ITERATED FUNCTION SYSTEMS IN MIXED EUCLIDEAN AND $\mathfrak{p}$-ADIC SPACES

BERND SING
Fakultät für Mathematik
Universität Bielefeld
Universitätsstraße 25
33615 Bielefeld
Germany
E-mail: `sing@math.uni-bielefeld.de`
URL: `http://www.mathematik.uni-bielefeld.de/baake/sing/`

We investigate graph-directed iterated function systems in mixed Euclidean and $\mathfrak{p}$-adic spaces. Hausdorff measure and Hausdorff dimension in such spaces are defined, and an upper bound for the Hausdorff dimension is obtained. The relation between the Haar measure and the Hausdorff measure is clarified. Finally, we discus an example in $\mathbb{R} \times \mathbb{Q}_2$ and calculate upper and lower bounds for its Hausdorff dimension.

Keywords: Graph-Directed Iterated Function System, $\mathfrak{p}$-adic Spaces, Hausdorff Dimension, Affinity Dimension

1. Introduction and Setting

The main focus of this article is the following situation: Assume that a (finite) family $(\Omega_1, \ldots, \Omega_n)$ of subsets of a locally compact Abelian group $\mathbb{X}$, the topology of which is assumed to be generated by a metric, is implicitly given as the unique solution of a graph-directed iterated function system (GIFS). Can we define and calculate the Hausdorff measure and Hausdorff dimension of these sets, and determine their relation to the Haar measure in $\mathbb{X}$?

In the following, we assume that the space $\mathbb{X}$ is given by

$$\mathbb{X} = \mathbb{R}^r \times \mathbb{C}^s \times \mathbb{Q}_{\mathfrak{p}_1} \times \cdots \times \mathbb{Q}_{\mathfrak{p}_k}, \tag{1}$$

i.e., as a product of non-discrete locally compact fields (we shall expand on $\mathbb{Q}_{\mathfrak{p}}$ below). We call the number

$$\dim_{\text{metr}} \mathbb{X} = r + 2 \cdot s + k \tag{2}$$

the *metric dimension* of $\mathbb{X}$ (also see Section 3).

The organisation of this article is as follows: To keep everything as self-contained as possible, we briefly review $\mathfrak{p}$-adic spaces in Section 2. In Section 3, the relation between Hausdorff measure and Haar measure on $\mathbb{X}$ is clarified. Iterated function systems on $\mathbb{X}$ are introduced in Section 4. We define the affinity dimension for a GIFS and show that it is an upper bound for the Hausdorff dimension of the sets Ω_i. We also discus a condition for which we obtain a lower bound for the Hausdorff dimension. In the last Section, we explore a GIFS in $\mathbb{R} \times \mathbb{Q}_2$.

2. $\mathfrak{p}$-adic Spaces and their Visualisation

An *algebraic number field* K is a finite field extension of $\mathbb{Q}$ lying in $\mathbb{C}$, i.e., it is a simple extension of the form $K = \mathbb{Q}(\lambda)$. The *integral closure* of $\mathbb{Z}$ in an algebraic number field K is called the ring of *algebraic integers* $\mathfrak{o}_K$ of K. An ideal $\mathfrak{p}$ of the ring $\mathfrak{o}_K$ is called *prime* if the quotient $\mathfrak{o}_K/\mathfrak{p}$ is an integral domain. A key theorem[10] in algebraic number theory states that every (fractional) ideal of $\mathfrak{o}_K$ in K can be uniquely factored into prime ideals.

Let K^* be the multiplicative group of non-zero elements of K. A surjective homomorphism $v : K^* \to \mathbb{Z}$ with $v(x + y) \geq \min\{v(x), v(y)\}$ (and the convention $v(0) = \infty$) is called a *valuation*[18]. Every prime ideal $\mathfrak{p}$ yields a valuation of K, called the $\mathfrak{p}$-adic valuation $v_{\mathfrak{p}}$, and these are all possible valuations: For $x \in K$ let $v_{\mathfrak{p}}(x) = v_{\mathfrak{p}}(x\mathfrak{o}_K)$ (i.e., $x\mathfrak{o}_K$ is the (fractional) ideal generated by x) where a (fractional) ideal $\mathfrak{a}$ has the unique factorisation $\mathfrak{a} = \mathfrak{p}_1^{v_{\mathfrak{p}_1}(\mathfrak{a})} \cdots \mathfrak{p}_\ell^{v_{\mathfrak{p}_\ell}(\mathfrak{a})}$ into prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$.

Given a $\mathfrak{p}$-adic valuation $v_{\mathfrak{p}}$, one obtains an *ultrametric absolute value* (or, more precisely, a *non-Archimedean absolute value*) by $\|x\|_{\mathfrak{p}} = \eta^{-v_{\mathfrak{p}}(x)}$ for some $\eta > 1$ (where $\|0\|_{\mathfrak{p}} = 0$). The completion of $K = \mathbb{Q}(\lambda)$ with respect to such a $\mathfrak{p}$-adic absolute value yields the $\mathfrak{p}$*-adic number field* $\mathbb{Q}_{\mathfrak{p}}$, which is a locally compact field. We note that the completion of $\mathbb{Q}$ itself w.r.t. the prime ideal $p\mathbb{Z}$ yields the p-adic numbers $\mathbb{Q}_p$.

