The Kac problem for a gas of interacting Brownian loops
Le problème de Kac pour un gaz de lacets browniens en interaction

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Abstract. The following version of the inverse spectral problem of Kac is discussed for a system of interacting Brownian loops in a bounded admissible domain Λ of $E = \mathbb{R}^2$: Given the energy $\mathcal{U}$ of the system $\ln Z(\Lambda_R, z)$, the logarithm of the associated partition function in the delated region $R \cdot \Lambda$, $R \geq 1$, is expanded for small $z > 0$ as the sum of a volume term $p(\phi, z) \cdot R^2 \cdot |\Lambda|$, a boundary term $b(\phi, z) \cdot R \cdot |\partial \Lambda|$ and a rest term $o(R)$ as $R \to +\infty$.

Résumé. On considerè une version du problème inverse de Kac pour un système de lacets browniens en interaction dans un domaine convexe borné de $\mathbb{R}^2$ avec n trous convexes: Etant donnée l'énergie $\mathcal{U}$ du système, le logarithme de la fonction de partition associée $\ln Z(\Lambda_R, z)$ sur le domaine dilaté $R \cdot \Lambda$, $R \geq 1$, est développé pour $z > 0$ petit en une somme d'un terme volume $p(\phi, z) \cdot R^2 \cdot |\Lambda|$, d'un terme frontière $b(\phi, z) \cdot R \cdot |\partial \Lambda|$ et d'un reste $o(R)$, $R \to +\infty$.

1. Introduction

In 1966 Kac, in his famous paper [2], expanded for a given n-connected domain Λ of E the partition function $\rho_\beta(\mathcal{X}(\Lambda))$ as $\beta \searrow 0$. Here $\mathcal{X}(\Lambda)$ denotes the set of all continuous loops $x : [0, \beta] \to \Lambda$, and ρ_β is the restriction of the measure $\int_E du\ P^u$ to $\mathcal{X}(\Lambda)$, where P^u denotes the Brownian bridge measure on the set $\mathcal{X}^u$ of all loops starting and ending in u. (See [1], [6].) P^u is not normalized: $P^u(\mathcal{X}^u) = (\pi\beta)^{-1}$. This problem was considered in its dual form in the important work of Macris et al. [3]. The authors expanded for fixed $\beta > 0$ the function $\rho_\beta(\mathcal{X}(\Lambda_R))$ as $R \to +\infty$, where $\Lambda_R = R \cdot \Lambda$ is the delation of Λ by R.

To motivate our generalization of this dual problem, observe that $\rho = \rho_\beta$ generates the measure

$$(1.1) \quad \mathcal{W}_{\rho_\Lambda}(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{X}(\Lambda)} \overset{n}{\cdots} \int_{\mathcal{X}(\Lambda)} \varphi(\delta_{x_1} + \ldots + \delta_{x_n}) \rho(dx_1) \ldots \rho(dx_n),$$

where φ is a non-negative, measurable function on the space $\mathcal{M}(\Lambda)$ of finite subsets μ of $\mathcal{X}(\Lambda)$, which will be identified with finite sums of Dirac measures of its elements. Here $\rho_\Lambda = 1_{\mathcal{X}(\Lambda)} \cdot \rho$.

It is obvious that $\rho(\mathcal{X}(\Lambda_R)) = \ln \mathcal{W}_{\rho_{\Lambda_R}}(\mathcal{M}(\Lambda_R))$. Thus the partition function of Kac is the log-partition function of the socalled ideal gas of finitely many, non interacting Brownian loops in Λ_R. The aim of this paper is a partial generalization of Kac's theorem for interacting Brownian loops in Λ_R.

