p-Adic Confluence of q-Difference Equations

Andrea Pulita

November 26, 2006

Abstract. We develop the theory of p-adic confluence of q-difference equations. The main result is the surprising fact that in the p-adic framework a function is solution of a differential equation if and only if it is solution of a q-difference equation. This fact implies an equivalence between the category of differential equations and those of q-difference equations called "Confluence". We obtain this result by introducing a category of "sheaves" on the disk $\mathrm{D}^{-}(1,1)$, whose stalk at 1 is a differential equation, the stalk at q is a q-difference equation if q is not a root of unity ξ, and the stalk at a root of unity is a mixed object formed by a differential equation and an action of σ_{ξ}.

Contents

Introduction	2
0.1. Motivations	2
0.2. Results of the paper	3
0.3. Description of used methods	4
Leitfaden	6
Acknowledgments	6
1. Notations	6
1.1. Rings of function	6
1.2. Norms and radii of convergence	7
2. Discrete or analytic $\sigma-$modules and $(\sigma,\delta)-$modules	8
2.1. Discrete $\sigma-$modules	8
2.2. Discrete $(\sigma,\delta)-$modules	9
2.3. Analytic $\sigma-$modules	10
2.4. Analytic $(\sigma,\delta)-$modules	11
3. Solutions (formal definition)	12
3.1. Discrete $\sigma-$algebras and $(\sigma,\delta)-$algebras	12
3.2. Constant Solutions	12
4. Constant Confluence	13
4.1. Constant modules	13
4.2. Constant deformation and constant confluence	15
5. Taylor solutions	16
5.1. The $q-$algebras $\Omega\{T-c\}_{q,R}$ and $\Omega[\![T-c]\!]_q$	17

1991 *Mathematics Subject Classification.* Primary 12h25; Secondary 11S15; 11S20; 14F30.

Key words and phrases. p-adic q-difference equations, p-adic differential equations, Confluence, Deformation.

5.2. q-invariant Affinoïd 19
5.3. The formal Taylor solution 19
5.4. Taylor solutions of $(\sigma_q, \delta_q)-$modules 22
5.5. Twisted Taylor formula for $(\sigma, \delta)-$modules, and rough estimate of radius 23
5.6. Generic radius of convergence and solvability 24
5.7. Solvability over an annulus and over the Robba ring 25
5.8. Generic radius for discrete and analytic objects over $\mathcal{R}_K$ and $\mathcal{H}_K^\dagger$ 26
6. The Propagation Theorem 27
6.1. Taylor admissible modules 27
6.2. Propagation Theorem 28
6.3. Generalizing the Confluence Functor 29
6.4. Propagation Theorem for other rings 30
6.5. Classification of solvable Rank one $q-$difference equations 31
7. Quasi unipotence and $p-$adic local monodromy theorem 32
7.1. Frobenius Structure 32
7.2. Special coverings of $\mathcal{H}_K^\dagger$, canonical extension. 34
7.3. Quasi unipotence of differential equations 36
7.4. Quasi unipotence of $\sigma-$modules and $(\sigma, \delta)-$modules with Frob. structure 37
7.5. The confluence of André-Di Vizio 39
Index 41
References 42

Introduction

0.1. Motivations. Heuristically we say that a family of $q-$difference equations $\{\ \sigma_q(Y_q) = A(q,T)\cdot Y_q\ \}_{q\in D^-(1,\epsilon)}$, (where σ_q is the automorphism $f(T)\mapsto f(qT)$) is confluent to the differential equation $\delta_1(Y_q) = G(1,T)\cdot Y_q$, with $\delta_1 := T\frac{d}{dT}$, if one has $\lim_{q\to 1}\frac{A(q,T)-1}{q-1} = G(1,T)$ and, in some suitable meaning, one has

$$\lim_{q\to 1} Y_q = Y_1\ . \tag{0.0.1}$$

Recently in [ADV04], the authors have studied for the first time the phenomena of the confluence in the $p-$adic framework. If K is a discrete valuation field, they have found the existence of an equivalence between the category of $q-$difference equations with Frobenius structure over the Robba ring $\mathcal{R}_{K^{\mathrm{alg}}}$ (called $\sigma_q - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$), and the category of differential equations with Frobenius structure over the Robba ring $\mathcal{R}_{K^{\mathrm{alg}}}$ (called $\delta_1 - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$). They are subject to the restriction $|q-1| < |p|^{\frac{1}{p-1}}$. Indeed in the annulus $|q-1| = |p|^{\frac{1}{p-1}}$ one encounters the $p-$th root of unity and, if $\xi^p = 1$, then the category $\sigma_\xi - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ is not $K-$linear, and hence it can not be equivalent to the category of differential equations. They obtained this equivalence by describing the Tannakian group of $\sigma_q - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$, using a general result of André (cf. [And02]), and finding that this Tannakian group is the same of that of the category of differential equations. By composition with the Tannakian equivalences (T_q and T_1 below), they obtained then what they called

the *confluence functor* " Conf_q ":
(0.0.2)

$$\begin{array}{ccc} \sigma_q - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} & \overset{\mathrm{Conf}_q}{\underset{\cong}{\dashrightarrow}} & \delta_1 - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} \\ {\scriptstyle T_q,\ \cong}\searrow & & \swarrow{\scriptstyle T_1,\ \cong} \\ & \underline{\mathrm{Rep}}_{K^{\mathrm{alg}}}(\mathcal{I}_{k((t))} \times \mathbb{G}_a) & \end{array}$$

Their strategy consists in showing that every object of the category $\sigma_q - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ is quasi-unipotent, i.e. becomes unipotent after a special extension of $\mathcal{R}_K$ (cf. section 7.3). The proof of this fact needs a remarkable effort and is actually not less complicate than the classical $p-$adic local monodromy theorem for differential equations itself. Moreover these methods show the existence, but do not provide a satisfactory description of the confluence functor.

0.2. Results of the paper. In this paper we will generalize these results in many different ways. First we show that the functor Conf_q admits a very explicit and direct description in terms of solutions of those equations. Roughly speaking Conf_q sends a $q-$difference equation into the differential equation *having the same formal Taylor solution*. In other words, we show that in the $p-$adic framework the dream of the lazy mathematician is realized: solutions of $q-$difference equations are not only a "discretization" of solutions of differential equations, but they are actually equal, indeed we do not only have the relation (0.0.1), but one has the stronger equality

$$Y_q = Y_1 \ , \quad \text{for all } q \in \mathrm{D}^-(1,\varepsilon),$$

and the number ε depends on the radius of convergence of the solution. The most basic example is given by considering $Y_q := \exp(T)$, for all q. One has $A(q,T) = \sigma_q(Y_q) \cdot Y_q^{-1} = \frac{\exp((q-1)T)}{q-1}$, hence, if q is sufficiently close to 1, the radius of convergence of $A(q,T)$ tends to ∞, and then $A(q,T)$ lies actually in almost every ring of functions we need.

The definition of Conf_q becomes very easy and leads us to define this functor (and prove that it is an equivalence) over a very large class of ring of functions (analytic elements on an affinoïd, analytic functions on a bounded annulus, etc...), and not only over the Robba ring $\mathcal{R}_{K^{\mathrm{alg}}}$. Secondly, since we do not use Tannakian methods, we are not obliged to extend the scalars to K^{alg}. Thirdly we extend this equivalence to a large class of $q-$difference equations called *Taylor admissible*. Taylor admissible equations are a large class of equations containing for example solvable equations, equations with a Frobenius structure, and almost every class of equations studied until today. Lastly and most important, we describe the situation near the roots of unity, and we generalize these results to the general case $|q-1| < 1$.

We prove then the $p-$adic local monodromy theorem for $q-$difference equations (i.e. the fact that T_q is an equivalence) by composition between Conf_q and T_1. By the way, by using the results of [**ADV04**] that T_q is an equivalence, we obtain another proof of the $p-$adic local monodromy theorem for differential equations, by composing with Conf_q.

Our technics are quasi-completely deprived of computations. Moreover these results depends only on the definition (and formal properties) of the Taylor solution (cf. Lemma 5.20), they are hence independent on the theory developed until today and implies the main results of [**ADV04**] and [**DV04**] (cf. Remark 7.29). For these reasons our approach seems to be more efficacious than those of [**ADV04**] and [**DV04**].

On the other hand the fact that a function is solution of a differential equation if and only if it is solution of a q-difference equation for $|q-1|<1$ should be related to the Grothendieck-Katz conjecture (cf. [**And04**], [**Kat82**], [**DV02**], ...), since the q-analogue of this conjecture has already been proved by L. Di Vizio [**DV02**].

0.3. Description of used methods. Our point of view is the following. Let B be a ring of functions (say for example analytic elements over an affinoïd), and let $\mathcal{Q} = \mathcal{Q}(\mathrm{B})$ be the open subgroup of $K^\times$ consisting in elements $q \in K^\times$ such that $f(T) \mapsto f(qT)$ is an automorphism of B. For all open subset $U \subseteq \mathcal{Q}$, let us set $\mathcal{O}_U := \mathrm{B}[\{\sigma_q\}_{q\in U}]$. One sees easily that the co-variant functor $U \mapsto \mathcal{O}_U$ verifies the dual properties of a sheaf of (non commutative) rings. In other words, by reversing the arrows in the category of (non commutative) rings, the functor $U \mapsto \mathcal{O}_U$ becomes a sheaf. As a $q-$difference equation defines a B module with an action of σ_q, the notion of "family of $q-$difference equations" can be intended as a *sheaf of $\mathcal{O}_{\mathcal{Q}}-$modules*, and a single $q-$difference equation can be seen as the "*stalk at q*" of such an object. We define the category $\sigma - \mathrm{Mod}(\mathrm{B})^{\mathrm{an}}_U$ of analytic $\sigma-$modules, and we look at these objects as "sheaves" over U (with values in the dual category of the category of non commutative rings). Roughly speaking such an object is simply a free B$-$module together with an action of σ_q, for all $q \in U$, in order that, in a basis, the matrix of σ_q depends analytically on q. We only observe here that two analytic $\sigma-$modules can have isomorphic stalks at all q, without being isomorphic "globally". Indeed a base change in such an object should be seen as a simultaneous base change of all stalks by the same base change matrix.

The equations involved by our results are those called *Taylor admissible* (see Definition 6.1). Roughly speaking an equation is Taylor admissible if it admits a generic Taylor solution whose radius R_c at every point c verifies $|q-1||c| < R < R_c$, for all $q \in U$, where R is a real number which does not depend on c.

0.3.1. *The Propagation Theorem.* Our main result is then the following: given a Taylor admissible $q-$difference equation, with $|q-1|<1$ and $q \notin \boldsymbol{\mu}_{p^\infty}(K)$, there exists an open disk $\mathrm{D}^-_K(1,r)$, containing q, and a *canonical* analytic $\sigma-$module over $U = \mathrm{D}^-_K(1,r)$, whose stalk at q is our $q-$difference equation. Moreover this analytic $\sigma-$module over $\mathrm{D}^-_K(1,r)$ is characterized by the fact that all its stalks have the same generic Taylor solution, for all $q' \in \mathrm{D}^-_K(1,r)$. Heuristically this property tells us that these sheaves are "constant". The radius r depends on the radius R of the solution of the given equation. In particular, if the starting equation is solvable at 1 over the Robba ring (i.e. $R = 1$), then $r = 1$. The fiber functor

$$\sigma - \mathrm{Mod}(\mathrm{B})^{\mathrm{adm}}_{\mathrm{D}^-_K(1,r)} \xrightarrow{\mathrm{Res}_q^{\mathrm{D}^-_K(1,r)}} \sigma_q - \mathrm{Mod}(\mathrm{B})^{\mathrm{adm}} , \tag{0.0.1}$$

is easily seen to be fully faithful since every equation is determined by its Taylor solution. The above result then provides the essential surjectivity of the functor

$$\bigcup_{r>|q-1|} \sigma - \mathrm{Mod}(\mathrm{B})^{\mathrm{adm}}_{\mathrm{D}^-_K(1,r)} \xrightarrow[\sim]{\cup_r \mathrm{Res}_q^{\mathrm{D}^-_K(1,r)}} \sigma_q - \mathrm{Mod}(\mathrm{B})^{\mathrm{adm}} . \tag{0.0.2}$$

This stalk functor $\cup_r \mathrm{Res}_q^{\mathrm{D}_K^-(1,r)}$ is hence an equivalence. By composition, for all $q' \in \mathrm{D}^+(1,|q-1|) - \boldsymbol{\mu}_{p^\infty}(K)$, we found then an equivalence
(0.0.3)

$$\mathrm{Def}_{q,q'} \;:\; \sigma_q - \mathrm{Mod}(\mathrm{B})^{\mathrm{adm}} \xrightarrow{(\cup_r \mathrm{Res}_{q'}^{\mathrm{D}^-(1,r)})\circ(\cup_r \mathrm{Res}_q^{\mathrm{D}^-(1,r)})^{-1}} \sigma_{q'} - \mathrm{Mod}(\mathrm{B})^{\mathrm{adm}} \;.$$

This equivalence sends a $q-$difference equation into the $q'-$difference equation having "*the same generic Taylor solution*".

0.3.2. *The $q-$tangent operator.* What happens if $q \in \boldsymbol{\mu}_{p^\infty}(K)$? In the particular case in which $q = 1$, we expect that the "stalk at $q = 1$" of a Taylor admissible $\sigma-$module is a differential equation, and not simply a σ_1-module, which is a trivial data consisting just in a free B$-$module. In fact, given an analytic $\sigma-$module M, one can obtain a connection as follows:

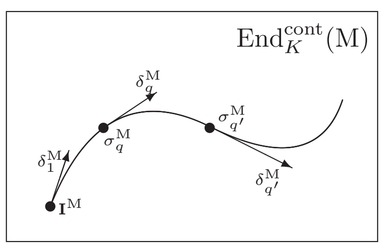
$$\delta_1^{\mathrm{M}} := \lim_{q\to 1} \frac{\sigma_q^{\mathrm{M}} - 1}{q-1} \;, \tag{0.0.4}$$

where $\sigma_q^{\mathrm{M}} : \mathrm{M} \to \mathrm{M}$ is the action of σ_q on M. The Propagation Theorem provides the existence of this limit in $M_n(\mathrm{B})$.

The idea is now that near the roots of unity, and more generally, near every point $q \in \mathcal{Q}$, one should have the same behavior. Actually, an analytic $\sigma-$module is canonically endowed, for all $q \in U$, with an action of the $q-$tangent operator

$$\delta_q^{\mathrm{M}} := q\frac{d}{dq}\Big(q \mapsto \sigma_q^{\mathrm{M}}\Big) = q \cdot \lim_{q'\to q} \frac{\sigma_{q'}^{\mathrm{M}} - \sigma_q^{\mathrm{M}}}{q'-q} = \sigma_q^{\mathrm{M}} \circ \delta_1^{\mathrm{M}} \;, \tag{0.0.5}$$

and every morphism between analytic $\sigma-$modules commutes also with δ:



We define then the categories of analytic $(\sigma,\delta)-$modules and admissible $(\sigma,\delta)-$modules, which are actually equal to the categories of analytic $\sigma-$modules and admissible $\sigma-$modules respectively. The stalk at every point q of an admissible $\sigma-$module is a B$-$module with an action of σ_q and δ_q (or actually δ_1, which is equivalent since $\delta_q = \sigma_q \circ \delta_1$), we will call such an object a $(\sigma_q,\delta_q)-$*module.* If $q = 1$, the stalk reduces to a differential equation, and if $q \notin \boldsymbol{\mu}_{p^\infty}(K)$, the action of δ_q^{M} can be removed since the category of admissible σ_q-modules is equivalent to that of admissible $(\sigma_q,\delta_q)-$modules (in other words, the connection can be recovered by the data of σ_q^{M}). But if $q = \xi \in \boldsymbol{\mu}_{p^\infty}(K)$, then, if we want that the stalk functor is an equivalence, one can not forget the connection nor the action of σ_ξ^{M}.

One can obtain for $(\sigma,\delta)-$modules the analogous of every previous equivalence, with the improvement that these equivalence exist also in the stalks at the root of unity. Indeed, if $\xi \in \boldsymbol{\mu}_{p^\infty}$, the category $(\sigma_\xi,\delta_\xi) - \mathrm{Mod}(\mathrm{B})^{\mathrm{adm}}$ is $K-$linear, while

$\sigma_\xi - \mathrm{Mod}(\mathrm{B})^{\mathrm{adm}}$ is not $K-$linear. One has hence the diagram:

$$\begin{array}{ccc} \sigma - \mathrm{Mod}(\mathrm{B})^{[r]}_{\mathrm{D}} & \xleftarrow[\sim]{\text{Forget } \delta} & (\sigma,\delta) - \mathrm{Mod}(\mathrm{B})^{[r]}_{\mathrm{D}} \\ {\scriptstyle \mathrm{Res}^{\mathrm{D}}_q}\downarrow & \odot & \wr\downarrow{\scriptstyle \mathrm{Res}^{\mathrm{D}}_q} \\ \sigma_q - \mathrm{Mod}(\mathrm{B})^{[r]} & \xleftarrow[\text{Forget } \delta_q]{} & (\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B})^{[r]} \end{array} \tag{0.0.6}$$

in which if $q \notin \boldsymbol{\mu}_{p^\infty}(K)$, then all rows are equivalences. On the other hand, if $q = \xi \in \boldsymbol{\mu}_{p^\infty}(K)$, then the functor "Forget δ" and the right hand "$\mathrm{Res}^{\mathrm{D}}_\xi$" are equivalences, while "Forget δ_ξ" is not an equivalence. One may have the feeling that the "information" on the category $\sigma_\xi - \mathrm{Mod}(\mathrm{B})^{[r]}$ is contained in the functor "Forget δ_ξ", but one can show actually (cf. Prop. 7.9) that in the important case of equations with Frobenius structure over $\mathcal{R}_K$, the image by "Forget δ_ξ" of such equations is always direct sum of the unit object, and this functor is a trivial data.

