Positivity preserving Hadamard matrix functions

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Abstract

For every positive real number p that lies between even integers $2(m-1)$ and $2m$ we demonstrate a matrix $A = [a_{ij}]$ of order $2(m+1)$ such that A is positive definite but the matrix with entries $|a_{ij}|^p$ is not.

1 Introduction

Let $A = [a_{ij}]$ be an $n \times n$ complex matrix, and let $|A|_\circ = [|a_{ij}|]$ be the matrix obtained by replacing each entry of A by its absolute value. Suppose A is positive semidefinite. When $n = 2$, this is equivalent to the conditions $a_{11} \geq 0$, $a_{22} \geq 0$, and $|a_{12}|^2 \leq a_{11}a_{22}$. This shows that $|A|_\circ$ is also positive semidefinite. A small calculation with determinants shows that if A is a 3×3 positive semidefinite matrix, then so is $|A|_\circ$. When $n = 4$, this is no longer true as can be seen from the instructive example [5, p.462]

$$A = \begin{bmatrix} 1 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 1 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 1. \end{bmatrix}. \tag{1}$$

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $n \times n$ matrices, we denote by $A \circ B$ their *Hadamard product* (entrywise product) $[a_{ij}\, b_{ij}]$. By a famous theorem of I. Schur, if A and B are positive semidefinite, then so is the product $A \circ B$. As a consequence, all *Hadamard powers* $A^{\circ m} = [a_{ij}^m]$ of a positive semidefinite matrix A share the same property. Obviously, the matrix $\overline{A} = [\overline{a}_{ij}]$ is positive semidefinite along with A. Hence by Schur's theorem all matrices $[|a_{ij}|^{2m}]$, $m = 0, 1, 2, \ldots,$ are positive semidefinite.

Let p be any nonnegative real number. If $a_{ij} \geq 0$, we use the notation $A^{\circ p}$ for the matrix $[a_{ij}^p]$. If a_{ij} are complex numbers we let $|A|_\circ^{\circ p}$ stand for the matrix $[|a_{ij}|^p]$.

The following fact is known. It seems remarkable and is the object of this note.

Theorem 1. *Let p be a positive real number not equal to any even integer. Then there exists a positive integer n, and an $n \times n$ matrix A such that A is positive semidefinite but $|A|_\circ^{\circ p}$ is not.*

Though this will be subsumed in the ensuing discussion, the reader may check that for the matrix A given in (1), none of the matrices $|A|_\circ^{\circ p}$ is positive semidefinite for $0 < p < 2$.

Theorem 1 can be derived from two other theorems of wider range and greater depth. The first of them addresses the most general question of the type we are discussing. Let

φ be any function of a complex variable. When does φ have the property that whenever $[a_{ij}]$ is a positive semidefinite matrix (of any size n), then so is the matrix $[\varphi(a_{ij})]$? The answer, following from the work of I. J. Schoenberg [9], C. Herz [4], and W. Rudin [8] is that φ satisfies this (rather stringent) requirement if and only if it has a series expansion of the form

$$\varphi(z) = \sum_{k,\ell=0}^{\infty} b_{k\ell} z^k \overline{z}^\ell, \tag{2}$$

with coefficients $b_{k\ell} \geq 0$.

It is not difficult to see that if $\varphi(z) = |z|^p$, then φ can be expressed in this way if and only if $p = 2m$. Theorem 1 follows as a corollary.

The second theorem from which we can derive Theorem 1 concerns positive definite functions. A function $f : \mathbb{R} \to \mathbb{C}$ is said to be *positive definite* if for all n and for all choices of points $x_1, x_2, \ldots, x_n$ in $\mathbb{R}$ the matrix $[f(x_i - x_j)]$ is positive semidefinite. Continuous positive definite functions with the normalisation $f(0) = 1$ are characteristic functions of probability distributions and have been extensively studied in that context. It is known that if a positive definite function f is analytic on an open interval of the form $(-a, a)$, then it is analytic on $\mathbb{R}$. See [6]. So, if g is a positive definite function such that $g(x) = f(x)$ for $-a < x < a$, then $g = f$ everywhere on $\mathbb{R}$.

The function $f(x) = \cos x$ is positive definite and the general theorem we have just cited can be used to show that $|\cos x|^p$ is a positive definite function if and only if $p = 2m$. See [2, Theorem 2.2]. We remark here that the entries of the matrix (1) are $a_{ij} = \cos(x_i - x_j)$ with $x_1 = 0$, $x_2 = \pi/4$, $x_3 = \pi/2$, and $x_4 = 3\pi/4$.