We define the $\mathfrak{p}$-adic integers $\mathbb{Z}_{\mathfrak{p}} = \{x \in \mathbb{Q}_{\mathfrak{p}} \mid \|x\|_{\mathfrak{p}} \leq 1\}$ and the related ideal $\mathfrak{m}_{\mathfrak{p}} = \{x \in \mathbb{Q}_{\mathfrak{p}} \mid \|x\|_{\mathfrak{p}} < 1\}$. Then $\mathbb{Z}_{\mathfrak{p}}$ is a *discrete valuation ring*, i.e., it is a principal ideal domain that has a unique non-zero prime ideal, namely $\mathfrak{m}_{\mathfrak{p}}$. Furthermore, the *residue field* $k_{\mathfrak{p}} = \mathbb{Z}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$ is finite, and the choice $\eta = [\mathbb{Z}_{\mathfrak{p}} : \mathfrak{m}_{\mathfrak{p}}]$ in the above definition of the $\mathfrak{p}$-adic absolute value yields the so-called *normalised* $\mathfrak{p}$*-adic absolute value* (which has nice properties w.r.t. the Haar measure on $\mathbb{Q}_{\mathfrak{p}}$, see Section 3).

An element π which generates $\mathfrak{m}_{\mathfrak{p}}$, i.e., $\mathfrak{m}_{\mathfrak{p}} = \pi\mathbb{Z}_{\mathfrak{p}}$, is called a *uniformizer*. By the uniqueness of $\mathfrak{m}_{\mathfrak{p}}$, the non-zero ideals of $\mathbb{Z}_{\mathfrak{p}}$ are given by $\pi^m\mathbb{Z}_{\mathfrak{p}}$ ($m \in \mathbb{N}_0$). If S is a system of representatives of $k_{\mathfrak{p}}$ (including 0 for simplicity), every element $x \in \mathbb{Q}_{\mathfrak{p}}$ can be written uniquely as a convergent series (w.r.t. the $\mathfrak{p}$-adic absolute value)

$$x = \sum_{j=m}^{\infty} s_j\, \pi^j, \tag{3}$$

with $s_j \in S$ and $m \in \mathbb{Z}$. If $x \in \mathbb{Z}_{\mathfrak{p}}$, then one can take $m = 0$ and we simply write (with obvious meaning) $x = s_0 s_1 s_2 \ldots$.

One can visualise $\mathbb{Z}_{\mathfrak{p}}$ (and also $\mathbb{Q}_{\mathfrak{p}}$) as a Cantor set[16]. For example, if we take $\mathbb{Q}_2$, every $x \in \mathbb{Q}_2$ can be written as $x = \sum_{j=m}^{\infty} s_j\, 2^j$, where $S = \{0, 1\}$. Therefore, $\mathbb{Z}_2$ can be identified with the set of all 0-1-sequences, i.e., $\mathbb{Z}_2 = \{0, 1\}^{\mathbb{N}_0}$. But this is also a coding of points in the Cantor set[a], and points which are close in the Cantor set are also close w.r.t. the 2-adic metric (also, both the Cantor set and $\mathbb{Q}_2$ are totally disconnected). In Section 5, we will use the Cantor set to visualise sets in $\mathbb{Z}_2$, where (for reasons of representation) we take a factor of $\frac{1}{2}$ instead of $\frac{1}{3}$ in the construction of the "Cantor set" (of course, one then obtains the whole interval $[0, 1]$).

3. Haar and Hausdorff Measures

Given an Abelian topological group G, a measure μ on the family $\mathcal{B}$ of Borel sets in G is called a *Haar measure* if it satisfies the following conditions[6,12,5]:

H1 μ is a regular measure.
H2 If C is compact, $\mu(C) < \infty$.
H3 μ is not identically zero.
H4 μ is invariant under translations, i.e., $\mu(B + t) = \mu(B)$ for all $B \in \mathcal{B}$ and $t \in G$.

Haar measures are unique up to a multiplicative constant. They are obtained by a so-called "Method I Construction"[12].

The Haar measure on $\mathbb{X}$ is the product measure of the Haar measures of its factors. We remark that the (1-dimensional) Lebesgue measure on $\mathbb{R}$ and the 2-dimensional Lebesgue measure on $\mathbb{R}^2 \simeq \mathbb{C}$ are Haar measures. We also note that we have for a Haar measure μ on $\mathbb{R}$, resp. $\mathbb{C}$, resp. $\mathbb{Q}_{\mathfrak{p}}$

- $\mu(\alpha B) = |\alpha| \cdot \mu(B)$ if $\alpha \in \mathbb{R}$ and $B \subset \mathbb{R}$.
- $\mu(\alpha B) = |\alpha|^2 \cdot \mu(B)$ if $\alpha \in \mathbb{C}$ and $B \subset \mathbb{C}$.
- $\mu(\alpha B) = \|\alpha\|_{\mathfrak{p}} \cdot \mu(B)$ if $\alpha \in \mathbb{Q}_{\mathfrak{p}}$ and $B \subset \mathbb{Q}_{\mathfrak{p}}$ (where $\|\cdot\|_{\mathfrak{p}}$ denotes the normalised $\mathfrak{p}$-adic absolute value).

