Im z > 0$ and $P \geq 0$, $N \geq 1$ be integers. Then

(3.33)

$$\begin{aligned}
&\int_0^\infty |z+\theta|^{2P} e^{-N\Re(z+\theta)^2/2} d\theta \\
&= \int_{\Re z}^\infty |\xi + i\Im z|^{2P} e^{-N\Re(\xi+i\Im z)^2/2} d\xi \leq \int_{\Im z}^\infty |\xi + i\Im z|^{2P} e^{-N\Re(\xi+i\Im z)^2/2} d\xi \\
&= \int_{\Im z}^\infty (\xi^2 + (\Im z)^2)^P e^{-N(\xi^2-(\Im z)^2)/2} d\xi \\
&= e^{N(\Im z)^2} \int_{\Im z}^\infty (\xi^2 + (\Im z)^2)^P e^{-N(\xi^2+(\Im z)^2)/2} d\xi \\
&\leq (\Im z)^{-1} e^{N(\Im z)^2} 2^P N^{-(P+1)} \\
&\int_{\Im z}^\infty \big(N\big(\xi^2 + (\Im z)^2\big)/2\big)^P e^{-N(\xi^2+(\Im z)^2)/2} d\big(N\big(\xi^2 + (\Im z)^2\big)/2\big) \\
&= (\Im z)^{-1} e^{N(\Im z)^2} 2^P N^{-(P+1)} \int_{N(\Im z)^2}^\infty \eta^P e^{-\eta} d\eta \\
&= (\Im z)^{-1} e^{N(\Im z)^2} 2^P N^{-(P+1)} \Gamma(P+1) \int_{N(\Im z)^2}^\infty f_{1,P}(\eta) d\eta \\
&= (\Im z)^{-1} e^{N(\Im z)^2} 2^P N^{-(P+1)} \Gamma(P+1) e^{-N(\Im z)^2} \\
&\left(1 + \frac{N(\Im z)^2}{1!} + \cdots + \frac{[N(\Im z)^2]^P}{P!}\right) \\
&= (\Im z)^{-1} 2^P N^{-(P+1)} \Gamma(P+1) \left(1 + \frac{N(\Im z)^2}{1!} + \cdots + \frac{[N(\Im z)^2]^P}{P!}\right),
\end{aligned}$$

where we used a known formula ([6], p.11) while integrating $f_{1,P}$.

If $|\Im z| \geq \beta$ then $N(\Im z)^2 \geq 1$ for $N \geq \beta^{-2}$, and the above bound gives

$$\int_0^\infty |z+\theta|^{2P} e^{-N\Re(z+\theta)^2/2} d\theta \quad \leq e 2^P N^{-1} \Gamma(P+1) (\Im(z))^{2P-1}. \tag{3.34}$$

In particular for $P = N - 1$ and $P = N$ we have

$$\int_0^\infty |z+\theta|^{2(N-1)} e^{-N\Re(z+\theta)^2/2} d\theta \quad \leq e 2^{N-1} N^{-1} \Gamma(N) (\Im(z))^{2N-3}. \tag{3.35}$$

and

$$\int_0^\infty |z+\theta|^{2N} e^{-N\Re(z+\theta)^2/2} d\theta \quad \le e2^N \Gamma(N)(\Im(z))^{2N-1} \tag{3.36}$$

In view of (3.21), (3.22) we see that for $\Im z \ge \beta$

(3.37)

