ON THE DISTRIBUTION OF ROOTS OF RANDOM POLYNOMIALS

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ABSTRACT. In this article we obtain a simple condition for the coefficients of a random polynomial. This condition appears to be necessary and sufficient for the roots of the polynomial to concentrate asymptotically near the unit circumference with probability one as the degree of the polynomial increases. It is also shown that this condition is sufficient, but not necessary for the roots to be asymptotically uniformly distributed in argument.

Key words and concepts: roots of a random polynomial, roots concentration, random series.

1. Introduction. In this article we are interested in asymptotic distribution on the complex plane of the roots of the following polynomials:

$$G_n(z) = \xi_0 + \xi_1 z + \cdots + \xi_{n-1}z^{n-1} + \xi_n z^n$$

when $n \to \infty$. Here $\xi_0, \xi_1, \dots, \xi_n, \dots$ are independent identically distributed complex-valued random variables. It is always supposed that

$$\mathbf{P}(\xi_i = 0) < 1.$$

Denote $R_n(a,b)$ the number of the roots z_{jn} of G_n belonging to the ring $\{z \in \mathbb{C} : a \le |z| \le b\}$. Denote $S_n(\alpha,\beta), 0 \le \alpha < \beta \le 2\pi$, the number of the roots of G_n belonging to the sector $\{z \in \mathbb{C} : \alpha \le \arg z \le \beta\}$. The roots are counted according to their multiplicity. Evidently the function $R_n(a,b)$ describes the distribution of the absolute values $|z_{jn}|$ of z_{jn} and $S_n(\alpha,\beta)$ describes the distribution of the arguments $\arg z_{jn}$ of z_{jn}. Below we study the asymptotic behavior of the functions R_n, S_n when $n \to \infty$.

The question of distribution of the complex roots of G_n originated from Hammersley [1]. The asymptotic study of the functions R_n, S_n has been initiated by Shparo and Shur in [6]. To describe their results let us introduce the functions

$$f(t) = \left[\underbrace{\log^+ \log^+ \dots \log^+ t}_{m+1}\right]^{1+\varepsilon} \prod_{k=1}^{m} \underbrace{\log^+ \log^+ \dots \log^+ t}_{k},$$

where $\log^+ s = \max(1, \log s)$. Here $\varepsilon > 0, m = 0, 1, \dots$. We assume $f(t) = (\log^+ t)^{1+\varepsilon}$ at $m = 0$.

Shparo and Shur have proved that if for certain $\varepsilon > 0, m \in \mathbb{Z}^+$

$$\mathbf{E}f(|\xi_i|) < \infty, \tag{1}$$

then for any $\delta \in (0,1)$ and any α, β

$$\frac{1}{n}R_n(1-\delta, 1+\delta) \xrightarrow{\mathbf{P}} 1, \quad n \to \infty, \tag{2}$$

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$$\frac{1}{n}S_n(\alpha,\beta) \xrightarrow{\mathbf{P}} \frac{\beta-\alpha}{2\pi}, \quad n\to\infty. \tag{3}$$

The first relation means that under quite weak constraints imposed on the coefficients of a random polynomial, almost all its roots "concentrate uniformly" near the unit circumference with high probability; the second relation means that the arguments of the roots are asymptotically uniformly distributed.

Later Shepp and Vanderbei [5] and Ibragimov and Zeitouni [2] under additional conditions imposed on the coefficients of G_n got more precise asymptotic formulas for R_n.

In this work we show, that the condition (1) could be substituted by a weaker one:

$$\mathbf{E}\log(1+|\xi_i|)<\infty, \tag{*}$$

and the convergence in probability in (2) and (3) could be substituted by convergence almost surely.

We will provide a proof that the condition (*) is, in fact, not only sufficient but also necessary for (2) to hold almost surely. It appears, however, that (*) is not a necessary condition for (3) to hold almost surely. In order to show the latter we provide a corresponding example of the distribution of coefficients of G_n.

Remark. *Let $\tau=\inf\{n:\xi_0=0,\dots,\xi_{n-1}=0,\xi_n\neq 0\}$. Then*

$$\mathbf{P}(\tau=k)=\mathbf{P}^k(\xi_0=0)(1-\mathbf{P}(\xi_0=0))$$

and by Borel-Cantelli's lemma $\tau<\infty$ with probability 1. Below (in the proofs of the theorems 1,2) for the sake of simplicity we suppose that $\tau=0$ with probability 1 (i.e. $\mathbf{P}(\xi_0=0)=0$). To treat the general case it is enough to study in the same way the behavior of R_n, S_n on the sets $\{\tau=k\}$.