On the other hand, $\mathbb{X}$ is also a separable metric space, where we take the maximum metric d_∞, i.e., for $x, y \in \mathbb{X}$ with

$$x = (x_1, \ldots, x_r, x_{r+1}^{(1)} + i \cdot x_{r+1}^{(2)}, \ldots, x_{r+s}^{(1)} + i \cdot x_{r+s}^{(2)}, x_{r+s+1}, \ldots, x_{r+s+k}) \qquad (4)$$

(where $x_1, \ldots, x_r, x_{r+1}^{(1)}, \ldots, x_{r+s}^{(1)}, x_{r+1}^{(2)}, \ldots, x_{r+s}^{(2)} \in \mathbb{R}$, while $x_{r+s+j} \in \mathbb{Q}_{\mathfrak{p}_j}$ for all $1 \le j \le k$) we have

$$\begin{aligned} d_\infty(x, y) = \max\{&|x_1 - y_1|, \ldots, |x_r - y_r|, |x_{r+1}^{(1)} - y_{r+1}^{(1)}|, \ldots, |x_{r+s}^{(1)} - y_{r+s}^{(1)}|, \\ &|x_{r+1}^{(2)} - y_{r+1}^{(2)}|, \ldots, |x_{r+s}^{(2)} - y_{r+s}^{(2)}|, \|x_{r+s+1} - y_{r+s+1}\|_{\mathfrak{p}_1}, \ldots, \\ &\|x_{r+s+k} - y_{r+s+k}\|_{\mathfrak{p}_k}\}. \end{aligned} \qquad (5)$$

---

[a] Indeed, the Cantor set is given by $\left\{x = \frac{2}{3} \cdot \sum_{j=0}^{\infty} s_j \left(\frac{1}{3}\right)^j \mid s_j \in \{0, 1\}\right\}$.

Therefore, we can define the *diameter* of a set $B \subset \mathbb{X}$ by $\mathrm{diam}(B) = \sup_{x,y \in B} d_\infty(x,y)$ with the convention $\mathrm{diam}(\varnothing) = 0$. Then, the measure obtained by the so-called "Method II Construction"[12,17] from the set function $\tau(B) = [\mathrm{diam}(B)]^d$ is a measure and called the *d-dimensional Hausdorff measure* $h^{(d)}$.

Generalising Theorem 30 in Ref. 17, we can show that the two measures are related as follows.

Theorem 3.1. *Let the space* $\mathbb{X}$ *be given as in Eq.* (1)*, and let* $d = \dim_{\mathrm{metr}} \mathbb{X}$*. Then, the* d*-dimensional Hausdorff measure* $h^{(d)}$ *is a Haar measure. Furthermore,* $h^{(d)}$ *equals the Haar measure constructed as product measure where we assign measure* 1 *to the unit interval (in* $\mathbb{R}$*) resp. to* $\mathbb{Z}_\mathfrak{p}$ *(in* $\mathbb{Q}_\mathfrak{p}$*).* □

As usual, it is clear that $h^{(d)}(B)$ is non-increasing for a given subset $B \subset \mathbb{X}$ as d increases from 0 to ∞. Furthermore, there is a unique value $\dim_{\mathrm{Hd}} B$, called the *Hausdorff dimension* of B, such that $h^{(d)}(B) = \infty$ if $0 \leq d < \dim_{\mathrm{Hd}} B$ and $h^{(d)}(B) = 0$ if $d > \dim_{\mathrm{Hd}} B$.

Note that one can see from this property that Hausdorff dimension is a metric concept rather than a topological one[7] (therefore we have chosen the name *metric* dimension; the (topological) dimension of $\mathbb{X}$ is $r + 2 \cdot s$, because $\mathfrak{p}$-adic spaces $\mathbb{Q}_\mathfrak{p}$ are totally disconnected).

4. Graph-Directed Iterated Function Systems

Let us consider the following subspace $\mathcal{L}$ of linear mappings[b] from $\mathbb{X}$ to $\mathbb{X}$: For each $T \in \mathcal{L}$, there are numbers $a_1, \ldots, a_{r+s+k}$ such that

$$T(x) = T((x_1, \ldots, x_{r+s+k})) = (a_1 \cdot x_1, \ldots, a_{r+s+k} \cdot x_{r+s+k}), \tag{6}$$

where $x_1, \ldots, x_r, a_1, \ldots, a_r \in \mathbb{R}$, while $x_{r+1}, \ldots, x_{r+s}, a_{r+1}, \ldots, a_{r+s} \in \mathbb{C}$ and $x_{r+s+j}, a_{r+s+j} \in \mathbb{Q}_{\mathfrak{p}_j}$ $(1 \leq j \leq k)$.