$$\begin{aligned}
&|N^{-1}\tilde{K}_N\big(z_1,z_2,N^{-1}\big)| \\
&= |N^{-1/2}K_N\big(\sqrt{N}z_1,\sqrt{N}z_2\big)| \\
&= (2N)^{-1/2}\Big|\int_0^\infty \big(\varphi_N(\sqrt{N}z_1+\tau)\varphi_{N-1}(\sqrt{N}z_2+\tau) \\
&+ \varphi_{N-1}(\sqrt{N}z_1+\tau)\varphi_N(\sqrt{N}z_2+\tau)\big)d\tau\Big| \\
&= 2^{-1/2}\Big|\int_0^\infty \big(\varphi_N(\sqrt{N}(z_1+\theta))\varphi_{N-1}(\sqrt{N}(z_2+\theta)) \\
&+ \varphi_{N-1}(\sqrt{N}(z_1+\theta))\varphi_N(\sqrt{N}(z_2+\theta))\big)d\theta\Big| \\
&\le C^2(\beta)N^{-1/2}M^{2N-1}\int_0^\infty \big(|z_1+\theta|^N e^{-N\Re(z_1+\theta)^2/4}|z_2+\theta|^{N-1}e^{-N\Re(z_2+\theta)^2/4} \\
&+ |z_1+\theta|^{N-1}e^{-N\Re(z_1+\theta)^2/4}|z_2+\theta|^N e^{-N\Re(z_2+\theta)^2/4}\big)d\theta \\
&\le C^2(\beta)N^{-1/2}M^{2N-1} \\
&\Bigg[\bigg(\int_0^\infty |z_1+\theta|^{2N}e^{-N\Re(z_1+\theta)^2/2}d\theta\int_0^\infty |z_2+\theta|^{2(N-1)}e^{-N\Re(z_2+\theta)^2/2}d\theta\bigg)^{1/2} \\
&+\bigg(\int_0^\infty |z_1+\theta|^{2(N-1)}e^{-N\Re(z_1+\theta)^2/2}\int_0^\infty \big(|z_2+\theta|^{2N}e^{-N\Re(z_2+\theta)^2/2}d\theta\bigg)^{1/2}\Bigg] \\
&= C^2(\beta)N^{-1/2}M^{2N-1} \\
&\Bigg[\bigg(\int_0^\infty |z_1+\theta|^{2N}e^{-N\Re(z_1+\theta)^2/2}d\theta\int_0^\infty |z_2+\theta|^{2(N-1)}e^{-N\Re(z_2+\theta)^2/2}d\theta\bigg)^{1/2} \\
&+\bigg(\int_0^\infty |z_1+\theta|^{2(N-1)}e^{-N\Re(z_1+\theta)^2/2}\int_0^\infty |z_2+\theta|^{2N}e^{-N\Re(z_2+\theta)^2/2}d\theta\bigg)^{1/2}\Bigg].
\end{aligned}$$

Combining this bound with (3.36) and (3.35) we find that for $\Im z \ge \beta$

$$\begin{aligned}
&|N^{-1}\tilde{K}_N\big(z_1,z_2,N^{-1}\big)| \\
&\le e2^{1/2}\beta^{-2}C^2(\beta)N^{-3/2}M^{2N-1}2^N\Gamma(N)\Im(z_1))^N(\Im(z_2))^N,
\end{aligned} \tag{3.38}$$

which proves the lemma. □

Corollary 3. *Let $\alpha < 2$, $A > 0$ be positive numbers. Then there exist such $C(\alpha, A)$ and $N_0 = N_0(\alpha, A)$ that for every natural $N \geq N_0$ and real $u \in (0, 2-\alpha]$, $t_1, t_2 \in [-uA, uA]$, and $H > H_{\beta/|u|}$ we have*

$$\begin{aligned} &|N^{-1}\tilde{K}_N\big((u+t_1/N)d(H),(u+t_2/N)d(H),N^{-1}\big)| \\ &\quad \leq C(\alpha,A)N^{-3/2}M(\alpha)^{2N-1}2^N\Gamma(N)H^{2N} \end{aligned} \tag{3.39}$$

Proof. Let $N_0 \geq A$. It is clear from 3.9 and 3.10 that $\Re d(H) \geq \Im d(H)$. We set $z_i = (u + t_i/N)d(H)$, $(i = 1, 2)$ and observe that $\Re z_i \geq \Im z_i$ and

$$\sqrt{\frac{\sqrt{1+H^2}-1}{2}} \leq \frac{H}{2}.$$

Then we see for $N \geq N_0$ and $i = 1, 2$ that

$$\Im(z_i) \leq u(1+(A/N))\sqrt{\frac{\sqrt{1+H^2}-1}{2}} \leq (1+(A/N))H$$

and

$$\Im(z_i)^N \leq C(A)H^N.$$

Then by Lemma 4

$$\begin{aligned} &|N^{-1}\tilde{K}_N\big((u+t_1/N)d(H),(u+t_2/N)d(H),N^{-1}\big)| \\ &\quad \leq C(\alpha,A)N^{-3/2}M(\alpha)^{2N-1}2^N\Gamma(N)H^{2N}. \end{aligned}$$