We carry further the idea behind this example to obtain another proof of Theorem 1. The merits of this proof are that it is constructive, does not depend on deeper general theorems, gives more information about the dependence of n on p, and suggests further questions. Our main result is the following.

Theorem 2. *Let $n = 2m$ be an even integer, $m = 2, 3, \ldots,$ and for $0 \leq j \leq n-1$ let $x_j = \frac{j\pi}{n}$. Let A be the $n \times n$ matrix with entries $a_{ij} = \cos(x_i - x_j)$. Then the matrix $|A|_\circ^{\circ p}$ is not positive semidefinite for any p in the range $2(m-2) < p < 2(m-1)$.*

2 The Proof

To avoid cluttering the proof of our theorem we record separately the principal ideas and facts that we use. A *circulant matrix* C is one whose rows are cyclic permutations of its first row. Thus

$$C = \begin{bmatrix} c_0 & c_1 & \dots & c_{n-1} \\ c_{n-1} & c_0 & \dots & c_{n-2} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ c_1 & c_2 & \dots & c_0 \end{bmatrix}.$$

Let $\omega = e^{2\pi i/n}$. Then the eigenvalues of C are given by the formula

$$\lambda_j = c_0 + c_1\omega^j + c_2\omega^{2j} + \cdots + c_{n-1}\omega^{(n-1)j}, \tag{3}$$

$0 \le j \le n-1$.

The following lemma is well-known. See [7, p46] for a more general statement. For the reader's convenience we outline a quick proof.

Lemma 1. *Let* $\lambda_0 > \lambda_1 > \cdots > \lambda_n > 0$ *be any positive real numbers and for* $x \in \mathbb{R}$ *let*

$$f(x) = a_0\lambda_0^x + a_1\lambda_1^x + \cdots + a_n\lambda_n^x,$$

where $a_j \in \mathbb{R}$, $a_0 \neq 0$. *Then* f *has at most* n *zeros on the real line.*

Proof We use induction. When $n = 1$, we have

$$f(x) = a_0\lambda_0^x + a_1\lambda_1^x = \lambda_1^x\left(a_0\mu_0^x + a_1\right) = \lambda_1^x g(x),$$

where $g(x) = a_0\mu_0^x + a_1$, and $\mu_0 = \lambda_0/\lambda_1$. Since $\mu_0 \neq 1$ and $a_0 \neq 0$, $g(x)$ has at most one zero. Since $f(x) = 0$ if and only if $g(x) = 0$, the statement of the Lemma is true for $n = 1$. Suppose it is true for some $n \geq 1$. Then consider

$$f(x) = a_0\lambda_0^x + a_1\lambda_1^x + \cdots + a_{n+1}\lambda_{n+1}^x = \lambda_{n+1}^x g(x),$$

where $g(x) = a_0\mu_0^x + a_1\mu_1^x + \cdots + a_n\mu_n^x + a_{n+1}$, and $\mu_j = \lambda_j/\lambda_{n+1}$. Note that $\mu_0 > \mu_1 > \cdots > \mu_n > 1$. Differentiating, we get

$$g'(x) = b_0\mu_0^x + b_1\mu_1^x + \cdots + b_n\mu_n^x,$$

where $b_j = a_j \log \mu_j$. We have $b_0 \neq 0$. So, by the induction hypothesis g' has at most n zeros. By Rolle's theorem g has at most $n+1$ zeros. The functions f and g are zero at exactly the same points. ■

We will use the vanishing of a special trigonometric sum. This is given by

Lemma 2. *Let p and n be positive integers such that $p < n$ and $p + n$ is even. Then*

$$\sum_{j=0}^{n-1} (-1)^j \left(\cos \frac{j\pi}{n}\right)^p = 0. \tag{4}$$

Proof Let $\omega = e^{\pi i/n}$. Then $\omega^n = -1$, and we have

$$\begin{aligned}
\sum_{j=0}^{n-1} (-1)^j \left(\cos \frac{j\pi}{n}\right)^p &= \sum_{j=0}^{n-1} \omega^{nj} \left(\frac{\omega^j + \omega^{-j}}{2}\right)^p \\
&= \frac{1}{2^p} \sum_{j=0}^{n-1} \omega^{nj} \sum_{k=0}^{p} \binom{p}{k} \omega^{j(p-k)} (\omega^{-j})^k \\
&= \frac{1}{2^p} \sum_{k=0}^{p} \binom{p}{k} \sum_{j=0}^{n-1} \omega^{j(n+p-2k)} \\
&= \frac{1}{2^p} \sum_{k=0}^{p} \binom{p}{k} S_k, \qquad (5)
\end{aligned}$$

where

$$S_k = \sum_{j=0}^{n-1} \omega^{j(n+p-2k)}, \quad 0 \leq k \leq p. \tag{6}$$

Now note that $n + p - 2k$ is an even number smaller than $2n$. So ω is a primitive root of unity of order $2n$. Hence $S_k = 0$ for all $0 \leq k \leq p$. ■

