Edge-reinforced random walk on one-dimensional periodic graphs

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Abstract

In the present paper, linearly edge-reinforced random walk is studied on a large class of one-dimensional periodic graphs satisfying a certain reflection symmetry. It is shown that the edge-reinforced random walk is recurrent. Estimates for the position of the random walker are given. The edge-reinforced random walk has a unique representation as a random walk in a random environment, where the random environment is given by random weights on the edges. It is shown that these weights decay exponentially in space. The distribution of the random weights equals the distribution of the asymptotic proportion of time spent by the edge-reinforced random walker on the edges of the graph.

The results generalize work of the authors in [MR05c], [Rol06], and [MR05a] to a large class of graphs and to periodic initial weights with a reflection symmetry. [4] [5]

1 Introduction

Linearly edge-reinforced random walk (errw) on a locally finite connected graph $G = (V, E)$ with vertex set V and set of undirected edges E is defined as follows: Initially, every edge is assigned in a non-random way a strictly positive number as a weight. The edge-reinforced random walk is a discrete-time nearest-neighbor random walk on G. In each discrete time step, the random walker jumps from its current location to a nearest-neighbor point with probability proportional to the weight of the chosen edge. Each time an edge is traversed, its weight is increased by 1.

The model was introduced by Diaconis in 1986. For the edge-reinforced random walk on $\mathbb{Z}^d$, $d \geq 2$, with identical initial weights, many questions are unsolved. For instance, it is not known whether on $\mathbb{Z}^2$ the process returns to its starting point infinitely often with probability one.

The area of random walks with reinforcement has attracted considerable attention during the past years. Many different models have been studied. For overviews on the subject see e.g. [MR06], [Pem01], and [Dav99].

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In the present article, we study the edge-reinforced random walk on a large class of one-dimensional reflection-symmetric periodic graphs. Special cases are strips of finite width, i.e. $\mathbb{Z} \times \{1, 2, \dots, d\}$ for $d \in \mathbb{N}$, and $\mathbb{Z} \times G'$ with G' an arbitrary finite graph. The initial weights of the edge-reinforced random walk are assumed to be periodic and reflection-symmetric. As a special case, all initial weights being equal to the same strictly positive constant b are allowed in all our results.

In [MR05c], it was shown that the edge-reinforced random walk on the ladder $\mathbb{Z} \times \{1, 2\}$ with all initial weights equal to the same constant $b > 3/4$ is recurrent. For the edge-reinforced random walk on $\mathbb{Z} \times T$ with a finite tree T and constant large initial weights, recurrence was shown in [Rol06]. The present paper generalizes these results: We prove recurrence, among others, for the edge-reinforced random walk on $\mathbb{Z} \times G'$ for any arbitrary finite graph G' with constant initial weights $b > 0$. Our result covers also certain periodic initial weights.

In [MR05a], a more detailed analysis of the edge-reinforced random walk with large constant initial weights on $\mathbb{N}_0 \times \{1, 2\}$ and on $\mathbb{N}_0 \times T$ with a finite tree T was given. Some of the results of that paper are generalized below, among others, to the edge-reinforced random walk with identical initial weights on $\mathbb{Z} \times G'$ with an arbitrary finite graph G'.

The techniques used below are different from the methods in [MR05c], [Rol06], and [MR05a]. To analyze the edge-reinforced random walk on the infinite periodic graph, we first study the process on finite periodic graphs. On these finite graphs, there is a known representation of the edge-reinforced random walk as a random walk in a random environment, given by random weights on the edges. The distribution of the random environment is explicitly known. In [MR05c], [Rol06], and [MR05a], this random environment was studied using a transfer operator method, which was technically quite involved due to the complicated dependence structure of the random environment. In the present paper, we pursue a different approach: we study a symmetric interpolation of the random environments for two different starting points using a deformation argument. One advantage of this new approach is that we do not need the sophisticated bounds for the density of the random environment from [MR05c] and [Rol06]. Another advantage of the current approach is that it needs only a few existing results about the edge-reinforced random walk: The explicit form of the distribution of the random environment is used, but only some simple calculations have to be performed with it. Furthermore, we use a tail estimate from [MR05d] for quotients of the random weights constituting the random environment.

The method we present below, is quite flexible. A more complicated variation of the argument was used in [MR05b] to prove results for the edge-reinforced random walk on $\mathbb{Z}^2$ with constant initial weights. There, some decay properties of the random environment are derived and estimates for some hitting probabilities of the edge-reinforced random walk are presented.

2 Main results

2.1 Definition of the underlying graph

Let $G = (V, E)$ be a connected, undirected, locally finite graph without direct loops. In the following, we assume that there is a free group action $\Gamma : \mathbb{Z} \times V \to V$ of the group $(\mathbb{Z}, +)$ by automorphisms of G. Furthermore, we assume that the group action Γ has only finitely many orbits. In more concrete terms, G can be described – up to isomorphism – as follows:

(G1) Let W be a finite set and define the vertex set $V = \mathbb{Z} \times W$.

Thus, the vertex set V consists of infinitely many copies of W. Intuitively speaking, W labels the finitely many orbits.

(G2) Take a finite set $\mathcal{E} \subset W \times W \times \mathbb{Z}$ such that for all $(u, w, z) \in \mathcal{E}$, one has $(w, u, -z) \in \mathcal{E}$, and define the set of edges

$$E = \{\{(i, u), (z + i, w)\} : (u, w, z) \in \mathcal{E}, i \in \mathbb{Z}\}. \tag{2.1}$$

(G3) Define the graph $G = (V, E)$.

The group action $\Gamma : \mathbb{Z} \times V \to V$ is then given by $\Gamma(z, (i, u)) = (z + i, u)$. Define the *level* of $(i, w) \in V$ to be i and write $\text{level}(i, w) = i$. The set of vertices on level 0 forms a set of representatives for the orbits of the group action Γ.

The condition (G2) guarantees that the i-th copy of u in V is connected to the $(z+i)$-th copy of w in V if *and only if* $(u, w, z) \in \mathcal{E}$. The only reason why we require $(u, w, z) \in \mathcal{E}$ iff $(w, u, -z) \in \mathcal{E}$ is to simplify the notation. Because of (G2), G is periodic. It is automatically undirected and locally finite. But we need to assume that

(G4) G is connected and has no direct loops (i.e. for all $w \in W$, $(w, w, 0) \notin \mathcal{E}$).

Fix a vertex $0 \in V$ on level 0; it will be the starting point of the edge-reinforced random walk. In addition to the above, we assume that the graph G has the following reflection symmetry:

(G5) For all $i \in \mathbb{Z}$, there are $w \in W$ and an automorphism ϕ of the graph G that exchanges 0 and (i, w).

2.2 Definition of the edge-reinforced random walk

Initially, every edge of the graph G is assigned a weight in the following reflection-symmetric way:

(A1) For every $(u, w, z) \in \mathcal{E}$, let $a(u, w, z) > 0$ be such that $a(u, w, z) = a(w, u, -z)$, and define the initial weight of the edges $e_i = \{(i, u), (z + i, w)\}$ by $a_{e_i} := a(u, w, z)$ for every $i \in \mathbb{Z}$.

(A2) Assume that for all $i \in \mathbb{Z}$, there is an automorphism ϕ as in (G5) such that $a_{\{u,u'\}} = a_{\{\phi(u),\phi(u')\}}$ for all $\{u, u'\} \in E$.

Thus, the weights $(a_e)_{e\in E}$ may be non-constant, but they are assumed to be periodic with some symmetry property.

The edge-reinforced random walk on G with initial weights $a = (a_e)_{e\in E} \in (0, \infty)^E$, starting in 0 is a stochastic process $(X_t)_{t\in\mathbb{N}_0}$ with law $P = P^G_{0,a}$ satisfying the following properties:

$$P[X_0 = 0] = 1, \tag{2.2}$$

$$P[X_{t+1} = v|X_0, \dots, X_t] = \frac{w_{\{X_t,v\}}(t)}{\sum_{e\ni X_t} w_e(t)} 1_{\{X_t,v\}\in E} \tag{2.3}$$

for $t \in \mathbb{N}_0$ and $v \in V$, where

$$w_e(t) = a_e + \sum_{s=0}^{t-1} 1_{\{X_s,X_{s+1}\}=e} \tag{2.4}$$

denotes the weight of the edge $e \in E$ at time t. We realize X_t as canonical projection to the t-th coordinate on the set of admissible paths $\subseteq V^{\mathbb{N}_0}$.

2.3 Special graphs

Let us give some examples for W and $\mathcal{E}$ such that the graph $G = (V, E)$ defined by (G1)–(G3) satisfies (G4) and (G5):

(a) For $W = \{1\}$ and $\mathcal{E} = \{(1, 1, 1), (1, 1, -1)\}$, $(X_t)_{t\in\mathbb{N}_0}$ is a nearest-neighbor edge-reinforced random walk on $\mathbb{Z}$. This process is well-understood.

(b) For $W = \{1\}$ and $\mathcal{E} = \{(1, 1, i) : i \in \mathcal{I}\}$ with some finite set $\mathcal{I} = -\mathcal{I} \subset \mathbb{Z} \setminus \{0\}$ with $|\mathcal{I}| \geq 4$, $(X_t)_{t\in\mathbb{N}_0}$ is an edge-reinforced random walk on $\mathbb{Z}$ with bounded jumps. To our knowledge, this process has not been studied before.