Leitfaden. After some notations (section 1) we introduce, in section 2, the notion of discrete/analytic $\sigma-$modules and $(\sigma,\delta)-$modules. In section 3 we give the abstract definition of solution of such an object, and in section 4 we expose formally the confluence. In section 5 we recall the definition of *generic Taylor solution* and *generic radius of convergence*. In section 6 we define Taylor admissible objects and obtain the main theorem (cf. section 0.3.1). In the last section 7 we apply the previous theory to the Robba ring and to the $p-$adic local monodromy theorem. After recalling the p-adic local monodromy Theorem for differential equations, we show how to deduce, by Deformation, the q-analogue of this theorem.

Acknowledgments. We thank L. Di Vizio which has been always willing to talk and to explain the technical difficulties of her papers. The author wish to express also his gratitude to Gilles Christol for his guidance and constant encouragements. A particular thanks goes to B.Chiarellotto for his support and for general corrections. We thank also Y. André and J.P. Ramis for useful discussions.

1. Notations

We refer to [DM] for the definitions concerning Tannakian categories. In the sequel if we say that a given category $\mathcal{C}$ is (or is not) K-linear, we mean that the ring of endomorphisms of the unit object is (or is not) exactly *equal* to K.

We set $\mathbb{R}_{\geq} := \{r \in \mathbb{R} \mid r \geq 0\}$, and $\delta_1 := T\frac{d}{dT}$.

1.1. Rings of function. Let $R > 0$. The ring of analytic functions on the disk $\mathrm{D}^-(c,R)$, $c \in K$, is

$$\mathcal{A}_K(c,R) := \{\sum_{n\geq 0} a_n(T-c)^n \mid a_n \in K, \liminf_n |a_n|^{-1/n} \geq R\}\ . \tag{1.0.1}$$

Its topology is given by the family of norms $|\sum a_i(T-c)^i|_{(c,\rho)} := \sup |a_i|\rho^i$, for all $\rho < R$. Let $\emptyset \neq I \subseteq \mathbb{R}_{\geq 0}$ be some interval. We denote the annulus relative to I by $\mathcal{C}_K(I) := \{x \in K \mid |x| \in I\}$. By $\mathcal{C}(I)$, without the index K, we mean the annulus itself and not its $K-$valued points. The ring of analytic functions on $\mathcal{C}(I)$ is

$$\mathcal{A}_K(I) := \{\sum_{i\in\mathbb{Z}} a_iT^i \mid a_i \in K, \lim_{i\to\pm\infty} |a_i|\rho^i = 0,\ \text{for all } \rho \in I\}\ . \tag{1.0.2}$$

We set $|\sum_i a_i T^i|_\rho := \sup_i |a_i|\rho^i < +\infty$, for all $\rho \in I$. The ring $\mathcal{A}_K(I)$ is complete for the topology given by the family of norms $\{|.|_\rho\}_{\rho\in I}$. Set $I_\varepsilon :=]1-\varepsilon, 1[$, $0<\varepsilon<1$. The Robba ring is then defined as $\mathcal{R}_K := \bigcup_{\varepsilon>0} \mathcal{A}_K(I_\varepsilon)$, and is complete with respect to the limit Frechet topology.

Definition 1.1. A $K-$*affinoïd* is an analytic subset of $\mathbb{P}^1$ defined by $X := \mathrm{D}^+(c_0,R_0) - \bigcup_{i=1}^n \mathrm{D}^-(c_i,R_i)$, for some $R_0,\ldots,R_n > 0$, $c_1,\ldots,c_n \in \mathrm{D}^+_K(c_0,R)$. We denote by X the $K-$affinoïd itself, and for all ultrametric valued $K-$algebras $(L,|.|)$, we denote by $X(L)$ its $L-$rational points.

Let X be an affinoïd, let $H_K^{\mathrm{rat}}(X)$ be the ring of rational fractions $f(T)$ in $K(T)$, without poles in $X(K^{\mathrm{alg}})$, and let $\|\cdot\|_X$ be the norm on $H_K^{\mathrm{rat}}(X)$ given by $\|f(T)\|_X := \sup_{x\in X(K^{\mathrm{alg}})} |f(x)|$. We denote the completion of $(H_K^{\mathrm{rat}}(X), \|\cdot\|_X)$ by $\mathcal{H}_K(X)$. If $\rho_1,\rho_2 \in |K^{\mathrm{alg}}|$, and if $X = \mathrm{D}^+(0,\rho_2) - \mathrm{D}^-(0,\rho_1)$, then $\mathcal{H}_K(X) = \mathcal{A}_K(I)$, whit $I = [\rho_1,\rho_2]$.

Lemma 1.2. *Let $X = \mathrm{D}^+(c_0,R_0) - \cup_{i=1,\ldots,n} \mathrm{D}^-(c_i,R_i)$ be an affinoïd. Let $r_X := \min(R_0,\ldots,R_n)$. Then $\|\frac{d}{dT}f(T)\|_X \le r_X^{-1}\|f(T)\|_X$.*

Proof : This follows easily from the Mittag-Leffler decomposition of $f(T)$. □

Let $\varepsilon > 0$. If $X = \mathrm{D}^+(c_0,R_0) - \bigcup_{i=1}^n \mathrm{D}^-(c_i,R_i)$, we set $X_\varepsilon := \mathrm{D}^+(c_0,R_0+\varepsilon) - \bigcup_{i=1}^n \mathrm{D}^-(c_i,R_i-\varepsilon)$. We set then $\mathcal{H}_K^\dagger(X) := \bigcup_{\varepsilon>0}\mathcal{H}_K(X_\varepsilon)$. The ring $\mathcal{H}_K^\dagger(X)$ is complete with respect to the limit topology. We set $\mathcal{H}_K := \mathcal{H}_K(X_1)$, $\mathcal{H}_K^\dagger := \mathcal{H}_K^\dagger(X_1)$, where $X_1 := \{x \mid |x| = 1\}$.

1.2. Norms and radii of convergence.

1.2.1. *Logarithmic properties.* Let $r \mapsto N(r) : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be a function. The log-function attached to N is defined by $\widetilde{N}(t) := \log(N(\exp(t)))$:

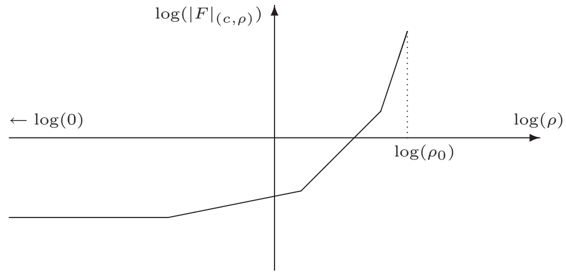
$$\widetilde{N} \;:\; \mathbb{R}\cup\{-\infty\} \xrightarrow[\sim]{\exp} \mathbb{R}_{\ge 0} \xrightarrow{N} \mathbb{R}_{\ge 0} \xrightarrow[\sim]{\log} \mathbb{R}\cup\{-\infty\}\;. \tag{1.2.1}$$

We will say that N has logarithmically a given property if $\widetilde{N}$ has that property.

1.2.2. *Norms.* Let $(\Omega,|\cdot|)/(K,|\cdot|)$ be an arbitrary extension of valued fields. The absolute value on Ω will be extended to a norm on $M_{n\times n}(\Omega) = M_n(\Omega)$, by setting $|(a_{i,j})_{i,j}|_\Omega := \max_{i,j}|a_{i,j}|_\Omega$. Let $f(T) = \sum_{i\in\mathbb{Z}} a_i(T-c)^i$, $a_i \in K$, be a formal power series. We set $|f|_{(c,\rho)} := \sup_i |a_i|\rho^i$, this number can be equal to $+\infty$. If $F(T) = (f_{h,k}(T))_{h,k}$, is a matrix, we set $|F(T)|_{(c,\rho)} := \max_{h,k}|f_{h,k}|_{(c,\rho)}$.

Lemma 1.3 ([CR94, ch.II]). *Let $F(T) \in M_n(K[[T-c]])$. Suppose that $|F|_{(c,\rho_0)} < \infty$. Then $|F|_{(c,\rho)} < \infty$, for all $\rho < \rho_0$. Moreover, the function $\rho \mapsto |F|_{(c,\rho)} : [0,\rho_0] \longrightarrow \mathbb{R}_{\ge 0}$ is log-convex, piecewise log-affine, and log-increasing. Moreover*

$$|F(T)|_{(c,\rho)} = \sup_{|x-c|\le\rho,\, x\in K^{\mathrm{alg}}} |F(x)|_{K^{\mathrm{alg}}}\;. \tag{1.3.1}$$



1.2.3. *Radii.* Let $f(T)=\sum_{i\geq 0}a_i(T-c)^i$, $a_i\in K$ be a formal power series. The radius of convergence of $f(T)$ at c is $Ray_c(\ f(T)\ ):=\liminf_{i\geq 0}|a_i|^{-1/i}$. If $F(T)=(f_{h,k}(T))_{h,k}$, is a matrix, then we set $Ray_c(F(T)):=\min_{h,k}\ Ray_c(\ f_{h,k}(T)\ )$.

2. Discrete or analytic $\sigma-$modules and $(\sigma,\delta)-$modules

Definition 2.1. Let B be one of the rings of section 1.1. We denote by

$$\mathcal{Q}(\mathrm{B}) = \{q\in K \mid \sigma_q : f(T)\mapsto f(qT) \text{ is an automorphism of B}\} \tag{2.1.1}$$

$$\mathcal{Q}_1(\mathrm{B}) = \mathcal{Q}(\mathrm{B})\cap \mathrm{D}^-(1,1)\ . \tag{2.1.2}$$

We will write $\mathcal{Q}$ and $\mathcal{Q}_1$ when no confusion is possible.

Remark 2.2. $\mathcal{Q}(\mathrm{B})\subset(K^\times,|.|)$ is a topological group and contains always a disk $\mathrm{D}^-(1,\tau_0)$, for some $\tau_0>0$. One has $\mathcal{Q}(\mathcal{A}_K(I))=\mathcal{Q}(\mathcal{R}_K)=\mathcal{Q}(\mathcal{H}_K^\dagger)=\{q\in K\mid |q|=1\}$. Since, by Definition 1.1, X is *bounded* (i.e. contained in $\mathrm{D}^-(0,r)$, for some $r>0$), and since $x\mapsto qx$ is assumed to be bijective on X, in order that σ_q is bijective on $\mathcal{H}_K(X)$, then $\mathcal{Q}(\mathcal{H}_K(X))\subseteq\{q\in K\mid |q|=1\}$.

Definition 2.3. Let $S\subseteq\mathcal{Q}$ be a subset. We denote by $\langle S\rangle$ the subgroup of $\mathcal{Q}$ generated by S. Let $\boldsymbol{\mu}(\mathcal{Q})$ be the set of all roots of 1 belonging to $\mathcal{Q}$. Then we set

$$S^\circ := S-\boldsymbol{\mu}(Q)\ . \tag{2.3.1}$$

2.1. Discrete $\sigma-$modules. By assumption, every finite dimensional free B$-$module M has the product topology.

Definition 2.4 (discrete $\sigma-$modules). Let $S\subset\mathcal{Q}$ be an arbitrary subset. An object of σ–$\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ is a finite dimensional free B$-$module M, together with a group morphism

$$\sigma^{\mathrm{M}}:\langle S\rangle \xrightarrow[q\mapsto\sigma_q^{\mathrm{M}}]{} \mathrm{Aut}_K^{\mathrm{cont}}(\mathrm{M})\ , \tag{2.4.1}$$

such that, for all $q\in S$, the operator σ_q^{M} is σ_q-semi-linear, that is

$$\sigma_q^{\mathrm{M}}(fm)=\sigma_q(f)\cdot\sigma_q^{\mathrm{M}}(m)\ , \tag{2.4.2}$$

for all $f\in\mathrm{B}$, and all $m\in\mathrm{M}$. Objects (M,σ^{M}) in $\sigma-\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ will be called *discrete $\sigma-$modules over S*. A morphism between $(\mathrm{M},\sigma^{\mathrm{M}})$ and $(\mathrm{N},\sigma^{\mathrm{N}})$ is a B$-$linear map $\alpha:\mathrm{M}\to\mathrm{N}$ such that

$$\alpha\circ\sigma_q^{\mathrm{M}}=\sigma_q^{\mathrm{N}}\circ\alpha\ , \tag{2.4.3}$$

for all $q\in S$. We will denote the $K-$vector space of morphisms by $\mathrm{Hom}_S^\sigma(\mathrm{M},\mathrm{N})$.

NOTATION 2.5. If $S = \{q\}$ is reduced to a point, then the category of discrete $\sigma-$modules over $\{q\}$ is the usual category of $q-$difference modules. We will use then a simplified notation:

$$\sigma_q - \mathrm{Mod}(\mathrm{B}) \ := \ \sigma - \mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_{\{q\}} \ . \tag{2.5.1}$$

REMARK 2.6. 1.— Conditions (2.4.2) and (2.4.3) for $q \in S$ imply the same conditions for every $q \in \langle S \rangle$.

2.— If $\mathrm{M} \neq 0$, the map $\sigma^{\mathrm{M}} : \langle S \rangle \to \mathrm{Aut}^{\mathrm{cont}}_K(\mathrm{M})$ is injective. Indeed, since B is a domain and M is free, the equality $\sigma^{\mathrm{M}}_q(fm) = \sigma^{\mathrm{M}}_{q'}(fm)$, $\forall\, f \in \mathrm{B}$, $\forall\, m \in \mathrm{M}$, implies $\sigma_q(f)\sigma^{\mathrm{M}}_q(m) = \sigma_{q'}(f)\sigma^{\mathrm{M}}_{q'}(m)$, and hence the contradiction: $\sigma_q(f) = \sigma_{q'}(f), \forall f \in \mathrm{B}$.

3.— The morphism σ^{M} on $\langle S \rangle$ is determined by its restriction to the set S. Reciprocally, if a map $S \to \mathrm{Aut}^{\mathrm{cont}}_K(\mathrm{M})$ is given, then this map extends to a group morphism $\langle S \rangle \to \mathrm{Aut}^{\mathrm{cont}}_K(\mathrm{M})$ if and only if the following conditions are verified:

i. $\sigma^{\mathrm{M}}_q \circ \sigma^{\mathrm{M}}_{q'} = \sigma^{\mathrm{M}}_{q'} \circ \sigma^{\mathrm{M}}_q$, for all $q, q' \in S$;
ii. If $\exists\, n, m \in \mathbb{Z}$, $\exists\, q_1, q_2 \in S$, such that $q_1^n = q_2^m$, then $(\sigma^{\mathrm{M}}_{q_1})^n = (\sigma^{\mathrm{M}}_{q_2})^m$;
iii. If $1 \in S$, then $\sigma^{\mathrm{M}}_1 = \mathrm{Id}$.

2.1.1. *Matrices of* σ^{M}. Let $\mathbf{e} = \{\mathrm{e}_1, \ldots, \mathrm{e}_n\} \subset \mathrm{M}$ be a basis over B. If $\sigma^{\mathrm{M}}_q(\mathrm{e}_i) = \sum_j a_{i,j}(q,T) \cdot \mathrm{e}_j$, then in this basis σ^{M}_q acts as

$$\sigma^{\mathrm{M}}_q(f_1, \ldots, f_n) = (\sigma_q(f_1), \ldots, \sigma_q(f_n)) \cdot A(q,T) \ , \tag{2.6.1}$$

where $A(q,T) := (a_{i,j}(q,T))_{i,j}$. By definition $A(1,T) = \mathrm{Id}$, and one has

$$A(qq',T) = A(q',qT) \cdot A(q,T) \ . \tag{2.6.2}$$

In particular $A(q^n,T) = A(q,q^{n-1}T) \cdot A(q,q^{n-2}T) \cdots A(q,T)$.

2.1.2. *Internal* Hom *and* $\otimes$. Let $(\mathrm{M}, \sigma^{\mathrm{M}})$, $(\mathrm{N}, \sigma^{\mathrm{N}})$ be two discrete $\sigma-$modules over S. We define a structure of discrete $\sigma-$module on $\mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{N})$ by setting $\sigma^{\mathrm{Hom}(\mathrm{M},\mathrm{N})}_q(\alpha) := \sigma^{\mathrm{N}}_q \circ \alpha \circ (\sigma^{\mathrm{M}}_q)^{-1}$, for all $q \in S$, and all $\alpha \in \mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{N})$. We define on $\mathrm{M} \otimes_{\mathrm{B}} \mathrm{N}$ a structure of discrete $\sigma-$module over S by setting $\sigma^{\mathrm{M}\otimes\mathrm{N}}_q(m \otimes n) := \sigma^{\mathrm{M}}_q(m) \otimes \sigma^{\mathrm{N}}_q(n)$, for all $q \in S$, and all $m \in \mathrm{M}$, $n \in \mathrm{N}$.

REMARK 2.7. If $S^\circ \neq \emptyset$ (cf. (2.3.1)), then the category $\sigma - \mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_S$ is $K-$linear. If B is a Bezout ring (i.e. every finitely generated ideal of B is principal), then $\sigma - \mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_S$ is Tannakian (cf. [**ADV04**, 12.3]). The ring $\mathcal{H}_K(X)$ is always principal. If K is spherically closed, then $\mathcal{A}_K(I)$, $\mathcal{R}_K$, $\mathcal{H}^\dagger_K$ are Bezout rings.

2.2. Discrete $(\sigma,\delta)-$modules. Let $S \subset \mathcal{Q}(\mathrm{B})$ be an arbitrary subset.

DEFINITION 2.8 (discrete $(\sigma,\delta)-$modules). An object of

$$(\sigma,\delta)\text{–}\mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_S \tag{2.8.1}$$

is a discrete $\sigma-$module over S, together with a connection[1] $\delta^{\mathrm{M}}_1 : \mathrm{M} \to \mathrm{M}$. Objects $(\mathrm{M}, \sigma^{\mathrm{M}}, \delta^{\mathrm{M}}_1)$ of $(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_S$ will be called *discrete* $(\sigma,\delta)-$*modules over* S. A morphism between $(\mathrm{M}, \sigma^{\mathrm{M}}, \delta^{\mathrm{M}}_1)$ and $(\mathrm{N}, \sigma^{\mathrm{N}}, \delta^{\mathrm{N}}_1)$ is a morphism $\alpha : (\mathrm{M}, \sigma^{\mathrm{M}}) \to (\mathrm{N}, \sigma^{\mathrm{N}})$ of discrete $\sigma-$modules satisfying also

$$\alpha \circ \delta^{\mathrm{M}}_1 = \delta^{\mathrm{N}}_1 \circ \alpha \ . \tag{2.8.2}$$

We will denote the $K-$vector space of morphisms by $\mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M},\mathrm{N})$.

