LIMIT CORRELATION FUNCTIONS AT ZERO FOR FIXED TRACE RANDOM MATRIX ENSEMBLES

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ABSTRACT. The large-N limit of the eigenvalue correlation functions is examined in the neighborhood of zero for the spectrum of $N \times N-$Hermitian matrices chosen at random from the Hilbert-Schmidt sphere of appropriate radius. Dyson's famous $\sin \pi(t_1 - t_2)/\pi(t_1 - t_2)$-kernel asymptotics is extended to this class of matrix ensembles.

1. Introduction

Let $\mathcal{H}_N$ be the set of all $N \times N$ (complex) Hermitian matrices. $\mathcal{H}_N$ is a real Hilbert space of dimension N^2 with respect to the bilinear form $(A, B) \mapsto trAB$. Here trA denotes the trace of a square matrix $A = (a_{ij})_{i,j=1}^N$ so that $trA = \sum_{i=1}^N a_{ii}$. The Hilbert space structure determines the normalized Lebesgue measure l_N on $\mathcal{H}_N$ satisfying $l_N(Q) = 1$ for every cube $Q \subset \mathcal{H}_N$ with the edges of length 1. By definition, the standard Gaussian probability measure μ_N on $\mathcal{H}_N$ is specified by its density

$$g_N(A) = \frac{1}{(2\pi)^{N^2/2}} \exp\left(-\frac{1}{2}\, trA^2\right) \tag{1.1}$$

with respect to l_N. The set $\mathcal{H}_N$ supplied with the measure μ_N is known as the Gaussian Unitary Ensemble (GUE).

Let X be a random $N \times N$ Hermitian matrix (that is a random variable taking values in $\mathcal{H}_N$). We consider the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ of the random matrix X as exchangeable random variables whose joint distribution is a probability measure P_N^X on $\mathbb{R}^N$ which is invariant respective to all permutations of the coordinate axes. For every $n, 1 \leq$

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$n \leq N$, we denote by $P^X_{n,N}$ the projection of P^X_N to $\mathbb{R}^n$ induced by the orthogonal projection of $\mathbb{R}^N$ to its subspace spanned by any n of the coordinate axes of $\mathbb{R}^N$. The probability measure $P^X_{n,N}$ on $\mathbb{R}^n$ is also permutation invariant, and represents the joint distribution of n eigenvalues arbitrarily chosen from the whole collection of N eigenvalues of X. In particular, the measure $P^X_{1,N}$ describes the distribution of a single eigenvalue. By definition, the *n-point correlation measure* $\rho^X_{n,N}$ of a random matrix X is a non-normalized measure defined by the relation

$$\rho^X_{n.N} = \frac{N!}{(N-n)!} P^X_{n,N}. \tag{1.2}$$

If the measure $\rho^X_{n.N}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$, its Radon-Nikodym derivative $R^X_{n.N}$ is called the *n-point correlation function* of the random matrix X.

In particular, the measure $\rho^X_{1.N}$ has the total mass N. For a measurable set $E \subset \mathbb{R}^1$, the amount $\rho^X_{1.N}(E)$ expresses the expected number of eigenvalues belonging to E. The corresponding density with respect to the Lebesgue measure in $\mathbb{R}^1$, if it exists, is called the *eigenvalue density* or the *density of states* (sometimes the same names denote as well the *normalized* eigenvalue density whose integral over the line equals 1).

Now let X_N be a random $N \times N$ matrix from GUE. The classical results for GUE (valid also for many other ensembles) says that for $Y_N = \frac{1}{\sqrt{N}} X_N$ we have

$$P^{Y_N}_{1,N} \underset{N\to\infty}{\to} W, \tag{1.3}$$

where the measures converge in the weak sense and W is the standard Wigner measure on $[-2, 2]$ defined by the density