Proof of Theorem 2 For any real number p, the matrix $|A|_\circ^{\circ p}$ in the statement of the theorem is a Hermitian circulant matrix. The entries on its first row are

$$1, \quad \left|\cos \frac{\pi}{n}\right|^p, \quad \left|\cos \frac{2\pi}{n}\right|^p, \ldots, \left|\cos \frac{(n-1)\pi}{n}\right|^p.$$

Since $n = 2m$, one of the n th roots of unity is -1. So one of the eigenvalues of the matrix $|A|_\circ^{\circ p}$ is

$$f(p) = 1 - \left|\cos\frac{\pi}{n}\right|^p + \left|\cos\frac{2\pi}{n}\right|^p - \cdots - \left|\cos\frac{(n-1)\pi}{n}\right|^p. \qquad (7)$$

Using the relation $\cos\theta = \cos(\pi - \theta)$, this can be expressed also as

$$f(p) = 1 - 2\left(\cos\frac{\pi}{2m}\right)^p + 2\left(\cos\frac{2\pi}{2m}\right)^p - \cdots + (-1)^{m-1}2\left(\cos\frac{(m-1)\pi}{2m}\right)^p. \qquad (8)$$

An application of Lemma 1 shows that the function $f(p)$ is zero for at most $m-1$ values of p. On the other hand from the expression (7) and Lemma 2 we see that $f(p)$ is zero when $p = 2, 4, \ldots, 2(m-1)$. These are then all the zeros of $f(p)$. At $p = \infty$, $f(p)$ is equal to 1. The last sign change of $f(p)$ occurs at $p = 2(m-1)$. Thus $f(p)$ is positive for $p > 2(m-1)$, and negative for $2(m-2) < p < 2(m-1)$.

We have shown that one of the eigenvalues of the $n \times n$ matrix $|A|_\circ^{\circ p}$ is negative when $2(m-2) < p < 2(m-1)$, and hence this matrix cannot be positive semidefinite.

■

3 Remarks

- An interesting question raised by our analysis is the following. Let p be a positive real number not equal to any even integer. What is the smallest n for which there exists an $n \times n$ positive semidefinite matrix A such that $|A|_\circ^{\circ p}$ is not positive semidefinite?

- Lemma 2 is included in a more general result given as Formula (18.1.8) on page 257 of [3]. This says that if n is any positive integer, then

$$\sum_{j=0}^{n-1}(-1)^j\left(\cos\frac{j\pi}{n}\right)^p = \begin{cases} \frac{1}{2}\left[1-(-1)^{n+p}\right], & p = 0, 1, \ldots, n-1 \\ n/2^{n-1}, & p = n. \end{cases} \qquad (9)$$

  The case $p = n$ with a proof similar to ours is given in [10]. We give here a proof of all the cases for completeness. Proceed as in the proof of our Lemma 2 upto (5) and (6). Let $p = n$. Then

$$S_k = \sum_{j=0}^{n-1}\omega^{j2(n-k)}.$$

This sum is zero for $0 < k < n$ and it is equal to n when $k = 0$ or n. This proves (9) when $p = n$. Let $p < n$. The only case left to consider is that when $n + p$ is odd. In this case

$$S_k = \frac{1 - \omega^{n(n+p-2k)}}{1 - \omega^{n+p-2k}} = \frac{2}{1 - \omega^{n+p-2k}}, \tag{10}$$

since $\omega^n = -1$ and $n + p - 2k$ is odd. Using the identity

$$\frac{1}{1-x} + \frac{1}{1-1/x} = 1, \quad (x \neq 1)$$

we have

$$\frac{1}{1 - \omega^{n+p-2k}} + \frac{1}{1 - \omega^{n+p-2(p-k)}} = 1.$$

So, from (10) we have

$$S_k + S_{p-k} = 2 \quad \text{for} \quad 0 \le k \le p.$$

Note also that when p is even we have from (10)

$$S_{p/2} = \frac{2}{1 - \omega^n} = 1.$$

Thus when $n + p$ is odd we have

$$\sum_{k=0}^{p} \binom{p}{k} S_k = \sum_{k=0}^{p} \binom{p}{k} = 2^p,$$

and the expression (5) reduces to 1. This establishes (9) in all cases.

- Positivity preserving maps of various kinds have been studied extensively in the last few decades. The recent book [1] describes some of the major results.

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