(c) Let $G' = (W, E')$ be a finite connected undirected graph with vertex set W and edge set E'. Assume that G' has no direct loops. We denote the graph G defined by (G1)–(G3) with $\mathcal{E} = \{(w, w, 1), (w, w, -1) : w \in W\} \cup \{(u, w, 0) : \{u, w\} \in E'\}$ by $\mathbb{Z} \times G'$. If G' consists of the vertices 1 and 2 and the edge $\{1, 2\}$ connecting them, then, $\mathbb{Z}\times G'$ is just the ladder graph; we denote it by $\mathbb{Z}\times\{1, 2\}$. If G' is the line graph with vertex set $W = \{1, \dots, d\}$ and set of edges $E' = \{\{i, i+1\} : i = 1, 2, \dots, d-1\}\}$, then $\mathbb{Z} \times G'$ is a d-level ladder.

In the above examples, any constant initial weights satisfy the conditions (A1) and (A2).

2.4 Results

Throughout this section, we assume that the graph G and the initial weights a satisfy (G1)–(G5), (A1), and (A2).

We call the edge-reinforced random walk *recurrent* if almost all its paths visit all vertices infinitely often. Although $(X_t)_{t\in\mathbb{N}_0}$ is not a Markov chain, it was shown in [MR05d] that the edge-reinforced random walk is recurrent iff it returns to the starting point at least once with probability one.

Theorem 2.1 (Recurrence) *The edge-reinforced random walk on G with periodic initial weights is recurrent.*

This fact for edge-reinforced random walk is similar to the behavior of simple random walk on G. However, the qualitative behavior of the edge-reinforced random walk differs considerably from the behavior of ordinary random walk. For example, the edge-reinforced random walk is much more localized. This is stated formally in the following theorem:

For $v = (i, w) \in V$, set $|v| := |i|$.

Theorem 2.2 (Estimates for the position of the random walk) *For the edge-reinforced random walk on G with periodic initial weights a, there exist strictly positive constants $c_1 = c_1(G, a)$, $c_2 = c_2(G, a)$, and $c_3 = c_3(G, a)$ such that the following estimates hold:*

(a) For all times $t \in \mathbb{N}_0$ and all vertices $v \in V$, one has

$$P[X_t = v] \leq c_1 e^{-c_2|v|}. \tag{2.5}$$

(b) As a consequence, one has

$$P\left[\max_{0\leq s\leq t} |X_s| \leq c_3 \log t \textit{ for all but finitely many } t\right] = 1. \tag{2.6}$$

Here is another difference between ordinary random walks and edge-reinforced random walks: For simple random walk, the probability to reach a vertex v far away from the starting point 0 before returning to 0 behaves roughly like const$/|v|$. On the other hand, for edge-reinforced random walk, the corresponding probability decays at least exponentially as $|v| \to \infty$, as the following theorem shows:

Theorem 2.3 (Hitting probabilities) *Consider the edge-reinforced random walk on G with periodic initial weights a. For $v \in V$, let $\tau_v := \inf\{t \geq 1 : X_t = v\}$ denote the first time ≥ 1 when the random walker visits the vertex v. For all $v \in V$, the probability that the edge-reinforced random walker visits v before it returns to its starting point 0 satisfies the bound*

$$P[\tau_v < \tau_0] \leq c_1 e^{-c_2|v|} \tag{2.7}$$

with the same constants $c_1 > 0$ and $c_2 > 0$ as in Theorem 2.2.

In [MR05d], it is shown that the edge-reinforced random walk on any graph with arbitrary initial weights has the same distribution as a mixture of reversible Markov chains. For the periodic graphs defined above with periodic initial weights, we can say more: Denote by $\Omega := (0,\infty)^E$ the set of strictly positive edge weights for the graph G. For $x = (x_e)_{e\in E} \in \Omega$, set

$$x_v := \sum_{e\ni v} x_e. \tag{2.8}$$

Let $Q_{0,x}$ be the distribution of the Markovian nearest-neighbor random walk on G induced by the weights x, starting at 0. More precisely,

$$Q_{0,x}[X_0 = 0] = 1, \tag{2.9}$$

$$Q_{0,x}[X_{t+1} = v | X_0, \ldots, X_t] = \frac{x_{\{X_t,v\}}}{x_{X_t}} 1_{\{X_t,v\}\in E} \tag{2.10}$$

for $t \in \mathbb{N}_0$ and $v \in V$.

Theorem 2.4 (Unique mixture of positive-recurrent Markov chains) *The edge-reinforced random walk on G with periodic initial weights is a unique mixture of positive recurrent Markov chains: There exists a probability measure $\mathbb{Q} := \mathbb{Q}^G_{0,a}$ on Ω such that*

$$P[A] = \int_\Omega Q_{0,x}[A] \, \mathbb{Q}(dx) \tag{2.11}$$

holds for any event $A \subseteq V^{\mathbb{N}_0}$ of admissible paths. Furthermore, for $\mathbb{Q}$-almost all $x \in \Omega$, the Markov chain with distribution $Q_{0,x}$ is positive recurrent.

If one imposes additionally either the normalization condition $\sum_{e\in E} x_e = 1$ $\mathbb{Q}$-almost surely or $x_{e_0} = 1$ $\mathbb{Q}$-almost surely for some fixed reference edge $e_0 \in E$, the measure $\mathbb{Q}$ is unique.

By Theorem 2.4, the edge-reinforced random walk has the same distribution as a random walk in a random environment, given by random time-independent weights on the edges. Furthermore, $\sum_{u\in V} x_u < \infty$ for $\mathbb{Q}$-almost all $x \in \Omega$. Hence, $v \mapsto x_v/(\sum_{u\in V} x_u)$ is the probability function of a reversible probability measure for the Markovian random walk induced by the edge weights $(x_e)_{e\in E}$. As the following theorem shows, the x_v decay almost surely exponentially in space.

Theorem 2.5 (Exponential decay of the random weights) *There exists a constant $c_4 = c_4(G, a) > 0$ such that for any mixing measure $\mathbb{Q}$ from Theorem 2.4, for $\mathbb{Q}$-almost all $x \in \Omega$, the bound*

$$x_v \le x_0 \exp(-c_4|v|) \tag{2.12}$$

holds for all but finitely many vertices $v \in V$.

Since $x_e \leq x_v$ for $e \in v$, Theorem 2.5 provides a bound for the edge weights x_e, $e \in E$.

The distribution $\mathbb{Q}$ can be obtained from the edge-reinforced random walk in the following way: Let $k_e(t) := w_e(t) - a_e$ denote the number of times the edge-reinforced random walker traverses the edge e up to time t.

Theorem 2.6 (Recovering the random environment) *For the edge-reinforced random walk on G with periodic initial weights, the limit*

$$\left(\lim_{t\to\infty} k_e(t)/t\right)_{e\in E} \tag{2.13}$$

of the proportions of time spent on the edges of the graph G exists P-almost surely. Its distribution is the mixing measure $\mathbb{Q}$ normalized in such a way that it is supported on $\{(x_e)_{e\in E} \in \Omega : \sum_{e\in E} x_e = 1\}$.

Note that as a consequence of Theorems 2.6 and 2.4 the limit in (2.13) is almost surely positive. This was not known before for some of the models in Section 2.3 (b) and (c).

Combining Theorems 2.6 and 2.5, we get an upper bound on the asymptotic proportions of time spent on the vertices of the graph G. The bound c_4 for the decay rate in (2.12) can be described explicitly: For $a = (a_e)_{e\in E}$, set

$$\alpha = \alpha(a) := \max_{v\in V} \frac{a_v + 1}{2}, \tag{2.14}$$

where $a_v := \sum_{e\ni v} a_e$. Since the initial weights a are periodic, α is finite. The *local range of $\mathcal{E}$* is defined to be

$$R := \max\{|z| : (u, w, z) \in \mathcal{E}\}. \tag{2.15}$$

It measures how much the level of two vertices can at most differ if they are connected by an edge. Using this notation, in Theorem 2.5 we can choose

$$c_4 = (96|W|R^2\alpha)^{-1}. \tag{2.16}$$

Note that c_4 is a decreasing function of the initial weights $a = (a_e)_{e\in E}$.

How this paper is organized

In Section 3, we introduce finite periodic analogues $G^{(N)}$ of the graph G. It is well-known that the edge-reinforced random walk on $G^{(N)}$ has the same distribution as a mixture of the Markov chains $Q_{0,x}$ with an explicitly known mixing measure $\mathbb{Q}_0^{(N)}$. The core argument consists in proving bounds for the random weights x_v, $v \in V^{(N)}$, when $(x_e)_{e\in E^{(N)}}$ is distributed according to $\mathbb{Q}_0^{(N)}$. The bounds are uniform in the size N of the graph $G^{(N)}$. The proofs of essentially all other results use these estimates. To bound the x_v, $v \in V^{(N)}$, in Section 4, we consider some symmetric interpolation $\mathbb{P}_{0,\ell}^{(N)}$ of the mixing measures $\mathbb{Q}_0^{(N)}$ and $\mathbb{Q}_\ell^{(N)}$ for the edge-reinforced random walk with starting points 0 and ℓ, respectively. In Section 5, we extend the bound for the x_v to a mixing measure $\mathbb{Q}$ for the edge-reinforced random walk on the infinite graph G. Then, we deduce the uniqueness of $\mathbb{Q}$ and give proofs of the main results.