$$w(x) = (2\pi)^{-1}\sqrt{(4-x^2)_+}, x \in \mathbb{R}. \tag{1.4}$$

In terms of the correlation measures the same relation reads

$$\frac{1}{N}\rho^{Y_N}_{1,N} \underset{N\to\infty}{\to} W. \tag{1.5}$$

For the $n-$point correlation measures we have a similar relation

$$\frac{1}{N^n}\rho^{Y_N}_{n,N} \underset{N\to\infty}{\to} \underbrace{W \times \cdots \times W}_{n \text{ times}}, \tag{1.6}$$

which means that the eigenvalues get independent in the limit. However, the study of finer asymptotics near a point on the diagonal of $[-2, 2]^n$ for $n \geq 2$ shows that here the dependence takes a rather specific form known as a determinantal point process. Observe, that for the correlation function $R^X_{n.N}$ of GUE an explicit formula is known for every $n \leq N$. This formula, along with classical results on the asymptotic properties of the Hermite functions, makes it possible to determine the limit behavior of the $n-$point correlation function in a properly contracting neighborhood of a point of the form $(u,\dots,u)\in(-2,2)^n$. The following relation holds true for $R_{n,N}$ ([4, 3]): for every $t_1,\dots,t_n\in\mathbb{R}^1$

(1.7)
$$\lim_{N\to\infty}\frac{1}{(Nw(u))^n}R^{Y_N}_{n,N}\left(u+\frac{t_1}{Nw(u)},\dots,u+\frac{t_n}{Nw(u)}\right) = \det\left(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\right)^n_{i.j=1}.$$

The right hand side of this relation represents an example of the correlation function of a so-called determinantal (or fermionic) point process [7] defined by the kernel $(t_1,t_2)\mapsto(\sin\pi(t_1-t_2))/(\pi(t_1-t_2))$ in this particular case.

In the present note we are interested in a nondeterminantal ensemble of random Hermitian matrices known as the fixed trace ensemble. Let

$$S^r_N=\{A\in\mathcal{H}_N : trA^2=r^2\} \tag{1.8}$$

be the sphere in $\mathcal{H}_N$ of the radius $r>0$ centered at the origin. The sphere S^r_N carries a unique probability measure ν^r_N invariant with respect to all orthogonal linear transformations in the space $\mathcal{H}_N$. A well known result [4, 6] says that the random matrices $(Z_N)_{(N\geq1)}$, where Z_N is distributed according to $\nu^{\sqrt{N}}_N$, satisfy

$$\frac{1}{N}\rho^{Z_N}_{1,N}\underset{N\to\infty}{\to}W. \tag{1.9}$$

In this note we prove that the correlation functions $R^{Z_N}_{n,N}$ have the same determinantal limit with the kernel $(t_1,t_2)\mapsto\sin\pi(t_1-t_2)/\pi(t_1-t_2)$ as the GUE correlation functions. This result may be regarded as an instance of the measure concentration phenomenon or, in physicists language, Gibbs' equivalence of ensembles. However, we can not directly deduce from these concentration ideas the expected determinantal limit with the sin-kernel on the whole interval $(-2,2)$. The goal can be achieved in the form of a tauberian theorem making use the local form of the concentration phenomenon along with the asymptotics of the Hermite reproducing kernel in the complex plane. This is the subject of a companion paper [2] by the first two authors. Curiously enough, the method of the latter paper does not work at 0 while a much more elementary approach does (with some drawbacks discussed below), which is the raison d'être for the present note.

Thus, here we establish the sin-kernel asymptotics at 0, combining the concentration idea in the form of well known large deviation estimates with simple scaling transformations.

Note that we are dealing here with a sequence of finite measures whose total charges tend to infinity. These measures converge to a locally finite measure in $\mathbb{R}^n$. In the paper [2] we prove the uniform convergence of densities on compact subsets of $\mathbb{R}^n$. In the present note we deal with much weaker topology: this is the convergence of integrals for every compactly supported continuous function. We conjecture that in fact near zero we have the uniform convergence of the densities as well, and this particular role of 0 is a consequence of the techniques used. Moreover, in the literature the point 0 was sometimes considered as the most agreeable one to the studying the spectral asymptotics.

Thus, we prove here the following result.