We now look at the family (complex numbers $a_{r+1}, \ldots, a_{r+s}$ taken twice) of the $r + 2 \cdot s + k$ numbers

$$\begin{aligned}(\, & |a_1|, \ldots, |a_r|, |a_{r+1}|, |a_{r+1}|, |a_{r+2}|, \ldots, |a_{r+s-1}|, \\ & |a_{r+s}|, |a_{r+s}|, \|a_{r+s+1}\|_{\mathfrak{p}_1}, \ldots, \|a_{r+s+k}\|_{\mathfrak{p}_k} \,),\end{aligned} \tag{7}$$

called the *singular values* of T. We order them in descending order $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{r+2s+k}$, where $(\alpha_1, \ldots, \alpha_{r+2s+k})$ is a permutation of $(|a_1|, \ldots, \|a_{r+s+k}\|_{\mathfrak{p}_k})$. We are only interested in maps $T \in \mathcal{L}$ which are contracting $(\alpha_1 < 1)$ and non-singular $(\alpha_{r+2s+k} > 0)$. We denote the subspace of non-singular and contracting maps of $\mathcal{L}$ by $\mathcal{L}'$.

---

[b] One can also consider more general linear mappings[2]; the ones considered here then correspond to the case where the coordinate axes and the principal axes coincide.

The *singular value function* $\Phi^q(T)$ of $T \in \mathcal{L}'$ is defined[2,3] for $q \geq 0$ as follows:

$$\Phi^q(T) = \begin{cases} 1 & \text{if } q = 0 \\ \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{j-1} \cdot \alpha_j^{q-j+1} & \text{if } j-1 < q \leq j \\ (\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{r+2s+k})^{q/(r+2s+k)} & \text{if } q > r + 2 \cdot s + k \end{cases} \tag{8}$$

Then, $\Phi^q(T)$ is continuous and strictly decreasing in q. Moreover, for fixed q, the singular value function is submultiplicative, i.e., $\Phi^q(T \circ U) \leq \Phi^q(T) \cdot \Phi^q(U)$ for $T, U \in \mathcal{L}'$. Note that we have $\Phi^q(T^n) = [\Phi^q(T)]^n$.

We now look at a *graph-directed iterated function system* (GIFS) ($1 \leq i \leq n$):

$$\Omega_i = \bigcup_{i=1}^{n} \bigcup_{f_{ij}^{(\ell)} \in F_{ij}} f_{ij}^{(\ell)}(\Omega_j), \tag{9}$$

where F_{ij} is a (finite) set of affine contracting mappings, i.e., $f_{ij}^{(\ell)}(x) = T_{f_{ij}^{(\ell)}}(x) + t_{f_{ij}^{(\ell)}}$ with $T_{f_{ij}^{(\ell)}} \in \mathcal{L}'$ and $t_{f_{ij}^{(\ell)}} \in \mathbb{X}$. A GIFS can be visualised by a directed multi-graph $G_{(\Omega_1,\ldots,\Omega_n)}$, where the vertices are the sets Ω_i. If $F_{ij} \neq \varnothing$, we draw $|F_{ij}|$ directed edges from Ω_i to Ω_j, labelling each edge with exactly one of the maps $f_{ij}^{(\ell)}$. We denote by $\boldsymbol{F}$ the matrix[c] $\boldsymbol{F} = (|F_{ij}|)_{1 \leq i,j \leq n}$ (with the convention $|\varnothing| = 0$) and by $\rho(\boldsymbol{F})$ its spectral radius.

We define the *path space* E^∞ as the set of all infinite paths in the graph along directed edges that start at some vertex. Each path (and its starting point) is (uniquely, maybe after renaming) indexed by the sequence of the edges $\omega = (\omega_1 \omega_2 \ldots)$ it runs along. We also define the sets $E^{(0)} = \varnothing$ (paths of length 0), and the set $E_{ij}^{(\ell)}$ of all paths of length ℓ that start at Ω_i and end at Ω_j (then $\omega_1 \in \bigcup_{m=1}^n F_{im}$ and $\omega_\ell \in \bigcup_{m=1}^n F_{mj}$). We also set $E^{(\ell)} = \bigcup_{i=1}^n \bigcup_{j=1}^n E_{ij}^{(\ell)}$ (all paths of length ℓ), $E^{\text{fin}} = \bigcup_{\ell \geq 0} E^{(\ell)}$ (all finite paths) and $E^* = E^{\text{fin}} \cup E^\infty$.

For $\omega \in E^{\text{fin}}$ and $\varpi \in E^*$, we denote by $\omega\varpi$ the sequence obtained by concatenation (or juxtaposition) if $\omega\varpi \in E^*$. If ω is a prefix of ϖ, i.e., $\varpi = \omega \ldots$, we write $\omega < \varpi$. By $\omega \wedge \varpi$ we denote the maximal sequence such that both $(\omega \wedge \varpi) < \omega$ and $(\omega \wedge \varpi) < \varpi$. We can topologise E^∞ in a natural way using the ultrametric $d(\omega, \varpi) = \eta^{-|\omega \wedge \varpi|}$ for some $\eta > 1$. Then E^∞ is a compact space and the sets $N(\varpi) = \{\omega \in E^\infty \mid \varpi < \omega\}$ with $\varpi \in E^{\text{fin}}$ form a basis of clopen sets for E^∞.

For $\omega = (\omega_1 \ldots \omega_\ell) \in E^{\text{fin}}$, we define $T_\omega = T_{\omega_1} \circ \ldots \circ T_{\omega_\ell}$ (with $T_\varnothing(x) = x$), i.e., we are only interested in the linear part of each map $\omega_i(x) = T_{\omega_i}(x) + t_{\omega_i}$. By the "Method II Construction" with the set function $\tau^q(N(\omega)) = \Phi^q(T_\omega)$ (with $\tau^q(\varnothing) = 0$), we obtain a measure $\nu^{(q)}$ on E^∞. Then, we can generalise Proposition 4.1 of Ref. 2.