□

Proof of Theorem 1. As explained in Section 1 we need to show that

$$\begin{aligned} &\frac{1}{(Nw(u))^n}\int_{-\infty}^{\infty}\phi_{N^2}(h)R_{n,N}^{\mathrm{GUE},1/((1-ih\sqrt{2}/N)N)}\left(u+\frac{t_1}{Nw(u)},\dots,u+\frac{t_n}{N}\right)dh \\ &\qquad \underset{N\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi}}\det\left(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\right)_{i,j=1}^n \end{aligned} \tag{3.40}$$

uniformly in $u, t_1, \dots, t_n$ subject to the conditions formulated in the the statement of the Theorem. Throughout the proof we will assume these conditions satisfied and omit the arguments of $R_{n,N}^{\mathrm{GUE},\cdot}$ whenever this is possible. The existence of the integral in the left hand side of the last relation will be a consequence of our estimates. Note that $R_{n,N}^{\mathrm{GUE},s}(-u_1,\dots,-u_n) = R_{n,N}^{\mathrm{GUE},s}(u_1,\dots,u_n)$ since $K_N(-u_1,-u_2) = K_N(u_1,u_2)$. The latter property holds because φ_{2k} and φ_{2k+1} are, respectively, even and odd for every $k = 0, 1, \dots$. In view of this property it suffices to prove (3.40) for $u > 0$ only. Observe that for real h

$$\phi_{N^2}(-h) = \overline{\phi_{N^2}(h)}$$

and

$$R_{n,N}^{\mathrm{GUE},1/((1+ih\sqrt{2}/N)N)} = \overline{R_{n,N}^{\mathrm{GUE},1/((1-ih\sqrt{2}/N)N)}},$$

which shows that (3.40) can be established if we prove

$$\text{(3.41)} \quad \frac{1}{(Nw(u))^n}\int_0^\infty \Re\big(\phi_{N^2}(h)R_{n,N}^{\mathrm{GUE},1/((1+ih\sqrt{2}/N)N)}\big)dh \underset{N\to\infty}{\to} \frac{1}{2\sqrt{2\pi}}\det\Big(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\Big)_{i,j=1}^n.$$

It follows from the central limit theorem for densities that

$$\text{(3.42)} \quad \int_0^\infty \Re\big(\phi_{N^2}(h)\big)dh \underset{N\to\infty}{\to} \frac{1}{2\sqrt{2\pi}}.$$

Here $\phi_{N^2}(\cdot)$ is a prelimiting characteristic function and $1/2\sqrt{2\pi}$ is half the value of the limiting standard Gaussian density at 0. Thus, to prove (3.41) it suffices to check the following relation

$$\text{(3.43)} \quad \frac{1}{(Nw(u))^n}\int_0^\infty \Re\big(\phi_{N^2}(h)R_{n,N}^{\mathrm{GUE},1/((1+ih\sqrt{2}/N)N)}\big)dh - \det\Big(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\Big)_{i,j=1}^n\int_0^\infty \Re\big(\phi_{N^2}(h)\big)dh \underset{N\to\infty}{\to} 0$$

under the same uniformity constraints as above.

Let, for a given $\alpha\in(0,2)$, $\beta=\beta(\alpha)$ be a number existing according to Lemma 2. Let $\epsilon>0$ be a positive number, and $\delta=\delta(\epsilon)\in(0,\beta)$ be small enough to ensure

(3.44)
$$\left|\det\Big(\frac{\sin\big(\pi(t_i-t_j)d(H)\big)}{\pi(t_i-t_j)d(H)}\Big)_{i,j=1}^n - \det\Big(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\Big)_{i,j=1}^n\right| \le \epsilon\sqrt{2}/\pi$$

for all $u\in[-2+\alpha,2-\alpha]$, $|H|\le H_{\delta/u}$ and $t_1,\dots,t_n$ satisfying $|t_1|\le A|u|,\dots,|t_n|\le A|u|$. Such $\delta(\epsilon)$ exists because of continuity of the sin −kernel and the Lipschitz property at 0 of the function $d(\cdot)$ (recall that $d(0)=1$ and $d(\cdot)$ is smooth at 0). Performing the change