2. The distribution of absolute values.

Theorem 1. *The limit relation*

$$\frac{1}{n}R_n(1-\delta,1+\delta) \overset{n\to\infty}{\longrightarrow} 1 \quad a.s. \tag{4}$$

holds for any $\delta\in(0,1)$ if and only if (*) holds.

Remark. *It follows from the proof of the theorem that if the coefficients of a random polynomial do not satisfy (*), then for any $\delta\in(0,1)$ the limit relation*

$$\frac{1}{n}R_n(1-\delta,1+\delta) \overset{n\to\infty}{\longrightarrow} 1$$

does not hold with probability one.

Proof. Suppose condition (*) holds. Let us prove that the radius of convergence of the series

$$G(z)=\sum_{n=0}^{\infty}\xi_n z^n \tag{5}$$

is equal to one with probability one.

Consider $\rho>0$, such that $\mathbf{P}\{|\xi_i|>\rho\}>0$. Using the Borel-Cantelli lemma we obtain that the sequence $\xi_0,\xi_1,\dots,\xi_n,\dots$ with probability one contains infinitely many members of absolute value larger than ρ. Consequently, radius of convergence of the series (5) does not exceed 1 almost surely.

On the other hand, since for any non-negative random variable η

$$\sum_{n=1}^{\infty}\mathbf{P}(\eta \geqslant n) \leqslant \mathbf{E}\eta \leqslant 1+\sum_{n=1}^{\infty}\mathbf{P}(\eta \geqslant n), \tag{6}$$

it follows from (*) that

$$\sum_{n=1}^{\infty}\mathbf{P}(|\xi_n| \geqslant e^{\gamma n}) < \infty,$$

where γ — is an arbitrary positive constant. It follows from the Borel-Cantelli lemma that with probability one $|\xi_n| < e^{\gamma n}$ for any sufficiently large n. This means, that according to the Cauchy-Hadamard formula [4], the radius of convergence of the series $G(z)$ is at least 1 almost surely.

Thus, with probability one, the series $G(z)$ is an analytic function for $|z| < 1$. For $0 \leqslant a < b < 1$ denote $R(a,b)$ to be the number of the roots of G from the set $\{z \in \mathbb{C} : a \leqslant |z| \leqslant b\}$. It follows from the Hurwitz theorem [4], that $R_n(0, 1-\delta) \leqslant R(0, 1-\delta/2)$ almost surely for all sufficiently large n which implies that

$$\frac{1}{n}R_n(0,1-\delta) \overset{n\to\infty}{\longrightarrow} 0 \quad \text{a.s.}$$

In order to conclude the proof of (4) it remains to show that

$$\frac{1}{n}R_n(1+\delta,\infty) \overset{n\to\infty}{\longrightarrow} 0 \quad \text{a.s.}$$

In other words, we need to prove that $\mathbf{P}(A) = 0$, where A is an event, implying that there exists $\varepsilon > 0$, such that

$$R_n(1+\delta,\infty) \geqslant \varepsilon n \tag{7}$$

holds for infinitely many values n.

Denote B to be the event $\{G(z)$ is an analytic function for $|z| < 1\}$. Define random variable η_j, $j = 1, 2, \ldots$, as

$$\eta_j = \sup_n \left|\xi_n e^{-\frac{n}{j}}\right|.$$

Let us denote C_j to be an event that $\eta_j < \infty$. It was shown above that

$$\mathbf{P}(B) = \mathbf{P}(C_j) = 1, \quad j \in \mathbb{N},$$

therefore in order for $\mathbf{P}(A) = 0$ to hold it is sufficient that for a certain j

$$\mathbf{P}(ABC_j) = 0$$

holds.

Let us take a certain j, the exact value of which will be chosen later. Suppose the event ABC_j occurred. Index the roots $z_{jn} = z_j$ of the polynomial $G_n(z)$ in the order of magnitude of its absolute values:

$$|z_1| \leqslant |z_2| \leqslant \cdots \leqslant |z_n|.$$

Let us fix an arbitrary number $C > 1$ (we will choose the exact value later). Consider indices k, l, such that:

$$|z_k| < 1 - \frac{\delta}{C}, \quad |z_{k+1}| \geqslant 1 - \frac{\delta}{C},$$
$$|z_l| \leqslant 1+\delta, \quad |z_{l+1}| > 1+\delta.$$

(For $|z_1| \geqslant 1 - \delta/C$ we take $k = 0$, for $|z_n| \leqslant 1+\delta$ we take $l = n$.)