[1] i.e. δ^{M}_1 verifies $\delta^{\mathrm{M}}_1(fm) = \delta_1(f) \cdot m + f \cdot \delta^{\mathrm{M}}_1(m)$, $\forall\, f \in \mathrm{B}$, $\forall\, m \in \mathrm{M}$. Recall that $\delta_1 := T\frac{d}{dT}$.

Remark 2.9. By analogy with (2.5.1), if $S = \{q\}$, then we set:

$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B}) \quad := \quad (\sigma, \delta) - \mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_{\{q\}} \ . \tag{2.9.1}$$

Remark 2.10. Suggested by (0.0.5), we introduce the operator

$$\delta_q^{\mathrm{M}} := \sigma_q^{\mathrm{M}} \circ \delta_1^{\mathrm{M}} \ . \tag{2.10.1}$$

Then, for all $f \in \mathrm{B}$, all $m \in \mathrm{M}$, and all $q \in \langle S \rangle$, one has

$$\delta_q^{\mathrm{M}}(f \cdot m) = \sigma_q(f) \cdot \delta_q^{\mathrm{M}}(m) + \delta_q(f) \cdot \sigma_q^{\mathrm{M}}(m) \ . \tag{2.10.2}$$

Moreover, for all $\alpha \in \mathrm{Hom}^{(\sigma,\delta)}(\mathrm{M},\mathrm{N})$, and all $q \in \langle S \rangle$, one has $\alpha \circ \delta_q^{\mathrm{M}} = \delta_q^{\mathrm{N}} \circ \alpha$. Heuristically we imagine M as endowed with the map $q \mapsto \delta_q^{\mathrm{M}} : \langle S \rangle \to \mathrm{End}_K^{\mathrm{cont}}(\mathrm{M})$, this justifies notations (2.8.1) and (2.9.1).

2.2.1. *Matrices of* δ_q^{M}. Let $\mathbf{e} = \{\mathrm{e}_1, \ldots, \mathrm{e}_n\} \subset \mathrm{M}$ be a basis over B. Let $A(q,T) \in GL_n(\mathrm{B})$ be the matrix of σ_q^{M} in the basis $\mathbf{e}$ (cf. (2.6.1)). If $\delta_q^{\mathrm{M}}(\mathrm{e}_i) = \sum_j g_{i,j}(q,T) \cdot \mathrm{e}_j$, and if $G(q,T) = (g_{i,j}(q,T))_{i,j}$, then δ_q^{M} acts in the basis $\mathbf{e}$ as:

$$\delta_q^{\mathrm{M}}(f_1, \ldots, f_n) = (\delta_q(f_1), \ldots, \delta_q(f_n)) \cdot A(q,T) + (\sigma_q(f_1), \ldots, \sigma_q(f_n)) \cdot G(q,T) \ . \tag{2.10.3}$$

One has moreover

$$G(q' \cdot q, T) = G(q', qT) \cdot A(q,T) \ . \tag{2.10.4}$$

2.2.2. *Internal* Hom *and* $\otimes$. Let $(\mathrm{M}, \sigma^{\mathrm{M}}, \delta^{\mathrm{M}})$, $(\mathrm{N}, \sigma^{\mathrm{N}}, \delta^{\mathrm{N}})$ be two discrete $(\sigma,\delta)-$modules over S. We define a structure of discrete $(\sigma,\delta)-$module on $\mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{N})$ by setting

$$\delta_q^{\mathrm{Hom(M,N)}}(\alpha) := \Big(\delta_q^{\mathrm{N}} \circ \alpha - \sigma_q^{\mathrm{Hom(M,N)}}(\alpha) \circ \delta_q^{\mathrm{M}} \Big) \circ (\sigma_q^{\mathrm{M}})^{-1} \ . \tag{2.10.5}$$

This definition gives the relation $\delta_q^{\mathrm{N}}(\alpha \circ m) = \sigma_q^{\mathrm{H}}(\alpha) \circ \delta_q^{\mathrm{M}}(m) + \delta_q^{\mathrm{H}}(\alpha) \circ \sigma_q^{\mathrm{M}}(m)$, for all $\alpha \in \mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{N})$, and all $m \in \mathrm{M}$, where $\mathrm{H} := \mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{N})$. We define on $\mathrm{M} \otimes_{\mathrm{B}} \mathrm{N}$ a structure of discrete $(\sigma,\delta)-$module over S by setting

$$\delta_q^{\mathrm{M}\otimes\mathrm{N}}(m \otimes n) := \delta_q^{\mathrm{M}}(m) \otimes \sigma_q^{\mathrm{N}}(n) + \sigma_q^{\mathrm{M}}(m) \otimes \delta_q^{\mathrm{N}}(n) \ , \tag{2.10.6}$$

for all $q \in S$, and all $m \in \mathrm{M}$, $n \in \mathrm{N}$.

Remark 2.11. If B is Bezout, $(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_S$ is $K-$linear and Tannakian.

2.3. Analytic $\sigma-$modules. Analytic $\sigma-$modules are defined only if the ring B is equal to one of the following rings: $\mathcal{A}_K(I)$, $\mathcal{H}_K(A)$, $\mathcal{H}_K^{\dagger}(X)$, $\mathcal{H}_K$, $\mathcal{H}_K^{\dagger}$, $\mathcal{R}_K$.

Remark 2.12. Let $U \subset \mathcal{Q}(\mathrm{B})$ be an open subset. Then the subgroup $\langle U \rangle \subseteq \mathcal{Q}(\mathrm{B})$ generated by U is open, i.e. $\langle U \rangle$ contains a disk $\mathrm{D}_K^-(1,\tau)$, for some $\tau > 0$.

Definition 2.13. Let $\mathrm{B} := \mathcal{H}_K(X)$. Let $(\mathrm{M}, \sigma^{\mathrm{M}})$ be a discrete $\sigma-$module over U. Let $A(q,T) \in GL_n(\mathrm{B})$ be the matrix of σ_q^{M} in a fixed basis. We will say that $(\mathrm{M}, \sigma^{\mathrm{M}})$ is an *analytic $\sigma-$module* if, for all $q \in U$, there exist a disk $\mathrm{D}^-(q,\tau_q) = \{q' \mid |q'-q| < \tau_q\}$, with $\tau_q > 0$, and a matrix $A_q(Q,T)$ such that:

(1) $A_q(Q,T)$ is an analytic element on the domain $(Q,T) \in \mathrm{D}^-(q,\tau_q) \times X$;
(2) For all $q' \in \mathrm{D}_K^-(q,\tau_q)$, one has $A_q(Q,T)_{|Q=q'} = A(q',T)$.

This definition does not depend on the chosen basis $\mathbf{e}$. We define

$$\sigma - \mathrm{Mod}(\mathrm{B})^{\mathrm{an}}_U \tag{2.13.1}$$

as the full sub-category of $\sigma - \mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_U$, whose objects are analytic $\sigma-$modules.

Let $I \subset \mathbb{R}_{\geq 0}$ be an interval. We give the same definition over the ring $\mathrm{B} := \mathcal{A}_K(I)$, namely the point (1) is replaced by

(1′) $A_q(Q,T)$ is an analytic function on the domain $(Q,T) \in \mathrm{D}^-(q,\tau_q) \times \mathcal{C}(I)$.

Remark 2.14. If $(\mathrm{M},\sigma^{\mathrm{M}})$ and $(\mathrm{N},\sigma^{\mathrm{N}})$ are two analytic $\sigma-$modules over U, then $(\mathrm{Hom}(\mathrm{M},\mathrm{N}),\sigma^{\mathrm{Hom}(\mathrm{M},\mathrm{N})})$ and $(\mathrm{M}\otimes\mathrm{N},\sigma^{\mathrm{M}\otimes\mathrm{N}})$ are analytic. This follows from the explicit dependence of the matrices of $\sigma^{\mathrm{Hom}(\mathrm{M},\mathrm{N})}$ and $\sigma^{\mathrm{M}\otimes\mathrm{N}}$ on terms of the matrices of σ^{M} and σ^{N}.

2.3.1. *Discrete and analytic $\sigma-$modules over $\mathcal{A}_K(I)$, $\mathcal{R}_K$ and $\mathcal{H}_K^\dagger(X)$.* If $I_1 \subset I_2$, then the restriction functor $\sigma-\mathrm{Mod}(\mathcal{A}_K(I_2))_U^{\mathrm{an}} \to \sigma-\mathrm{Mod}(\mathcal{A}_K(I_1))_U^{\mathrm{an}}$ is fully faithful. Indeed the equality $f_{|_{I_1}} = g_{|_{I_1}}$ implies $f=g$, for all $f,g \in \mathcal{A}_K(I_2)$ (analytic continuation [**CR94**, 5.5.8]).

Definition 2.15. Let $S \subseteq \mathcal{Q}$ be a subset, and let $U \subseteq \mathcal{Q}$ be an open subset. We set

$$\sigma-\mathrm{Mod}(\mathcal{R}_K)_U^{\mathrm{an}} := \bigcup_{\varepsilon>0} \sigma-\mathrm{Mod}(\mathcal{A}_K(]1-\varepsilon,1[))_U^{\mathrm{an}} \;; \tag{2.15.1}$$

$$\sigma-\mathrm{Mod}(\mathcal{R}_K)_S^{\mathrm{disc}} := \bigcup_{\varepsilon>0} \sigma-\mathrm{Mod}(\mathcal{A}_K(]1-\varepsilon,1[))_S^{\mathrm{disc}} \,. \tag{2.15.2}$$

We define analogously $\sigma-\mathrm{Mod}(\mathcal{H}_K^\dagger(X))_U^{\mathrm{an}}$ and $\sigma-\mathrm{Mod}(\mathcal{H}_K^\dagger(X))_S^{\mathrm{disc}}$.

Remark 2.16. Since U is open, one has $U^\circ \neq \emptyset$ (cf. (2.3.1)). By remark 2.7, if B is one of the previous rings (and if it is a Bezout ring), then $\sigma-\mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}}$ is $K-$linear and Tannakian.

2.4. Analytic $(\sigma,\delta)-$modules.

We maintain the previous notations. In section 2.4.1 below we define a *fully faithful* functor (cf. remark 2.19) $\sigma-\mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}} \to (\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_U^{\mathrm{disc}}$, which is a "local" section of the functor which "forgets" δ.

Definition 2.17. We call $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}}$ the essential image of that functor.

By definition, the functor which "forgets" the action of δ is hence an equivalence

$$(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}} \xrightarrow[\sim]{\text{Forget }\delta} \sigma-\mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}} \,. \tag{2.17.1}$$

2.4.1. *Construction of δ.* Let $(\mathrm{M},\sigma^{\mathrm{M}})$ be an analytic $\sigma-$module. We shall define a (σ,δ)-module structure on M. It follows by the definitions 2.13 and 2.15 that the map $q \mapsto \sigma_q^{\mathrm{M}} : \langle U\rangle \to \mathrm{Aut}_K(\mathrm{M})$ is *derivable*, in the sense that, for all $q \in \langle U\rangle$, the limit

$$\delta_q^{\mathrm{M}} := q \cdot \lim_{q'\to q} \frac{\sigma_{q'}^{\mathrm{M}} - \sigma_q^{\mathrm{M}}}{q'-q} = \text{“ } (q\frac{d}{dq}\sigma^M)(q) \text{ ”} \tag{2.17.2}$$

exists in $\mathrm{End}_K^{\mathrm{cont}}(\mathrm{M})$, with respect to the simple convergence topology (cf. (2.20.1)). Moreover, for all $q \in \langle U\rangle$, the rule (2.10.2) holds, and $\delta_q^{\mathrm{M}} = \sigma_q^{\mathrm{M}} \circ \delta_1^{\mathrm{M}}$.

Remark 2.18. A morphism between analytic $(\sigma,\delta)-$modules is, by definition, a morphism of *discrete* $(\sigma,\delta)-$modules.

Remark 2.19. Let $\alpha : (\mathrm{M},\sigma^{\mathrm{M}}) \to (\mathrm{N},\sigma^{\mathrm{N}})$ be a morphism of analytic $\sigma-$modules, that is $\alpha \circ \sigma_q^{\mathrm{M}} = \sigma_q^{\mathrm{N}} \circ \alpha$, for all $q \in U$. Passing to the limit in the definition (2.17.2), one shows that α commutes with δ_q^{M}, for all $q \in U$. Hence the inclusion $\mathrm{Hom}_U^{(\sigma,\delta)}(\mathrm{M},\mathrm{N}) \subseteq \mathrm{Hom}_U^{\sigma}(\mathrm{M},\mathrm{N})$ is an equality.

Remark 2.20. If $\mathbf{e} = \{\mathrm{e}_1, \dots, \mathrm{e}_n\} \subset \mathrm{M}$ is a basis in which the matrix of σ_q^{M} is $A(q,T)$, then the matrix of δ_q^{M} is (cf. (2.10.3), Def. 2.13 and 2.15)

$$(2.20.1) \qquad G(q,T) \;:=\; q \cdot \lim_{q' \to q} \frac{A(q',T) - A(q,T)}{q' - q} \;=\; \Big(\partial_Q\big(A_q(Q,T)\big)\Big)_{|_{Q=q}} ,$$

where ∂_Q is the derivation $Q\frac{d}{dQ}$, and $A_q(Q,T)$ is the matrix of Definition 2.13.

3. Solutions (formal definition)

3.1. Discrete $\sigma-$algebras and $(\sigma,\delta)-$algebras. Let $S \subseteq \mathcal{Q}(\mathrm{B})$ be a subset.

Definition 3.1 (Discrete $\sigma-$algebra over S). A B-*discrete $\sigma-$algebra over S*, or simply a *discrete $\sigma-$algebra over S* is a B$-$algebra C such that:

(1) C is an *integral domain*,
(2) there exists a group morphism $\sigma^{\mathrm{C}} : \langle S \rangle \to \mathrm{Aut}_K(\mathrm{C})$ such that σ_q^{C} is a ring automorphism extending σ_q^{B}, for all $q \in \langle S \rangle$;
(3) one has $\mathrm{C}_S^{\sigma} = K$, where $\mathrm{C}_S^{\sigma} := \{c \in \mathrm{C} \mid \sigma_q(c) = c \;,\; \text{for all } q \in S\}$.

We will call C_S^{σ} *the sub-ring of $\sigma-$constants of* C. We will write σ_q instead of σ_q^{C}, when no confusion is possible.

Remark 3.2. If a discrete $\sigma-$algebra C is free and of finite rank as B$-$module, then it is a discrete $\sigma-$module.

Remark 3.3. Observe that no topology is required on C. The word *discrete* is employed, here and later on, to emphasize that we do not ask "continuity" with respect to q.

Remark 3.4. Suppose that $S = \{\xi\}$, with $\xi \in \boldsymbol{\mu}(\mathcal{Q})$. Since $\mathrm{B}_S^{\sigma} = \mathrm{B}^{\sigma_\xi} \neq K$, then B itself is not a discrete $\sigma-$algebra over S. Hence there is no discrete $\sigma-$algebra over $S = \{\xi\}$. On the other hand, if $S^\circ \neq \emptyset$ (cf. (2.3.1)), then $\mathrm{B}_S^{\sigma} = K$, and B is a discrete $\sigma-$algebra over S.

Definition 3.5 (Discrete $(\sigma,\delta)-$algebra over S). A *discrete $(\sigma,\delta)-$algebra* C *over S* is a B$-$algebra such that:

(1) C verifies the properties (1) and (2) of Definition 3.1,
(2) there exists a derivation δ_1^{C}, extending $\delta_1 = T\frac{d}{dT}$ on B, and commuting with σ_q^{C}, for all $q \in \langle S \rangle$,
(3) one has $\mathrm{C}_S^{(\sigma,\delta)} = K$, where $\mathrm{C}_S^{(\sigma,\delta)} := \{f \in \mathrm{C} \mid f \in \mathrm{C}_S^{\sigma} \;,\; \text{and } \; \delta_1(f) = 0\}$.

We will call $\mathrm{C}_S^{(\sigma,\delta)}$ the *sub-ring of $(\sigma,\delta)-$constants of* C . We will write δ_1 instead of δ_1^{C}, if no confusion is possible.

Remark 3.6. The operator $\delta_q^{\mathrm{C}} := \sigma_q^{\mathrm{C}} \circ \delta_1^{\mathrm{C}}$ satisfies analogous property of (2.10.2). Since $\mathrm{B}_S^{(\sigma,\delta)} = K$, then B is always a $(\sigma,\delta)-$algebra over S, for all arbitrary sub-set $S \subseteq \mathcal{Q}(\mathrm{B})$, even for $S = \{\xi\}$, with $\xi \in \boldsymbol{\mu}(\mathcal{Q}(\mathrm{B}))$.

3.2. Constant Solutions.

Definition 3.7 (Constant solutions on S). Let $(\mathrm{M}, \sigma^{\mathrm{M}})$ (resp.$(\mathrm{M}, \sigma^{\mathrm{M}}, \delta^{\mathrm{M}})$) be a *discrete* $\sigma-$module (resp. $(\sigma,\delta)-$module) over S, and let C be a discrete $\sigma-$algebra (resp. $(\sigma,\delta)-$algebra) over S. A *constant solution* of M, with values in C, is a $B-$linear morphism

$$\alpha : \mathrm{M} \longrightarrow \mathrm{C}$$

such that $\alpha \circ \sigma_q^{\mathrm{M}} = \sigma_q^{\mathrm{C}} \circ \alpha$, for all $q \in S$ (resp. α verifies simultaneously $\alpha \circ \delta_1^{\mathrm{M}} = \delta_1^{\mathrm{C}} \circ \alpha$, and $\alpha \circ \sigma_q^{\mathrm{M}} = \sigma_q^{\mathrm{C}} \circ \alpha$, for all $q \in S$). We denote by $\mathrm{Hom}_S^{\sigma}(\mathrm{M},\mathrm{C})$ (resp. $\mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{C})$) the $K-$vector space of the solutions of M in C.