Proposition 4.1. *For a GIFS (with strongly connected directed graph), the following numbers exist and are all equal:*

---

[c] This is also the adjacency matrix of the graph $G_{(\Omega_1,\ldots,\Omega_n)}$.

(1) $\inf\{q \mid \sum_{\omega\in E^{\mathrm{fin}}} \Phi^q(T_\omega) < \infty\} = \sup\{q \mid \sum_{\omega\in E^{\mathrm{fin}}} \Phi^q(T_\omega) = \infty\}$.

(2) $\inf\{q \mid \nu^{(q)}(E^\infty) = 0\} = \sup\{q \mid \nu^{(q)}(E^\infty) = \infty\}$.

(3) the unique $q > 0$ *such that*[d]

$$\lim_{\ell\to\infty}\left(\rho\left(\left[\sum_{\omega\in E^{\ell}_{ij}} \Phi^q(T_\omega)\right]_{1\le i,j\le n}\right)\right)^{1/\ell} = 1. \tag{10}$$

We denote the common value by $\dim_{\mathrm{aff}} G_{(\Omega_1,\ldots,\Omega_n)}$ *and call it the* affinity dimension *(or* Falconer dimension*) of the GIFS.* □

By a covering argument, we get an upper bound for the Hausdorff dimension of the sets Ω_i, compare Proposition 5.1 of Ref. 2 and Theorem 9.12 of Ref. 3.

Proposition 4.2. *If* $\nu^{(q)}(E^\infty) < \infty$, *then* $h^{(q)}(\Omega_i) < \infty$ *for all* $1 \le i \le n$. *In particular, we have* $\dim_{\mathrm{Hd}} \Omega_i \le \dim_{\mathrm{aff}} G_{(\Omega_1,\ldots,\Omega_n)}$ *for all* $1 \le i \le n$. □

In general, it is difficult to decide whether equality holds in this last inequality for a self-affine GIFS, although in a certain sense equality is the generic case – at least in $\mathbb{R}^r$ (see Theorem 5.3 of Ref. 2 and Theorem 9.12 of Ref. 3). And contrary to the well-studied self-similar case (where $\alpha_1 = \ldots = \alpha_r$) in $\mathbb{R}^r$, even the *open set condition*[e] (OSC) does not ensure the equality sign (cf. Ref. 11 and Examples 9.10 & 9.11 in Ref. 3).

We now define a second singular value function $\Psi^q(T)$ of $T \in \mathcal{L}'$ for $q \ge 0$ as follows[4,13]:

$$\Psi^q(T) = \begin{cases} 1 & \text{if } q = 0 \\ \alpha_{r+2s+k}\cdot\ldots\cdot\alpha_{r+2s+k-j+2}\cdot\alpha^{q-j+1}_{r+2s+k-j+1} & \text{if } j-1 < q \le j \\ (\alpha_1\cdot\alpha_2\cdot\ldots\cdot\alpha_{r+2s+k})^{q/(r+2s+k)} & \text{if } q > r+2\cdot s+k \end{cases} \tag{11}$$

Again, $\Psi^q(T) = [\Phi^q(T^{-1})]^{-1}$ is continuous and strictly decreasing in q, but supermultiplicative for fixed q. Just as in Proposition 4.1, we define the *lower affinity dimension*

$$\underline{\dim}_{\mathrm{aff}} G_{(\Omega_1,\ldots,\Omega_n)} = \inf\{q \mid \sum_{\omega\in E^{\mathrm{fin}}} \Psi^q(T_\omega) < \infty\} = \sup\{q \mid \sum_{\omega\in E^{\mathrm{fin}}} \Psi^q(T_\omega) = \infty\} \tag{12}$$

of the GIFS. Then, with the help of the "mass distribution principle" (see Proposition 4.2 in Ref. 3), we obtain the following lower bound for the Hausdorff dimension of the sets Ω_i, compare Proposition 2 of Ref. 4.

Proposition 4.3. *Let* $(\Omega_1,\ldots,\Omega_n)$ *be the solution of a (strongly connected) GIFS*

---

[d] As a reminder: $\rho(\boldsymbol{F})$ denotes the spectral radius of the matrix $\boldsymbol{F}$.

[e] The OSC is satisfied if there exist disjoint non-empty bounded open sets $(U_1,\ldots,U_n)$ such that $U_i \supset \bigcup_{i=1}^n \bigcup_{f_{ij}^{(\ell)}\in F_{ij}} f_{ij}^{(\ell)}(U_j)$, with the unions disjoint.

$\Omega_i = \bigcup_{i=1}^{n} \bigcup_{f_{ij}^{(\ell)} \in F_{ij}} f_{ij}^{(\ell)}(\Omega_j)$, *where all unions are disjoint. If the sets* $(\Omega_1, \ldots, \Omega_n)$ *are also pairwise disjoint, then* $\underline{\dim}_{\mathrm{aff}} G_{(\Omega_1,\ldots,\Omega_n)} \le \dim_{\mathrm{Hd}} \Omega_i$ *for all* $1 \le i \le n$. $\square$

We remark that this disjointness condition is often easy to check in the cases we are interested in, since $\mathfrak{p}$-adic spaces are totally disconnected.