If

$$|z| < \min\left(1, \frac{|\xi_0|}{n\max_{i=1\dots n}|\xi_i|}\right),$$

then

$$|\xi_0| > |\xi_1 z| + |\xi_2 z^2| + \dots + |\xi_n z^n|.$$

Therefore, such z can not be a root of the polynomial G_n. Taking in the account the fact that the event C_j occurred, we obtain a lower bound for the absolute values of the roots of the polynomial for all sufficiently large n:

$$|z_1| \geqslant \min\left(1, \frac{|\xi_0|}{n\max_{i=1\dots n}|\xi_i|}\right) \geqslant \frac{|\xi_0|}{n\eta_j e^{\frac{n}{j}}} \geqslant |\xi_0|\eta_j^{-1}e^{-\frac{2n}{j}}.$$

Therefore, for any integer m, satisfying $l+1 \leqslant m \leqslant n$, we have

$$\begin{aligned}|z_1\dots z_m| = |z_1\dots z_k||z_{k+1}\dots z_l||z_{l+1}\dots z_m|&\\ \geqslant |\xi_0|^k\eta_j^{-k}e^{-\frac{2n}{j}k}&\left(1-\frac{\delta}{C}\right)^{l-k}(1+\delta)^{m-l}\end{aligned}$$

for all sufficiently large n. Since A occurred, it follows, that $n-l \geqslant n\varepsilon$ for infinitely many values of n. Therefore, for m, satisfying $n-\sqrt{n} \leqslant m \leqslant n$, inequalities $l+1 \leqslant m \leqslant n, m-l \geqslant n\varepsilon/2$ hold for infinitely many values of n. Further, according to the Hurwitz theorem, $k \leqslant R_n(0, 1-\delta/C) \leqslant R(0, 1-\delta/(2C))$ for all sufficiently large n. Therefore, for infinitely many values of n the following inequalities hold

$$|z_1\dots z_m| \geqslant \left(\frac{|\xi_0|}{\eta_j}\right)^{R(0,1-\delta/(2C))} e^{-\frac{2n}{j}R(0,1-\delta/(2C))}\left(1-\frac{\delta}{C}\right)^n(1+\delta)^{\frac{n\varepsilon}{2}}.$$

Choose now C large enough to yield

$$\left(1-\frac{\delta}{C}\right)(1+\delta)^{\frac{\varepsilon}{2}} > 1.$$

Next, holding C constant, choose j such that

$$e^{-\frac{2}{j}R(0,1-\delta/(2C))}\left(1-\frac{\delta}{C}\right)(1+\delta)^{\frac{\varepsilon}{2}} = b > 1.$$

Since

$$\left(\frac{|\xi_0|}{\eta_j}\right)^{\frac{R(0,1-\delta/(2C))}{n}} \to 1$$

as $n \to \infty$, we obtain that there exists a random variable $a > 1$, such that

$$|z_1\dots z_m| \geqslant \left(\frac{|\xi_0|}{\eta_j}\right)^{R(0,1-\delta/(2C))} b^n = \left(b\left(\frac{|\xi_0|}{\eta_j}\right)^{\frac{R(0,1-\delta/(2C))}{n}}\right)^n \geqslant a^n$$

holds for infinitely many values of n. On the other hand it follows from Vieta's theorem that

$$|z_{m+1}\dots z_n| \geqslant \binom{n}{n-\sqrt{n}}^{-1}\left|\sum_{i_1<\dots<i_{n-m}} z_{i_1}\dots z_{i_{n-m}}\right| = \binom{n}{n-\sqrt{n}}^{-1}\frac{|\xi_m|}{|\xi_n|}.$$