3.2.1. *Matrices of solutions.* Let M be a discrete $\sigma-$module (resp. $(\sigma,\delta)-$module). Let C be a discrete $\sigma-$algebra (resp. $(\sigma,\delta)-$algebra) over S. Recall that, if $S = \{\xi\}$, with $\xi^n = 1$, then there is no discrete $\sigma-$algebra, over S (cf. remark 3.4).

Let $\mathbf{e} = \{\mathrm{e}_1, \dots, \mathrm{e}_n\}$ be a basis of M, and let $A(q,T)$ (resp. $G(q,T)$) be the matrix of σ_q^{M} (resp. δ_q^{M}) in this basis (cf. (2.10.3)). We identify a morphism $\alpha : \mathrm{M} \to \mathrm{C}$ with the vector $(y_i)_i \in \mathrm{C}^n$, given by $y_i := \alpha(\mathrm{e}_i)$. In this way constant solutions become solutions in the usual *vector form*. Indeed

$$\begin{pmatrix} \sigma_q(y_1) \\ \vdots \\ \sigma_q(y_n) \end{pmatrix} = A(q,T) \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} , \quad \text{for all } q \in S ,$$

$$\Bigg(\text{resp. } \begin{pmatrix} \delta_q(y_1) \\ \vdots \\ \delta_q(y_n) \end{pmatrix} = G(q,T) \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} , \quad \text{for all } q \in S \Bigg) .$$

By a *fundamental matrix of solutions* of M (in the basis $\mathbf{e}$) we mean a matrix $Y \in GL_n(\mathrm{C})$ satisfying *simultaneously*

$$\sigma_q(Y) = A(q,T) \cdot Y , \quad \text{for all } q \in S , \tag{3.7.1}$$

(resp. satisfying *simultaneously*

$$\begin{cases} \sigma_q(Y) = A(q,T) \cdot Y , & \text{for all } q \in S , \\ \delta_1(Y) = G(1,T) \cdot Y . & \end{cases}) . \tag{3.7.2}$$

By this identification, one has $\mathrm{B}_S^{\sigma} \cong \mathrm{Hom}_S^{\sigma}(\mathbb{I},\mathrm{B})$ (resp. $\mathrm{B}_S^{(\sigma,\delta)} \cong \mathrm{Hom}_S^{(\sigma,\delta)}(\mathbb{I},\mathrm{B})$) (cf. def. 3.1, 3.5, 3.7), where $\mathbb{I} = \mathrm{B}$ is the unit object.

3.2.2. *Dimension of the space of solutions.*

Remark 3.8. Let $F := \mathrm{Frac}(\mathrm{C})$ be the fraction field of C, then both σ_q and δ_1 extend to F (cf. [**vdPS03**, Ex.1.5]).

Lemma 3.9. *Let* M *be a* $(\sigma,\delta)-$*module (resp.* $\sigma-$*module) over* S*, and let* C *be a discrete* $(\sigma,\delta)-$*algebra (resp.* $\sigma-$*algebra) over* S*. One has*

$$\dim_K \mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{C}) \leq \mathrm{rk}_{\mathrm{B}}(\mathrm{M}) . \tag{3.9.1}$$

(resp. if $S^\circ \neq \emptyset$ *(cf. (2.3.1)), then* $\dim_K \mathrm{Hom}_S^{\sigma}(\mathrm{M},\mathrm{C}) \leq \mathrm{rk}_{\mathrm{B}}(\mathrm{M})$*.)*

Proof : One has $\dim_K \mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{C}) \leq \dim_K \mathrm{Hom}^{\delta_1}(\mathrm{M},\mathrm{C}) \leq \mathrm{rk}_{\mathrm{B}}(\mathrm{M})$. On the other hand, if $q \in S^\circ$, then $\mathrm{Hom}^{\sigma_q}(\mathrm{M},\mathrm{C}) \leq \mathrm{rk}_{\mathrm{B}}(\mathrm{M})$. Hence $\dim_K \mathrm{Hom}_S^{\sigma}(\mathrm{M},\mathrm{C}) \leq \dim_K \mathrm{Hom}^{\sigma_q}(\mathrm{M},\mathrm{C}) \leq \mathrm{rk}_{\mathrm{B}}(\mathrm{M})$. □

4. Constant Confluence

4.1. Constant modules. Let B be one of the rings defined in Section 1.1, let $S \subset \mathcal{Q}(\mathrm{B})$ be a subset, and $U \subset \mathcal{Q}(\mathrm{B})$ be an open subset.

Definition 4.1 (Constant modules). Let M be a discrete $\sigma-$module over S. We will say that M is *constant* on S, or equivalently that M *is trivialized by* C, if there exists a discrete $\sigma-$algebra C over S such that

$$\dim_K \mathrm{Hom}_S^{\sigma} \; (\mathrm{M},\mathrm{C}) = \mathrm{rk}_{\mathrm{B}} \mathrm{M} . \tag{4.1.1}$$

We give the analogous definition for (σ,δ)-modules. The full sub-category of $\sigma-\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ (resp. $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$), whose objects are trivialized by C, will be denoted by

$$\sigma-\mathrm{Mod}(\mathrm{B},\mathrm{C})_S^{\mathrm{const}} \qquad (\text{resp. } (\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})_S^{\mathrm{const}}\ ) . \tag{4.1.2}$$

The full subcategory of $\sigma-\mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{const}}$ (resp. $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{const}}$) whose objects are analytic will be denoted by

$$\sigma-\mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an,const}} \qquad (\text{resp. } (\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an,const}}\ ) . \tag{4.1.3}$$

Remark 4.2. Let $n=\mathrm{rk}_{\mathrm{B}}\mathrm{M}$, then M is trivialized by C if there exists $Y\in GL_n(\mathrm{C})$ such that Y is *simultaneously* solution, for all $q\in S$, of the family of equations (3.7.1) (resp. both the conditions of (3.7.2)). Roughly speaking, M is constant on S if it admits a basis of $q-$solutions which "does not depend on $q\in S$".

Lemma 4.3. *Let* M*,* N *be two discrete* $\sigma-$*modules (resp.* $(\sigma,\delta)-$*modules). If* M*,* N *are both trivialized by* C*, then* $\mathrm{M}\otimes\mathrm{N}$ *and* $\mathrm{Hom}(\mathrm{M},\mathrm{N})$ *are trivialized by* C*. In particular* $M^\vee$ *and* $N^\vee$ *are trivialized by* C*.*

Proof : The fundamental matrix solution of $\mathrm{M}\otimes\mathrm{N}$ (resp. $\mathrm{Hom}(\mathrm{M},\mathrm{N})$) is obtained by taking products of entries of the two matrices of solutions of M and N respectively. Hence "it does not depend on $q\in S$".□

Lemma 4.4. *Let* $S'\subseteq S$ *be a non empty subset. Let* C *be a discrete* $(\sigma,\delta)-$*algebra over* S*. Then the restriction functor, sending* $(\mathrm{M},\sigma^{\mathrm{M}},\delta_1^{\mathrm{M}})$ *into* $(\mathrm{M},\sigma^{\mathrm{M}}_{|_{\langle S'\rangle}},\delta_1^{\mathrm{M}})$*:*

$$\mathrm{Res}^S_{S'}:(\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})_S^{\mathrm{const}}\longrightarrow(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_{S'}^{\mathrm{disc}} \tag{4.4.1}$$

is fully faithful and its image is contained in the category $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})_{S'}^{\mathrm{const}}$*. The same fact is true for discrete* $\sigma-$*modules under the assumption:* $(S')^\circ\neq\emptyset$*.*

Proof : The proof is the same in both cases, here we give the proof in the case of $(\sigma,\delta)-$modules. We must show that the inclusion $\mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{N})\to\mathrm{Hom}_{S'}^{(\sigma,\delta)}(\mathrm{M},\mathrm{N})$ is an isomorphism, for all M,N in $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})_S^{\mathrm{const}}$. In other words we have to show that if $\alpha:\mathrm{M}\to\mathrm{N}$ commutes with $\sigma_{q'}$, for all $q'\in S'$, then it commutes also with σ_q, for all $q\in S$. One has

$$\begin{aligned}\mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{N}) &= \mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{B})\ ;\\ \mathrm{Hom}_{S'}^{(\sigma,\delta)}(\mathrm{M},\mathrm{N}) &= \mathrm{Hom}_{S'}^{(\sigma,\delta)}(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{B})\ .\end{aligned} \tag{4.4.2}$$

Observe that $\mathrm{M}\otimes\mathrm{N}^\vee$ is the dual of the "internal hom" $\mathrm{Hom}(\mathrm{M},\mathrm{N})$. By lemma 4.3, $\mathrm{M}\otimes\mathrm{N}^\vee$ is trivialized by C. The restriction of $\mathrm{M}\otimes\mathrm{N}^\vee$ to S' is obviously constant on S', since it is trivialized by C. This implies that

$$\mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{C})=\mathrm{Hom}_{S'}^{(\sigma,\delta)}(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{C})\ . \tag{4.4.3}$$

This shows that a morphism with values in $\mathrm{B}\subseteq\mathrm{C}$ commutes with all σ_q and δ_q, for all $q\in S$, if and only if it commutes with all σ_q and δ_q, for all $q\in S'$. Hence

$$\mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{B})=\mathrm{Hom}_{S'}^{(\sigma,\delta)}(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{B})\ .\ \square \tag{4.4.4}$$

Remark 4.5. By the previous lemma one sees that if $\xi\in S\cap\boldsymbol{\mu}(\mathcal{Q})$, then

$$\mathrm{Res}^S_{\{\xi\}}:(\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})_S^{\mathrm{const}}\longrightarrow(\sigma_\xi,\delta_\xi)-\mathrm{Mod}(\mathrm{B}) \tag{4.5.1}$$

is again fully faithful. On the other hand, if $S^\circ\neq\emptyset$, then the restriction

$$\mathrm{Res}^S_{\{\xi\}}:\sigma-\mathrm{Mod}(\mathrm{B},\mathrm{C})_S^{\mathrm{const}}\longrightarrow\sigma_\xi-\mathrm{Mod}(\mathrm{B}) \tag{4.5.2}$$

is *not* fully faithful, since $\sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S^{\mathrm{const}}$ is $K-$linear, while $\sigma_\xi - \mathrm{Mod}(\mathrm{B})$ is not $K-$linear (i.e. $K \subset \mathrm{End}(\mathbb{I})$, but $K \neq \mathrm{End}(\mathbb{I})$, cf. Section 1).

REMARK 4.6. If U is open, then the condition $U^\circ \neq \emptyset$ is automatically verified.

REMARK 4.7. By Lemma 4.4, if $S \subset U$ is a (non empty) subset, the restriction

$$\mathrm{Res}_S^U : (\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an,const}} \longrightarrow (\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S^{\mathrm{const}} \tag{4.7.1}$$

is fully faithful. The same is true for $\sigma-$modules, under the assumption $S^\circ \neq \emptyset$. In particular if $U' \subset U$ is an open subset, then the restriction functor is fully faithful:

$$\mathrm{Res}_{U'}^U : (\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an,const}} \longrightarrow (\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_{U'}^{\mathrm{an,const}} . \tag{4.7.2}$$

4.2. Constant deformation and constant confluence. As usual $S \subseteq \mathcal{Q}(\mathrm{B})$ is an arbitrary subset, and $U \subseteq \mathcal{Q}(\mathrm{B})$ is an open subset.

DEFINITION 4.8 (Extensible objects). Let $q \in S$. Let C be a discrete $\sigma-$algebra over S. A $q-$difference module M is said *extensible to* S if it belongs to the essential image of the restriction functor

$$\mathrm{Res}_{\{q\}}^S : \sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S^{\mathrm{const}} \quad \longrightarrow \quad \sigma_q - \mathrm{Mod}(\mathrm{B}) .$$

The *full sub-category* of $\sigma_q - \mathrm{Mod}(\mathrm{B})$ whose objects are extensible to S, will be denoted by $\sigma_q - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S$.

If $q \in U$, we will denote by $\sigma_q - \mathrm{Mod}(B,\mathrm{C})_U^{\mathrm{an}}$ the full sub-category of $\sigma_q - \mathrm{Mod}(\mathrm{B})_U$ whose objects belong to the essential image of $\sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an,const}}$.

We give analogous definitions for (σ,δ)-modules.

Lemma 4.4 and definition 4.8 give easily the following formal statement:

COROLLARY 4.9. *In the notations of Lemma 4.4, one has an equivalence*

$$(\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S^{\mathrm{const}} \xrightarrow[\sim]{\mathrm{Res}_{\{q\}}^S} (\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S . \tag{4.9.1}$$

The same fact is true for $\sigma-$modules, under the additional hypothesis: $q \in S^\circ$. □

DEFINITION 4.10. 1.– Let $S \subseteq \mathcal{Q}(\mathrm{B})$ be a subset and let $q,q' \in \langle S\rangle$. We will call the *constant deformation functor*, denoted by

$$\mathrm{Def}_{q,q'} : (\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S \xrightarrow{\sim} (\sigma_{q'},\delta_{q'}) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S , \tag{4.10.1}$$

the equivalence obtained by composition with the restriction functor (4.9.1):

$$\mathrm{Def}_{q,q'} := \mathrm{Res}_{\{q'\}}^S \circ (\mathrm{Res}_{\{q\}}^S)^{-1} . \tag{4.10.2}$$

2.– We will call the *constant confluence functor*, the equivalence

$$\mathrm{Conf}_q := \mathrm{Def}_{q,1} : (\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S \xrightarrow{\sim} (\sigma_1,\delta_1) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S . \tag{4.10.3}$$

3.– Suppose that $q \in S^\circ$ and $q' \in S$, then we will call again the *constant deformation functor*, denoted again by

$$\mathrm{Def}_{q,q'} : \sigma_q - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S \longrightarrow \sigma_{q'} - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S , \tag{4.10.4}$$

the functor obtained by composition with the restriction functor (4.9.1): $\mathrm{Def}_{q,q'} := \mathrm{Res}_{\{q'\}}^S \circ (\mathrm{Res}_{\{q\}}^S)^{-1}$. If $q' \in S^\circ$, then $\mathrm{Def}_{q,q'}$ is an equivalence.

Remark 4.11. It follows from Corollary 4.9, that if $q, q' \in U$, one has an *equivalence*, again called $\mathrm{Def}_{q,q'}$

$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an}} \xrightarrow[\sim]{\mathrm{Def}_{q,q'}} (\sigma_{q'}, \delta_{q'}) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an}} \; . \tag{4.11.1}$$

The same fact is true for analytic $\sigma-$modules under the condition $q, q' \notin \boldsymbol{\mu}(\mathcal{Q})$.

Remark 4.12. a.— Let $q \in U$. Consider the following diagram:

$$\begin{array}{ccc} \bigcup_U \ \sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an,const}} & \overset{\text{Def. } 2.17.1}{=\!=\!=} & \bigcup_U \ (\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an,const}} \\ \Big\downarrow {\scriptstyle \bigcup_U \mathrm{Res}^U_{\{q\}}} & \odot & {\scriptstyle \wr}\Big\downarrow {\scriptstyle \bigcup_U \mathrm{Res}^U_{\{q\}}} \\ \bigcup_U \ \sigma_q - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U & \xleftarrow[\text{Forget } \delta_q]{} & \bigcup_U \ (\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U \\ \Big\downarrow {\scriptstyle i_\sigma} & \odot & \Big\downarrow {\scriptstyle i_{(\sigma,\delta)}} \\ \sigma_q - \mathrm{Mod}(\mathrm{B}) & \xleftarrow[\text{Forget } \delta_q]{} & (\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B}) \end{array} \tag{4.12.1}$$

where U runs in the set of open neighborhoods of q, and where i_σ and $i_{(\sigma,\delta)}$ are the trivial inclusions of full sub-categories. In the sequel we will study the full subcategory of $\sigma_q - \mathrm{Mod}(\mathrm{B})$ (resp. $(\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B})$) formed by *Taylor admissible objects*, this category is contained in the essential image of i_σ (resp. $i_{(\sigma,\delta)}$) (see Th. 6.2). In this case we will obtain an analogous diagram (see Cor. 6.4)) in which $i_{(\sigma,\delta)}$ is an equivalence (for all $q \in U$), and i_σ is an equivalence only if q is not a root of unity.

If q is not a root of unity, then all the arrows of this diagram will be equivalences, hence the data of δ_q is superfluous. If q is a root of unity, then the right hand side vertical arrows will be equivalences, while the left hand side one will be not. In this last case the $q-$tangent operator is necessary to "*preserve the information in the neighborhood of* q". The good notion of "*stalk at* q" of an analytic σ-module is in this case the notion of (σ_q, δ_q)-module and not simply the notion of σ_q-module.

One may have the feeling that the functor "Forget δ_q" contains an "information" if q is a root of unity, but we will see (Prop. 7.9) that, if $\mathrm{B} = \mathcal{R}_K$ or if $\mathrm{B} = \mathcal{H}_K^\dagger$, then this functor sends every $(\sigma, \delta)-$module with Frobenius structure into a direct sum of the unit object.

5. Taylor solutions

In this section $\mathrm{B} = \mathcal{H}_K(X)$, for some affinoïd $X = \mathrm{D}^+(c_0, R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i, R_i)$, and $S = \{q\} \in \mathcal{Q}(\mathcal{H}_K(X)) \subseteq \{q \in K \mid |q| = 1\}$ is reduced to a point. Let $(\Omega, |.|)/(K, |.|)$ be an arbitrary extension of complete valued fields. Let $c \in X(\Omega)$ and let $\rho_{c,X} > 0$ be the largest real number such that $\mathrm{D}^-(c, \rho_{c,X}) \subseteq X$. One has

$$\rho_{c,X} = \min_{i=0,\ldots,n} |c - c_i|. \tag{5.0.2}$$

Note that c can be equal to a generic point (cf. Definition 5.31). We want to find solutions of $q-$difference equations converging in a disc centered at c, i.e. matrix solutions in the form (3.7.1), with values in the σ_q-algebra $\mathrm{C} := \mathcal{A}_K(c, R)$, for some $0 < R \le \rho_{c,X}$.