The following **theorem** is shown: Given an admissible domain Λ (see §2) the energy $\mathcal{U} = \mathcal{U}^\phi$, defined in (2.1) by means of a translation invariant potential $\phi \in \mathcal{P}_l$, $l > 16$, the associated log-partition function $Z(\Lambda_R, z) = \mathcal{W}_{z \cdot \rho_{\Lambda_R}}(\exp -\mathcal{U}^\phi)$ (defined in §2) is expanded for $0 < z < \pi \cdot [4^{l+3} \cdot e^{4\beta B} \cdot p_l(\phi)]^{-1}$ as

$$(1.2) \quad \ln Z(\Lambda_R, z) = p(\phi, z) \cdot R^2 \cdot |\Lambda| + b(\Lambda, \phi, z) \cdot R + o(R) \text{ as } R \to +\infty.$$

$p(\phi, z)$ respectively $b(\Lambda, \phi, z)$ are explicitly expressed as functional integrals. If ϕ is rotation invariant then b has a simpler form: $b(\Lambda, \phi, z) = \tilde{b}(\phi, z) \cdot |\partial\, \Lambda|$. The theorem shows that $|\Lambda|$ and $|\partial\, \Lambda|$ are determined by the interaction ϕ. Thus the gas of interacting loops gives informations about the geometry of the domain in which it is living.

The third term in the above expansion is expected to be of the form $c(\phi, z) \cdot \chi(\Lambda)$ where $\chi(\Lambda)$ is the Euler characteristic. This final result be presented in a forthcoming paper.

The model which we consider is actually a quantum gas with Maxwell-Boltzmann statistic in the "loop representation" (see [1]). Hence (1.2) gives the first two terms of the asymptotic expansion of the log-partition function of a quantum gas in thermodynamic limit. For classical gases a complete expansion of ln $Z(\Lambda_R, z)$ was obtained in [5].

2. The model of the gas considered

An admissible domain Λ for the theorem above is described as follows: Λ is an open, convex, bounded subset of E with n convex closed holes. We assume that the boundaries are one dimensional, closed $\mathcal{C}^2$-manifolds. Thus for any $u \in \Lambda^{bd} = \{u \in \Lambda \mid d(u, \partial \Lambda) < \delta\}$ there is a unique $r \in \partial \Lambda$ with $d(u, r) = d(u, \partial \Lambda)$, for $\delta > 0$ small enough.

The interaction ϕ is an even, measurable, real function ϕ on E which is stable, i.e. there exists $B \geq 0$ with $\sum_{u,v\in\mu,u\neq v} \phi(u-v) \geq -2 \cdot |\mu| \cdot B$ for any $\mu \in \mathcal{M}(\mathbb{R}^2)$. ($|\mu| = card\,\mu$). We also assume that $p_l(\phi) := \int_{\mathbb{R}^2} |\phi|(u) \cdot (1 + |u|)^l du < +\infty$. $l \geq 0$ will be chosen later. Let $\mathcal{P}_l$ denote the class of these potentials.

By means of a potential $\phi \in \mathcal{P}_l$ the energy $\mathcal{U}$ is defined by

$$\mathcal{U}(\mu) = \frac{1}{2} \cdot \sum_{x,y\in\mu,x\neq y} \int_0^\beta \phi(x(s) - y(s))ds, \mu \in \mathcal{M}(E). \tag{2.1}$$

The associated Boltzmann factor $f = \exp -\mathcal{U}$ is $\mathcal{W}_{z\cdot\rho_\Lambda}$-integrable. Given an admissible domain Λ and a positive parameter $z > 0$ we then consider the finite measure $\mathcal{W}_{z\cdot\rho_\Lambda}$ on the set $\mathcal{M}(\Lambda)$ of finite subsets of $\mathcal{X}(\Lambda)$. $(\mathcal{M}(\Lambda), \mathcal{W}_{z\cdot\rho_\Lambda}, \phi)$ defines the underlying model of a "gas" in Λ. The main object to be considered in this model is the partition function $Z(\Lambda, z) = \mathcal{W}_{z\cdot\rho_\Lambda}(f)$.