We combine these two inequalities to obtain for infinitely many values of n

$$\begin{aligned}
\frac{|\xi_0|}{|\xi_n|} &= |z_1 \dots z_n| \geqslant a^n \binom{n}{n-\sqrt{n}}^{-1} \frac{|\xi_m|}{|\xi_n|} \\
&\geqslant c_1 a^n \frac{(\sqrt{n})^{\sqrt{n}+\frac{1}{2}} (n-\sqrt{n})^{n-\sqrt{n}+\frac{1}{2}}}{n^{n+\frac{1}{2}}} \frac{|\xi_m|}{|\xi_n|} \geqslant c_2 a^n (\sqrt{n})^{-\sqrt{n}} \left(1 - \frac{1}{\sqrt{n}}\right)^n \frac{|\xi_m|}{|\xi_n|} \\
&\geqslant c_3 \exp\left(n \log a - \frac{\sqrt{n}\log n}{2} - \sqrt{n}\right) \frac{|\xi_m|}{|\xi_n|} \geqslant e^{\alpha n} \frac{|\xi_m|}{|\xi_n|},
\end{aligned}$$

where α is a positive random variable. Multiplying left and right parts by $|\xi_n|$, we get that

$$ABC_j \subset \bigcup_{i=1}^{\infty} D_i,$$

where D_i denotes the event $\{|\xi_0| > e^{\frac{n}{i}} \max_{n-\sqrt{n} \leqslant m \leqslant n} |\xi_m|$ for infinitely many values of $n\}$.

To complete the proof it is sufficient to show that $\mathbf{P}(D_i) = 0$. Having in mind to apply the Borel-Cantelli lemma, let us introduce the following events:

$$H_{in} = \{|\xi_0| > e^{\frac{n}{i}} \max_{n-\sqrt{n} \leqslant m \leqslant n} |\xi_m|\}.$$

Considering such $\theta > 0$ that $\mathbf{P}(|\xi_0| \leqslant \theta) = F(\theta) < 1$, we have

$$H_{in} \subset \{|\xi_0| > \theta e^{\frac{n}{i}}\} \cup \{\max_{n-\sqrt{n} \leqslant m \leqslant n} |\xi_m| \leqslant \theta\},$$

consequently,

$$\sum_{n=1}^{\infty} \mathbf{P}(H_{in}) \leqslant \sum_{n=1}^{\infty} \mathbf{P}(|\xi_0| > \theta e^{\frac{n}{i}}) + \sum_{n=1}^{\infty} (F(\theta))^{\sqrt{n}} < \infty$$

and, according to the Borel-Cantelli lemma, $\mathbf{P}(D_i) = 0$.

We prove the implication (4) ⇒ (*) arguing by contradiction. Suppose (*) does not hold, i.e.

$$\mathbf{E} \log(1 + |\xi_i|) = \infty.$$

It follows from (6) that

$$\sum_{n=1}^{\infty} \mathbf{P}(|\xi_n| \geqslant e^{\gamma n}) = \infty \tag{8}$$

for an arbitrary positive γ. For $k \in \mathbb{N}$ introduce an event F_k, implying that $|\xi_n| \geqslant e^{kn}$ holds for infinitely many indices n. It follows from (8) and the Borel-Cantelli lemma, that $\mathbf{P}(F_k) = 1$ and, consequently, $\mathbf{P}(\cap_{k=1}^{\infty} F_k) = 1$. This yields

$$\limsup_{n\to\infty} |\xi_n|^{1/n} = \infty \quad \text{a.s.}$$

This means that with probability one for infinitely many values of n

$$|\xi_n|^{1/n} > \max_{i=0,\dots,n-1} |\xi_i|^{1/i}, \quad |\xi_n|^{1/n} > \frac{3}{\varepsilon}, \quad |\xi_0| < 2^{n-1},$$

where $\varepsilon > 0$ is an arbitrary fixed value. Let us hold one of those n. Suppose $|z| \geqslant \varepsilon$. Then

$$\begin{aligned}|\xi_0 + \xi_1 z + \dots + \xi_{n-1} z^{n-1}| &\leqslant 2^{n-1} + |\xi_n z^n|^{\frac{1}{n}} + |\xi_n z^n|^{\frac{2}{n}} + \dots + |\xi_n z^n|^{\frac{n-1}{n}} \\ &= \frac{2^n}{2} - 1 + \frac{|\xi_n z^n| - 1}{|\xi_n^{\frac{1}{n}} z| - 1} \leqslant \frac{|\xi_n^{\frac{1}{n}} z|^n}{2} - 1 + \frac{|\xi_n z^n| - 1}{\frac{3}{\varepsilon} \times \varepsilon - 1} < |\xi_n z^n|.\end{aligned}$$

We obtain that with probability one for infinite number of indices n all the roots of the polynomial G_n are located inside the circle $\{z : |z| = \varepsilon\}$, where ε is an arbitrary positive constant. This means that (4) does not hold for any $\delta \in (0, 1)$. □

2. The distribution of arguments.

Theorem 2. *Let (*) hold. Then the limit relations*

$$\frac{1}{n} S_n(\alpha, \beta) \overset{n\to\infty}{\longrightarrow} \frac{\beta - \alpha}{2\pi} \quad a.s. \tag{9}$$

hold for any given α, β, *such that* $0 \leqslant \alpha < \beta \leqslant 2\pi$.