Theorem. *Let $\rho_{n,N}^{\nu,\sqrt{N}}$ be the $n-$point correlation measure of the eigenvalues for a random matrix uniformly distributed on the sphere $S_N^{\sqrt{N}}$. Then for every continuous function f on $\mathbb{R}^n$ with compact support we have*

$$\lim_{N\to\infty}\int_{\mathbb{R}^n} f(Nt_1/\pi,\ldots,Nt_n/\pi)\rho_{n,N}^{\nu,\sqrt{N}}(dt_1\times\cdots\times dt_n)$$
$$=\int_{\mathbb{R}^n} f(t_1,\ldots,t_n)\det\left(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\right)_{i,j=1}^n dt_1\cdots dt_n.$$

Remark 1. Here we give some comments on the statement of Theorem. Relation (1.10) is the specialization to the case $u=0$ of the more general relation

$$\lim_{N\to\infty}\int_{\mathbb{R}^n} f(Nw(u)(t_1-u),\ldots,Nw(u)(t_n-u))\rho_{n,N}^{\nu,\sqrt{N}}(dt_1\times\cdots\times dt_n)$$
$$=\int_{\mathbb{R}^n} f(t_1,\ldots,t_n)\det\left(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\right)_{i.j=1}^n dt_1\cdots dt_n,$$

which is expected to be valid for every $u\in(-2,2)$. For a quite different class of ensembles the result of this form is proved in [3] for u from some symmetric subinterval of $(-2,2)$.

Remark 2. Note that the correlation measure $\rho_{n,N}^{\nu,\sqrt{N}}$ is absolutely continuous with respect to the Lebesgue measure if $n<N$ (this follows from the explicit formula for $\rho_{n,N}^{\nu,\sqrt{N}}$: see, for example, [1]). Denote by $R_{n,N}^{\nu,\sqrt{N}}$ the corresponding correlation function. Relation (1.10) can be rewritten in terms of $R_{n,N}^{\nu,\sqrt{N}}$, and after the change of the integration

variables it takes the form

$$\big(\pi/N\big)^n \int_{\mathbb{R}^n} f(t_1,\dots,t_n) R^{\nu,\sqrt{N}}_{n,N}\big(\pi t_1/N,\dots,\pi t_n/N\big) dt_1,\cdots dt_n$$
$$\underset{N\to\infty}{\to} \int_{\mathbb{R}^n} f(t_1,\dots,t_n) \det\left(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\right)^n_{i.j=1} dt_1\cdots dt_n.$$

2. Proof

Let X_N be, as above, a random matrix with the standard Gaussian probability distribution μ_N (GUE) on $\mathcal{H}_N$. We set $T_N = trX_N^2$ and denote by $\alpha_N(\cdot)$ the probability density of T_N. We call a measure on $\mathcal{H}_N$ orthogonally invariant if it is invariant under every orthogonal linear transformation in the real Hibert space $\mathcal{H}_N$. The standard Gaussian probability measure μ_N on $\mathcal{H}_N$ (GUE) admits the integral representation of the form

$$\mu_N = \int_0^\infty \nu_N^{\sqrt{s}} \alpha_N(s) ds, \tag{2.1}$$

where for every $r > 0$ ν_N^r is a (unique) orthogonally invariant probability measures on the sphere S_n^r and $\alpha_N(\cdot)$ was introduced above. The random variable T_N can be represented as a sum of N^2 independent standard Gaussian random variables. Thus, T_N has the familiar $\chi^2_{N^2}$ distribution. Observe that

$$ET_N = N^2,\ DT_N^2 = E(T_N - ET_N)^2 = 3N^2.$$

Lemma 1. *Let ξ be a random variable with the standard Gaussian distribution and $q \in [0,1)$ be a real number. Then for every real number $t \in [-q/2, q/2]$*

$$E\exp\left(t(\xi^2-1)\right) \le \exp\left(\frac{1}{2}\frac{2}{1-q}t^2\right). \tag{2.2}$$