3. Sketch of the proof of the theorem

The first important step is the representation of the log-partition function $\mathcal{I}(R) = \ln \mathcal{W}_{z\cdot\rho_{\Lambda_R}}(f)$, $R \geq 1$, in terms of the Ursell function g, which is defined by

$$g(\mu) = \sum_{\gamma\in\mathcal{G}(\mu)} \prod_{e\in\gamma} q(e), \mu \in \mathcal{M}(E), \tag{3.1}$$

where $\mathcal{G}(\mu)$ is the set of all connected graphs built on μ, $e \in \gamma$ denotes an edge $e = (x, y)$ of γ and $q(x, y) = \exp(-\int_0^\beta \phi(x(s) - y(s))ds) - 1$. The proof of the following representation of $\mathcal{I}$ uses the technique of strong cluster estimates. (See [7].)

Lemma 1 For $\phi \in \mathcal{P}_0$ and $0 < \pi \cdot [e^{4\beta B+1} \cdot p_0(\phi)]^{-1}$ then the Ursell function g is $\mathcal{W}_{z\cdot\rho_{\Lambda_R}}$-integrable and $\mathcal{I}(R) = \mathcal{W}_{z\cdot\rho_{\Lambda_R}}(g), R \geq 1$.

The second important step is the representation of $\mathcal{W}_{z\cdot\rho_{\Lambda_R}}(g)$ by means of the Campbell measures $\mathcal{C}_\rho$ of $\mathcal{W}_\rho$ defined by

$$(3.2) \qquad \mathcal{C}_\rho(h) = \int_{\mathcal{M}} \int_{\mathcal{X}} h(x,\mu)\mu(dx)\mathcal{W}_\rho(d\mu), h \geq 0, \text{ measurable.}$$

(See [4].) Here and below $\mathcal{M} = \mathcal{M}(\mathcal{X})$. $\mathcal{C}_\rho$ is a measure on $X \times \mathcal{M}$ which is concentrated on the measurable subset $\{(x,\mu) \mid x \in \mu\}$. $\mathcal{C}_\rho$ will be used in connection with the following well known partial integration formula (see [4]):

$$(3.3) \qquad \mathcal{C}_\rho(h) = \int_{\mathcal{M}} \int_{\mathcal{X}} h(x,\mu+\delta_x)\rho(dx)\mathcal{W}_\rho(d\mu), h \in \mathcal{L}^1(\mathcal{C}_\rho).$$

A fundamental role in our analysis is played by the main lemma, which describes the decay of correlations in our model: Let

$$(3.4) \qquad \mathcal{H}^u(\varphi) = \int_{\mathcal{X}^u} P^u(dx^u) \int_{\mathcal{M}} \mathcal{W}_\rho(d\mu)\ (\varphi \cdot \tilde{g}_z)(\mu + \delta_{x^u}), u \in \mathbb{R}^2,$$

$\varphi \geq 0$ measurable. Here $\tilde{g}_z(\eta) = z^{|\eta|} \cdot \frac{g(\eta)}{|\eta|}$, if $\eta \neq 0$.

Main lemma. (α) If $\phi \in \mathcal{P}_0$ and $0 < z < \pi[e^{4\beta B+1} \cdot p_0(\phi)]^{-1}$ then $\mathcal{H}^u$ defines a finite signed measure on $\mathcal{M} \setminus \{0\}$ with $\mathcal{H}^u(1)$ independent of u.

(β) If $\phi \in \mathcal{P}_l$ and $0 < z < \pi \cdot [4^{l+3} \cdot e^{4\beta B} \cdot p_l(\phi)]^{-1}$ then there exists a constant $c = C(\beta, l, \phi, z)$ with

$$(3.5) \qquad |\mathcal{H}^u|(\mathcal{M}^c(K_R(u))) \leq \frac{c}{(1+R)^l} \text{ uniformly in } u.$$

Here $K_R(u)$ is the closed ball in $\mathbb{R}^2$ centered in u with radius R; and $|\mathcal{H}^u|$ is defined by (3.4) with $\tilde{g}_z$ replaced by $|\tilde{g}_z|$. The proof of this lemma is contained in [6] and is a combination of corollary 5.1 and lemma 5.5 there.