Proof. Consider a set of sequences of reals

$$\{a_{11}\}, \{a_{12}, a_{22}\}, \dots, \{a_{1n}, a_{2n}, \dots a_{nn}\}, \dots$$

where all $a_{jn} \in [0, 1]$. We say that $\{a_{jn}\}$ are uniformly distributed in $[0, 1]$ if for any $0 \leqslant a < b \leqslant 1$

$$\lim_{n\to\infty} \frac{\sharp\{j : a_{jn} \in [a, b], j = 1, 2, \cdots, n\}}{n} = b - a.$$

The definition is an insignificant generalization of the notion of uniformly distributed sequences (see for example [3], ch.1). It is easy to see that the Weyl criterion ([3], ch.1) continues to be valid in this case:

the set of sequences $\{a_{jn}, j = 1, \dots, n\}$, $n = 1, 2, \dots$ *is uniformly distributed if and only if for all* $l = 1, 2, \dots$

$$\frac{1}{n} \sum_{j=1}^{n} e^{2\pi l a_{jn}} \to 0, \quad n \to \infty.$$

Let $z_{jn} = r_{jn} e^{i\theta_{jn}}$ be a zero of $G_n(z)$, $r_{jn} = |z_{jn}|$, $\theta_{jn} = \arg z_{jn}$, $0 \leqslant \theta_{jn} < 2\pi$. The theorem 2 is equivalent to the statement that the set of sequences $\{\frac{\theta_{jn}}{2\pi}\}$ is uniformly distributed. Thus, according to Weyl's criterion, it is enough to show that for any $l = 1, 2, \dots$

$$\lim_{n} \frac{1}{n} \sum_{j=1}^{n} e^{il\theta_{jn}} = 0$$

with probability 1.

Consider the random polynomial

$$\tilde{G}_n = \xi_n + \xi_{n-1} z + \dots + \xi_1 z^{n-1} + \xi_0 z^n.$$

Its roots are z_{jn}^{-1}. According to Newton's formulas

$$\sum_{j=1}^{n} \frac{1}{z_{jn}^l} = \varphi_l \left(\frac{\xi_1}{\xi_0}, \dots \frac{\xi_l}{\xi_0} \right),$$

where $\varphi_l(x_1, \dots x_l)$ are polynomials which do not depend on n. (For example, $\varphi_1(x) = -x$). It follows that

$$\frac{1}{n}\sum_{j=1}^{n} e^{-il\theta_{jn}} = \frac{1}{n}\sum_{j=1}^{n} e^{-il\theta_{jn}}\left(1 - \frac{1}{r_{jn}^{l}}\right) + \frac{\varphi_l}{n}. \tag{10}$$

For $|z| < 1$ the polynomials $G_n(z)$ converge to the analytical function $G(z) = \sum_{k=0}^{\infty} \xi_k z^k$, $\xi_0 \neq 0$, with probability 1. The function $G(z)$ has no zeros inside a circle $\{z : |z| \leqslant \rho\}$, $\mathbf{P}(\rho > 0) = 1$. Hence for $n \geqslant N$, $\mathbf{P}(N < \infty)$, the polynomials $G_n(z)$ have no zeros inside $\{z : |z| \leqslant \rho\}$. Let $\gamma > 0$ be a positive number. It follows from (10) that

$$\left|\frac{1}{n}\sum_{j=1}^{n} e^{il\theta_{jn}}\right| \leqslant (l+1)\frac{\gamma}{(1-\gamma)^l} + \\ + \frac{1}{n}\left(1 + \frac{1}{\rho}\right)\sharp\{j : |r_{jn} - 1| > \gamma, i = 1, \dots n\} + \frac{\varphi_l}{n}.$$