If the linear part of all maps $f_{ij}^{(\ell)}$ is the same, i.e., $T = T_{f_{ij}^{(\ell)}}$ for all i, j, ℓ, we finally obtain the following theorem.

Theorem 4.1. *For a (strongly connected) GIFS with (unique non-empty compact) solution* $(\Omega_1, \ldots, \Omega_n)$, *where all maps* $f_{ij}^{(\ell)}$ *have the same linear part* T, *the affinity dimension* $\dim_{\mathrm{aff}} G_{(\Omega_1,\ldots,\Omega_n)}$ *is given by the unique value* $q > 0$ *such that* $\Phi^q(T) \cdot \rho(\boldsymbol{F}) = 1$. *The Hausdorff dimension of the sets* Ω_i *is bounded by the affinity dimension of the GIFS, i.e.,* $\dim_{\mathrm{Hd}} \Omega_i \le \dim_{\mathrm{aff}} G_{(\Omega_1,\ldots,\Omega_n)}$ *for all* $1 \le i \le n$. *Furthermore, if the unions in the GIFS are disjoint and the sets* $(\Omega_1, \ldots, \Omega_n)$ *are pairwise disjoint, the Hausdorff dimension of the sets* Ω_i *is bounded from below by the lower affinity dimension of the GIFS, i.e.,* $\underline{\dim}_{\mathrm{aff}} G_{(\Omega_1,\ldots,\Omega_n)} \le \dim_{\mathrm{Hd}} \Omega_i$ *for all* $1 \le i \le n$, *where* $\underline{\dim}_{\mathrm{aff}} G_{(\Omega_1,\ldots,\Omega_n)}$ *is given by the unique value* $q > 0$ *such that* $\Psi^q(T) \cdot \rho(\boldsymbol{F}) = 1$. $\square$

5. An Example

Our motivation for this work are so-called “Rauzy fractals”[15], which are used to prove pure pointedness of the dynamical system of certain 1-dimensional sequences over a finite alphabet, obtained by a substitution rule. “Rauzy fractals” yield a geometric representation[14] (or so-called windows for models sets[1]) for such sequences.

Here, we look at the substitution $a \mapsto aaba$, $b \mapsto aa$ (we obtain a two-sided infinite sequence by applying the substitution repeatedly (we denote the zeroth position by $|$): $a|a \mapsto aaba|aaba \mapsto \ldots aaaba|aabaaabaaaaaba\ldots$). From such a substitution, one can obtain a GIFS (see the above literature[15,14,1] and references therein), in this case in the space $\mathbb{R} \times \mathbb{Q}_2$:

$$\begin{aligned} \Omega_a &= T(\Omega_a) \cup T(\Omega_b) \cup T(\Omega_a) + \frac{1}{2}t_1 \cup T(\Omega_b) + \frac{1}{2}t_1 \cup T(\Omega_a) + t_2 \\ \Omega_b &= T(\Omega_a) + t_1 \end{aligned} \tag{13}$$

where $T((x_1, x_2)) = (\kappa{\cdot}x_1, \lambda{\cdot}x_2)$, $t_1 = (\kappa, \lambda)$, $t_2 = (\kappa{+}1, \lambda{+}1)$, $\kappa = \frac{3-\sqrt{17}}{2} \approx -0.562$ and $\lambda = \frac{3+\sqrt{17}}{2} \approx 3.562$, which in the 2-adic expansion starts as $\lambda = 01101\ldots$. We have $|\kappa| = \frac{2}{\lambda}$, $\|\lambda\|_2 = \frac{1}{2}$ and $\rho(\boldsymbol{F}) = \lambda$, and therefore the affinity dimension[f] $\dim_{\mathrm{aff}} G_{(\Omega_a,\Omega_b)} = 2 = \dim_{\mathrm{metr}} \mathbb{R} \times \mathbb{Q}_2$. Indeed, one can show that the Haar measure of the sets Ω_a and Ω_b is positive and the intersection $\Omega_a \cap \Omega_b$ has Haar measure 0.

It is more interesting to calculate the Hausdorff dimension of the boundaries $\partial\Omega_a$ and $\partial\Omega_b$. For the boundary, one can also derive a GIFS with the same contraction

---

[f] We also have $\underline{\dim}_{\mathrm{aff}} G_{(\Omega_a,\Omega_b)} = 2$, but the sets Ω_a and Ω_b are not disjoint. Therefore Proposition 4.3 does not apply here.