5.1. The $q-$algebras $\Omega\{T-c\}_{q,R}$ and $\Omega[\![T-c]\!]_q$.

REMARK 5.1. Without precise mention, we will not assume that $q \notin \boldsymbol{\mu}(\mathcal{Q})$. The following results generalize the analogous constructions of [DV04] to the case of root of unity.

LEMMA 5.2. *Let $0 < R \leq \rho_{c,X}$. The algebra $\mathcal{A}_\Omega(c,R)$ is a $\mathcal{H}_\Omega(X)$-discrete $\sigma-$algebra over $S=\{q\}$, if and only if both the following conditions are verified:*

$$|q-1||c| < R\ , \qquad \text{and} \qquad |q|=1\ . \tag{5.2.1}$$

Proof : Let $x \in \mathrm{D}^-(c,R)$, then $|qx-c| = |qx-qc+qc-c| \leq \max(|q||x-c|,|q-1||c|)$. If both conditions hold, then $|qx-c|<R$, hence the disc $\mathrm{D}^-(c,R)$ is $q-$invariant. This shows the sufficiency. If $x=c$, one sees that the condition $|q-1| < R\cdot|c|^{-1}$ is necessary. Suppose now that $|q-1|<R\cdot|c|^{-1}$, and that $|q|>1$. Since $|q|=|q-1|$, hence $1 \leq |q| < R|c|^{-1}$, and $|q-1||c| = |q||c| < R$. If $|x-c|$ tends to R, then $|qx-c|=|q||x-c|$ is larger than R and, by the argument above, the disc $\mathrm{D}^-(c,R)$ is not $q-$invariant. The condition $|q|\leq 1$ is then necessary, and since by definition σ_q must be bijective with inverse $\sigma_{q^{-1}}$, one finds $|q|=1$. □

DEFINITION 5.3. For all $q \in \mathcal{Q}$ one defines

$$(T-c)_{q,n} := (T-c)(T-qc)(T-q^2c)\cdots(T-q^{n-1}c)\ , \tag{5.3.1}$$

$$[n]^!_q := \frac{(q-1)(q^2-1)(q^3-1)\cdots(q^n-1)}{(q-1)^n}\ . \tag{5.3.2}$$

REMARK 5.4. 1. If q is a $m-$th root of 1, then $[n]^!_q = 0$, for all $n \geq m$.

2. The family $\{(T-c)_{q,n}\}_{n\geq 0}$ is adapted to the $q-$derivation

$$d_q := \frac{\sigma_q - 1}{(q-1)T} = \frac{\Delta_q}{T} \tag{5.4.1}$$

in the sense that, for all $n \geq 1$ one has $d_q((T-c)_{q,n}) = [n]_q \cdot (T-c)_{q,n-1}$.

3. One has $d_q(fg) = \sigma_q(f)d_q(g) + d_q(f)g$, and more generally

$$d_q^n(fg)(T) = \sum_{i=0}^{n} \binom{n}{i}_q d_q^{n-i}(f)(q^iT)d_q^i(g)(T)\ . \tag{5.4.2}$$

LEMMA 5.5. *Let $|q-1||c| < R$, $|q|=1$, and let $f(T)=\sum_{n\geq 0} a_n(T-c)^n \in \mathcal{A}_\Omega(c,R)$, then:*

1. *$f(T)$ can be written uniquely as $f(T) = \sum_{n\geq 0}\widetilde{a}_n(T-c)_{q,n} \in \mathcal{A}_\Omega(c,R)$;*
2. *for all $|q-1||c| < \rho < R$ one has $|f(T)|_{(c,\rho)} = \sup_{n\geq 0}|\widetilde{a}_n|\rho^n$;*
3. *one has $Ray_c(f) = \liminf_n |a_n|^{-1/n} = \liminf_n |\widetilde{a}_n|^{-1/n}$;*
4. *if moreover $q \notin \boldsymbol{\mu}(\mathcal{Q})$, then one has the q-Taylor expansion*

$$f(T) = \sum_{n\geq 0} d_q^n(f)(c)\frac{(T-c)_{q,n}}{[n]^!_q}\ . \tag{5.5.1}$$

Proof : Since $\mathcal{A}_\Omega(c,R) = \varprojlim_{r\to R^-}\mathcal{H}_\Omega(\mathrm{D}^+(c,r))$, then we are reduced to show the proposition for $\mathcal{H}_\Omega(\mathrm{D}^+(c,r))$, with $|q-1||c| < r < R$. Write $(T-q^ic) = (1-q^i)c + (T-c)$, then, since $|q^i-1| \leq |q-1|$, one has $(T-c)_{q,n} = \sum_{i=0}^n b_{n,i}(T-c)^i$, with , $b_{n,n}=1$, $b_{n,0}=0$, and $b_{n,i} = c^{n-i}\cdot\sum_{1\leq k_1<\cdots<k_{n-i}\leq n-1}(1-q^{k_1})(1-q^{k_2})\cdots(1-q^{k_{n-i}})$. Hence $|b_{n,i}| \leq (|q-1||c|)^{n-i} < r^{n-i}$, and $|(T-c)_{q,n}|_{(c,r)} = |(T-c)^n|_{(c,r)} = r^n$. One finds then $a_n = \sum_{k\geq 0}\widetilde{a}_{n+k}b_{n+k,n}$, the sum converges since $|b_{n+k,n}| < r^k$. A symmetric argument proves that $\widetilde{a}_n = \sum_{k\geq 0} a_{n+k}\widetilde{b}_{n+k,n}$, with $\widetilde{b}_{n,n} = 1$, and $|\widetilde{b}_{n,i}| < r^{n-i}$. This shows that $|a_n|r^n \leq \sup_{j\geq n}(|\widetilde{a}_j|r^j)$ and $|\widetilde{a}_n|r^n \leq \sup_{j\geq n}(|a_j|r^j)$, hence $|a_n|r^n$ tends to 0 if and only if $|\widetilde{a}_n|r^n$ tends to 0. Moreover $\sup_{n\geq 0}|a_n|r^n = \sup_{n\geq 0}|\widetilde{a}_n|r^n$. This shows the uniqueness since if $\sum_{n\geq 0}\widetilde{a}_n(T-c)_{q,n} = \sum_{n\geq 0}\widetilde{a}'_n(T-c)_{q,n}$, then $\sum_{n\geq 0}(\widetilde{a}_n - \widetilde{a}'_n)(T-c)_{q,n} = 0$, and hence $\sup_n(|\widetilde{a}_n - \widetilde{a}'_n|r^n) = 0$, then $\widetilde{a}_n = \widetilde{a}'_n$, for all $n \geq 0$. Clearly the radius of convergence of $f(T)$ is equal to both $\sup_{n\geq 0}\{r \geq 0 \mid |a_n|r^n \text{ is bounded}\}$ and $\sup_{n\geq 0}\{r \geq 0 \mid |\widetilde{a}_n|r^n \text{ is bounded}\}$. Hence, by classic arguments on the radius, one has $Ray_c(f) = \liminf_n |a_n|^{-1/n} = \liminf_n |\widetilde{a}_n|^{-1/n}$. □

Remark 5.6. If $f(T) = \sum_{n\geq 0} f_n(T-c)_{q,n}$, and if $g(T) = \sum_{n\geq 0} g_n(T-c)_{q,n}$, then $f(T)g(T) = \sum_{n\geq 0} h_n(T-c)_{q,n}$, where $h_n = h_n(q;c;f_0,\ldots,f_n;g_0,\ldots,g_n)$ is a polynomial in $\{q,c,f_0,\ldots,f_n,g_0,\ldots,g_n\}$. Indeed one has $(T-c)_{q,n}\cdot(T-c)_{q,m} = \sum_{k=\max(n,m)}^{n+m}\alpha_k^{(n,m)}(T-c)_{q,k}$, with $\alpha_k^{(n,m)} = \alpha_k^{(n,m)}(q,c) \in \Omega$. This shows also that if $v_{q,c}(f) := \min\{n \mid f_n \neq 0\}$, then one has

$$v_{q,c}(fg) \geq \max(v_{q,c}(f), v_{q,c}(g)) \ . \tag{5.6.1}$$

If moreover $q \notin \boldsymbol{\mu}(\mathcal{Q})$, then, by using equation (5.4.2) and the twisted Taylor formula (5.5.1), one has

$$h_n = \sum_{j=0}^{n}\sum_{s=0}^{j}\frac{[n]^!_q[j]^!_q[s+n-j]^!_q}{([s]^!_q)^2[n-j]^!_q}\cdot q^{\frac{s(s-1)}{2}}(q-1)^s c^s f_{s+n-j}g_j \ . \tag{5.6.2}$$

Definition 5.7. For all $q \in \mathcal{Q}(X)$ we set

$$\Omega[\![T-c]\!]_q \ := \ \{\sum_{n\geq 0} f_n(T-c)_{q,n} \mid f_n \in \Omega\} \ . \tag{5.7.1}$$

$$\Omega\{T-c\}_{q,R} \ := \ \{\sum_{n\geq 0} f_n(T-c)_{q,n} \mid f_n \in \Omega,\ \liminf_n |f_n|^{-1/n} \geq R\ \} \ . \tag{5.7.2}$$

We define a multiplication on $\Omega[\![T-c]\!]_q$ and $\Omega\{T-c\}_{q,R}$ by the rule given by Remark 5.6.

Lemma 5.8. *$\Omega[\![T-c]\!]_q$ and $\Omega\{T-c\}_{q,R}$ are commutative Ω-algebras, $\forall\, q \in \mathcal{Q}$.*

Proof : The proof is easy, we prove only the associativity. We have to prove that $(fg)h = f(gh)$. By Lemma 5.5 the assertion is proved if $f,g,h \in \Omega\{T-c\}_{q,R}$, with $|q-1||c| < R$, since in this case $\Omega\{T-c\}_{q,R} \cong \mathcal{A}_\Omega(c,R)$. On the other hand one can assume that f,g,h are polynomial since, by Remark 5.6, the n-th coefficient of $(fg)h$ and of $f(gh)$ is a polynomial on q and on the fist n coefficients of f,g,h. □

Remark 5.9. If there exists a (smallest) integer k_0 such that $|\overline{q}^{k_0} - 1||c| < R$, then one shows that $\Omega\{T-c\}_{q,R} = \prod_{i=0}^{k_0-1}\mathcal{A}_\Omega(q^ic,\widetilde{R})$, where $\widetilde{R}$ depends explicitly on R, c, and q (cf. [DV04, 15.3]). In this case $\Omega\{T-c\}_{q,R}$ is not a domain and hence is not a $\mathcal{H}_\Omega(X)-$discrete $\sigma-$algebra over $S = \{q\}$.

Remark 5.10. If x,y are variables, then $\Omega[\![x-y]\!]_q$ is not an algebra, but merely a vector space. Indeed the multiplication law involves y in the coefficients "h_n" of Remark 5.6. This mistake occurs many times in [DV04].

5.2. q-invariant Affinoïd. Let $|q| = 1$, $q \in K$. Let $X := \mathrm{D}^+(c_0, R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i, R_i)$, $c_1, \ldots, c_n \in \mathrm{D}_K^+(c_0, R_0)$, $c_0 \in K$, be a K-affinoïd. Then X is q-invariant if and only if $|q-1||c_0| < R_0$, and the map $x \mapsto qx$ permutes the family of disks $\{ \mathrm{D}^-(c_i, R_i) \}_{i=1,\ldots,n}$. This happens if and only if for all $i = 1, \ldots, n$ there exists (a smallest) $k_i \geq 1$, such that $|q^{k_i} - 1||c_i| < R_i$, and moreover the family of disks $\{ \mathrm{D}^-(q^k c_i, R_i) \}_{k=1,\ldots,k_i}$ is finite and contained in $\{ \mathrm{D}^-(c_i, R_i) \}_{i=1,\ldots,n}$. If k_0 is the minimum common multiple of the k_i's, then $x \mapsto q^{k_0}x$ leaves globally fixed every disk and, by Lemmas 5.2 and 5.5, one has (cf. Lemma 1.2)

$$\|d_{q^{k_0}}(f)\|_X \;\leq\; r_X^{-1}\|f\|_X \;, \tag{5.10.1}$$

for all $f \in \mathcal{H}(X)$. Indeed by Mittag-Lefler decomposition [CR94], we are reduced to show that every series $f = \sum_{j\leq -1} a_j (T-c_i)^j$, such that $|a_j|R_i^j$ tends to zero, satisfies $|d_{q^{k_0}}(f)|_{(c_i,R_i)} \leq R_i^{-1} \cdot |f|_{(c_i,R_i)}$, which is true by Lemma 5.5.

Such a bound does not exist for d_q itself. One can easily construct counterexamples via the Mittag-Leffler decomposition.

5.3. The formal Taylor solution. We recall the definition of the classical Taylor solution of a differential equation

DEFINITION 5.11. Let $\delta_1 - G(1,T)$, be a differential equation. Let $G_{[n]}(T)$ be the matrix of $(d/dx)^n$. We set

$$Y_{G(1,T)}(x,y) := \sum_{n\geq 0} G_{[n]}(y)\frac{(x-y)^n}{n!} \;. \tag{5.11.1}$$

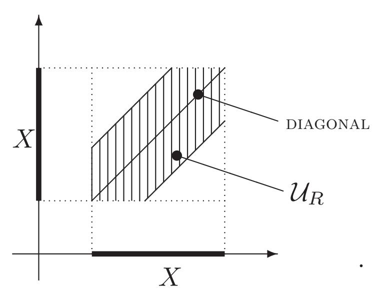
REMARK 5.12. By induction on the rule $G_{[n+1]} = G'_{[n]} + G_{[n]}G_{[1]}$, one shows that $\|G_{[n]}\|_X \leq \max(\|G_{[1]}\|_X, r_X^{-1})^n$, and hence

$$R_c := Ray_c(Y_G(T,c)) = \liminf_n \Big(\frac{|G_{[n]}(c)|_\Omega}{|n!|}\Big)^{-1/n} \geq \frac{|p|^{\frac{1}{p-1}}}{\max(r_X^{-1}, \|G_{[1]}\|_X)} \;. \tag{5.12.1}$$

In other words $Y_G(x,y)$ is an analytic element over a neighborhood $\mathcal{U}_R$ of the diagonal of the type

$$\mathcal{U}_R := \{(x,y) \in X \times X \mid |x-y| < R\} \;, \tag{5.12.2}$$

for some $R > 0$.



LEMMA 5.13. *One has* $Y_G(x,x) = \mathrm{Id}$, *and for all* $(x,y) \in \mathcal{U}_R$:

$$d/dy\;(Y_G(x,y)) = -Y_G(x,y)\cdot G_{[1]}(y) \;, \tag{5.13.1}$$

$$Y_G(x,y)\cdot Y_G(y,z) = Y_G(x,z) \;, \tag{5.13.2}$$

$$Y_G(x,y)^{-1} = Y_G(y,x) \;, \tag{5.13.3}$$

$$d/dx\;(Y_G(x,y)) = G_{[1]}(x)\cdot Y_G(x,y) \;. \tag{5.13.4}$$

Proof : See [CM02, p.137] (cf. Lemma 5.20).□

DEFINITION 5.14. Let $q \in \mathcal{Q} - \boldsymbol{\mu}(\mathcal{Q})$. Consider the $q-$difference equation

$$\sigma_q(Y) = A(q,T)\cdot Y\ , \qquad A(q,T) \in GL_n(\mathcal{H}_K(X))\ . \tag{5.14.1}$$

Let H_n be defined by $d_q^n(Y) = H_n \cdot Y$. We formally set

$$Y_{A(q,T)}(x,y) = \sum_{n\geq 0} H_n(y)\frac{(x-y)_{q,n}}{[n]^!_q}\ . \tag{5.14.2}$$

We will omit the index $A(q,T)$ if no confusion is possible.

REMARK 5.15 (Transfer principle). For all $c \in X(\Omega)$ we let

$$R_c := \liminf_n (|H_n(c)|_\Omega/[n]^!_q)^{-1/n}\ . \tag{5.15.1}$$

As in the differential framework, if $X' := \mathrm{D}^+(c'_0,R'_0) - \cup_{i=1}^n \mathrm{D}^-(c'_i,R'_i) \subseteq X$ is a $q-$invariant subaffinoïd, such that every disk $\mathrm{D}^-(c'_i,R'_i)$ is $q-$invariant too in order to have the estimation (5.10.1) (cf. Remark 5.2), then, by induction on the rule $H_{n+1} = d_q(H_n) + \sigma_q(H_n)H_1$, one shows that $\|H_n\|_{X'} \leq \max(\|H_1\|_{X'}, r_{X'}^{-1})^n$, hence
(5.15.2)

$$R_{X'} := \liminf_n (\|H_n\|_{X'}/[n]^!_q)^{-1/n} = \min_{x\in X'(K^{\mathrm{alg}})} R_x \ \geq\ \frac{\liminf_n([n]^!_q)^{1/n}}{\max(r_{X'}^{-1}, \|H_1\|_{X'})}\ .$$

In particular if $X' = \mathrm{D}^+(c,\rho) \subseteq X$, with $|q-1||c| < \rho \leq \rho_{c,X}$, is a $q-$invariant disk, then

$$R_c \ \geq\ R_{\mathrm{D}^+(c,\rho)} = \min_{x\in\ \mathrm{D}^+_{K^{\mathrm{alg}}}(c,\rho)} R_x \ \geq\ \frac{\liminf_n([n]^!_q)^{1/n}}{\max(\rho^{-1}, |H_1|_{(c,\rho)})}\ . \tag{5.15.3}$$

REMARK 5.16. If $|q-1||c| < R_c$, then $Y_A(x,c)$ belongs to $M_n(\mathcal{A}_\Omega(c,R_c))$, but it is invertible only in $GL_n(\mathcal{A}_\Omega(c,\widetilde{R}_c))$, with $\widetilde{R}_c := \min(\ \rho_{c,A}\ ,\ R_c\ )$ (cf. Lemmas 5.19 and 5.20 below).

REMARK 5.17. The formal matrix solution $Y_A(x,y)$ is not always a function in a neighborhood of type $\mathcal{U}_R$ of the diagonal of $X \times X$.