Theorem 1 implies that the second member on the right goes to zero when $n \to \infty$ with probability 1. Hence

$$\frac{1}{n}\sum_{j=1}^{n} e^{il\theta jn} \to 0,\ n \to \infty$$

with probability 1 and the theorem follows. $\square$

4. An example

We show that in contrast with theorem 1 the conditions of theorem 2 are not necessary. Namely, we construct an example of a sequence of i.i.d. real valued random variables $\xi_0, \xi_1, \dots, \xi_n, \dots$ such that

$$\mathbf{E}\log(1 + |\xi_i|) = \infty,$$

and at the same time, if $z_{jn} = z_j$ are the zeros of

$$G_n(z) = \xi_0 + \xi_1 z + \dots + \xi_{n-1}z^{n-1} + \xi_n z^n,$$

the arguments $\arg z_j$ are asymptotically uniformly distributed:

$$\frac{1}{n}S_n(\alpha, \beta) \overset{n\to\infty}{\longrightarrow} \frac{\beta - \alpha}{2\pi} \quad \text{a.s.} \tag{11}$$

for all α, β such that $0 \leqslant \alpha < \beta \leqslant 2\pi$.

We begin with the observation that if R is a sufficiently large positive number, $S_n(\alpha, \beta)$ is equal to the number of the roots of G_n inside the region $B(\alpha, \beta; R)$ bounded by the contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$, where γ_1, γ_2 are the radii

$$\gamma_1 = \{z : z = re^{i\alpha}, 0 \leqslant r \leqslant R\}, \gamma_1 = \{z : z = re^{i\beta}, 0 \leqslant r \leqslant R\}$$

and γ_3 is the arc

$$\gamma_3 = \{z : z = Re^{i\theta}, \alpha \leqslant \theta \leqslant \beta\}.$$

By the argument principle $S(\alpha, \beta)$ is equal to the change in the argument of G_n as z traverses the curve γ divided by 2π. If R is sufficiently large, the change of the argument of G_n as z traverses the arc γ_3 is $(\beta - \alpha)n + o(n)$. Further, if $\mathrm{Re}G_n(re^{i\alpha})$ as a function of r has on the interval $[o, R]$ p zeros, we divide the interval $[o, R]$ on $(p + 1)$ parts in a such way that $\mathrm{Re}G_n(re^{i\alpha}) \geqslant 0$ or $\mathrm{Re}G_n(re^{i\alpha}) \leqslant 0$ in any part.

The change in the argument of $G_n(z)$ when z traverses such part is π or less. Hence the change in the argument of G_n when z traverses γ_1 is not greater than $(p+1)\pi$. The same arguments can be applied to γ_2. Thus to constructed the example it is enough to construct such random variables $\{\xi_j\}$ that all polynomials

$$G_n^\alpha(r) = \mathrm{Re} G_n(re^{i\alpha}) = \xi_0 + \xi_1 r \cos\alpha + \cdots + \xi_n r^n \cos n\alpha$$

will have small number of positive roots, in fact, $o(n)$ is enough for our aims. The second author have studied a similar problem in [7]. The construction below repeats some ideas from [7].

We find a distribution for the coefficients ξ_j such that the largest coefficient would be radically different from others with high probability. Consider a discrete distribution

$$\mathbf{P}(\xi_j = b_k) = a_k, \quad k \in \mathbb{N}, \tag{12}$$

where

$$\sum_{k=1}^{\infty} a_k = 1,$$

$$0 < b_1 < \cdots < b_k < \ldots,$$

$$b_k \to \infty$$

and search for corresponding a_k and b_k.

At first, we build such a sequence a_k that with a probability of order $1 - O(n^{-2}))$ among the coefficients $\xi_0, \xi_1, \ldots, \xi_n$ there is exactly one maximum equal to a certain b_m with sufficiently large m.

We hold $\lambda \in \left(0, \frac{1}{8}\right)$, and consider a sequence of numbers

$$\begin{aligned} p_1 &= \lambda, \\ p_2 &= \left(\frac{\lambda}{4}\right)^{\frac{1}{2}} - \lambda, \\ &\ldots, \\ p_k &= \left(\frac{\lambda}{k^2}\right)^{\frac{1}{k}} - \left(\frac{\lambda}{(k-1)^2}\right)^{\frac{1}{k-1}}, \\ &\ldots. \end{aligned}$$

Set

$$a_1 = a_2 = \cdots = a_{N_1} = \frac{p_1}{N_1},$$

$$a_{N_1+1} = a_{N_1+2} = \cdots = a_{N_1+N_2} = \frac{p_2}{N_2},$$

$$\cdots$$

$$a_{N_1+\cdots+N_{k-1}+1} = a_{N_1+\cdots+N_{k-1}+2} \cdots = a_{N_1+\cdots+N_k} = \frac{p_k}{N_k},$$

$$\ldots,$$

where a strictly increasing sequence of natural numbers $N_1, N_2, \ldots$ will be defined later. Since the function

$$g(x) \overset{def}{=} \left(\frac{\lambda}{x^2}\right)^{\frac{1}{x}}$$

strictly increases for $x > e\lambda^{\frac{1}{2}}$ and $\lim_{x\to\infty} g(x) = 1$, all $p_k > 0$ and

$$\sum_{k=1}^{\infty} p_k = 1,$$

consequently, the same holds for a_j.