3 Results for errw on finite periodic graphs

For $N \in \mathbb{N}$ with $N \geq R$, we define a finite periodic analogue $G^{(N)} = (V^{(N)}, E^{(N)})$ of G by

$$V^{(N)} =(\mathbb{Z}/N\mathbb{Z}) \times W \quad \text{and} \tag{3.1}$$

$$E^{(N)} =\{\{(i \bmod N, u), ((z+i) \bmod N, w)\} : (u,w,z) \in \mathcal{E}, i \in \mathbb{Z}\}. \tag{3.2}$$

The edges of $G^{(N)}$ are assigned periodic weights in analogy to (A1):

(A') The weight of the edges $e_i = \{(i \bmod N, u), (z+i \bmod N, w)\}$ is defined by $a_{e_i} := a(u,w,z)$ for every $i \bmod N \in \mathbb{Z}/N\mathbb{Z}$.

In this section, we state results for the edge-reinforced random walk on $G^{(N)}$, starting in 0, with initial weights defined by (A'). The results will be used in Section 4 to prove the theorems about the edge-reinforced random walk on the infinite graph G.

For a vertex $v = (i \bmod N, w) \in V^{(N)}$ with i being the unique representative of $i \bmod N$ in the interval $[\lfloor N/2 \rfloor - N + 1, \lfloor N/2 \rfloor]$, we define $|v| := |i|$.

Pick a reference edge $e_0 \in E^{(N)}$ adjacent to the starting vertex 0. We endow the set $\Omega^{(N)} := (0,\infty)^{E^{(N)}}$ of edge weights for $G^{(N)}$ with the reference measure $\rho = \rho^{(N)}$ defined by

$$\rho(dx) = \delta_1(dx_{e_0}) \prod_{e \in E^{(N)} \setminus \{e_0\}} \frac{dx_e}{x_e}, \tag{3.3}$$

where δ_1 denotes the Dirac measure on $(0,\infty)$ with unit mass at 1. Let $\mathcal{T}^{(N)}$ be the set of all spanning trees of $G^{(N)}$, viewed as subsets of the set of edges $E^{(N)}$. For $v_0 \in V^{(N)}$ and initial weights $a = (a_e)_{e \in E^{(N)}}$ given by (A'), define $\mathbb{Q}_{v_0}^{(N)} = \mathbb{Q}_{v_0,a}^{(N)}$ to be the probability measure on $\Omega^{(N)}$ with the following density with respect to ρ:

$$\frac{d\mathbb{Q}_{v_0}^{(N)}}{d\rho}(x) = \frac{1}{z_{v_0}^{(N)}} \frac{\prod_{e \in E^{(N)}} x_e^{a_e}}{x_{v_0}^{a_{v_0}/2} \prod_{v \in V^{(N)} \setminus \{v_0\}} x_v^{(a_v+1)/2}} \sqrt{\sum_{T \in \mathcal{T}^{(N)}} \prod_{e \in T} x_e} \tag{3.4}$$

with some normalizing constant $z_{v_0}^{(N)} > 0$. It follows from Fact 3.1 below that $\mathbb{Q}_{v_0}^{(N)}$ is indeed a well-defined probability measure.

For $v_0 \in V$ and $x \in \Omega^{(N)}$, define $Q_{v_0,x}$ in analogy to (2.9) and (2.10) by just replacing the starting point 0 by v_0 and the underlying graph G by $G^{(N)}$.

Fact 3.1 (Lemma A.1 of [MR05b]) *The edge-reinforced random walk on $G^{(N)}$, starting at v_0, with the initial weights a is a mixture of the Markovian random walks $Q_{v_0,x}$. If we normalize the edge weights in such a way that the reference edge e_0 gets weight 1, then the unique mixing measure is the measure $\mathbb{Q}_{v_0}^{(N)}$.*

The key estimate of this paper is a bound for x_v, $v \in V^{(N)}$, if the random weights $(x_e)_{e\in E^{(N)}}$ are distributed according to $\mathbb{Q}_0^{(N)}$. First, we bound $E_{\mathbb{Q}_0^{(N)}}\left[(x_\ell/x_0)^{1/4}\right]$ for vertices ℓ that can be exchanged with 0 without changing the structure of the weighted graph. The conditions (G5) and (A2) guarantee the existence of at least one such vertex ℓ per level. Recall the definition (2.15) of the local range R.

Theorem 3.2 *Let $N \geq 6R$ and let $\ell \in V^{(N)}$ be a vertex with $|\ell| \geq 3R$ such that there exists an automorphism of the weighted graph $G^{(N)}$ that interchanges 0 and ℓ. Then, with the constant $c_4 > 0$ from (2.16) and $c_5 = \exp(c_4 R)$, one has*

$$E_{\mathbb{Q}_0^{(N)}}\left[\left(\frac{x_\ell}{x_0}\right)^{1/4}\right] \leq c_5 \exp(-c_4|\ell|). \tag{3.5}$$

The following theorem shows that with probability exponentially close to one, x_v/x_0 gets exponentially small in $|v|$. Here, it is not required that there exists an automorphism that interchanges 0 and v.

Theorem 3.3 *Let c_4 be as in (2.16). There are constants $c_6 = c_6(G,a) > 0$ and $c_7 = c_7(G,a) > 0$ such that for all $N \geq 6R$ and for all $v \in V^{(N)}$ with $|v| \geq 3R$, the following bound holds:*

$$\mathbb{Q}_0^{(N)}\left[\frac{x_v}{x_0} \geq e^{-c_4|v|}\right] \leq c_6 e^{-c_7|v|}. \tag{3.6}$$

4 Analyzing the random environment for finite graphs

In this section, we prove Theorems 3.2 and 3.3, the key bounds for the quotients x_v/x_0 in the random environment for the edge-reinforced random walk on $G^{(N)}$. The hard work consists in proving Theorem 3.2.

Before starting formally with the proof, let us explain the physical idea behind it: We are interested in an upper bound for

$$Z_{0,\ell}^{(N)} = E_{\mathbb{Q}_0^{(N)}}\left[\left(\frac{x_\ell}{x_0}\right)^{1/4}\right] \qquad \text{((4.46), below)}. \tag{4.1}$$

Let us interpret this expression as a partition sum $Z_{0,\ell}^{(N)} = E_{\mathbb{Q}_0^{(N)}}[\exp(\beta H)]$ with Hamiltonian

$$-H = -\log\frac{x_\ell}{x_0} \tag{4.2}$$

at temperature $1/\beta = 4$. We introduce the thermal measure (Gibbs measure)

$$d\mathbb{P}_{0,\ell}^{(N)} = \frac{e^{\beta H}}{Z_{0,\ell}^{(N)}}\, d\mathbb{Q}_0^{(N)} \qquad \text{((4.14), below)}. \tag{4.3}$$

The main reason to consider $\mathbb{P}_{0,\ell}^{(N)}$ is its symmetry for exchanging $0 \leftrightarrow \ell$ which implies

$$E_{\mathbb{P}_{0,\ell}^{(N)}}\left[\log \frac{x_\ell}{x_0}\right] = 0 \qquad ((4.49), \text{ below}); \tag{4.4}$$

this symmetry is also the reason to consider the expectation of $(x_\ell/x_0)^{1/4}$. We can reverse the roles of the original measure $\mathbb{Q}_0^{(N)}$ and the Gibbs measure $\mathbb{P}_{0,\ell}^{(N)}$, writing

$$d\mathbb{Q}_0^{(N)} = Z_{0,\ell}^{(N)} e^{-\beta H}\, d\mathbb{P}_{0,\ell}^{(N)}. \tag{4.5}$$

Thermodynamics motivates us to write

$$\text{free energy} = \text{internal energy} - \text{temperature} \cdot \text{entropy}, \tag{4.6}$$

i.e.

$$-\frac{1}{\beta}\log\frac{1}{Z_{0,\ell}^{(N)}} = E_{\mathbb{Q}_0^{(N)}}[H] + \frac{1}{\beta}E_{\mathbb{Q}_0^{(N)}}\left[\log\frac{d\mathbb{Q}_0^{(N)}}{d\mathbb{P}_{0,\ell}^{(N)}}\right]. \tag{4.7}$$

By the variational principle for free energies, the functional

$$\Pi \mapsto E_\Pi[H] + \frac{1}{\beta}E_\Pi\left[\log\frac{d\Pi}{d\mathbb{P}_{0,\ell}^{(N)}}\right] \tag{4.8}$$

is minimized for $\Pi = \mathbb{Q}_0^{(N)}$. We choose a probability measure $\Pi = \Pi_\gamma$ reasonably close to $\mathbb{Q}_0^{(N)}$ such that one can express $E_{\Pi_\gamma}[H] = \gamma$ and estimate

$$E_{\Pi_\gamma}\left[\log\frac{d\Pi_\gamma}{d\mathbb{P}_{0,\ell}^{(N)}}\right] \le \frac{c_8\gamma^2}{l} \qquad ((4.26), \text{ below}) \tag{4.9}$$

with some constant c_8; the measure Π_γ is obtained by a deformation of $\mathbb{P}_{0,\ell}^{(N)}$ that tries to mimic $\mathbb{Q}_0^{(N)}$. This yields the upper bound

$$\frac{1}{\beta}\log Z_{0,\ell}^{(N)} \le E_{\Pi_\gamma}[H] + \frac{1}{\beta}E_{\Pi_\gamma}\left[\log\frac{d\Pi_\gamma}{d\mathbb{P}_{0,\ell}^{(N)}}\right] \le \gamma + \frac{c_8\gamma^2}{\beta l} \qquad ((4.52) \text{ below}). \tag{4.10}$$

Here, the energy term $E_{\Pi_\gamma}[H]$ is *linear* in the deformation parameter γ, while the entropy term (the second summand in (4.10)) is *quadratically* bounded in γ and arbitrarily small as l gets large. Thus, for negative γ close to zero, the linear term dominates. Optimizing over γ yields our upper bound for $Z_{0,\ell}^{(N)}$.