T. This is possible, because the above GIFS for (Ω_a, Ω_b) can be dualised[9] to obtain a point set equation for point sets (X_a, X_b):

$$\begin{aligned} X_a &= T^{-1}(X_a) \cup T^{-1}(X_a) + T^{-1}(\frac{1}{2}t_1) \cup \\ &\qquad T^{-1}(X_b) + T^{-1}(t_1) \cup T^{-1}(X_a) + T^{-1}(t_2) \\ X_b &= T^{-1}(X_a) \cup T^{-1}(X_a) + T^{-1}(\frac{1}{2}t_1) \end{aligned} \tag{14}$$

where $T^{-1}((x_1, x_2)) = (\frac{1}{\kappa} \cdot x_1, \frac{1}{\lambda} \cdot x_2)$. Starting this iteration with $X_a = \{(0,0)\} = X_b$, one obtains a fixed point for (X_a, X_b) and one can show that $J = (X_a + \Omega_a) \cup (X_b + \Omega_b)$ is a tiling with the prototiles Ω_a and Ω_b of the whole space $\mathbb{R} \times \mathbb{Q}_2$ (for purely Euclidean spaces, this is now well established[8]). With the help of this tiling J, one obtains the following GIFS for the boundary:

$$\begin{aligned} \Xi_{(a,b,0)} &= T(\Xi_{(a,a,1)}) \\ \Xi_{(b,a,0)} &= T(\Xi_{(a,a,-1)}) + t_1 \\ \Xi_{(a,a,1)} &= T(\Xi_{(a,a,-1)}) + t_1 \cup T(\Xi_{(a,a,\frac{\lambda}{2}-1)}) \cup T(\Xi_{(b,a,\frac{\lambda}{2}-1)}) \\ \Xi_{(a,a,-1)} &= T(\Xi_{(a,a,1)}) \cup T(\Xi_{(a,a,1-\frac{\lambda}{2})}) + \tfrac{1}{2}t_1 \cup T(\Xi_{(a,b,1-\frac{\lambda}{2})}) + \tfrac{1}{2}t_1 \\ \Xi_{(a,a,\frac{\lambda}{2}-1)} &= T(\Xi_{(a,a,1)}) + \tfrac{1}{2}t_1 \\ \Xi_{(a,a,1-\frac{\lambda}{2})} &= T(\Xi_{(a,a,-1)}) + t_2 \\ \Xi_{(a,b,1-\frac{\lambda}{2})} &= T(\Xi_{(a,a,\frac{\lambda}{2}-1)}) \cup T(\Xi_{(b,a,\frac{\lambda}{2}-1)}) \\ \Xi_{(b,a,\frac{\lambda}{2}-1)} &= T(\Xi_{(a,a,1-\frac{\lambda}{2})}) + t_1 \cup T(\Xi_{(a,b,1-\frac{\lambda}{2})}) + t_1 \end{aligned} \tag{15}$$

Here, $\Xi_{(a,a,1-\frac{\lambda}{2})} = \Omega_a \cap \Omega_a + (1 - \frac{\kappa}{2}, 1 - \frac{\lambda}{2})$ and similarly for the other sets. The boundaries are therefore given by

$$\begin{aligned} \partial\Omega_a &= \Xi_{(a,b,0)} \cup \Xi_{(a,a,1)} \cup \Xi_{(a,a,-1)} \cup \Xi_{(a,a,\frac{\lambda}{2}-1)} \cup \Xi_{(a,a,1-\frac{\lambda}{2})} \cup \Xi_{(a,b,1-\frac{\lambda}{2})} \\ \partial\Omega_b &= \Xi_{(b,a,0)} \cup \Xi_{(b,a,\frac{\lambda}{2}-1)}. \end{aligned} \tag{16}$$

To obtain a strongly connected GIFS which fulfills the disjointness condition from the GIFS in Eq. (15), we observe that $\Xi_{(a,b,0)} = \Xi_{(b,a,0)}$, $\Xi_{(a,a,1)} = \Xi_{(a,a,1-\frac{\lambda}{2})} \cup \Xi_{(a,b1-\frac{\lambda}{2})}$ and $\Xi_{(a,a,-1)} = \Xi_{(a,a,\frac{\lambda}{2}-1)} \cup \Xi_{(a,b,0)}$. So we arrive at the GIFS

$$\begin{aligned} \Xi_{(a,b,0)} &= T(\Xi_{(a,a,1-\frac{\lambda}{2})}) \cup T(\Xi_{(a,b,1-\frac{\lambda}{2})}) \\ \Xi_{(a,a,\frac{\lambda}{2}-1)} &= T(\Xi_{(a,a,1-\frac{\lambda}{2})}) + \tfrac{1}{2}t_1 \cup T(\Xi_{(a,b,1-\frac{\lambda}{2})}) + \tfrac{1}{2}t_1 \\ \Xi_{(a,a,1-\frac{\lambda}{2})} &= T(\Xi_{(a,a,\frac{\lambda}{2}-1)}) + t_2 \cup T(\Xi_{(a,b,0)}) + t_2 \\ \Xi_{(a,b,1-\frac{\lambda}{2})} &= T(\Xi_{(a,a,\frac{\lambda}{2}-1)}) \cup T(\Xi_{(b,a,\frac{\lambda}{2}-1)}) \\ \Xi_{(b,a,\frac{\lambda}{2}-1)} &= T(\Xi_{(a,a,1-\frac{\lambda}{2})}) + t_1 \cup T(\Xi_{(a,b,1-\frac{\lambda}{2})}) + t_1. \end{aligned} \tag{17}$$

For this GIFS, the spectral radius $\rho(\boldsymbol{F})$ equals 2. Consequently, we obtain $\dim_{\mathrm{aff}} G_{(\Xi_{(a,b,0)}, \Xi_{(a,a,\frac{\lambda}{2}-1)}, \ldots, \Xi_{(b,a,\frac{\lambda}{2}-1)})} = \frac{\log(\sqrt{17}-3)}{\log 2} + 1 \approx 1.1675$ and $\underline{\dim}_{\mathrm{aff}} G_{(\Xi_{(a,b,0)}, \Xi_{(a,a,\frac{\lambda}{2}-1)}, \ldots, \Xi_{(b,a,\frac{\lambda}{2}-1)})} = 1$. Using the total disconnectedness of $\mathbb{Q}_2$, one can show that the disjointness condition for the sets in Eq. (17) holds, wherefore these are the upper and lower bounds for the Hausdorff dimension of the boundaries $\partial\Omega_a$ and $\partial\Omega_b$. We end this article with pictures of the GIFS in Eq. (17) and of the sets Ω_a, Ω_b and their boundaries.