We define a sequence b_k by recursion:

$$b_1 = 1,$$
$$b_k = ((k+1)!b_{k-1})^k.$$

Now, let the distribution of the coefficients ξ_j of the polynomial G_n be described by (12). Obviously, these coefficients satisfy the condition

$$\mathbf{E}\log(1+|\xi_i|) = \infty.$$

Nevertheless, as it would be shown momentarily, (11) holds.

We say that coefficient ξ_i belongs to k-th group, if $\xi_i = b_j$ and $a_j = \frac{p_k}{N_k}$.

We introduce an event A_n, implying, that all coefficient $\xi_0, \xi_1, \dots, \xi_n$ have their group number less or equal $n+1$:

$$A_n = \{\xi_i \leqslant b_{N_1+\dots+N_{n+1}}, \quad i = 0, \dots, n\}.$$

The probability of this event is equal to

$$\mathbf{P}(A_n) = (p_1 + \dots + p_{n+1})^{n+1} = \frac{\lambda}{(n+1)^2}. \tag{13}$$

Denote B_n the event that the sequence $\xi_0, \xi_1, \dots, \xi_n$ has at least two maximums. Let us estimate $\mathbf{P}(B_n \setminus A_n)$. In order to do this introduce the events

$$C_{kn} = \{b_{N_1+\dots+N_{k-1}} < \max(\xi_0, \xi_1, \dots, \xi_n) \leqslant b_{N_1+\dots+N_k}\}$$

In other words C_{kn} express the fact that the largest coefficient belongs to the k-th group. Let D_{kn}^{ij} denote the event $\{\xi_i, \xi_j$ belong to the k-th group and $\xi_i = \xi_j\}$. Then

$$B_n \cap C_{kn} \subset \bigcup_{0\leqslant i<j\leqslant n} D_{kn}^{ij},$$

and hence

$$\mathbf{P}(B_n \cap C_{kn}) \leqslant \sum_{0\leqslant i<j\leqslant n} \mathbf{P}(D_{kn}^{ij}) = \frac{n(n+1)}{2} N_k \left(\frac{p_k}{N_k}\right)^2.$$

Take $N_k = [\lambda^{-1}k^4 2^k] + 1$, then

$$\begin{aligned} \mathbf{P}(B_n \setminus A_n) &= \sum_{k=n+2}^{\infty} \mathbf{P}(B_n \cap C_{kn}) \\ &\leqslant \frac{n(n+1)}{2} \sum_{k=n+2}^{\infty} \frac{\lambda p_k^2}{k^4 2^k} \\ &\leqslant \frac{\lambda n(n+1)}{2(n+2)^4} \sum_{k=n+2}^{\infty} \frac{1}{2^k} \leqslant \frac{\lambda}{(n+1)^2}. \end{aligned} \tag{14}$$

It follows that

$$\mathbf{P}(A_n \cup B_n) \leqslant \mathbf{P}(A_n) + \mathbf{P}(B_n \setminus A_n) \leqslant \frac{2\lambda}{(n+1)^2}.$$

The Borel-Cantelli lemma implies that there exists a random variable $N, \mathbf{P}\{N < \infty\} = 1$, such that for $n > N$ the sequence $\xi_0, \xi_1, \dots, \xi_n$ has only one maximal member ξ_τ and

$$\xi_\tau > b_{N_1+\dots+N_{n+1}}.$$

Obviously, we may suppose, that α has the form

$$\alpha = 2\pi\frac{q}{p},$$

where p is a prime and q runs the values $0, 1, \dots, p-1$ and consider the polynomials

$$G_n^\alpha(t) = \mathrm{Re}G_n(te^{i\alpha}) = \xi_0 + \xi_1\cos\alpha + \dots + \xi_n t^n \cos n\alpha.$$

Show that if $n > N$, $G_n^\alpha(t)$ has at most $2p$ positive zeros. We begin with the observation that

$$\inf_{\nu,q\in\mathbb{Z}^+} |\cos\nu\alpha| = \inf_{r=0,1\dots,p-1}\left|\cos\frac{2\pi r}{p}\right| \geqslant \frac{c_0}{p},$$

where c_0 is a positive constant.