Proof. It is easily seen that for every real $t < 1$

$$E\exp\left(t\xi^2\right) = \frac{1}{\sqrt{1-2t}}. \tag{2.3}$$

We have

$$\log E\exp\left(t(\xi^2-1\right) = -t - \frac{1}{2}\log(1-2t). \tag{2.4}$$

Assuming $t \in [0, q/2]$ we obtain

$$\frac{d}{dt}\left(-t - \frac{1}{2}\log(1-2t)\right) = -1 - \frac{-2}{2(1-2t)} = \frac{2t}{1-2t} \le \frac{2t}{1-q},$$

which gives for $t \in [0, \leq q/2]$ the bound

$$-t - \frac{1}{2}\log(1-2t) \leq \frac{t^2}{1-q}.$$

Combining this inequality with (2.4) gives

$$E \exp\left(t(\xi^2 - 1)\right) \leq \exp\left(\frac{1}{2}\frac{2}{1-q}t^2\right) \tag{2.5}$$

for $t \in [0, \leq q/2]$.

Quite similarly, setting $t = -u$ for $u \in [-q/2, 0]$, we obtain

$$\begin{aligned}\log E \exp\left(u(\xi^2 - 1\right) &= \log E \exp\left(t(1 - \xi^2\right) \\ &= t - \frac{1}{2}\log(1+2t) \leq t^2 < \frac{t^2}{1-q},\end{aligned}$$

which completes the proof. □

In the sequel we shall need the following result by V.V. Petrov (a part of Th.15, Ch.3 in [5]) strengthening the Bernstein inequality.

Lemma 2. *Assume that for independent random variables $\eta_1, \dots, \eta_m$ there exist positive constants $g_1, \dots, g_m, T$ such that*

$$E \exp t(\eta_k) \leq \exp\left(\frac{1}{2}g_k t^2\right) \quad (k = 1, \dots, m)$$

for $0 \leq t \leq T$. Set $S = \sum_{k=1}^m \eta_k$ and $G = \sum_{k=1}^m g_k$. Then

$$P(S \geq x) \leq \exp\left(-x^2/(2G)\right)$$

if $0 \leq x \leq GT$.

Lemma 3. *For every $y > 0$ we have*

$$P\left(\left|{T_N}^{1/2}/N - 1\right| > y\right) \leq 2\exp\left(-\frac{N^2 y^2}{4(1+y)}\right). \tag{2.6}$$

Proof. Let $q \in [0, 1)$ be a number to specify later. We apply Lemma 2 to the array $(\eta_k = \xi_k^2 - 1)_{k=1,\dots,N^2}$, where $(\xi_k)_{k=1,\dots,N^2}$ are independent standard Gaussian variables. Thus, we set $m = N^2$, $S = T_N$ and $T = q/2$. Let $y \in [0, q/(1-q)]$ be a real number. According to Lemma 1 we can take $g_k = 2/(1-q)$ for $k = 1, \dots, N^2$. Then $G = 2N^2/(1-q)$. Finally, we set $x = N^2 y$ and observe that

$$x \leq N^2 q/(1-q) = (2N^2/(1-q))(q/2) = GT.$$

Now by Lemma 2 we have

$$P(T_N - N^2 > N^2 y) \leq \exp\left(-N^2 y^2 (1-q)/4\right)$$

and

$$P(T_N - N^2 < -N^2 y) \le \exp\left(-N^2 y^2 (1-q)/4\right).$$

Thus, we established the inequality

$$P\left(\left|\frac{T_N}{N^2} - 1\right| > y\right) \le 2\exp\left(-N^2 y^2(1-q)/4\right). \tag{2.7}$$

Settiing now $q = y/(1+y)$ in (2.7), we obtain

$$P\big(\left|T_N/N^2 - 1\right| > y\big) \le 2\exp\left(-\frac{N^2 y^2}{4(1+y)}\right),$$

which is equivalent to

$$P\Big(\big\{{T_N}^{1/2}/N < \sqrt{(1-y)_+}\big\} \cup \big\{{T_N}^{1/2}/N > \sqrt{1+y}\big\}\Big) \le 2\exp\left(-\frac{N^2 y^2}{4(1+y)}\right).$$