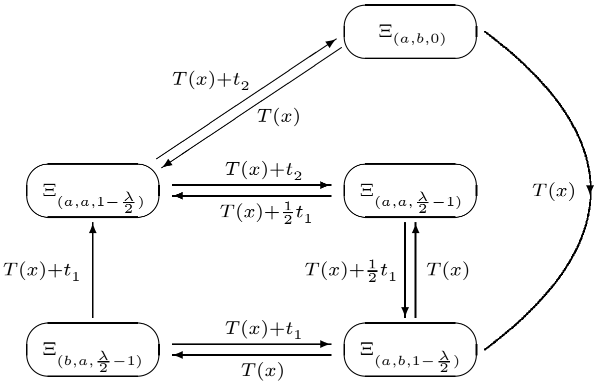

Figure 1. The directed graph $G_{(\Xi_{(a,b,0)},\Xi_{(a,a,\frac{\lambda}{2}-1)},\dots,\Xi_{(b,a,\frac{\lambda}{2}-1)})}$ associated to the GIFS in Eq. (17).

Acknowledgments

The author thanks the referees and Michael Baake for helpful comments. Furthermore, the author expresses his thanks to the Cusanuswerk for financial support, as well as to the German Research Council, Collaborative Research Centre 701.

References

1. M. Baake and B. Sing, “Kolakoski-$(3,1)$ is a (deformed) model set”, *Canad. Math. Bull.* **47** (2004) 168–190; `math.MG/0206098`.
2. K. Falconer, “The Hausdorff dimension of self-affine fractals”, *Math. Proc. Camb. Phil. Soc.* **103** (1988) 339–350.
3. K. Falconer, *Fractal Geometry*, John Wiley & Sons, Chicester, 1990.
4. K. Falconer, “The Hausdorff dimension of self-affine fractals II”, *Math. Proc. Camb. Phil. Soc.* **111** (1992) 169–179.
5. P.R. Halmos, *Measure Theory*, Springer, New York, 1974.
6. E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis I*, Springer, Berlin, 1963.
7. W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, Princeton, NJ, 1948.
8. S. Ito and H. Rao, “Atomic surfaces, tilings and coincidence I. Irreducible case”, Preprint.

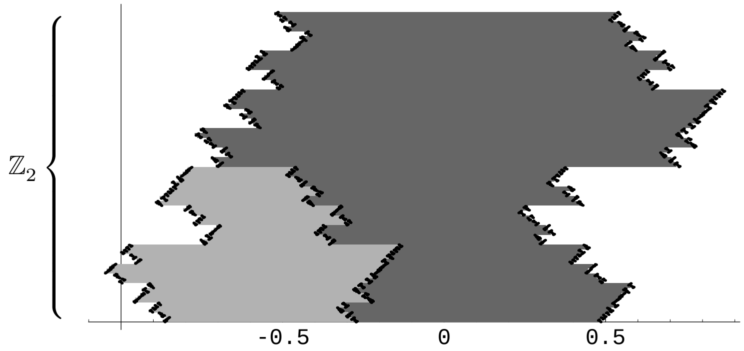

Figure 2. The sets Ω_a (dark gray) and Ω_b (light gray) and their boundaries (black) in $\mathbb{R} \times \mathbb{Q}_2$.

9. J.C. Lagarias and Y. Wang, “Substitution Delone sets”, *Discr. Comput. Geom.* **29** (2003) 175–209.
10. S. Lang, *Algebraic Number Theory*, Addison-Wesley, Reading, MA, 1970.
11. C. McMullen, “The Hausdorff dimension of general Sierpiński carpets”, *Nagoya Math. J.* **96** (1984) 1–9.
12. M.E. Munroe, *Measure and Integration*, Addison-Wesley, Reading, MA, 1971.
13. W.H. Paulsen, “Lower bounds for the Hausdorff dimension of n-dimensional self-affine sets”, *Chaos, Solitons & Fractals* **5** (1995) 909–931.
14. N. Pytheas-Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, (*Lect. Notes Math.* **1784**, edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel), Springer, Berlin, 2002.
15. G. Rauzy, “Nombres algébriques et substitutions”, *Bull. Soc. math. France* **110** (1982) 147–178.
16. A.M. Robert, *A Course in p-adic Analysis*, Springer, New York, 2000.
17. C.A. Rogers, *Hausdorff Measures*, Cambridge University Press, Cambridge, 1998 (Reissue of the 1970 edition).
18. J.-P. Serre, *Local Fields*, Springer, New York, 1979.