Altogether, these ideas yield below the proof of Theorem 3.2. Finally, Theorem 3.3 is deduced from Theorem 3.2 and some tail estimates for quotients of weights $x_e/x_{e'}$.

4.1 Deforming the random environment

Throughout this section, we fix

(D1) a deformation parameter $\gamma \in \mathbb{R}$,

(D2) a size $N \in \mathbb{N}$ with $N \geq 6R$, and

(D3) a vertex $\ell \in V^{(N)}$ with $|\ell| \geq 3R$. We assume that there exists an automorphism of the weighted graph $G^{(N)}$ that interchanges ℓ and the starting point 0.

There exists at least one vertex ℓ per level satisfying (D3) because of the assumptions (G5) and (A2). Set

$$l := |\ell| - R \geq 2R. \tag{4.11}$$

We define a probability measure $\mathbb{P}_{0,\ell}^{(N)}$ on $\Omega^{(N)}$ which interpolates between the random environments $\mathbb{Q}_0^{(N)}$ and $\mathbb{Q}_\ell^{(N)}$ for the edge-reinforced random walk on $G^{(N)}$, starting in the vertices 0 and ℓ, respectively: Define $\mathbb{P}_{0,\ell}^{(N)}$ to have the following density with respect to the reference measure ρ:

$$\frac{d\mathbb{P}_{0,\ell}^{(N)}}{d\rho} := \frac{1}{Z_{0,\ell}^{(N)}} \left(\frac{d\mathbb{Q}_0^{(N)}}{d\rho} \right)^{1/2} \left(\frac{d\mathbb{Q}_\ell^{(N)}}{d\rho} \right)^{1/2} \tag{4.12}$$

with the normalizing constant

$$Z_{0,\ell}^{(N)} := \int_{\Omega^{(N)}} \left(\frac{d\mathbb{Q}_0^{(N)}}{d\rho} \right)^{1/2} \left(\frac{d\mathbb{Q}_\ell^{(N)}}{d\rho} \right)^{1/2} d\rho. \tag{4.13}$$

Note that $Z_{0,\ell}^{(N)}$ is finite by Hölder's inequality. Since by (D3) there is an automorphism of the weighted graph $G^{(N)}$ which interchanges 0 and ℓ, the normalizing constants for the densities of $\mathbb{Q}_0^{(N)}$ and $\mathbb{Q}_\ell^{(N)}$ in (3.4) agree: $z_0^{(N)} = z_\ell^{(N)}$. Using this fact together with (3.4), one sees that $\mathbb{P}_{0,\ell}^{(N)}$ is absolutely continuous with respect to $\mathbb{Q}_0^{(N)}$ with

$$\frac{d\mathbb{P}_{0,\ell}^{(N)}}{d\mathbb{Q}_0^{(N)}} = \frac{1}{Z_{0,\ell}^{(N)}} \left(\frac{x_\ell}{x_0} \right)^{1/4} \tag{4.14}$$

and $\mathbb{P}_{0,\ell}^{(N)}$ is absolutely continuous with respect to ρ with

$$\frac{d\mathbb{P}_{0,\ell}^{(N)}}{d\rho}(x) = \frac{1}{z_0^{(N)} Z_{0,\ell}^{(N)}} \frac{\left(\prod_{e \in E^{(N)}} x_e^{a_e} \right) \sqrt{\sum_{T \in \mathcal{T}^{(N)}} \prod_{e \in T} x_e}}{x_0^{a_0/2+1/4} x_\ell^{a_\ell/2+1/4} \prod_{v \in V^{(N)} \setminus \{0,\ell\}} x_v^{(a_v+1)/2}}. \tag{4.15}$$

Next, we define a deformation map Ξ_γ on $\Omega^{(N)}$: For $e = \{u, v\}$, set $|e| := \min\{|u|, |v|\}$.

Definition 4.1 (Deformation map) *For any edge $e \in E^{(N)}$, we define*

$$D_e = D_{e,l} := \begin{cases} |e|/l & \text{if } 0 \le |e| \le l, \\ 1 & \text{if } |e| \ge l+1. \end{cases} \tag{4.16}$$

Furthermore, we define the deformation map $\Xi_\gamma = \Xi_{\gamma,l}^{(N)} : \Omega^{(N)} \to \Omega^{(N)}$ by

$$\Xi_\gamma(x) = \left(e^{\gamma D_e} x_e\right)_{e \in E^{(N)}}. \tag{4.17}$$

The map Ξ_γ leaves the weights of the edges at level 0 unchanged, whereas it multiplies the weights of the edges e with $|e| \ge l$ by e^γ. This has the following consequences:

Lemma 4.2 (Properties of the deformation map) *(a) One has*

$$x_u \circ \Xi_\gamma = x_u \quad \text{for any vertex } u \text{ with } |u| = 0 \qquad \text{and} \tag{4.18}$$

$$x_v \circ \Xi_\gamma = e^\gamma x_v \quad \text{for any vertex } v \text{ with } |v| = l + R. \tag{4.19}$$

(b) The reference measure ρ is invariant with respect to Ξ_γ.

Proof. Let u be a vertex with $|u| = 0$. Then, for any $e \ni u$, we have $|e| = 0$ and consequently, $(\Xi_\gamma(x))_e = x_e$. Hence,

$$x_u \circ \Xi_\gamma = \sum_{e \ni u} (\Xi_\gamma(x))_e = \sum_{e \ni u} x_e = x_u. \tag{4.20}$$

Let v be a vertex with $|v| = l + R$. By the definitions of the edge set $E^{(N)}$ and the local range R, any $e \ni v$ satisfies $|e| \ge |v| - R = l$. Since $D_e = 1$ for all these edges,

$$x_v \circ \Xi_\gamma = \sum_{e \ni v} (\Xi_\gamma(x))_e = \sum_{e \ni v} e^\gamma x_e = e^\gamma x_v. \tag{4.21}$$

This completes the proof of part (a).

To prove part (b), recall that by assumption $0 \in e_0$. Hence, $|e_0| = 0$ and $(\Xi_\gamma(x))_{e_0} = x_{e_0}$. Thus, the coordinate x_{e_0} remains unchanged under the map Ξ_γ. All other components x_e are multiplied under Ξ_γ by the factor $e^{\gamma D_e}$ which depends on e, but not on x. Such transformations leave the measure ρ invariant. ■

Definition 4.3 (Deformed measure) *We define the deformed measure $\Pi_\gamma = \Pi_{\gamma,0,\ell}^{(N)}$ to be the image measure of $\mathbb{P} = \mathbb{P}_{0,\ell}^{(N)}$ under the map $\Xi_{\gamma,l}^{(N)}$:*

$$\Pi_\gamma := \Xi_\gamma \mathbb{P}. \tag{4.22}$$

Lemma 4.4 *The measure Π_γ is absolutely continuous with respect to $\mathbb{P}$ with the Radon-Nikodym derivative*

$$\frac{d\Pi_\gamma}{d\mathbb{P}} = \left(\frac{d\mathbb{P}}{d\rho} \circ [\Xi_\gamma]^{-1}\right) \cdot \left(\frac{d\mathbb{P}}{d\rho}\right)^{-1}. \tag{4.23}$$

Proof. By (4.15), $\mathbb{P}$ is absolutely continuous with respect to ρ with a Radon-Nikodym derivative which is strictly positive everywhere on $\Omega^{(N)}$. Note that the deformation map Ξ_γ is invertible. Since the reference measure ρ is invariant under Ξ_γ by Lemma 4.2(b), we obtain

$$\frac{d\Pi_\gamma}{d\rho} = \frac{d(\Xi_\gamma \mathbb{P})}{d(\Xi_\gamma \rho)} = \frac{d\mathbb{P}}{d\rho} \circ [\Xi_\gamma]^{-1}. \tag{4.24}$$

The claim follows by dividing the last identity by the strictly positive Radon-Nikodym derivative $d\mathbb{P}/d\rho$. ■

A key estimate in the proof of Theorem 3.2 is the following entropy bound:

Proposition 4.5 (Entropy bound) *Set*

$$c_8 = c_8(G, a) := 3|W|R^2\alpha/2 \tag{4.25}$$

with α defined in (2.14). Under the assumptions (D1)–(D3), the following entropy bound holds:

$$E_{\Pi^{(N)}_{\gamma,0,\ell}}\left[\log \frac{d\Pi^{(N)}_{\gamma,0,\ell}}{d\mathbb{P}^{(N)}_{0,\ell}}\right] \leq \frac{c_8\gamma^2}{l} \tag{4.26}$$

with $l = |\ell| - R$.