$H = h\sqrt{2}/N$ of variable we may write for every $\delta \in (0, \beta)$

(3.45)

$$\begin{aligned} I(u,N) =& \frac{1}{(Nw(u))^n}\int_0^\infty \Re\big(\phi_{N^2}(h)R_{n,N}^{\mathrm{GUE},1/((1+ih\sqrt{2}/N)N)}dh \\ &= \frac{N/\sqrt{2}}{(Nw(u))^n}\int_0^\infty \Re\big(\phi_{N^2}(NH/\sqrt{2})R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}\big)dH \\ &= \frac{N/\sqrt{2}}{(Nw(u))^n}\int_0^{H_{\delta/u}} \Re\big(\phi_{N^2}(NH/\sqrt{2})R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}\big)dH \\ &+ \frac{N/\sqrt{2}}{(Nw(u))^n}\int_{H_{\delta/u}}^{H_{\beta/u}} \Re\big(\phi_{N^2}(NH/\sqrt{2})R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}\big)dH \\ &+ \frac{N/\sqrt{2}}{(Nw(u))^n}\int_{H_{\beta/u}}^{\infty} \Re\big(\phi_{N^2}(NH/\sqrt{2})R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}\big)dH \\ &= I_1(\epsilon,u,N) + I_2(\epsilon,u,N) + I_3(\epsilon,u,N). \end{aligned}$$

In view of formulas (**??**) and (3.4) we have

$$R_{n,N}^{\mathrm{GUE},\sigma s}(x_1,\dots,x_n) = \sigma^{-n/2}R_{n,N}^{\mathrm{GUE},s}(\sigma^{-1/2}x_1,\dots,\sigma^{-1/2}x_n) \tag{3.46}$$

It follows from (2.6) that

$$R_{n,N}^{\mathrm{GUE},\sigma s}(x_1,\dots,x_n) = \int_0^\infty R_{n,N}^{\mathrm{HSE},u/N^2}\gamma_{N^2,\sigma s}(u)du, \tag{3.47}$$

where

(3.48)

$$\gamma_{m,s}(u) = \begin{cases} (2^{m/2}s^{-m/2}\Gamma(m/2))^{-1}u^{(m/2)-1}\exp\left(-u/(2s)\right), \text{if } u \ge 0, \\ 0, \text{if } u < 0. \end{cases}$$

Formula (3.46), on the one hand, and formulae (3.47) along with (3.48), on the other hand, can be used to obtain analytic continuations of $R_{n,N}^{\mathrm{GUE},\sigma s}(x_1,\dots,x_n)$ in the parameter σ to the domain $\Re\sigma > 0$. Since these continuations coincide on the real half-line $\sigma > 0$ they agree for $\Re\sigma > 0$. Thus we have

(3.49)

$$R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}(x_1,\dots,x_n) = d^n(H)R_{n,N}^{\mathrm{GUE},1/N}(x_1d(H),\dots,x_nd(H)),$$

and we see that

(3.50)
$$\begin{aligned} &N^{-n}|R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}(u+t_1/N,\dots,u+t_n/N)| \\ &= N^{-n}|d(H)|^n R_{n,N}^{\mathrm{GUE},1/N)}((u+t_1/N)d(H),\dots,(u+t_1/N)d(H))| \\ &\le n!|d(H)|^n \max_{0\le i<j\le n} |N^{-1}\tilde{K}((u+t_i/N)d(H),(u+t_j/N)d(H))|^n. \end{aligned}$$

By Corollary 2 we obtain for every $\epsilon \in (0,\beta)$

$$\begin{aligned} &I_2(\epsilon,u,N) \\ &\le \frac{N/\sqrt{2}}{(w(u))^n} n! C(\alpha,A)^n d^n(H_{e/u}) \int_{H_{\epsilon/u}}^{H_{\beta/u}} |\phi_{N^2}(NH/\sqrt{2})|dH \\ &\le \frac{N/\sqrt{2}}{(w(u))^n} n! C(\alpha,A)^n d^n(H_{e/u}) \int_{H_{\epsilon/u}}^{\infty} |\phi_{N^2}(NH/\sqrt{2})|dH. \end{aligned} \tag{3.51}$$