Then the assertion of the Lemma follows from the relations

$$\begin{aligned} &P\big(\left|{T_N}^{1/2}/N - 1\right| > y\big) \\ &= P\Big(\big\{{T_N}^{1/2}/N < (1-y)_+\big\} \cup \big\{{T_N}^{1/2}/N > 1+y\big\}\Big) \\ &\le P\Big(\big\{{T_N}^{1/2}/N < \sqrt{(1-y)_+}\big\} \cup \big\{{T_N}^{1/2}/N > \sqrt{1+y}\big\}\Big). \end{aligned}$$

□

Proof of Theorem. Remind that μ_N is the distribution of a GUE random $N \times N-$ matrix X_N. Further, let $X_N^\sigma = \sigma X_N$ be a multiple of X_N, and μ_N^σ be the corresponding distribution (so that $X_N = X_N^1$ and $\mu_N = \mu_N^1$). We shall denote by $\rho_{n,N}^{\mu,\sigma}$ and $\rho_{n,N}^{\nu,r}$ the correlation measures for the eigenvalue distributions corresponding to the measures μ_N^σ and ν_N^r on $\mathcal{H}_N$. The random matrix $X_N^{1/\sqrt{N}}$ is distributed according to the measure $\mu_N^{1/\sqrt{N}}$ which disintegrates in the form

$$\mu_N^{1/\sqrt{N}} = E\nu_N^{\sqrt{T_N/N}}. \tag{2.8}$$

The relation (2.8) implies a similar integral relation for the correlation measures:

$$\rho_{n,N}^{\mu,1/\sqrt{N}} = E\rho_{n,N}^{\nu,\sqrt{T_N/N}}. \tag{2.9}$$

Let f be a continuous real-valued function on $\mathbb{R}^n$ with a compact support contained in an Eucidean ball B_R of radius R centered at 0, and let $M = \sup_{x\in\mathbb{R}^n} |f(x)|$. Let us given an $\epsilon > 0$. Since the function f is uniformly continuous, there exists such $\delta' > 0$ that $|f(x) - f(y)| \le \epsilon$ whenever $|x - y| < \delta'$ (here $|\cdot|$ denotes the Euclidean norm). Set

$\delta = \min(\delta'/(3R), 1/2)$, and let $\chi^{(1)}_{\delta,N}$ and $\chi^{(2)}_{\delta,N}$ be the indicators of the events $\{|\sqrt{T_N}/N - 1| > \delta\}$ and $\{|\sqrt{T_N}/N - 1| \le \delta\}$. We have

$$\int_{\mathbb{R}^n} f(Nt_1/\pi, \dots, Nt_n/\pi)\rho^{\mu,1/\sqrt{N}}_{n,N}(dt_1 \times \cdots \times dt_n)$$

$$= E\int_{\mathbb{R}^n} f(Nt_1/\pi, \dots, Nt_n/\pi)\rho^{\nu,\sqrt{T_N/N}}_{n,N}(dt_1 \times \cdots \times dt_n),$$

and we can write

$$\begin{aligned}\int_{\mathbb{R}^n} f(Nt_1/\pi, \dots, Nt_n/\pi)\rho^{\mu,1/\sqrt{N}}_{n,N}(dt_1 \times \cdots \times dt_n)& \\ = I^f_{1,N}(\delta) + I^f_{2,N}(\delta),&\end{aligned} \tag{2.10}$$

where

$$I^f_{1,N}(\delta) = E\int_{\mathbb{R}^n} \chi^{(1)}_{\delta,N} f(Nt_1/\pi, \dots, Nt_n/\pi)\rho^{\nu,\sqrt{T_N/N}}_{n,N}(dt_1 \times \cdots \times dt_n), \tag{2.11}$$

and

$$I^f_{2,N}(\delta) = E\int_{\mathbb{R}^n} \chi^{(2)}_{\delta,N} f(Nt_1/\pi, \dots, Nt_n/\pi)\rho^{\nu,\sqrt{T_N/N}}_{n,N}(dt_1 \times \cdots \times dt_n). \tag{2.12}$$