At first, we show that if $n > N$, the number of roots of $G_n^\alpha(t)$ on the interval $[0, 1]$ doesn't exceed p. If $n > N$, there exists an index $\tau \leqslant n$ such that $\xi_\tau = b_{n+s}$ and $\xi_j \leqslant b_{n+s-1}, j \neq \tau, j \leqslant n$, for some natural number s. There exists an integer $m, 0 \leqslant m \leqslant p-1$, such that $\cos m\alpha = \cos\tau\alpha$. According to the Roll's theorem, it is sufficient to show that the m-th derivative of $G_n^\alpha(t)$

$$(G_n^\alpha(t))^{(m)}(t) = \sum_{j=m}^{n}\frac{j!}{(j-m)!}\xi_j t^{j-m}\cos j\alpha$$

has no zeros in $[0, 1]$.

Consider separately the following three cases.

1. If $m = \tau$, then for $t \in [0, 1]$ and $n \geqslant n_0$

$$|\xi_\tau\cos\tau\alpha| \geqslant \frac{c_0}{p}b_{n+s} > nn!b_{n+s-1} \geqslant \left|\sum_{j=m+1}^{n}\frac{j!}{(j-m)!}\xi_j t^{j-m}\cos j\alpha\right|$$

and $(G_n^\alpha(t))^{(m)}(t)$ has no zeros in $[0, 1]$.

2. Let $m < \tau$ and $1 \geqslant t > ((n+s+1)!b_{n+s-1})^{-1} = t_0$. If $n > n_0$,

$$\begin{aligned}\left|m!\xi_m\cos m\alpha + \frac{\tau!}{(\tau-m)!}\xi_\tau t^{\tau-m}\cos\tau\alpha\right| &> \xi_\tau t^n|\cos\tau\alpha| = b_{n+s}t^n|\cos\tau\alpha|\\ \geqslant \frac{c_0}{p}b_{n+s}t^n &\geqslant \frac{c_0 b_{n+s}}{p((n+s+1)!b_{n+s-1})^n} \geqslant \frac{c_0 b_{n+s}}{p((n+s+1)!b_{n+s-1})^{n+s-1}}\\ = \frac{c_0}{p}(n+s+1)!b_{n+s-1} &> nn!b_{n+s-1} \geqslant \left|\sum_{j\in\{m+1,\dots,n\}\setminus\{\tau\}}\frac{j!}{(j-m)!}\xi_j t^{j-m}\right|\end{aligned}$$

and $(G_n^\alpha(t))^{(m)}(t)$ has no zeros in $[t_0, 1]$.

3. Let $m < \tau$ and $0 \leqslant t \leqslant t_0$. Then for $n > n_0$

$$\left|\xi_m \cos m\alpha + \frac{\tau!}{(\tau-m)!}\xi_\tau t^{\tau-m}\cos\tau\alpha\right| \geqslant \xi_m|\cos m\alpha| \geqslant \frac{c_0}{p}$$

$$\geqslant \frac{c_0}{p}(n+s+1)!b_{n+s-1}t \geqslant nn!b_{n+s-1}t > \left|\sum_{j\in\{m+1,\dots,n\}\setminus\{\tau\}} \frac{j!}{(j-m)!}\xi_j t^{j-m}\right|$$

and $(G_n^\alpha(t))^{(m)}(t)$ has no zeros in $[0,t_0]$.

The same argumentation we can apply to the polynomial

$$\tilde{G}_n^\alpha(t) = \xi_n \cos n\alpha + \xi_{n-1} t\cos(n-1)\alpha + \cdots + \xi_0 t^n.$$

Since $G_n^\alpha(t) = 0 \Leftrightarrow \tilde{G}_n^\alpha(t^{-1}) = 0$ for $t > 0$, the number of the roots of G_n^α on $[1,\infty)$ is equal to the number of the roots of $\tilde{G}_n^\alpha$ on $(0,1]$. Therefore, G_n^α has at most p roots on $[1,\infty)$ too.

Remark. *The same construction allows us to find example of* $\{\xi_i\}$ *such that for a given function* φ

$$\mathbf{E}|\varphi(|\xi_1|)| = \infty$$

holds and the arguments of z_i *will be asymptotically uniformly distributed.*

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