When proving Proposition 4.5, the following notations will be useful: For $v \in V^{(N)}$, we denote by $E^{(N)}_v := \{e \in E^{(N)} : e \ni v\}$ the set of edges incident to v. For $x \in \Omega^{(N)}$, set

$$x_{v,\gamma} = \sum_{e \in E^{(N)}_v} e^{\gamma D_e} x_e, \tag{4.27}$$

and define the probability measure

$$\mu^{(N)}_{v,x,\gamma} := \sum_{e \in E^{(N)}_v} \frac{e^{\gamma D_e} x_e}{x_{v,\gamma}} \delta_e \tag{4.28}$$

on the set of edges $E^{(N)}_v$. Consider $D_\bullet : E^{(N)}_v \to \mathbb{R}$, $e \mapsto D_e$, as a random variable on the probability space $(E^{(N)}_v, \mathcal{P}(E^{(N)}_v), \mu^{(N)}_{v,x,\gamma})$, where $\mathcal{P}(E^{(N)}_v)$ denotes the power set of $E^{(N)}_v$. Furthermore, for $T \in \mathcal{T}^{(N)}$, set

$$Y_{T,\gamma}(x) := \prod_{e \in T} \left(e^{\gamma D_e} x_e\right) \qquad \text{and} \qquad \Delta_T := \sum_{e \in T} D_e. \tag{4.29}$$

It is convenient to consider $\Delta_\bullet : \mathcal{T}^{(N)} \to \mathbb{R}$, $T \mapsto \Delta_T$, as a random variable on the probability space $(\mathcal{T}^{(N)}, \mathcal{P}(\mathcal{T}^{(N)}), \nu^{(N)}_{x,\gamma})$ with the probability measure $\nu^{(N)}_{x,\gamma}$ on $\mathcal{T}^{(N)}$ defined by

$$\nu^{(N)}_{x,\gamma} := \sum_{T \in \mathcal{T}^{(N)}} \frac{Y_{T,\gamma}(x)}{\sum_{T' \in \mathcal{T}^{(N)}} Y_{T',\gamma}(x)} \delta_T. \tag{4.30}$$

The following identities will be used below:

$$\frac{\partial}{\partial\gamma}x_{v,\gamma} = \sum_{e\ni v}\left(D_e e^{\gamma D_e} x_e\right) \qquad \text{and} \qquad \frac{\partial}{\partial\gamma}Y_{T,\gamma}(x) = \sum_{e\in T} D_e Y_{T,\gamma}(x) = \Delta_T Y_{T,\gamma}(x). \quad (4.31)$$

Lemma 4.6 *(a) The function*

$$\mathbb{R} \ni \gamma \mapsto f_\gamma := \log\left(\frac{d\Pi_\gamma}{d\mathbb{P}} \circ \Xi_\gamma\right) \quad (4.32)$$

is twice continuously differentiable.

(b) There is a constant $c_9 = c_9(G, a, N) > 0$ such that for $j = 1, 2$, one has

$$\sup_{x\in\Omega^{(N)}} \sup_{\gamma\in\mathbb{R}} \left|\frac{\partial^j f_\gamma}{\partial^j \gamma}(x)\right| \le c_9. \quad (4.33)$$

(c) For the second derivative, we have the following quantitative bound with c_8 as in (4.25):

$$\sup_{x\in\Omega^{(N)}} \sup_{\gamma\in\mathbb{R}} \frac{\partial^2 f_\gamma}{\partial^2 \gamma}(x) \le \frac{2c_8}{l}. \quad (4.34)$$

Proof. First, we rewrite f_γ using Lemma 4.4:

$$f_\gamma = \log\frac{d\mathbb{P}}{d\rho} - \log\left(\frac{d\mathbb{P}}{d\rho} \circ \Xi_\gamma\right). \quad (4.35)$$

We calculate the last difference inserting the explicit form of the Radon-Nikodym derivative $d\mathbb{P}/d\rho$ given in (4.15). By (4.11), we have $|\ell| = l + R$. Hence, using Lemma 4.2(a), it follows

$$\frac{d\mathbb{P}}{d\rho}\left(\Xi_\gamma(x)\right) = \frac{1}{z_0^{(N)} Z_{0,\ell}^{(N)}} \frac{\prod_{e\in E^{(N)}}\left(e^{\gamma D_e a_e} x_e^{a_e}\right)\sqrt{\sum_{T\in\mathcal{T}^{(N)}}\prod_{e\in T}\left(e^{\gamma D_e} x_e\right)}}{x_0^{a_0/2+1/4}\left(e^\gamma x_\ell\right)^{a_\ell/2+1/4}\prod_{v\in V^{(N)}\setminus\{0,\ell\}}\left[\sum_{e\ni v}\left(e^{\gamma D_e} x_e\right)\right]^{(a_v+1)/2}}. \quad (4.36)$$

We combine this formula with (4.15) to obtain

$$\begin{aligned} f_\gamma(x) = &-\sum_{e\in E^{(N)}}(\gamma D_e a_e) + \gamma\left(\frac{a_\ell}{2} + \frac{1}{4}\right) \\ &+ \sum_{v\in V^{(N)}\setminus\{0,\ell\}} \frac{a_v + 1}{2}\log\frac{x_{v,\gamma}}{x_v} - \frac{1}{2}\log\frac{\sum_{T\in\mathcal{T}^{(N)}} Y_{T,\gamma}(x)}{\sum_{T\in\mathcal{T}^{(N)}} Y_{T,0}(x)}. \end{aligned} \quad (4.37)$$

Using this expression for f_γ together with (4.31), we find that f_γ has the following derivatives:

$$\begin{aligned}\frac{\partial f_\gamma}{\partial\gamma}(x) =& -\sum_{e\in E^{(N)}}(D_e a_e) + \left(\frac{a_\ell}{2}+\frac{1}{4}\right)\\ &+ \sum_{v\in V^{(N)}\setminus\{0,\ell\}}\left(\frac{a_v+1}{2x_{v,\gamma}}\sum_{e\ni v}\left(D_e e^{\gamma D_e}x_e\right)\right) - \frac{1}{2}\frac{\sum_{T\in\mathcal{T}^{(N)}}\Delta_T Y_{T,\gamma}(x)}{\sum_{T\in\mathcal{T}^{(N)}}Y_{T,\gamma}(x)}\\ =& -\sum_{e\in E^{(N)}}(D_e a_e) + \left(\frac{a_\ell}{2}+\frac{1}{4}\right)\\ &+ \sum_{v\in V^{(N)}\setminus\{0,\ell\}}\left(\frac{a_v+1}{2}E_{\mu^{(N)}_{v,x,\gamma}}[D_\bullet]\right) - \frac{1}{2}E_{\nu^{(N)}_{x,\gamma}}[\Delta_\bullet],\end{aligned} \tag{4.38}$$

$$\frac{\partial^2 f_\gamma}{\partial^2\gamma}(x) = \sum_{v\in V^{(N)}\setminus\{0,\ell\}}\frac{a_v+1}{2}\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet) - \frac{1}{2}\mathrm{Var}_{\nu^{(N)}_{x,\gamma}}(\Delta_\bullet). \tag{4.39}$$

By (4.38) and (4.39), f_γ is twice continuously differentiable, which proves part (a).

Note that a_e and a_v are bounded by 2α, uniformly in e and v. Since $|D_e|\le 1$ for all e and $\Delta_T\le|E^{(N)}|$ for all $T\in\mathcal{T}^{(N)}$, it follows from (4.38) and (4.39) that we can find $c_9(G,a,N)>0$ such that the estimate (4.33) holds for $j=1,2$.