Let $S(\Lambda) = \{x \in \mathcal{X} \mid x(0) \in \Lambda\}$. Part ($\alpha$) of the main lemma implies that $1_{S(\Lambda)\times\mathcal{M}} \cdot \hat{g}_z$ is $\mathcal{C}_\rho$-integrable for any bounded, measurable $\Lambda \subseteq \mathbb{R}^2$. Thus also $1_{\mathcal{X}(\Lambda)\times\mathcal{M}(\Lambda)} \cdot \hat{g}_z \in \mathcal{L}^1(\mathcal{C}_\rho)$. Here $\hat{g}_z(x,\mu) = \tilde{g}_z(\mu)$. This implies (see Matthes et al. [4]) the inversion formula

$$(3.6) \qquad \mathcal{W}_{z\cdot\rho_\Lambda}(g) = \mathcal{C}_\rho(1_{\mathcal{X}(\Lambda)\times\mathcal{M}(\Lambda)} \cdot \hat{g}_z).$$

Hence

$$\mathcal{I}(R) = \mathcal{C}_\rho(1_{S(\Lambda_R)\times\mathcal{M}(\Lambda_R)} \cdot \hat{g}_z), R \geq 1. \tag{3.7}$$

This will be the starting point of the expansion of $\mathcal{I}$. For notational convenience we shall write $\mathcal{I}_R = \mathcal{C}(S(\Lambda_R) \times \mathcal{M}(\Lambda_R))$, where $\mathcal{C} = \hat{g}_z \cdot \mathcal{C}_\rho$.

The expansion of $\mathcal{I}$ proceeds by decomposition of $\mathcal{C}(S(\Lambda_R) \times \mathcal{M}(\Lambda_R))$. The initial decomposition is

$$\mathcal{I}_R = \mathcal{C}(S(\Lambda_R) \times \mathcal{M}) - \mathcal{C}(S(\Lambda_R) \times \mathcal{M}^c(\Lambda_R)). \tag{3.8}$$

By partial integration the first term is given by

(3.9)
$$\mathcal{C}(S(\Lambda_R) \times \mathcal{M}) = R^2 \cdot |\Lambda| \cdot \int_{\mathcal{X}^0} P^0(dx^0) \int_{\mathcal{M}} \mathcal{W}_\rho(d\mu)\ \tilde{g}_z(\mu + \delta_{x^0}) = R^2 \cdot |\Lambda| \cdot p(\phi, z).$$

The functional integral $p(\phi, z)$ appearing here is called the pressure. It remains to expand the second term $\mathcal{T}_R = \mathcal{C}(S(\Lambda_R) \times \mathcal{M}^c(\Lambda_R))$.

Decomposing Λ_R into a boundary region $\Lambda_R^{bd} = \{u \in \Lambda_R \mid d(u, \partial\,\Lambda_R) \leq \delta \cdot R^\varepsilon\}$ and its interior $\Lambda_R^{int} = \Lambda_R \setminus \Lambda_R^{bd}$, with $\varepsilon, \delta > 0$ to be chosen later, we get

$$\mathcal{T}_R = \mathcal{C}(S(\Lambda_R^{bd}) \times \mathcal{M}^c(\Lambda_R)) + \mathcal{C}(S(\Lambda_R^{int}) \times \mathcal{M}^c(\Lambda_R)). \tag{3.10}$$

The second term on the right hand side is $\mathcal{T}_R^{int} = \int_{\Lambda_R^{int}} du\ \mathcal{H}^u(\mathcal{M}^c(\Lambda_R))$ by partial integration. The main lemma then implies

$$|\mathcal{T}_R^{int}| \leq R^2 \cdot |\Lambda^{int}| \frac{c}{(1+\delta R^\varepsilon)^l} = o(1) \text{ as } R \to \infty, \tag{3.11}$$

if $\varepsilon \cdot l > 2$. From now on we choose $\varepsilon = \frac{1}{8}$ and $l > 16$. Thus all relevant information of $\mathcal{T}$ is contained in $\mathcal{T}_R^{bd} := \mathcal{C}(S(\Lambda_R^{bd}) \times \mathcal{M}^c(\Lambda_R))$.