In view of Lemma 3 the summand $I^f_{1,N}(\delta)$ can be estimated as

$$\begin{aligned}I^f_{1,N}(\delta) &\le M\frac{N!}{(N-n)!}P\{|\sqrt{T_N}/N - 1| > \delta\} \\ &\le 2M\frac{N!}{(N-n)!}\exp\left(-\frac{N^2\delta^2}{4(1+\delta)}\right),\end{aligned}$$

which implies

$$I^f_{1,N}(\delta) \underset{N\to\infty}{\to} 0. \tag{2.13}$$

It follows from relations (2.10), (2.13) and the well-known results for the GUE (see, for example, [3]) that

$$I^f_{2,N}(\delta) \underset{N\to\infty}{\to} \int_{\mathbb{R}^n} f(t_1, \dots, t_n)\det\left(\frac{\sin\pi(t_i - t_j)}{\pi(t_i - t_j)}\right)^n_{i.j=1} dt_1 \cdots dt_n. \tag{2.14}$$

On the other hand we have

$$I^f_{2,N}(\delta) = E\int_{\mathbb{R}^n} \chi^{(2)}_{\delta,N} f(Nt_1/\pi,\dots,Nt_n/\pi)\rho^{\nu,\sqrt{T_N/N}}_{n,N}(dt_1\times\cdots\times dt_n)$$

$$=P\{|\sqrt{T_N}/N-1|\le\delta\}\int_{\mathbb{R}^n} f(Nt_1/\pi,\dots,Nt_n/\pi)\rho^{\nu,\sqrt{N}}_{n,N}(dt_1\times\cdots\times dt_n)$$

$$+E\Bigg(\int_{\mathbb{R}^n} \chi^{(2)}_{\delta,N} f(Nt_1/\pi,\dots,Nt_n/\pi)\rho^{\nu,\sqrt{T_N/N}}_{n,N}(dt_1\times\cdots\times dt_n)$$

$$-\chi^{(2)}_{\delta,N}\int_{\mathbb{R}^n}\big(f(Nt_1/\pi,\dots,Nt_n/\pi)\rho^{\nu,\sqrt{N}}_{n,N}(dt_1\times\cdots\times dt_n)\Bigg),$$

where $P(|\sqrt{T_N}/N-1|\le\delta)\underset{N\to\infty}{\to}1$ by Lemma 3. Comparing the last two relations and (2.14), we see that the conclusion of Theorem holds whenever

(2.15)

$$\lim_{\delta\to 0}\limsup_{N\to\infty} E\Bigg(\int_{\mathbb{R}^n} \chi^{(2)}_{\delta,N} f(Nt_1/\pi,\dots,Nt_n/\pi)\rho^{\nu,\sqrt{T_N/N}}_{n,N}(dt_1\times\cdots\times dt_n)$$

$$-\chi^{(2)}_{\delta,N}\int_{\mathbb{R}^n}\big(f(Nt_1/\pi,\dots,Nt_n/\pi)\rho^{\nu,\sqrt{N}}_{n,N}(dt_1\times\cdots\times dt_n)\Bigg)=0.$$

To complete the proof of Theorem we need a one more lemma. Remind that B_R denotes the ball of the radius R in $\mathbb{R}^n$ centered at zero.

Lemma 4. *For any $R>0$ the sequence $(\rho^{\nu,\sqrt{N}}_{n,N}(B_{R/N}))_{N=1.2,\dots}$ is bounded.*

Proof. Let $\delta\in(0,1)$ be a real number and let γ be a smooth decreasing function on $[0,\infty)$ such that $\gamma(v)=1$ for $v\in[0,R)$ and $\gamma(v)=0$ for $v\ge(1+\delta)R$. Set $\varphi(x)=\gamma(|x|)$ for $x=(x_1,\dots,x_n)\in\mathbb{R}^n$. It follows from (2.14) and the definition of $I^f_{2,N}(\delta)$ that for every $\delta\in(0.1)$ and $k>0$