If for all $c \in X(K^{\mathrm{alg}})$ one has $|q-1||c| < R \leq \min(\rho_{c,A}, R_c)$, then, by Lemma 5.5, and by transfer principle (cf. Equation (5.15.3)), $Y_A(x,y)$ defines actually an *invertible* function on $\mathcal{U}_R$ (cf. Lemmas 5.19 and 5.20). If $X = \mathrm{D}^+(c_0,R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i,R_i)$, and if the previous condition is satisfied, then R verifies
(5.17.1)

$$|q-1|\sup(R_0,|c_0|) = |q-1|\max_{c\in X}|c| \ <\ R\ \leq\ \min_{c\in X}\rho_{c,X} = \min(R_0,\ldots,R_n) = r_X\ .$$

In particular, since $r_X = \min(R_0,\ldots,R_n)$, this will be possible only if

$$|q-1| < 1\ , \quad \text{i.e. if } q \in \mathcal{Q}_1(X)\ . \tag{5.17.2}$$

HYPOTHESIS 5.18. From now on, without precise mention, we suppose $q \in \mathcal{Q}_1$.

LEMMA 5.19. *Let $q \in \mathcal{Q}_1(X) - \boldsymbol{\mu}(\mathcal{Q}_1(X))$. Let $f(x,y)$ be an analytic function in a neighborhood of type $\mathcal{U}_R \subset X \times X$ of the diagonal of $X \times X$. Assume that*[2]

$$|q-1|\max(|c_0|,R_0) < R \leq r_X\ . \tag{5.19.1}$$

[2]i.e. assume that $|q-1|\max(|c_0|,R_0) < R \leq \rho_{c,X}$ for all $c \in X(K^{\mathrm{alg}})$.

Suppose that $f(x,y)$ satisfies $f(x,qy)=a(y)\cdot f(x,y)$, with $a(y)\in\mathcal{H}_K(X)^\times$, then $f(x,y)$ is invertible too.

Proof : Since f is an analytic function, it is sufficient to prove that f has no zeros in $\mathcal{U}_R$. We are reduced to show that for all $c\in X(\Omega)$, the function $g_c(y):=f(c,y)$ has no zeros in $\mathrm{D}^-(c,R)$. One has $d_q(g_c(y))=h(y)\cdot g_c(y)$, with $h(y)=\frac{a(y)-1}{(q-1)y}$. Assume that $g_c(\widetilde{c})=0$, for some $\widetilde{c}\in\mathrm{D}^-(c,R)=\mathrm{D}^-(\widetilde{c},R)$, then, by Lemma 5.5, $g_c(y)=\sum_{n\geq 0}a_k(y-\widetilde{c})_{q,n}$, with $a_0=0$. Since $q\notin\boldsymbol{\mu}(\mathcal{Q})$, then $[n]_q a_n=0$ if and only if $a_n=0$. Hence, by Remark 5.6 one has $v_{q,\widetilde{c}}(d_q(g_c))=v_{q,\widetilde{c}}(g_c)-1$. On the other hand, $v_{q,\widetilde{c}}(hg_c)\geq v_{q,\widetilde{c}}(g_c)$, which is in contradiction with $d_q(g_c)=hg_c$. □

Lemma 5.20. *Let $q\in\mathcal{Q}_1(X)-\boldsymbol{\mu}(Q_1(X))$, and let*

(5.20.1)
$$\begin{array}{llllll}\sigma_q^x & : & f(x,y)\mapsto f(qx,y)\ , & \sigma_q^y & : & f(x,y)\mapsto f(x,qy)\ ,\\ d_q^x & := & \frac{\sigma_q^x-1}{(q-1)x}\ , & d_q^y & := & \frac{\sigma_q^y-1}{(q-1)y}\ .\end{array}$$

Suppose that $Y_A(x,y)$ converges on $\mathcal{U}_R$, with

(5.20.2)
$$|q-1|\max(|c_0|,R_0)<R\leq r_X\ ,$$

(cf. Remark 5.17). Then $Y_A(x,y)$ is invertible on $\mathcal{U}_R$ and verifies:

(5.20.3) $$Y_A(x,x) = \mathrm{Id}\ ,$$
(5.20.4) $$d_q^y\, Y_A(x,y) = -\sigma_q^y(Y_A(x,y))\cdot H_1(y)\ ,$$
(5.20.5) $$\sigma_q^y\, Y_A(x,y) = Y_A(x,y)\cdot A(q,y)^{-1}\ ,$$
(5.20.6) $$Y_A(x,y)\cdot Y_A(y,z) = Y_A(x,z)\ ,$$
(5.20.7) $$Y_A(x,y)^{-1} = Y_A(y,x)\ ,$$
(5.20.8) $$d_q^x\, Y_A(x,y) = H_1(x)\cdot Y_A(x,y)\ .$$
(5.20.9) $$\sigma_q^x\, Y_A(x,y) = A(q,x)\cdot Y_A(x,y)\ .$$

Proof : The relation (5.20.3) is evident, while (5.20.4) is easy to compute explicitly, and is equivalent to (5.20.5). Since $Y(x,y)$ converges on $\mathcal{U}_R$, hence by (5.20.5), the determinant $d(x,y)$ of $Y(x,y)$ verifies $d(x,qy)=a(y)d(x,y)$, with $a(y)=\det(A(q,y)^{-1})\in\mathcal{H}_K(X)^\times$. By Lemma 5.19, $d(x,y)$ is then invertible on $\mathcal{U}_R$, and hence also $Y(x,y)$ is invertible. By point 3 of Remark 5.4, and since $q\notin\boldsymbol{\mu}(\mathcal{Q})$, then the relation $d_q^y(Y(x,y)Y(x,y)^{-1})=0$ gives $d_q^y(Y(x,y)^{-1})=-\sigma_q^y(Y(x,y)^{-1})\cdot d_q^y(Y(x,y))\cdot Y(x,y)^{-1}$. Hence, for all x,y,z such that $|x-y|,|z-y|<R$, one finds $d_q^y(Y(x,y)\cdot Y(z,y)^{-1})=0$. Since $q\notin\boldsymbol{\mu}(\mathcal{Q})$, this implies, by Lemma 5.5, that the function $Y(x,y)Y(z,y)^{-1}$ does not depend on y. For $y=x$, and $y=z$, one find $Y(x,z)=Y(z,x)^{-1}$, and $Y(x,y)\cdot Y(y,z)=Y(x,z)$. Then, by the above expression of $d_q^x(Y(y,x)^{-1})=d_q^x(Y(x,y))$, the relations (5.20.8) and (5.20.9) follow from (5.20.7) and (5.20.4). □

Remark 5.21. If for a $c\in X$ one has $|q-1||c|\geq R_c$, then Lemma 5.20 does not apply, but it may happen (cf. Remark 5.2) that there exists a (smallest) $k_0\geq 0$ such that condition (5.20.2) holds for q^{k_0} instead of q, and for $Y_{A(q^{k_0},T)}(x,y)$ instead of $Y_{A(q,T)}(x,y)$. There exists then a Taylor solution $Y_c\in M_n(\mathcal{A}_\Omega(c,R))$ of the iterated system. In this case, for all $c\in X(\Omega)$, we can recover a solution $\widetilde{Y}$ of the system $\sigma_q(\widetilde{Y})=A(q,T)\widetilde{Y}$ itself in the algebra of analytic functions over the disjoint union of disks $\bigcup_{i=0}^{k_0-1}\mathrm{D}^-(q^ic,R'_c)$. Indeed σ_q acts on the algebra $\prod_{i\in\mathbb{Z}/k_0\mathbb{Z}}M_n(\mathcal{A}_K(q^ic,R'_c))$ by $\sigma_q((M_{q^ic}(T))_{i\in\mathbb{Z}/k_0\mathbb{Z}})=(M_{q^{i+1}c}(qT))_{i\in\mathbb{Z}/k_0\mathbb{Z}}$, then one has

$$\widetilde{Y}(T)=(\ \widetilde{Y}_{q^ic}(T)\ )_i:=(\ A(q^i,q^{-i}T)\cdot Y_c(q^{-i}T)\ )_{i\in\mathbb{Z}/q^{k_0}\mathbb{Z}}\ , \tag{5.21.1}$$

indeed $A(q^{i+1},q^{-i}T)=A(q,T)A(q^i,q^{-i}T)$.

Remark 5.22. Note that the relations of Lemma 5.20 hold for $Y_A(x,y)$ as a function on $\mathcal{U}_R$, and not for $\widetilde{Y}(T)$ (cf. (5.21.1)). In other words the expression $\widetilde{Y}_A(x,y)$ has no meaning if $|x-y|\geq R$. In particular the expression (5.20.6), which is the main tool of the Propagation Theorem 6.3, holds only if $|x-y|,|z-y|<R$.

Remark 5.23. If $q\in\boldsymbol{\mu}(\mathcal{Q})$, then even if a solution $Y\in GL_n(\mathcal{A}_\Omega(c,R))$ may exist, the radius is not defined since we may have another solution with different radius (cf. Example 5.24 below). For this reason, the radius of convergence of the system (5.14.1) will be not defined if $q\in\boldsymbol{\mu}(\mathcal{Q})$.

Example 5.24. Let $q=\xi$ be a $p-$th root of unity, with $\xi\neq 1$. The solutions of the unit object at $t^p\in\Omega$ are the functions $y\in\mathcal{A}_\Omega(t^p,R)$ such that $y(\xi T)=y(T)$, every function in T^p has this property. For example the family of functions $\{\ y_\alpha:=\exp(\alpha(T^p-t^p))\ \}_{\alpha\in\Omega}$, is such that for different values of α one has different radius.

5.4. Taylor solutions of $(\sigma_q,\delta_q)-$modules. In this subsection q can be a root of unity. We preserve the previous notations. We consider now a system:

$$\begin{cases}\sigma_q(Y) &=& A(q,T)\cdot Y\ , & A(q,T)\in GL_n(\mathcal{H}_K(X))\ ,\\ \delta_q(Y) &=& G(q,T)\cdot Y\ , & G(q,T)\in\ M_n(\mathcal{H}_K(X))\ .\end{cases} \tag{5.24.1}$$

It can happen that a solution of σ_q^{M} is not a solution of δ_q^{M} as showed in the following example:

Example 5.25. Suppose that $q\in\mathrm{D}^-(1,1)$ is not a root of unity. Let $X:=\mathrm{D}^+(0,|p|^{\frac{1}{p-1}})$, $A(q,T):=\exp((q-1)T)\in\mathcal{H}_K(X)^\times$, $G(q,T):=0$. Let $c=0$, and $R<|p|^{\frac{1}{p-1}}$. Then every solution $y(T)\in\mathcal{A}_K(0,R)$ of the operator $\sigma_q-A(q,T)$ is of the form $y(T)=\lambda\cdot\exp(T)$, with $\lambda\in K$. If $\delta_q(y)=0$, then $y=0$. Hence, the $(\sigma_q,\delta_q)-$module defined by $A(q,T)$ and $G(q,T)$ has no (non trivial) solutions in $\mathcal{A}_K(0,R)$.

Remark 5.26. To guarantee the existence of solutions we need a *compatibility condition* between σ_q and δ_q, which should be written explicitly in terms of matrices of σ_q^n and δ_1^n. This obstruction will not appear in the sequel of the paper since this condition is automatically satisfied by analytic σ-modules (cf. Lemma 5.27).

Lemma 5.27. *Let $U\subseteq\mathcal{Q}(\mathcal{H}_K(X))$ be an open subset, and let* M *be an analytic $(\sigma,\delta)-$module on U, representing the family of equations $\{\sigma_q(Y)=A(q,T)\cdot Y\}_{q\in U}$, with $A(q,T)\in GL_n(\mathcal{H}_K(X))$, for all $q\in U$. Let $Y_c(T)\in GL_n(\mathcal{A}_\Omega(c,R))$, $|q-1||c|<R\leq\rho_{c,X}$, be a* simultaneous *solution of every equation of this family. Then $Y_c(T)$ is also solution of the equation*

$$\delta_q(Y)=G(q,T)\cdot Y\ , \tag{5.27.1}$$

where $G(q,T):=q\frac{d}{dq}(A(q,T))$ (cf. (2.20.1)). Hence $Y_c(T)$ is solution of the differential equation defined in section 2.4.1:

$$\delta_1(Y_c(T))=G(1,T)\cdot Y_c(T)\ , \tag{5.27.2}$$

where $G(1,T) = G(q,q^{-1}T)\cdot A(q,q^{-1}T)^{-1} \in M_n(\mathcal{H}_K(X))$ *(cf.* (2.10.4)*).*

Proof : In terms of modules, the columns of the matrix $Y_c(T)$ correspond to $\mathcal{H}_K(X)-$linear maps $\alpha : \mathrm{M} \to \mathcal{A}_\Omega(c,R)$, verifying $\sigma_q\circ\alpha = \alpha\circ\sigma_q^{\mathrm{M}}$, for all $q\in U$ (cf. Section 3.2.1). We must show that such an α commutes also with δ_q. This follows immediately by the continuity of α. Indeed, the inclusion $\mathcal{H}_K(X)\to\mathcal{A}_\Omega(c,R)$ is continuous, and hence every $\mathcal{H}_K(X)-$linear map $\mathcal{H}_K(X)^n\to\mathcal{A}_\Omega(c,R)$ is continuous. $\square$

Remark 5.28. Observe that Lemma 5.27 is not a formal consequence of the previous theory. Indeed, by Definition 3.5, the general $(\sigma,\delta)-$algebra C used in Definition 3.5 has the discrete topology, hence the morphism $\alpha : \mathrm{M}\to\mathrm{C}$ defining the solution is not continuous in general.

5.5. Twisted Taylor formula for $(\sigma,\delta)-$modules, and rough estimate of radius.

Remark 5.29. Let X be a q-invariant affinoïd. Let $D_q := \sigma_q\circ\frac{d}{dT} = \lim_{q'\to q}\frac{\sigma_{q'}-\sigma_q}{T(q'-q)} = \frac{1}{qT}\cdot\delta_q$. For all $q\in\mathcal{Q}(X)$ and all $f(T)\in\mathcal{H}_K(X)$, one has

$$\mathrm{D}_q(f\cdot g) = \sigma_q(f)\cdot\mathrm{D}_q(g) + \mathrm{D}_q(f)\cdot\sigma_q(g)\ , \tag{5.29.1}$$

$$(d/dT\circ\sigma_q) = q\cdot(\sigma_q\circ d/dT)\ , \tag{5.29.2}$$

$$\mathrm{D}_q^n = q^{n(n-1)/2}\cdot\sigma_q^n\circ(d/dT)^n\ , \tag{5.29.3}$$

$$\|\mathrm{D}_q^n(f(T))\|_X \le \frac{|n!|}{r_X^n}\cdot\|f(T)\|_X \qquad \text{(cf. Lemma 1.2)}\ . \tag{5.29.4}$$

Hence, for all $c\in K$, $\mathrm{D}_q^n(T-c)^i = \frac{i!}{(i-n)!}\cdot q^{n(n-1)/2}\cdot(q^nT-c)^{i-n}$ if $n\le i$, and $\mathrm{D}_q^n(T-c)^i = 0$ if $n>i$. This shows that if $f(T) := \sum_{i\ge0} a_i\cdot\frac{(T-c)^i}{(i!)\cdot q^{i(i-1)/2}}\in\mathcal{A}_\Omega(c,R)$ is a formal series, with $|q-1||c|<R\le\rho_{c,X}$, then $a_n = \mathrm{D}_q^n(f)(c/q^n)$, and the usual Taylor formula can be written as

$$f(T) = \sum_{n\ge0}\mathrm{D}_q^n(f)(c/q^n)\cdot\frac{(T-c)^n}{(n!)\cdot q^{n(n-1)/2}}\quad . \tag{5.29.5}$$

The following proposition gives the analogous of the same classical rough estimate for differential and $q-$difference equations (cf. [**Chr83**, 4.1.2], [**DV04**, 4.3]).

Proposition 5.30. *Let* $c\in X(\Omega)$*. Assume that the system* (5.24.1) *has a Taylor solution* $Y_c\in M_n(\mathcal{A}_\Omega(c,R_c))$*, with* $|q-1||c|<R_c\le\rho_{c,X}$*. For all q-invariant sub-affinoïd* $X'\subseteq X$*, containing* $\mathrm{D}^+(c,|q-1||c|)$*, one has*

$$R_c \ge \frac{|p|^{\frac{1}{p-1}}}{\max(\ r_{X'}^{-1}\|A(q,T)\|_{X'}\ ,\ \|G(q,T)/qT\|_{X'}\ )}\ . \tag{5.30.1}$$

In particular if X' is a disk $\mathrm{D}^+(c,\rho)$*, with* $|q-1||c|\le\rho\le\rho_{c,X}$*, then*

$$R_c \ge \frac{|p|^{\frac{1}{p-1}}\cdot\rho}{\max(\ |A(q,T)|_{(c,\rho)}\ ,\ \frac{|G(q,T)|_{(c,\rho)}}{\max(1,|c|/\rho)}\ )}\ . \tag{5.30.2}$$

Proof : The matrix $Y_c(T)$ verifies $\sigma_q^n(Y_c(T)) = A_{[n]}(q,T)\cdot Y_c(T)$, and $D_q^n(Y_c(T)) = F_{[n]}(q,T)\cdot Y_c(T)$, where $F_{[0]} = \mathrm{Id} = A_{[0]}$, $A_{[1]} := A(q,T)$, $F_{[1]} := \frac{1}{qT}G(q,T)$, and

$$A_{[n]} := \sigma_q^{n-1}(A_{[1]})\cdots\sigma_q(A_{[1]})\cdot A_{[1]}\ , \tag{5.30.3}$$

$$F_{[n+1]} := \sigma_q(F_{[n]})\cdot F_{[1]} + D_q(F_{[n]})\cdot A_{[1]}\ . \tag{5.30.4}$$

Hence one has

$$Y_c(T) := \sum_{i\geq 0} F_{[n]}(c/q^n)\frac{(T-c)^n}{(n!)\cdot q^{n(n-1)/2}} \quad , \tag{5.30.5}$$

which is an hybrid between the usual Taylor formula, and the Taylor formula for $q-$difference equations. Inequalities 5.30.1 follows then from the inequality

$$|F_{[n]}(c/q^n)|_\Omega \leq \|F_{[n]}\|_{X'} \leq \max\left(\|F_{[1]}\|_{X'}\ ,\ \frac{1}{r_{X'}}\cdot\|A_{[1]}\|_{X'}\right)^n \tag{5.30.6}$$

If $X' = \mathrm{D}^+(c,\rho)$, then this last is equal to $\frac{1}{\rho^n}\cdot\max\left(\frac{|G(q,T)|_{(c,\rho)}}{\max(1,|c|/\rho)}\ ,\ |A(q,T)|_{(c,\rho)}\right)^n$. Indeed $r_{\mathrm{D}^+(c,\rho)} = \rho$, $F_{[1]} = \frac{1}{qT}G(q,T)$, and $|T|_{(c,\rho)} = |(T-c)+c|_{(c,\rho)} = \max(\rho,|c|)$, hence $|F_{[1]}|_{(c,\rho)} = \frac{1}{|q|\max(|c|,\rho)}\cdot|G(q,T)|_{(c,\rho)}$, and $|q|=1$. □

5.6. Generic radius of convergence and solvability.

5.6.1. *Generic points.* In this section we will introduce the notion of *generic points* and of *generic radius of convergence*. We recall that a generic point defines a semi-norm on $\mathcal{H}_K(X)$, and hence defines a Berkovich point (cf. [**Ber90**]). The reader knowing the language of Berkovich will not find difficulties to translate the contents of this paper in the language of Berkovich.