To analyse $\mathcal{T}_R^{bd}$ we from now on represent an element $u \in \Lambda_R^{bd}$ in the local Gaussian coordinate system as $u = r + t \cdot \mathbf{n}$ and set up at each point $r \in \partial\,\Lambda$ local coordinates (ξ, η) where ξ is along the tangent vector $\mathbf{s} = \mathbf{s}(r)$ and η is along the inward drawn unit normal $\mathbf{n} = \mathbf{n}(r)$ to $\partial\,\Lambda$ at r. Then $\partial\,\Lambda$ is locally given by $\eta = f_r(\xi)$, $|\xi| < \delta R^\varepsilon$, where f_r is a function of class $\mathcal{C}^2$. Furthermore we associate to r the cylinder $\Pi_{r,\delta R^\varepsilon} = \{(\xi, \eta) \mid \|\xi\| < \delta R^\varepsilon\}$. This yields the decomposition $\mathcal{M}^c(\Lambda_R) = \mathcal{M}^c(\Lambda_R)\mathcal{M}(\Pi_{r,\delta R^\varepsilon}) + \mathcal{M}^c(\Lambda_R)\mathcal{M}^c(\Pi_{r,\delta R^\varepsilon})$. The standard argument by means of the main lemma shows that $\mathcal{C}(S(\Lambda_R^{bd}) \times \mathcal{M}^c(\Lambda_R)\mathcal{M}^c(\Pi_{r,\delta R^\varepsilon})) = o(1)$ as $R \to +\infty$. It remains to analyse $\overline{\mathcal{T}}_R^{bd} = \mathcal{C}(S(\Lambda_R^{bd}) \times \mathcal{M}^c(\Lambda_R)\mathcal{M}(\Pi_{r,\delta R^\varepsilon}))$, where r has to be considered as a function of x.

Decompose this term by means of $\mathcal{F}^+_{r,\delta R^\varepsilon} = \{(\xi,\eta) \in \Pi_{r,\delta R^\varepsilon} \bigm| \eta > R \cdot f_r\left(\frac{\xi}{R}\right)\}$:

(3.12)

$$\overline{\mathcal{T}}_R^{bd} = \mathcal{C}(S(\Lambda_R^{bd}) \times \mathcal{M}^c(\Lambda_R)\mathcal{M}(\mathcal{F}^+_{r,\delta R^\varepsilon})) + \mathcal{C}(S(\Lambda_R^{bd}) \times \mathcal{M}(\Pi_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})).$$

The first term is $o(1)$ by the main lemma. The second term τ_R is decomposed by means of $\Pi^+_{r,\delta R^\varepsilon} = \{(\xi,\eta) \in \Pi_{r,\delta R^\varepsilon} \bigm| \eta \geq 0\}$

(3.13)

$$\tau_R = \mathcal{C}(S(\Lambda_R^{bd}) \times \mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}(\Pi_{r,\delta R^\varepsilon})) + \mathcal{C}(S(\Lambda_R^{bd}) \times \mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})).$$

Remark. For shortness we consider only the case where r is a point of the convex part of the boundary of Λ_R: The other, concave case can be treated in a similar way with natural modifications starting from (3.13).