To prove the upper bound (4.34), first note that $(a_v+1)/2\le\alpha$ for all v and the variance of $\Delta_\bullet$ is non-negative. Hence, it follows from (4.39) that

$$\frac{\partial^2 f_\gamma}{\partial^2\gamma}(x) \le \alpha\sum_{v\in V^{(N)}\setminus\{0,\ell\}}\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet). \tag{4.40}$$

For the variance, we obtain the following upper bound:

$$\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet) \le \max_{e\ni v, e'\ni v}(D_e - D_{e'})^2 \le \max_{e\ni v, e'\ni v}\frac{(|e|-|e'|)^2}{l^2} \le \frac{R^2}{l^2}; \tag{4.41}$$

for the last inequality, we used that two edges incident to the same vertex can differ in their level by at most R. By definition, D_e is constant on $|e|\ge l$. Since for e incident to v, we have $|e|\ge|v|-R$, it follows from (4.41) that $\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet)=0$ for all vertices v at a level $|v|\ge l+R$. Hence, in (4.40), we get non-zero contributions only for at most $2|W|(l+R)$ vertices v. Hence, using $R\le l/2$ by (4.11), it follows from (4.40) and (4.41) that

$$\frac{\partial^2 f_\gamma}{\partial^2\gamma}(x) \le \alpha\cdot\frac{R^2}{l^2}\cdot 2|W|(l+R) \le \frac{3\alpha|W|R^2}{l} = \frac{2c_8}{l} \tag{4.42}$$

uniformly in $x\in\Omega^{(N)}$ and $\gamma\in\mathbb{R}$; in the last step we used the definition (4.25) of c_8. This completes the proof of part (c). ∎

Now, we can derive the entropy bound:

Proof of Proposition 4.5. Recall that Π_γ is the image measure of $\mathbb{P}$ under Ξ_γ, and recall the definition (4.32) of f_γ. By the transformation formula,

$$E_{\Pi_\gamma}\left[\log\frac{d\Pi_\gamma}{d\mathbb{P}}\right] = E_{\mathbb{P}}\left[\log\frac{d\Pi_\gamma}{d\mathbb{P}}\circ\Xi_\gamma\right] = E_{\mathbb{P}}[f_\gamma] =: g(\gamma). \tag{4.43}$$

First, observe that $g(\gamma)$ is finite: Since $0 \le D_e \le 1$, one has $e^{-|\gamma|} \le e^{\gamma D_e} \le e^{|\gamma|}$ for all $e \in E^{(N)}$. Hence, it follows from the definitions (4.27) and (4.29) of $x_{v,\gamma}$ and $Y_{T,\gamma}(x)$, respectively, that $e^{-|\gamma|}x_v \le x_{v,\gamma} \le e^{|\gamma|}x_v$ and $e^{-|\gamma|\cdot|E^{(N)}|}Y_{T,0} \le Y_{T,\gamma}(x) \le e^{|\gamma|\cdot|E^{(N)}|}Y_{T,0}$ for all $v \in V^{(N)}$ and all $T \in \mathcal{T}^{(N)}$. Thus, using (4.37), one can find a constant $c_{10} = c_{10}(\gamma, G, a, N) > 0$ such that $|f_\gamma| \le c_{10}$ holds uniformly on $\Omega^{(N)}$. Consequently, $g(\gamma)$ is finite.

By parts (a) and (b) of Lemma 4.6, it follows that g is twice continuously differentiable and the derivatives can be obtained by interchanging integration and differentiation. In particular, using (4.34), we obtain

$$g''(\gamma) = E_{\mathbb{P}}\left[\frac{\partial^2 f_\gamma}{\partial^2\gamma}\right] \le \frac{2c_8}{l}. \tag{4.44}$$

For $\gamma = 0$, the map Ξ_γ is the identity, and thus, $\Pi_\gamma = \mathbb{P}$. Consequently, $g(0) = 0$. Since entropies are always non-negative, $g(\gamma) \ge 0$, and it follows that $g'(0) = 0$. Hence, using (4.44), a Taylor expansion of g around 0 yields:

$$g(\gamma) = \int_0^\gamma g''(\tilde\gamma)(\gamma - \tilde\gamma)\,d\tilde\gamma \le \frac{c_8\gamma^2}{l}. \tag{4.45}$$

■

4.2 Proving bounds for the random environment

In this section, we prove Theorems 3.2 and 3.3.

Proof of Theorem 3.2. Let $N \ge 6R$, and let $\ell \in V^{(N)}$ be a vertex with $|\ell| \ge 3R$ such that there exists an automorphism of the weighted graph $G^{(N)}$ that interchanges 0 and ℓ. Set $l := |\ell| - R$. Integrating both sides of (4.14) with respect to $\mathbb{Q}_0^{(N)}$, we find that

$$E_{\mathbb{Q}_0^{(N)}}\left[\left(\frac{x_\ell}{x_0}\right)^{1/4}\right] = Z_{0,\ell}^{(N)}. \tag{4.46}$$

Hence, to prove the theorem, we need an upper bound for $Z_{0,\ell}^{(N)}$. Using (4.14) again and also the properties (4.18) and (4.19) of the deformation Ξ_γ for the vertices 0 and ℓ, one finds that

$$\frac{d\mathbb{Q}_0^{(N)}}{d\mathbb{P}_{0,\ell}^{(N)}}\circ\Xi_\gamma = Z_{0,\ell}^{(N)}\left(\frac{x_0}{x_\ell}\circ\Xi_\gamma\right)^{1/4} = Z_{0,\ell}^{(N)}\left(\frac{x_0}{x_\ell}\right)^{1/4} e^{-\gamma/4}. \tag{4.47}$$

Taking logarithms, solving for $\log Z_{0,\ell}^{(N)}$, and taking the expectation of both sides with respect to $\mathbb{P}_{0,\ell}^{(N)}$, the last identity implies:

$$\log Z_{0,\ell}^{(N)} = E_{\mathbb{P}_{0,\ell}^{(N)}}\left[\log \frac{d\mathbb{Q}_0^{(N)}}{d\mathbb{P}_{0,\ell}^{(N)}} \circ \Xi_\gamma\right] + \frac{1}{4} E_{\mathbb{P}_{0,\ell}^{(N)}}\left[\log \frac{x_\ell}{x_0}\right] + \frac{\gamma}{4}. \tag{4.48}$$

Note that as a consequence of the argument following (4.51) below, one sees that the first expectation in (4.48) is finite. Consequently, the second expectation is finite as well. We claim that

$$E_{\mathbb{P}_{0,\ell}^{(N)}}\left[\log \frac{x_\ell}{x_0}\right] = 0. \tag{4.49}$$

To see this, recall first that there exists an automorphism of the weighted graph $G^{(N)}$ that interchanges 0 and ℓ. Hence, the distributions of $\log(x_\ell/x_0)$ under $\mathbb{P}_{0,\ell}^{(N)}$ and of $\log(x_0/x_\ell)$ under $\mathbb{P}_{\ell,0}^{(N)}$ agree. Second, by the definition of $\mathbb{P}_{0,\ell}^{(N)}$ (see (4.12) and (4.13)), one has $\mathbb{P}_{0,\ell}^{(N)} = \mathbb{P}_{\ell,0}^{(N)}$. Consequently,

$$E_{\mathbb{P}_{0,\ell}^{(N)}}\left[\log \frac{x_\ell}{x_0}\right] = E_{\mathbb{P}_{\ell,0}^{(N)}}\left[\log \frac{x_0}{x_\ell}\right] = -E_{\mathbb{P}_{0,\ell}^{(N)}}\left[\log \frac{x_\ell}{x_0}\right]. \tag{4.50}$$

Consequently, (4.49) follows.

Next, we estimate the first term on the right-hand side of (4.48): Using the positivity of the relative entropy $E_{\Pi_\gamma}[\log(d\Pi_\gamma/d\mathbb{Q}_0^{(N)})]$, we obtain

$$\begin{aligned}
E_{\mathbb{P}_{0,\ell}^{(N)}}\left[\log \frac{d\mathbb{Q}_0^{(N)}}{d\mathbb{P}_{0,\ell}^{(N)}} \circ \Xi_\gamma\right] &= E_{\Pi_\gamma}\left[\log \frac{d\mathbb{Q}_0^{(N)}}{d\mathbb{P}_{0,\ell}^{(N)}}\right] \\
&\leq E_{\Pi_\gamma}\left[\log \frac{d\mathbb{Q}_0^{(N)}}{d\mathbb{P}_{0,\ell}^{(N)}}\right] + E_{\Pi_\gamma}\left[\log \frac{d\Pi_\gamma}{d\mathbb{Q}_0^{(N)}}\right] \\
&= E_{\Pi_\gamma}\left[\log \frac{d\Pi_\gamma}{d\mathbb{P}_{0,\ell}^{(N)}}\right].
\end{aligned} \tag{4.51}$$

By the entropy bound (4.26) from Proposition 4.5, the last term is bounded above by $c_8\gamma^2/l$; note that the assumptions (D1)–(D3) are satisfied. Inserting this bound together with (4.49) in (4.48) yields

$$\log Z_{0,\ell}^{(N)} \leq \frac{c_8\gamma^2}{l} + \frac{\gamma}{4}. \tag{4.52}$$

The upper bound is minimal for $\gamma = -l/8c_8$. Inserting this value for γ and using the definition of c_8 from (4.25) gives the bound

$$\log Z_{0,\ell}^{(N)} \leq -\frac{l}{64c_8} = \frac{R - |\ell|}{64c_8} = \frac{R - |\ell|}{96|W|R^2\alpha} = \log c_5 - c_4|\ell| \tag{4.53}$$

with c_5 as defined in Theorem 3.2 and c_4 from (2.16). The claim follows from (4.46). ■