Let now $(\Omega,|.|)/(K,|.|)$ be a complete field extension such that $|\Omega| = \mathbb{R}_{\geq}$, and that k_Ω/k is not algebraic.

Proposition 5.31 ([**CR94**, 9.1.2]). *For all $c\in X(K)$ and all $0<\rho\leq\rho_{c,X}$, there exists a point $t_{c,\rho}\in X(\Omega)$, called* generic point of the disk $\mathrm{D}^+(c,\rho)$ *such that $|\,t_{c,\rho}-c\,|_\Omega = \rho$, and that $\mathrm{D}^-_\Omega(\ t_{c,\rho}\ ,\ \rho\ )\cap K^{\mathrm{alg}} = \emptyset$.* □

Remark 5.32. For all $f(T)\in\mathcal{H}_K(X)$, one has $|f(t_{c,\rho})|_\Omega = |f(T)|_{(c,\rho)} = \sup_{|x-c|\leq\rho\ ,\ x\in K^{\mathrm{alg}}}|f(x)|$. Hence, even if the point $t_{c,\rho}$ is not uniquely determined by the fact that $\mathrm{D}^-_\Omega(t_{c,\rho},\rho)\cap K^{\mathrm{alg}} = \emptyset$, the norm $|\cdot|_{(c,\rho)}$ (i.e. the Berkovich point $|\cdot|_{(c,\rho)}$) does not depend on the choice of $t_{c,\rho}$.

5.6.2. *Generic radius of convergence.*

Definition 5.33 (Generic radius of convergence). Let $q\in\mathcal{Q}(X)$ (resp. $q\in\mathcal{Q}(X)-\boldsymbol{\mu}(\mathcal{Q})$), let $c\in X(K^{\mathrm{alg}})$, and let $\mathrm{D}^+(c,\rho)$, $|q-1||c|<\rho\leq\rho_{c,X}$, be a q-invariant disk. Let M be the $(\sigma_q,\delta_q)-$module (resp. σ_q-module) defined by the system (5.24.1) (resp. (5.14.1)). Let $R_{t_{c,\rho}}$ be the radius of convergence[3] of $Y_{A(q,T)}(T,t_{c,\rho})$. Assume that

$$|q-1||t_{c,\rho}| < R_{t_{c,\rho}} \leq \rho_{c,X}\ .^{4} \tag{5.33.1}$$

Then we will call $(c,\rho)-$*generic radius of convergence of* M the real number

$$Ray_c(\mathrm{M},\rho) := \min\big(\ R_{t_{c,\rho}}\ ,\ \rho_{c,X}\ \big) > |q-1||c|\ . \tag{5.33.2}$$

[3]In the case of the q-difference equation (5.14.1), the radius $R_{t_{c,\rho}}$ is given by definition (5.15.1). In the case of the system (5.24.1) the radius $R_{t_{c,\rho}}$ is given indifferently by definition (5.12.1) or by definition (5.15.1), indeed under our assumptions these two definitions are equal since $Y_{A(q,T)}(x,y) = Y_{G(1,T)}(x,y)$. However observe that the definition (5.15.1) exists only if $q\in\mathcal{Q}-\boldsymbol{\mu}(\mathcal{Q})$, while definition (5.12.1) preserves its meaning on the root of unity.

[4]Observe that $\rho_{c,X} = \rho_{t_{c,\rho},X}$, indeed $\mathrm{D}^+(c,r) = \mathrm{D}^+(t_{c,\rho},r)$, for all $r\geq\rho$.

REMARK 5.34. The assumption (5.33.1) provides that the disk of convergence of $Y(x,y)$ is q-invariant, and that $Y(x,y)$ is invertible in that disk (cf. Lemma 5.19). Recall that $|t_{c,\rho}| = \min(|c|,\rho)$, and that, by the transfer principle, $R_{t_{c,\rho}} = R_{\mathrm{D}^+(c,\rho)} = \min_{x\in \mathrm{D}^+_{K^{\mathrm{alg}}}(c,\rho)} R_x$ (cf. Remark 5.15).

REMARK 5.35. The number $Ray_c(\mathrm{M},\rho)$ is invariant under change of basis in M, while the number $R_{t_{c,\rho}}$ depends on the chosen basis. Observe that $Ray_c(\mathrm{M},\rho)$ depends on the affinoïd X, and on the semi-norm $|.|_{(c,\rho)}$ defined by $t_{c,\rho}$, but not on the particular choice of $t_{c,\rho}$ (cf. Remark 5.32).

DEFINITION 5.36 (Solvability). Let M be a σ_q-module (resp. a $(\sigma_q,\delta_q)-$module) on $\mathcal{H}_K(X)$. We will say that M is solvable at $t_{c,\rho}$ if

$$Ray_c(\mathrm{M},\rho) = \rho_{c,X}\ .$$

5.7. Solvability over an annulus and over the Robba ring.

REMARK 5.37. In this section B $:= \mathcal{A}_K(I)$, with $I =]r_1,r_2[$, and M is a σ_q-module (resp. a $(\sigma_q,\delta_q)-$module) on $\mathcal{A}_K(I)$. We recall that $\mathcal{Q}(\mathcal{A}_K(I)) = \{q\in K \mid |q|=1\}$, and that $\mathcal{Q}_1(\mathcal{A}_K(I)) = \mathrm{D}^-_K(1,1)$ (cf. Def. (2.1.2)). For all $c\in K$, $|c|\in I$, one has $t_{c,|c|} = t_{0,|c|}$. For all affinoïd $X\subseteq \mathcal{C}(I)$ containing the disk $\mathrm{D}^-(c,|c|)$ one has $\rho_{c,X} = |c|$. Then the norm $|.|_{c,|c|} : \mathcal{A}_K(I)\longrightarrow \mathbb{R}_{\geq}$ and the generic radius $Ray_c(\mathrm{M},|c|)$, do not depend on the chosen c nor on X, but only on $|c|$. Hence, for all $\rho\in I$, we chose an arbitrary $c\in\Omega$, with $|c| = \rho\in I$, and we set

$$t_\rho := t_{c,\rho}\ ,\quad \text{and}\quad Ray(\mathrm{M},\rho) := Ray_c(\mathrm{M},\rho)\ . \tag{5.37.1}$$

REMARK 5.38. To define the Radius we need the assumption $|q-1||t_\rho| < \rho_{t_\rho,X} = \rho$ (cf. Definition 5.33). Since $|t_\rho| = \rho$, this assumption is equivalent to

$$|q-1| < 1\ . \tag{5.38.1}$$

DEFINITION 5.39 (solvability at ρ). Let $q\in\mathcal{Q}_1 - \boldsymbol{\mu}(\mathcal{Q}_1)$ (cf. Definition (2.1.2)). Let M be a σ_q-module on $\mathcal{A}_K(I)$. We will say that M is solvable at $\rho\in I$ if

$$Ray(\mathrm{M},\rho) = \rho\ . \tag{5.39.1}$$

The full subcategory of $\sigma_q - \mathrm{Mod}(\mathcal{A}_K(I))$, whose objects are solvable at ρ, will be denoted by $\sigma_q - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{sol}(\rho)}$.

5.7.1. *Solvability over $\mathcal{R}_K$ or $\mathcal{H}^\dagger_K$.*

REMARK 5.40. Let $q\in\mathcal{Q}_1 - \boldsymbol{\mu}(\mathcal{Q}_1)$. Let M be a σ_q-module over $\mathcal{R}_K$. By definition M comes, by scalar extension, from a module $\mathrm{M}_{\varepsilon_1}$ defined on an annulus $\mathcal{C}(]1-\varepsilon_1,1[)$. If $\varepsilon_2>0$, and if $\mathrm{M}_{\varepsilon_2}$ is another module on $\mathcal{C}(]1-\varepsilon_2,1[)$ satisfying $\mathrm{M}_{\varepsilon_2}\otimes_{\mathcal{A}_K(]1-\varepsilon_2,1[)}\mathcal{R}_K \xrightarrow{\sim} \mathrm{M}$, then there exists a $\varepsilon_3\leq\min(\varepsilon_1,\varepsilon_2)$ such that

$$\mathrm{M}_{\varepsilon_1}\otimes\mathcal{A}_K(]1-\varepsilon_3,1[) \xrightarrow{\ \sim\ } \mathrm{M}_{\varepsilon_2}\otimes\mathcal{A}_K(]1-\varepsilon_3,1[)\ . \tag{5.40.1}$$

Hence the limit $\lim_{\rho\to 1^-} Ray(\mathrm{M}_\varepsilon,\rho)$ is independent from the chosen module M_ε.

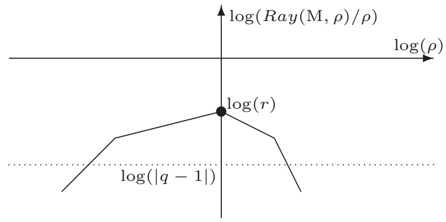
DEFINITION 5.41. Let $q\in\mathcal{Q}_1 - \boldsymbol{\mu}(\mathcal{Q}_1)$, and let $|q-1| < r\leq 1$. We define

$$\sigma_q - \mathrm{Mod}(\mathcal{H}^\dagger_K)^{[r]}\ , \tag{5.41.1}$$

as the full sub category of $\sigma_q - \mathrm{Mod}(\mathcal{H}^\dagger_K)$ whose objects verify

$$Ray(\mathrm{M},1)\geq r\ ,\qquad (r > |q-1|)\ , \tag{5.41.2}$$

as illustrated below in the log-graphic of the function $\log(\rho) \mapsto \log(Ray(\mathrm{M},\rho)/\rho)$ (cf. Section 1.2.1):



Objects in $\sigma_q - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{[1]}$ will be called *solvable*.

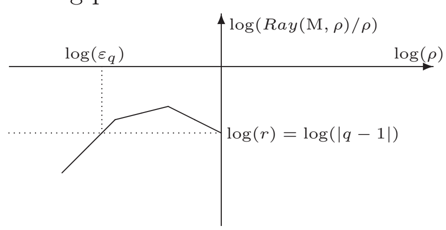
Definition 5.42. Let $q \in \mathcal{Q}_1 - \boldsymbol{\mu}(\mathcal{Q}_1)$, and let $|q-1| \leq r \leq 1$. We define

$$\sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{[r]} , \tag{5.42.1}$$

as the full sub category of $\sigma_q - \mathrm{Mod}(\mathcal{R}_K)$ whose objects verify

$$\lim_{\rho\to 1^-} Ray(\mathrm{M},\rho) \geq r , \quad (r \geq |q-1|) . \tag{5.42.2}$$

Moreover, in the particular case in which $\lim_{\rho\to1^-} Ray(\mathrm{M},\rho) = r = |q-1| < 1$, we ask that there exists $\varepsilon_q > 0$, such that $Ray(\mathrm{M}_{\varepsilon_q},\rho) > |q-1|\rho$, for all $1-\varepsilon_q < \rho < 1$, as illustrated in the following picture:



Objects in $\sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{[1]}$ will be called *solvable*.

Remark 5.43 (Analogous definitions for (σ_q,δ_q)-modules). In the case of (σ_q,δ_q)-modules, the generic radius of convergence is defined even if q is a root of unity. We give then analogous definitions for (σ_q,δ_q)-modules without restrictions on q: for all arbitrary $q \in \mathcal{Q}_1$, one defines $(\sigma_q,\delta_q) - \mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{sol}(\rho)}$, $(\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B})^{[r]}$, $(\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B})$, and solvable objects $(\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B})^{[1]}$.

5.8. Generic radius for discrete and analytic objects over $\mathcal{R}_K$ and $\mathcal{H}_K^\dagger$. In this section $\mathrm{B} = \mathcal{R}_K$ or $\mathrm{B} = \mathcal{H}_K^\dagger$.

Definition 5.44. For all $\varepsilon > 0$ let

$$I_\varepsilon := \begin{cases}]\,1-\varepsilon\ ,\ 1\,[\ , & \text{if} \quad \mathrm{B} = \mathcal{R}_K \\ \\ ]\,1-\varepsilon\ ,\ 1+\varepsilon\,[\ , & \text{if} \quad \mathrm{B} = \mathcal{H}_K^\dagger \end{cases} . \tag{5.44.1}$$

Definition 5.45. For all subset $S \subseteq \mathrm{D}^-(1,1) = \mathcal{Q}_1$, for all $0 < \tau < 1$, we set

$$S_\tau := S \cap \mathrm{D}^-(1,\tau) . \tag{5.45.1}$$

Definition 5.46. Let $0 < r \leq 1$. Let $S \subseteq \mathrm{D}^-(1,1)$, $S^\circ \neq \emptyset$. We denote by

$$\sigma - \mathrm{Mod}(\mathrm{B})_S^{[r]} , \tag{5.46.1}$$

the full subcategory of $\sigma - \mathrm{Mod}(\mathrm{B})_S$ whose objects M have the following properties:

(1) The restriction of M to every $q \in S$ belongs to $\sigma_q - \mathrm{Mod}(\mathrm{B})^{[r]}$;

(2) For all τ such that $0 < \tau < r$, there exists $\varepsilon_\tau > 0$ such that the restriction $\mathrm{Res}^S_{S_\tau}(\mathrm{M})$ comes, by scalar extension, from an object

$$\mathrm{M}_{\varepsilon_\tau} \in \quad \sigma - \mathrm{Mod}(\mathcal{A}_K(I_{\varepsilon_\tau}))^{\mathrm{disc}}_{S_\tau} \tag{5.46.2}$$

such that, for all $\rho \in I_{\varepsilon_\tau}$, and for all $q, q' \in S_\tau$ one has (cf. (5.14.2))

$$Y_{A(q,T)}(T, t_\rho) = Y_{A(q',T)}(T, t_\rho) \ . \tag{5.46.3}$$

Objects in $\sigma_q - \mathrm{Mod}(\mathrm{B})^{[1]}_S$ will be called *solvable*.

REMARK 5.47. Condition (1) implies implicitly $S \subseteq \mathrm{D}^-(1, r)$ if $\mathrm{B} = \mathcal{H}^\dagger_K$ (cf. Def. 5.41), and $S \subseteq \mathrm{D}^+(1, r)$ if $\mathrm{B} = \mathcal{R}_K$ (cf Def. 5.42).

REMARK 5.48 (Analogous definitions for (σ_q, δ_q)-modules). One defines analogously $(\sigma, \delta) - \mathrm{Mod}(\mathrm{B})^{[r]}_S$, *but without restrictions on* $S \subseteq \mathrm{D}^-(1, r)$, as the subcategory of $(\sigma, \delta) - \mathrm{Mod}(\mathrm{B})_S$, whose objects verify conditions (1) and (2), in which equation (5.46.3) is replaced by (cf. Definitions (5.11.1) and (5.14.2))

$$Y_{G(1,T)}(T, t_\rho) = Y_{A(q,T)}(T, t_\rho) \ , \tag{5.48.1}$$

for all $\rho \in I_{\varepsilon_\tau}$, and all $q \in S_\tau$.

EXAMPLE 5.49. This example justifies the condition (1) given in the above definition. Let $r := \omega := |p|^{\frac{1}{p-1}}$, and let $S = \mathrm{D}^-(1, \omega)$. Let M be the discrete σ-module over the Robba ring defined by the family of equations $\{\ \sigma_q - A(q, T)\ \}_{q \in S}$, where $A(q, T) := \exp((q^{-1} - 1)T^{-1})$. Then $Y(x, y) := \exp(x^{-1} - y^{-1})$ is the simultaneous solution of every equation of this family. Observe that $A(q, T) \in \mathcal{R}_K$ if and only if $|q^{-1} - 1| < \omega$, but if $|q - 1|$ tends to ω^-, then the matrices $A(q, T)$ do not belong all to the same annulus. Indeed $A(q, T) \in \mathcal{A}_K(I_\varepsilon)$ if and only if $|q^{-1} - 1| < \omega(1 - \varepsilon)$.

6. The Propagation Theorem

6.1. Taylor admissible modules.

DEFINITION 6.1 (Taylor admissible discrete modules on S). Let $X := \mathrm{D}^+(c_0, R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i, R_i)$ be an affinoïd, and let $S \subseteq \mathcal{Q}_1(X)$, be a subset with $S^\circ \neq \emptyset$ (cf. (2.3.1)). Let $(\mathrm{M}, \sigma^{\mathrm{M}})$ be a discrete σ-module defined by the family of equations

$$\{\sigma_q - A(q, T)\}_{q \in S} \ , \quad A(q, T) \in GL_n(\mathcal{H}_K(X)) \ , \ \forall\, q \in S \ . \tag{6.1.1}$$

We will say that $(\mathrm{M}, \sigma^{\mathrm{M}})$ is *Taylor admissible on* X if :

(1) there exists a matrix $Y(x, y)$, convergent in $\mathcal{U}_R$ (cf. (5.12.2)), with R satisfying, for all $q \in S$, the condition (5.17.1), that is

$$\sup(R_0, |c_0|) \cdot \sup_{q \in S} |q - 1| \ < \ R \ \leq \ r_X \ ; \tag{6.1.2}$$

(2) $Y(x, y)$ is simultaneous solution of every equation of the family (6.1.1).