Since

$$|\phi_{N^2}(NH/\sqrt{2})| = (1+H^2)^{-(N^2/4)}$$

and

$$\begin{aligned} &\int_L^\infty (1+H^2)^{-(N^2/4)} dH \\ &\qquad < \frac{1}{(1+L^2)^{N^2/4-1}} \int_0^\infty (1+H^2) dH \\ &= \frac{\pi}{2(1+L^2)^{N^2/4-1}} \end{aligned} \tag{3.52}$$

we have

$$\int_L^\infty |\phi_{N^2}(NH/\sqrt{2})|dH < \frac{\pi}{2(1+L^2)^{N^2/4-1}}, \tag{3.53}$$

and with this bound in hands we obtain

$$I_2(\epsilon,u,N) \le \frac{\pi N/2\sqrt{2}}{(w(u))^n} n! C(\alpha,A)^n \frac{d^n(H_{\epsilon/u})}{\left(1+H_{\epsilon/u}^2\right)^{N^2/4-1}}. \tag{3.54}$$

As

$$|d(H)| = (1+H^2)^{1/4},$$

$$\inf_{u\in[-2+\epsilon,2-\epsilon]} |w(u)| = w(2-\epsilon) \tag{3.55}$$

and, in view of (3.12),

$$\sup_{u\in[-2+\epsilon,2-\epsilon]} |d(H_{\epsilon/u})| = \sqrt{1+2\big(\epsilon/(2-\epsilon)\big)^2}$$

we conclude from (3.54) that for $N \geq 2$

$$I_2(\epsilon, u, N) \leq C(\alpha, \epsilon, A, n)(1 + H^2_{\epsilon/(2-\epsilon)})^{-N^2/4} \tag{3.56}$$

We also have by (3.50), (3.39) and (3.8) for every $\epsilon \in (0, \beta)$

$$\begin{aligned}
&I_3(\epsilon, u, N) \\
&\leq \frac{N/\sqrt{2}}{(w(u))^n} n! C(\alpha, A)^n N^{-3n/2} M^{n(2N-1)} e^{nN} \Gamma^n(N) \\
&\int_{H_{\beta/u}}^{\infty} d^n(H) H^{2nN} |\phi_{N^2}(NH/\sqrt{2})| dH \\
&= \frac{N/\sqrt{2}}{(w(u))^n} n! C(\alpha, A)^n N^{-3n/2} M^{n(2N-1)} e^{nN} \Gamma^n(N) \\
&\int_{H_{\beta/u}}^{\infty} H^{2nN} (1 + H^2)^{(n/4)-(N^2/4)} dH \\
&< \frac{N/\sqrt{2}}{(w(u))^n} n! C(\alpha, A)^n N^{-3n/2} M^{n(2N-1)} e^{nN} \Gamma^n(N) \\
&\int_{H_{\beta/u}}^{\infty} (1 + H^2)^{(n/4)+nN-(N^2/4)} dH.
\end{aligned} \tag{3.57}$$

Assuming $A > 0$ and $N^2 - 4nN - n - 2 > 0$ we may write

$$\begin{aligned}
&\int_A^{\infty} H^{2nN} (1 + H^2)^{(n/4)-(N^2/4)} dH \\
&\quad < \int_A^{\infty} (1+H^2)^{(n/4)+nN-(N^2/4)-(1/2)} H dH \\
&\qquad = \frac{1}{2} \int_{A^2}^{\infty} (1 + t)^{(n/4)+nN-(N^2/4)-(1/2)} dt \\
&\qquad = \frac{2}{(N^2 - 4nN - n - 2)(1 + A^2)^{(N^2-4nN-n-2)/4}},
\end{aligned} \tag{3.58}$$

which gives for $N \geq N_0(n)$, together with (3.57), (3.55) and the classical bound for $\Gamma(N)$, the relations