Consider in (3.13) the first term τ_R^1: To get rid of the restriction to the cylinder, use $\Pi_r^+ = \{(\xi,\eta) \bigm| \eta \geq 0\}$ to get $\tau_R^1 = \mathcal{C}(S(\Lambda_R^{bd}) \times \mathcal{M}^c(\Pi_r^+)) - \mathcal{C}(S(\Lambda_R^{bd}) \times \mathcal{M}^c(\Pi_r^+)\mathcal{M}^c(\Pi^+_{r,\delta R^\varepsilon}))$. The second term here is $o(1)$ by the main lemma. The first is given by partial integration as

$$\text{(3.14)} \qquad \mathcal{J}_R^\varepsilon = \int_{\partial\Lambda_R} \sigma_R(dr) \int_0^{\delta R^\varepsilon} dt\ [1 - t \cdot \kappa(r|R)] \cdot \mathcal{H}^{r+t\cdot\mathbf{n}}(\mathcal{M}^c(\Pi_r^+)).$$

where $\kappa(r|R)$ is the curvature of $\partial\Lambda_R$ at r.

Using once more the main lemma we can approximate $\mathcal{J}_R^\varepsilon$ by the integral $\mathcal{J}_R$, where now δR^ε is replaced by $+\infty$. Developing then the bracket, $\mathcal{J}_R$ is the difference of the following two functional integrals.

$$\text{(3.15)} \qquad \mathcal{J}_R^{bd} = R \cdot \int_{\partial\Lambda} \sigma(dr) \int_0^{+\infty} dt\ \mathcal{H}^{r+t\cdot\mathbf{n}}(\mathcal{M}^c(\Pi_r^+));$$

$$\text{(3.16)} \qquad \mathcal{J} = \int_{\partial\Lambda} \sigma(dr)\ \kappa(r|1) \int_0^{+\infty} dt\ t \cdot \mathcal{H}^{r+t\cdot\mathbf{n}}(\mathcal{M}^c(\Pi_r^+)).$$

Here we used $\sigma_R(dr) = R \cdot \sigma_R(dr)$ and $\kappa(r|R) = R^{-1} \cdot \kappa(r|1)$. Note that the second term does not depend on R and thus is of type $o(R)$ while the first term has the form $R \cdot b(\Lambda, \phi, z)$. We remark that if ϕ is Euclidean invariant then $\mathcal{J}_R^{bd} = R \cdot \sigma(\partial\, \Lambda) \cdot \int_0^\infty dt\, \mathcal{H}^{t \cdot \mathbf{n}_0}(\mathcal{M}^c(\Pi_0^+))$ where $\mathbf{n}_0$ denotes any fixed unit vector.

It remains to study the second term τ_R^2 on the right hand side of (3.13). By partial integration we obtain

$$(3.17) \qquad |\tau_R^2| \leq \int\limits_{S(\Lambda_R^{bd})\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})} \rho(dx) \int_{\mathcal{M}} \mathcal{W}_\rho(d\mu)|\tilde{g}_z(\mu + \delta_x)| + \int\limits_{S(\Lambda_R^{bd})\mathcal{X}(\mathcal{F}^+_{r,\delta R^\varepsilon})} \rho(dx) \int_{\mathcal{M}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{M}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})} \mathcal{W}_\rho(d\mu)|\tilde{g}_z(\mu + d_x)|.$$

¿From the proof of proposition 2 in [3] it follows that the first term on the right hand side is $o(R)$. Using arguments from [6] we can show that the second term $\tau_R^{2,2}$ in (3.17) can be estimated from above as $|\tau_R^{2,2}| \leq C \cdot R^{1/8} \cdot \int_{\partial\, \Lambda_R} \sigma(dr) \rho(\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon}))$. Using finally a modified technique from [3] we can show that $\rho(\mathcal{X}(\Pi^+_{r,\delta R^\varepsilon})\mathcal{X}^c(\mathcal{F}^+_{r,\delta R^\varepsilon})) \leq C \cdot R^{-5/8}$. This actually completes the proof of the theorem.

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