(2.16)

$$\limsup_{N\to\infty} E\int_{\mathbb{R}^n} \chi^{(2)}_{\delta,N}\varphi(kNt_1,\dots,kNt_n)\rho^{\nu,\sqrt{T_N/N}}_{n,N}(dt_1\times\cdots\times dt_n)<\infty$$

Then, making use of the fact that the function $\alpha\mapsto\varphi(\alpha x_1,\dots,\alpha x_n)$ decreases for $\alpha>0$, we have

$$\rho^{\nu,\sqrt{N}}_{n,N}(B_{R/N})\le\int_{\mathbb{R}^N}\varphi(Nt_1,\dots,Nt_n)\rho^{\nu,\sqrt{N}}_{n,N}(dt_1,\times\cdots\times dt_n)$$

$$\le\frac{E\int_{\mathbb{R}^n}\chi^{(2)}_{\delta,N}\varphi(\frac{\sqrt{T_N}}{N(1+\delta)}Nt_1,\dots,\frac{\sqrt{T_N}}{N(1+\delta)}Nt_n)\rho^{\nu,\sqrt{N}}_{n,N}(dt_1\times\cdots\times dt_n)}{P(\{|\frac{\sqrt{T_N}}{N}-1|\le\delta\})}$$

$$= \frac{E \int_{\mathbb{R}^n} \chi^{(2)}_{\delta,N} \varphi(Nt_1/(1+\delta), \dots, Nt_n/(1+\delta)) \rho^{\nu,\sqrt{T_N/N}}_{n,N}(dt_1 \times \cdots \times dt_n)}{P(\{|\sqrt{T_N}/N - 1| \le \delta\})}.$$

In the last fraction the numerator is bounded by (2.16) while the denominator tends to 1 by Lemma 3. □

Observe that under the above assumptions about f for every $\alpha \in [1-\delta, 1+\delta]$ and every $x \in \mathbb{R}^n$ we have

$$|f(\alpha x) - f(x)| \le \epsilon. \tag{2.17}$$

Indeed, the functions $f(\cdot)$, $f(\alpha \cdot)$ both vanish outside the ball $B_{(1-\delta)^{-1}R}$. On the orther hand, if at least one of the the relations $f(x) \neq 0$ and $f(\alpha x) \neq 0$ holds, then $|x| \le (1+\delta)(1-\delta)^{-1}R$. In the first case $f(\alpha x) = f(x) = 0$, while in the second case $|\alpha x - x| = |1-\alpha||x| \le \delta(1+\delta)(1-\delta)^{-1}R \le 3\delta R < \delta'$. Now we have

$$\begin{aligned} &\bigg| E\bigg(\int_{\mathbb{R}^n} \chi^{(2)}_{\delta,N} f(Nt_1/\pi, \dots, Nt_n/\pi) \rho^{\nu,\sqrt{T_N/N}}_{n,N}(dt_1 \times \cdots \times dt_n) \\ &\quad - \chi^{(2)}_{\delta,N} \int_{\mathbb{R}^n} \big(f(Nt_1/\pi, \dots, Nt_n/\pi) \rho^{\nu,\sqrt{N}}_{n,N}(dt_1 \times \cdots \times dt_n) \bigg) \bigg| \\ &\le \int_{\mathbb{R}^n} E\chi^{(2)}_{\delta,N} \big| f((\sqrt{T_N}/N)Nt_1/\pi, \dots, (\sqrt{T_N}/N)Nt_n/\pi) \\ &\quad - f(Nt_1/\pi, \dots, Nt_n/\pi) \big| \rho^{\nu,\sqrt{N}}_{n,N}(dt_1 \times \cdots \times dt_n) \\ &\le \epsilon\ \rho^{\nu,\sqrt{N}}_{n,N}(B_{(1-\delta)^{-1}R}) \le \epsilon\ \rho^{\nu,\sqrt{N}}_{n,N}(B_{2R}). \end{aligned} \tag{2.18}$$

In view of Lemma 4, this establishes (2.15) and completes the proof. □

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