$$
\begin{aligned}
&I_3(\epsilon, u, N) \\
&\leq \frac{N\sqrt{2}}{(w(u))^n} n! C(\alpha, A)^n N^{-3n/2} M^{n(2N-1)} e^{nN} \Gamma^n(N) \\
&\quad \times (N^2 - 4nN - n - 2)^{-1} \frac{1}{(1 + H^2_{\beta/u})^{(N^2-4nN-n-2)/4}} \\
&\qquad \leq C(\alpha, \epsilon, A, n) \frac{N^{-3n/2} M^{n(2N-1)} e^{nN} \Gamma^n(N)}{(1 + H^2_{\beta/(2-\epsilon)})^{(N^2-4nN)/4}} \\
&\qquad\quad \leq C(\alpha, \epsilon, A, n) \frac{\exp\big(c(\alpha, n) N \log N\big)}{(1 + H^2_{\beta/(2-\epsilon)})^{(N^2-4nN)/4}}.
\end{aligned} \tag{3.59}
$$

The latter expression clearly tends to 0 as $N \to \infty$. Note that this estimate implies that

$$\lim_{N\to\infty} I_3(\epsilon, u, N) = 0$$

uniformly in u and $t_1, \ldots, t_n$ subject to the conditions of the Theorem. Due to the bound (3.54) the same conclusion is also valid for $I_2(\epsilon, u, N)$.

Now we complete the proof by establishing (3.43) which can be rewritten as

$$I(u, N) - \int_0^\infty \Re\left(\det\left(\frac{\sin \pi(t_i - t_j)}{\pi(t_i - t_j)}\right)^n_{i,j=1} \phi_{N^2}(h)\right) dh \underset{N\to\infty}{\longrightarrow} 0.$$

Recall that we again omit the arguments of $R^{\mathrm{GUE},1/N)}_{n,N}$. All bounds and limit transitions hold to be uniformly with respect to $u \in [-2+\alpha, 2-\alpha]$ and $t_1, t_2, \ldots, t_n$ satisfying $|t_1| \leq uA, \ldots, |t_n| \leq |u|A$. Since

$$I(u, N) = I_1(\epsilon, u, N) + I_2(\epsilon, u, N) + I_3(\epsilon, u, N)$$

and

$$I_2(\epsilon, u, N) \to 0,\ I_3(\epsilon, u, N) \to 0$$

for every $\epsilon > 0$ as $N \to \infty$, we only need to bound

$$I_1(\epsilon, u, N) - \int_0^\infty \Re\left(\det\left(\frac{\sin \pi(t_i - t_j)}{\pi(t_i - t_j)}\right)^n_{i,j=1} \phi_{N^2}(h)\right) dh. \tag{3.60}$$

Observe that the sequence of functions $\big(|\phi_{N^2}(\cdot)|\big)_{N\geq 1}$ where $|\phi_{N^2}(h)| = 1/(1+2h^2/N^2)^{N^2/4}$ decreases in N. Therefore

$$
\begin{aligned}
&\frac{N}{\sqrt{2}}\int_0^\infty \phi_{N^2}(NH/\sqrt{2})dH \\
&\quad = \int_0^\infty \phi_{N^2}(h)dh = \int_0^\infty 1/(1+2h^2/N^2)^{N^2/4}dh \\
&\quad \leq \int_0^\infty 1/(1+h^2/2)dh = (1/\sqrt{2})\int_{-\infty}^\infty 1/(1+v^2)dv \\
&\quad = \pi/\sqrt{2}.
\end{aligned} \tag{3.61}
$$

Substituting $h = NH/\sqrt{2}$ in the integral and taking in view (3.45) and (3.44), we get from (3.60)

(3.62)