Proof of Theorem 3.3. Let $N \geq 6R$ and take $v \in V^{(N)}$ with $|v| \geq 3R$. Let $\ell \in V^{(N)}$ be a vertex on the same level as v such that there exists an automorphism of the weighted graph $G^{(N)}$ that interchanges 0 and ℓ. Such a vertex ℓ exists by our assumptions (G5) and (A2). Using the constant c_4 from (2.16), we obtain:

$$\mathbb{Q}_0^{(N)}\left[\frac{x_v}{x_0} \geq e^{-c_4|v|}\right] \leq \mathbb{Q}_0^{(N)}\left[\frac{x_v}{x_\ell} \geq e^{c_4|v|}\right] + \mathbb{Q}_0^{(N)}\left[\frac{x_\ell}{x_0} \geq e^{-2c_4|v|}\right]. \tag{4.54}$$

An application of Chebyshev's inequality together with the bound (3.5) from Theorem 3.2 gives the following bound for the second term on the right-hand side of (4.54):

$$\begin{aligned}\mathbb{Q}_0^{(N)}\left[\frac{x_\ell}{x_0} \geq e^{-2c_4|v|}\right] &\leq e^{c_4|v|/2} E_{\mathbb{Q}_0^{(N)}}\left[\left(\frac{x_\ell}{x_0}\right)^{1/4}\right] \\ &\leq e^{c_4|v|/2} \cdot c_5 e^{-c_4|\ell|} = c_5 e^{-c_4|v|/2}.\end{aligned} \tag{4.55}$$

It remains to bound the first term on the right-hand side of (4.54). Since the graph $G^{(N)}$ is periodic and connected, there is a constant L such that for all N and all vertices $w, w' \in V^{(N)}$ with $\operatorname{level}(w) = \operatorname{level}(w')$ there exists a path from w to w' in $G^{(N)}$ of length at most L. Hence, Theorem 2.4 of [MR05d] implies that there exist constants $c_{11} = c_{11}(a, L) > 0$ and $c_{12} = c_{12}(a, L) > 0$, depending only on a (but not on N) such that for all N, all vertices $w, w' \in V^{(N)}$ with $\operatorname{level}(w) = \operatorname{level}(w')$, all edges $e \ni w$, $e' \ni w'$, and all $M > 0$, one has

$$\mathbb{Q}_0^{(N)}\left[x_e \geq M x_{e'}\right] \leq c_{11} M^{-c_{12}}. \tag{4.56}$$

We apply this estimate to the first term on the right-hand side of (4.54): Let K denote the number of neighbors of the vertex v in the graph $G^{(N)}$. For any edge $e' \ni \ell$, we get

$$\mathbb{Q}_0^{(N)}\left[\frac{x_v}{x_\ell} \geq e^{c_4|v|}\right] \leq \sum_{e \ni v} \mathbb{Q}_0^{(N)}\left[\frac{x_e}{x_{e'}} \geq e^{c_4|v|/K}\right] \leq c_{13} e^{-c_{14}|v|} \tag{4.57}$$

with some constants $c_{13} = c_{13}(G, a) > 0$ and $c_{14} = c_{14}(G, a) > 0$. Combining (4.54), (4.55), and (4.57), the claim follows. ■

5 Analyzing the errw on the infinite graph G

In this section, we prove Theorems 2.1–2.6. First, we show that there exists a mixing measure $\mathbb{Q}$ for the edge-reinforced random walk on the infinite graph G such that the analogue of (3.6) holds for $\mathbb{Q}$:

Lemma 5.1 *There exists a measure $\mathbb{Q}$ on Ω with the following properties:*

- *The representation (2.11) of errw as a random walk in a random environment is satisfied for all events* $A \subseteq V^{\mathbb{N}_0}$ *of admissible paths.*
- *For any vertex* $v \in V$ *on level* $|v| \geq 3R$*, the estimate*

$$\mathbb{Q}\left[\frac{x_v}{x_0} > e^{-c_4|v|}\right] \leq c_6 e^{-c_7|v|} \tag{5.1}$$

holds with the same constants $c_4 > 0$*,* $c_6 > 0$*, and* c_7 *as in Theorem 3.3.*

Proof. The proof uses the same idea as the proof of Theorem 2.2 of [MR05d].

For $m \in \mathbb{N}$, let $\tilde{V}^{(m)}$ denote the set of all vertices $v \in V$ that can be connected to the vertex $0 \in V$ by a path in G of length at most m. Let

$$\tilde{E}^{(m)} := \{e \in E \; : \; e \subseteq \tilde{V}^{(m)}\}. \tag{5.2}$$

Note that by definition the graph $\tilde{G}^{(m)} = (\tilde{V}^{(m)}, \tilde{E}^{(m)})$ is connected, that the absolute value of the level of all vertices in $\tilde{V}^{(m)}$ is bounded by Rm, and that $\tilde{V}^{(m)} \uparrow V$ as $m \to \infty$.

For N large enough, more specifically for $N \geq 2(m+1)R$, the canonical map $\operatorname{mod} N$: $\tilde{V}^{(m)} \to V^{(N)}$, $(i, u) \mapsto (i \operatorname{mod} N, u)$ yields a graph isomorphism of $\tilde{G}^{(m)}$ to a full subgraph of $G^{(N)}$; we use this graph isomorphism to identify $\tilde{G}^{(m)}$ with its image.

Now fix $m \in \mathbb{N}$ and let $N \geq 2(m+1)R$. Theorem 2.4 of [MR05d], applied to the graph $G^{(N)}$, yields for all $e \in \tilde{E}^{(m)}$ and all $M > 0$ the bounds

$$\mathbb{Q}_0^{(N)}\left[x_{e_0} \geq M x_e\right] \leq c_{15} m M^{-c_{16}/m}, \qquad \mathbb{Q}_0^{(N)}\left[x_e \geq M x_{e_0}\right] \leq c_{15} m M^{-c_{16}/m} \tag{5.3}$$

with some positive constants $c_{15} = c_{15}(a)$ and $c_{16} = c_{16}(a) = \min\{a_e/2 : \; e \in E\}$ depending only on the initial weights a, but not on N; recall that e_0 denotes a reference edge adjacent to 0, and note that e_0 and e are connected by a path of at most m intermediate vertices. In particular, for any fixed m, the laws $\mathbb{Q}_0^{(N)}[(\log(x_e/x_{e_0}))_{e \in \tilde{E}^{(m)}} \in \cdot]$, $N \geq 2(m+1)R$, are tight. Using a compactness and diagonal sequence argument, we find a strictly increasing sequence $(N_k)_{k\in\mathbb{N}}$ with $N_k \geq 2(k+1)R$, such that for all $m \in \mathbb{N}$, the sequence $(\mathbb{Q}_0^{(N_k)}[(\log(x_e/x_{e_0}))_{e\in\tilde{E}^{(m)}} \in \cdot])_{k\geq m}$ converges weakly as $k \to \infty$; hence $(\mathbb{Q}_0^{(N_k)}[(x_e/x_{e_0})_{e\in\tilde{E}^{(m)}} \in \cdot])_{k\geq m}$ converges weakly as $k \to \infty$ to a distribution $\mathbb{Q}_{0,a,m}$ supported on $(0,\infty)^{\tilde{E}^{(m)}}$. By construction, the projection of $\mathbb{Q}_{0,a,m+1}$ to $(0,\infty)^{\tilde{E}^{(m)}}$ equals $\mathbb{Q}_{0,a,m}$; hence by Kolmogorov's extension theorem, there is a distribution $\mathbb{Q}$ on $\Omega = (0,\infty)^E$ having the marginals $\mathbb{Q}_{0,a,m}$ on $(0,\infty)^{\tilde{E}^{(m)}}$.

We claim that (2.11) holds for this probability measure $\mathbb{Q}$. Events of the form $A = \{(X_s)_{s=0,\dots,m-1} = \pi\}$, ($\pi \in V^m$ an admissible path, $m \in \mathbb{N}$), together with the empty set, generate the canonical σ-field on Ω and form a closed system with respect to intersection. Thus it suffices to check (2.11) for these events. Fix $m \in \mathbb{N}$ and $\pi \in V^m$. Without loss of generality we may assume that π is a path in G starting in 0. Let $N \geq 2(m+1)R$. Now every path in G of length m starting in 0 is contained in $\tilde{V}^{(m)}$, and every path in $G^{(N)}$ of length m starting in 0 is contained in $\tilde{V}^{(m)}$ as well. Thus the probability that the edge-reinforced random walk with initial weights a starting in 0 follows π up to time m is the same for the three graphs $G = (V, E)$, $\tilde{G}^{(m)} = (\tilde{V}^{(m)}, \tilde{E}^{(m)})$, and $G^{(N)} = (V^{(N)}, E^{(N)})$. Using the representation of the edge-reinforced random walk on $G^{(N)}$ as a mixture of Markov chains $Q_{0,x}$ with mixing measure $\mathbb{Q}_0^{(N)}(dx)$ (Fact 3.1), this implies

$$P[(X_s)_{s=0,\dots,m-1} = \pi] = \int_{\Omega^{(N)}} Q_{0,x}[(X_s)_{s=0,\dots,m-1} = \pi]\,\mathbb{Q}_0^{(N)}(dx) \tag{5.4}$$

for all $N \geq 2(m+1)R$. This yields (2.11) as follows:

$$\begin{aligned} P[(X_s)_{s=0,\dots,m-1} = \pi] &= \lim_{k\to\infty} \int_{\Omega^{(N_k)}} Q_{0,x}[(X_s)_{s=0,\dots,m-1} = \pi]\,\mathbb{Q}_0^{(N_k)}(dx) \\ &= \lim_{k\to\infty} \int_{\Omega^{(N_k)}} Q_{0,x/x_{e_0}}[(X_s)_{s=0,\dots,m-1} = \pi]\,\mathbb{Q}_0^{(N_k)}(dx) \\ &= \int_{\Omega} Q_{0,x}[(X_s)_{s=0,\dots,m-1} = \pi]\,\mathbb{Q}(dx). \end{aligned} \tag{5.5}$$