$$
\begin{aligned}
&\left| I_1(\epsilon,u,N) - \int_0^\infty \Re\bigg(\det\bigg(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\bigg)_{i,j=1}^n \phi_{N^2}(h)\bigg)dh\right| \\
&\leq \frac{N}{\sqrt{2}}\bigg(\int_0^{H_{\delta/u}}\left|\frac{R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}}{(Nw(u))^n} - \det\bigg(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\bigg)_{i,j=1}^n\right| \big|\phi_{N^2}(NH/\sqrt{2})\big|dH \\
&+ \det\bigg(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\bigg)_{i,j=1}^n \int_{H_{\delta/u}}^\infty \big|\phi_{N^2}(NH/\sqrt{2})\big|dH\bigg) \\
&\leq \frac{N}{\sqrt{2}}\int_0^{H_{\delta/u}}\left|\frac{R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}}{(Nw(u))^n}\right. \\
&\left. - \det\bigg(\frac{\sin\big(\pi(t_i-t_j)d(H)w(ud(H))/w(u)\big)}{\pi(t_i-t_j)d(H)}\bigg)_{i,j=1}^n\right| \big|\phi_{N^2}(NH/\sqrt{2})\big|dH \\
&+ \frac{N}{\sqrt{2}}\int_0^{H_{\delta/u}}\left|\det\bigg(\frac{\sin\big(\pi(t_i-t_j)d(H)w(ud(H))/w(u)\big)}{\pi(t_i-t_j)d(H)}\bigg)_{i,j=1}^n\right. \\
&\left. - \det\bigg(\frac{\sin\big(\pi(t_i-t_j)\big)}{\pi(t_i-t_j)}\bigg)_{i,j=1}^n\right| \big|\phi_{N^2}(NH/\sqrt{2})\big|dH \\
&+ \frac{N}{\sqrt{2}}\det\bigg(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\bigg)_{i,j=1}^n \int_{H_{\delta/u}}^\infty \big|\phi_{N^2}(NH/\sqrt{2})\big|dH \\
&= J_1(\epsilon,u,N) + J_2(\epsilon,u,N) + J_3(\epsilon,u,N).
\end{aligned}
$$

It follows from the determinantal formula (**??**) and Lemma 2 that

$$\sup_{0\le H\gamma\le H_{\delta/u}}\Bigg|\frac{R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}}{(Nw(u))^n} - \det\Bigg(\frac{\sin\big(\pi(t_i-t_j)d(H)w(ud(H))/w(u)\big)}{\pi(t_i-t_j)d(H)}\Bigg)_{i,j=1}^n\Bigg| \underset{N\to\infty}{\longrightarrow} 0 \tag{3.63}$$

uniformly in $u\in[-2+\alpha,2-\alpha]$, $|t_i|\le A|u|(i=1,\dots,n)$ and $H\in[-H_{\delta/|u|},H_{\delta/|u|}]$. Hence, we obtain

$$\begin{aligned}
&J_1(\epsilon,u,N)\\
&\le \sup_{0\le H\gamma\le H_{\delta/u}}\Bigg|\frac{R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}}{(Nw(u))^n} - \det\Bigg(\frac{\sin\big(\pi(t_i-t_j)d(H)w(ud(H))/w(u)\big)}{\pi(t_i-t_j)d(H)}\Bigg)_{i,j=1}^n\Bigg|\\
&\qquad \frac{N}{\sqrt{2}}\int_0^\infty \phi_{N^2}(NH/\sqrt{2})dH\\
&= \big(\pi/\sqrt{2}\big)\sup_{0\le H\le H_{\delta/u}}\Bigg|\frac{R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}}{(Nw(u))^n}\\
&\qquad - \det\Bigg(\frac{\sin\big(\pi(t_i-t_j)d(H)w(ud(H))/w(u)\big)}{\pi(t_i-t_j)d(H)}\Bigg)_{i,j=1}^n\Bigg| \underset{N\to\infty}{\longrightarrow} 0.
\end{aligned}$$

We also have by (3.53)

$$\begin{aligned}
&J_3(\epsilon,u,N)\\
&= \frac{N}{\sqrt{2}}\det\Bigg(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\Bigg)_{i,j=1}^n\int_{H_{\delta/u}}^\infty \big|\phi_{N^2}(NH/\sqrt{2})\big|dH\\
&\le \det\Bigg(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\Bigg)_{i,j=1}^n\frac{\pi N}{2\sqrt{2}(1+H_{\delta/u}^2)^{N^2/4-1}}\\
&\le \det\Bigg(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\Bigg)_{i,j=1}^n\frac{\pi N}{2\sqrt{2}(1+H_{\delta/(2-\alpha)}^2)^{N^2/4-1}}\\
&\qquad \underset{N\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Finally, we see from (3.44) and (3.61)

$$J_2(\epsilon,u,N)\le\epsilon.$$

Since ϵ is arbitrary small, this completes the proof.

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