The bound (5.1) is an immediate consequence of its finite-volume analogue (3.6) and the definition of $\mathbb{Q}$:

$$\mathbb{Q}\left[\frac{x_v}{x_0} > e^{-c_4|v|}\right] \leq \lim_{k\to\infty} \mathbb{Q}_0^{(N_k)}\left[\frac{x_v}{x_0} > e^{-c_4|v|}\right] \leq c_6 e^{-c_7|v|}. \tag{5.6}$$

Here we used that the event $\{x_v/x_0 > e^{-c_4|v|}\} \subseteq \Omega^{(N_k)}$ (k large enough) is *open* and depends only on a fixed, finite number of components x_e. ■

Lemma 5.2 *Theorem 2.5 holds for the measure $\mathbb{Q}$ from Lemma 5.1.*

Proof. It follows from (5.1) that $\sum_{v\in V} \mathbb{Q}\,[x_v > x_0 \exp(-c_4|v|)] < \infty$. Hence, the Borel-Cantelli lemma implies the claim. ■

Proof of Theorem 2.1 (Recurrence). Let $\mathbb{Q}$ denote the measure from Lemma 5.1, and let $\Omega_{\exp}$ denote the set of all weights $x = (x_e)_{e\in E} \in \Omega$ such that $x_v \leq x_0 \exp(-c_4|v|)$ holds for all but finitely many $v \in V$. By Lemma 5.2, $\mathbb{Q}(\Omega_{\exp}) = 1$.

Take a fixed $x \in \Omega_{\exp}$. Then, $\sum_{v\in V} x_v < \infty$. Consequently, there is a reversible probability measure for the Markovian random walk induced by the weights $(x_e)_{e\in E}$, namely the measure π with $\pi(v) = x_v/(\sum_{u\in V} x_u) > 0$ for all $v \in V$. Hence, the Markovian random walk induced by the weights x is positive recurrent. In particular, it is recurrent: $Q_{0,x}[(X_t)_{t\in\mathbb{N}_0}$ is recurrent$] = 1$. To prove that the edge-reinforced random walk is recurrent, we use the representation (2.11) as a mixture of Markov chains which is applicable by Lemma 5.1:

$$P[(X_t)_{t\in\mathbb{N}_0} \text{ is recurrent}] = \int_{\Omega_{\exp}} Q_{0,x}[(X_t)_{t\in\mathbb{N}_0} \text{ is recurrent}]\,\mathbb{Q}(dx) = 1. \tag{5.7}$$

■

Proof of Theorem 2.6 (Recovering the random environment). Fix $x \in \Omega_{\text{exp}}$. Then, for the Markovian random walk induced by the weights x, $(\lim_{t\to\infty} k_e(t)/t)_{e\in E}$ exists $Q_{0,x}$-almost surely and equals $x = (x_e)_{e\in E}$. We apply the representation of the edge-reinforced random walk as a mixture of Markov chains (2.11) with the measure $\mathbb{Q}$ from Lemma 5.1:

$$P\left[\left(\lim_{t\to\infty} k_e(t)/t\right)_{e\in E} \text{ exists}\right] = \int_{\Omega_{\text{exp}}} Q_{0,x}\left[\left(\lim_{t\to\infty} k_e(t)/t\right)_{e\in E} \text{ exists}\right] \mathbb{Q}(dx) = 1. \quad (5.8)$$

Furthermore, one gets for any measurable set $B \subseteq \Omega$:

$$\begin{aligned} P\left[\left(\lim_{t\to\infty} k_e(t)/t\right)_{e\in E} \in B\right] &= \int_{\Omega_{\text{exp}}} Q_{0,x}\left[\left(\lim_{t\to\infty} k_e(t)/t\right)_{e\in E} \in B\right] \mathbb{Q}(dx) \\ &= \int_{\Omega_{\text{exp}}} 1_B(\tilde{x})\, \mathbb{Q}(dx), \end{aligned} \quad (5.9)$$

where $\tilde{x} = (\tilde{x}_e)_{e\in E}$ are given by $\tilde{x}_e := x_e / \left(\sum_{e'\in E} x_{e'}\right)$. In other words, the weights $\tilde{x}$ are normalized in such a way that $\sum_{e\in E} \tilde{x}_e = 1$. Consequently, (5.9) implies that the distribution of $(\lim_{t\to\infty} k_e(t)/t)_{e\in E}$ equals the image measure of the mixing measure $\mathbb{Q}$ under the map $x \mapsto \tilde{x}$.

By (5.9), we have $P\left[(\lim_{t\to\infty} k_e(t)/t)_{e\in E} \in \Omega_{\text{exp}}\right] = 1$. Hence, for any mixing measure $\mathbb{Q}'$, the event Ω_{exp} occurs $\mathbb{Q}'$-almost surely. Consequently, (5.9) is valid for any mixing measure $\mathbb{Q}'$ and any measurable set B. In particular, this yields the uniqueness of the mixing measure up to multiplication of all edge weights by the same (x-dependent) constant. ■

Proof of Theorem 2.4 (Unique mixture of positive-recurrent Markov chains). Lemma 5.1 proves the existence of a measure $\mathbb{Q}$ such that (2.11) holds. The uniqueness was shown in the proof of Theorem 2.6. Furthermore, by the proof of Theorem 2.1, for any $x \in \Omega_{\text{exp}}$, the Markovian random walk induced by the weights x is positive recurrent and $\mathbb{Q}(\Omega_{\text{exp}}) = 1$. This completes the proof of Theorem 2.4. ■

Proof of Theorem 2.2 (Estimates for the position of the random walk). Using the reversibility of the Markovian random walk induced by the weights x, one observes that

$$Q_{0,x}[X_t = v] = \frac{x_v}{x_0} Q_{v,x}[X_t = 0] \le \min\left\{\frac{x_v}{x_0}, 1\right\} \quad (5.10)$$

holds for any $t \in \mathbb{N}_0$, $v \in V$, and $x \in \Omega$; for a proof, see formula (3.12) in [MR05b]. Let $A_v := \{x \in \Omega : x_v/x_0 < \exp(-c_4|v|)\}$. We apply the representation of the edge-reinforced random walk as a mixture of Markovian walks (Theorem 2.4): Using Lemma 5.1, for any $t \in \mathbb{N}_0$ and any vertex v on level $|v| \ge 3R$, we get:

$$\begin{aligned} P[X_t = v] &= \int_\Omega Q_{0,x}[X_t = v]\, \mathbb{Q}(dx) \\ &\le \int_{A_v} \frac{x_v}{x_0} \mathbb{Q}(dx) + \mathbb{Q}[A_v^c] \le e^{-c_4|v|} + c_6 e^{-c_7|v|}. \end{aligned} \quad (5.11)$$

Hence, the estimate (2.5) holds with appropriate constants $c_1 = c_1(G, a)$ and $c_2 = c_2(G, a)$. This proves part (a).

To prove part (b), take $t \in \mathbb{N}_0$ and $c > 0$. By part (a),

$$P\left[|X_s| > \log(t^c)\right] = \sum_{v \in V: |v| > \log(t^c)} P[X_s = v] \le c_{17} t^{-c_{18}c} \tag{5.12}$$

with appropriate constants $c_{17} = c_{17}(G, a) > 0$ and $c_{18} = c_{18}(G, a) > 0$. Consequently,

$$\begin{aligned} P\left[\max_{0 \le s \le t} |X_s| > c \log t\right] \le & \sum_{0 \le s \le t} P\left[|X_s| > \log t^c\right] \\ \le & (t+1) c_{17} t^{-c_{18}c}. \end{aligned} \tag{5.13}$$

If we choose $c = c_3(G, a) > 0$ in such a way that $-c_{18}c_3 + 1 < -1$ holds, then one has $\sum_{t=0}^{\infty} P\left[\max_{0 \le s \le t} |X_s| > c_3 \log t\right] < \infty$ and the claim (2.6) follows from the Borel-Cantelli lemma. ■

Proof of Theorem 2.3 (Hitting probabilities). From reversibility it follows that

$$Q_{0,x}[\tau_v < \tau_0] = \frac{x_v}{x_0} Q_{v,x}[\tau_0 < \tau_v] \tag{5.14}$$

holds for all $v \in V$; for a proof see (3.8) in [MR05b]. The claim (2.7) follows with arguments analog to (5.11). ■

Proof of Theorem 2.5 (Exponential decay of the random weights). By Theorem 2.4, the mixing measure $\mathbb{Q}$ is unique up to the multiplication of all weights by the same constant. Clearly, the bound (2.12) is independent of such normalizations of the weights. Hence the claim follows from Lemma 5.2. ■

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