The full subcategory of $\sigma - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{disc}}_S$ whose objects are Taylor admissible, will be denoted by

$$\sigma - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}_S \ . \tag{6.1.3}$$

We define analogously the category $(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K(X))_S^{\mathrm{adm}}$ of *admissible discrete $(\sigma,\delta)-$modules on S*. Namely the condition $S^\circ \neq \emptyset$ is suppressed, and if $(\mathrm{M},\sigma^{\mathrm{M}},\delta_1^{\mathrm{M}})$ is a discrete (σ,δ)-modules on S defined by a system of equations (cf. (3.7.2)), then the Taylor solution $Y_{G(1,T)}(x,y)$ (cf. (5.11.1)) of the differential equation defined by δ_1^{M} satisfies (6.1.2), and moreover is simultaneously solution of every equation defined by σ_q^{M}, for all $q\in S$.

6.2. Propagation Theorem.

Theorem 6.2 (Propagation Theorem first form). *Let X be an* affinoïd*. Then, if $q\in\mathcal{Q}_1(X)-\boldsymbol{\mu}(\mathcal{Q}_1(X))$, the natural restriction functor*

$$\bigcup_U \sigma-\mathrm{Mod}(\mathcal{H}_K(X))_U^{\mathrm{adm}} \xrightarrow{\bigcup_U \mathrm{Res}_q^U} \sigma_q-\mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}} \tag{6.2.1}$$

is an equivalence, where U runs in the set of all open neighborhood of q. The analogous fact is true for (σ,δ)-modules without supposing $q\notin\boldsymbol{\mu}(\mathcal{Q})$.

Proof : By Lemma 4.4, $\cup_U\mathrm{Res}_{\{q\}}^U$ is fully faithful. Indeed for all M,N modules over U, by admissibility, there exists a number R, with $|q-1|\max(|c_0|,R_0)<R\leq r_X$, such that, for all $c\in X(K)$, the algebra $\mathrm{C}:=\mathcal{A}_K(c,R)$ trivializes both M and N. The essential surjectivity of $\cup_U\mathrm{Res}_{\{q\}}^U$ will follow from theorem 6.3 below. □

Theorem 6.3 (Propagation Theorem second form). *Let $X=\mathrm{D}^+(c_0,R_0)-\cup_{i=1}^n\mathrm{D}^-(c_i,R_i)$. Let $q\in\mathcal{Q}_1(X)-\boldsymbol{\mu}(\mathcal{Q}_1(X))$. Let*

$$Y(q\cdot T)=A(T)\cdot Y(T)\ ,\quad A(T)\in GL_n(\mathcal{H}_K(X)) \tag{6.3.1}$$

be a Taylor admissible *$q-$difference equation (cf. Def. 6.1). Then there exists a matrix $A(Q,T)$ uniquely determined by the following properties:*

1. *$A(Q,T)$ is analytic and invertible in the domain*

$$\mathrm{D}^-\Big(\ 1\ ,\ \frac{R}{\max(|c_0|,R_0)}\ \Big)\times X\ \ \subset\ \mathbb{A}_K^2\ , \tag{6.3.2}$$

2. *The matrix $A(Q,T)$ specialized at (q,T) is equal to $A(T)$,*
3. *For all $q'\in\mathrm{D}^-(1,R/\max(|c_0|,R_0))$, the Taylor solution matrix $Y_A(x,y)$ of the equation (6.3.1) (cf. (5.14.2)) verifies simultaneously*

$$Y_A(q'\cdot T,y)\ =\ A(q',T)\cdot Y_A(T,y)\ \ . \tag{6.3.3}$$

Moreover the matrix $A(Q,T)$ is independent from the chosen solution $Y_A(x,y)$.

Proof : By equation (6.3.3), the matrix $A(Q,T)$ must be equal to

$$A(Q,T)=Y_A(Q\cdot T,y)\cdot Y_A(T,y)^{-1}=Y_A(Q\cdot T,y)\cdot Y_A(y,T)=Y_A(Q\cdot T,T)\ . \tag{6.3.4}$$

This makes sense since $Y_{A(q,T)}(x,y)$ is invertible in its domain of convergence (cf. Lemma 5.20). Hence $A(Q,T)$ converges in the domain of convergence of $Y_A(QT,T)$ and is invertible in that domain, since $Y_A(x,y)$ is. By admissibility, there exists $|q-1|\max(|c_0|,R_0)<R\leq r_X$ such that $Y_A(x,y)$ converges for all $(x,y)\in\mathcal{U}_R$, i.e. $|x-y|<R$ (cf. (5.12.2)). Then $Y_A(QT,T)$ converges for $|Q-1||T|<R$. Since $|T|\leq\sup_{c\in A}|c|=\max(|c_0|,R_0)$, then $Y(QT,T)$ converges for $|Q-1|<R/\max(|c_0|,R_0)$.□

COROLLARY 6.4. *If $\tau_q := |q-1|\max(|c_0|, R_0)$, then one has the diagram*
(6.4.1)

$$\begin{array}{ccc}
\bigcup_{R>\tau_q} \sigma - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}_{\mathrm{D}^-(1,R)} & \overset{2.17.1}{=\!=\!=} & \bigcup_{R>\tau_q} (\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}_{\mathrm{D}^-(1,R)} \\
\Big\downarrow {\scriptstyle \bigcup_{R>\tau_q} \mathrm{Res}^{\mathrm{D}^-(1,R)}_{\{q\}}} & \odot & \wr\Big\downarrow {\scriptstyle \bigcup_{R>\tau_q} \mathrm{Res}^{\mathrm{D}^-(1,R)}_{\{q\}}} \\
\sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}} & \xleftarrow[\text{Forget } \delta_q]{} & (\sigma_q,\delta_q) - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}
\end{array}$$

in which the right hand vertical functor $\bigcup_{R>\tau_q} \mathrm{Res}^{\mathrm{D}^-(1,R)}_{\{q\}}$ is always an equivalence, for every arbitrary value of $q \in \mathcal{Q}_1(A)$. Moreover if $q \in \mathcal{Q}_1(X) - \boldsymbol{\mu}(\mathcal{Q}_1(X))$, then the also the left hand functor $\bigcup_{R>\tau_q} \mathrm{Res}^{\mathrm{D}^-(1,R)}_{\{q\}}$ is an equivalence, and in this case every functor appearing in this diagram is an equivalence.$\Box$

REMARK 6.5. By the propagation Theorem, every object of $\sigma-\mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}_{\mathrm{D}^-(1,R)}$ and of $(\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}_{\mathrm{D}^-(1,R)}$ is analytic.

REMARK 6.6. Starting from an *admissible* σ_q-module M over B, one can *compute* the differential equation $\mathrm{Conf}_q(\mathrm{M}) \in \delta_1 - \mathrm{Mod}(\mathrm{B})^{(\phi)}$ by the relation

$$G(1,T) = \lim_{q\to 1} \frac{A(q,T)-\mathrm{Id}}{q-1} = \lim_{n\to+\infty} \frac{A(q^{p^n},T)-\mathrm{Id}}{q^{p^n}-1}\ , \tag{6.6.1}$$

where $A(q^{p^n},T) = A(q,q^{p^n-1}T)A(q,q^{p^n-2}T)\cdots A(q,T)$. The propagation theorem provides the convergence of this limit in $M_n(\mathrm{B})$; this is an highly non trivial fact.

REMARK 6.7. We can extend this result to all kind of ring of functions appearing in this paper. If $q \in \boldsymbol{\mu}(\mathcal{Q}_1)$, we will see in Proposition 7.9 that the functor "Forget δ_q" is not very interesting since it sends every $(\sigma_q,\delta_q)-$module with Frobenius structure into the trivial σ_q-module (i.e. direct sum of the unit object).

REMARK 6.8. It should be possible to generalize the main theorem to other kind of operators, different from σ_q. In other words it should be possible to "deform" differential equations into "$\sigma-$difference equations", where σ in an automorphism different from σ_q, but sufficiently close to the identity. In a work in progress we will study the action of a $p-$adic Lie group on differential equations.

6.3. Generalizing the Confluence Functor. Let $q \in \mathcal{Q}(X) - \boldsymbol{\mu}(\mathcal{Q}(X))$ be such that $q^{k_0} \in \mathcal{Q}_1(X)$, for some $k_0 \geq 1$.[5] By composing with the evident functor

$$\sigma_q - \mathrm{Mod}(\mathcal{H}_K(X)) \longrightarrow \sigma_{q^{k_0}} - \mathrm{Mod}(\mathcal{H}_K(X))\ , \tag{6.8.1}$$

one defines "k_0-*Taylor admissible objects*" of $\sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))$ as objects whose image is Taylor admissible in $\sigma_{q^{k_0}} - \mathrm{Mod}(\mathcal{H}_K(X))$. Since the sequence $\{q^{k_0p^n}\}_{n\geq 0}$ tends to 1, then, for k_0 sufficiently large, q^{k_0} satisfies the condition of section 5.2, in order that d_{k^0} verifies equality (5.10.1). We obtain then a Confluence Functor:

$$\sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))^{k_0-\mathrm{adm}} \longrightarrow \delta_1 - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}\ . \tag{6.8.2}$$

The converse of this fact (i.e. the deformation of a differential equation into a $q-$difference equation with $|q| = 1$ and $|q-1|$ large) remains an open problem.

[5]For an annulus centered at 0, the condition $q^{k_0} \in \mathcal{Q}_1(A) = \mathrm{D}^-(1,1)$ is equivalent to $\bar{q} \in \mathbb{F}_p^{\mathrm{alg}}$.

REMARK 6.9. There exist equations in $\sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))$ which are not k_0-Taylor admissible, for all $k_0 \geq 1$. For example consider the rank one equation $\sigma_q - a$, with $a \in K$, $|a| > 1$. Suppose also that $|q-1| < |p|^{\frac{1}{p-1}}$, in order that $\liminf_n |[n]_q^!|^{1/n} = |p|^{\frac{1}{p-1}}$. Then the radius is small and one can compute it explicitly by applying [**DV04**, Prop.4.6]. One has $Ray((\mathrm{M}, \sigma_q^{\mathrm{M}}), \rho) = |a|^{-1}|p|^{\frac{1}{p-1}}|q-1|\rho < |q-1|\rho$, and $Ray((\mathrm{M}, \sigma_{q^{k_0}}^{\mathrm{M}}), \rho) = |a|^{-k_0}|p|^{\frac{1}{p-1}}|q^{k_0}-1|\rho < |q^{k_0}-1|\rho$.

6.4. Propagation Theorem for other rings. The Propagation Theorem is true over every base ring B appearing in this paper, up to define correctly the notion of "Taylor admissible".

DEFINITION 6.10. We will say that a discrete σ-module over S is Taylor admissible over an annulus $\mathcal{C}(I)$ if its restriction to every sub-annulus $\mathcal{C}(J)$, with J compact, $J \subseteq I$, is Taylor admissible in the sense of Definition 6.1.

6.4.1. *Taylor admissibility over $\mathcal{R}_K$ and $\mathcal{H}_K^\dagger$.* We recall that $\mathrm{D}^-(1,1) = \mathcal{Q}_1(\mathcal{R}_K) = \mathcal{Q}_1(\mathcal{H}_K^\dagger)$. We preserve the notations of section 5.8. One defines Taylor admissibility over $\mathcal{R}_K$ and $\mathcal{H}_K^\dagger$ by reducing to the case of modules over a single annulus $\mathcal{C}(I_\varepsilon)$, for some $\varepsilon > 0$ sufficiently close to 0. One finds in this way exactly the Definition 5.46:

DEFINITION 6.11. Let $\mathrm{B} := \mathcal{H}_K^\dagger$, or $\mathrm{B} := \mathcal{R}_K$. Let $S \subseteq \mathrm{D}^-(1,1)$, with $S^\circ \neq \emptyset$. Let $\tau_S := \sup_{q \in S} |q-1|$. We set

$$\sigma - \mathrm{Mod}(\mathrm{B})_S^{\mathrm{adm}} := \sigma - \mathrm{Mod}(\mathrm{B})_S^{[\tau_S]} \ . \tag{6.11.1}$$

We give the same definition for (σ,δ)-modules, without assuming " $S^\circ \neq \emptyset$ " : $(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{\mathrm{adm}} := (\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{[\tau_S]}$.

PROPOSITION 6.12. *Let again* $\mathrm{B} := \mathcal{H}_K^\dagger$, *or* $\mathrm{B} := \mathcal{R}_K$, *let* $0 < r \leq 1$, *and let* $S \subseteq \mathrm{D}^-(1,r)$, *be a subset, with* $S^\circ \neq \emptyset$. *Let* $\mathrm{M} \in \sigma - \mathrm{Mod}(\mathrm{B})_S^{[r]}$ *(i.e. in particular* M *is admissible). Then* M *is the restriction to* S *of an* analytically constant *module over all the disk* $\mathrm{D}^-(1,r)$. *Moreover, the restriction functor is an equivalence:*

$$\sigma - \mathrm{Mod}(\mathrm{B})_{\mathrm{D}^-(1,r)}^{[r]} \xrightarrow[\sim]{\mathrm{Res}_S^{\mathrm{D}^-(1,r)}} \sigma - \mathrm{Mod}(\mathrm{B})_S^{[r]} \ . \tag{6.12.1}$$

In particular solvable modules extend to all the disk $\mathrm{D}^-(1,1)$. *The analogous assertion holds for* $(\sigma,\delta)-$*modules, without supposing* $S^\circ \neq \emptyset$*:*

$$(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_{\mathrm{D}^-(1,r)}^{[r]} \xrightarrow[\sim]{\mathrm{Res}_S^{\mathrm{D}^-(1,r)}} (\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{[r]} \ . \tag{6.12.2}$$

Proof : By Lemma 4.4, it suffices to prove the essential surjectivity of $\mathrm{Res}_S^{\mathrm{D}^-(1,r)}$. The proof is straighforward and essentially equal to the proof of the propagation Theorem 6.2.□

COROLLARY 6.13. *Let* $q, q' \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$. *Let* r *satisfies*

$$\max(|q-1|, |q'-1|) < r \leq 1 \ . \tag{6.13.1}$$

Then one has an equivalence of deformation

$$\sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \xrightarrow[\sim]{\mathrm{Def}_{q,q'}} \sigma_{q'} - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \ . \tag{6.13.2}$$

The same equivalence holds between $(\sigma_q,\delta_q)-\mathrm{Mod}(\mathcal{R}_K)^{[r]}$ *and* $(\sigma_{q'},\delta_{q'})-\mathrm{Mod}(\mathcal{R}_K)^{[r]}$, *without assuming* $q\notin\boldsymbol{\mu}_{p^\infty}$. *Moreover, if* $q\notin\boldsymbol{\mu}_{p^\infty}$, *and if* $|q-1|<r$, *then*

$$(\sigma_q,\delta_q)-\mathrm{Mod}(\mathcal{R}_K)^{[r]}\xrightarrow[\sim]{\text{“Forget }\delta_q\text{”}}\sigma_q-\mathrm{Mod}(\mathcal{R}_K)^{[r]} \tag{6.13.3}$$

is an equivalence.□

EXAMPLE 6.14. We shall compute the deformation of the differential module U_m defined by the equation

$$\delta_1(Y_{U_m})=\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\&&&\ddots&\\0&0&0&\cdots&1\\0&0&0&\cdots&0\end{pmatrix}\cdot Y_{U_m}\;,\qquad Y_{U_m}(x,y)=\begin{pmatrix}1&\ell_1&\cdots&\ell_{m-2}&\ell_{m-1}\\0&1&\ell_1&\cdots&\ell_{m-2}\\\vdots&&&&\vdots\\0&\cdots&0&1&\ell_1\\0&\cdots&0&0&1\end{pmatrix}\;, \tag{6.14.1}$$

where $\ell_n:=[\log(x)-\log(y)]^n/n!$. One has $\sigma_q^x(\ell_n(x,y))=[\log(qx)-\log(y)]^n/n!=(\log(q)+\log(x)-\log(y))^n/n!=\sum_{i=0}^n\frac{\log(q)^{n-k}}{(n-k)!}\cdot\ell_k$. The matrix of $\sigma_q^{U_m}$ is then

$$A(q,T)=\begin{pmatrix}1&\log(q)&\frac{\log(q)^2}{2}&\cdots&\frac{\log(q)^{m-1}}{(m-1)!}\\0&1&\log(q)&\cdots&\frac{\log(q)^{m-2}}{(m-2)!}\\\vdots&&&&\vdots\\0&0&\cdots&1&\log(q)\\0&0&\cdots&\cdots&1\end{pmatrix}\;. \tag{6.14.2}$$

6.5. Classification of solvable Rank one $q-$difference equations. In this section we classify rank one solvable $q-$difference equations over $\mathcal{R}_{K_\infty}$ by applying the deformation to the classification of the differential equations (cf. [**Pul**]). We recall the classification of the rank one solvable differential equations over $\mathcal{R}_{K_\infty}$.

We fix a Lubin-Tate group $\mathfrak{G}_P$ isomorphic to $\mathbb{G}_m$ over $\mathbb{Z}_p$. We recall that $\mathfrak{G}_P$ is defined by an uniformizer w of $\mathbb{Z}_p$, and by a series $P(X)\in X\mathbb{Z}_p[[X]]$, satisfying $P(X)\equiv \mathrm{w}\cdot X \pmod{X^2\mathbb{Z}_p[[X]]}$ and $P(X)\equiv X^p \pmod{p\mathbb{Z}_p[[X]]}$. Such a formal series is called a *Lubin-Tate series*. We fix now a sequence $\boldsymbol{\pi}:=(\pi_m)_{m\geq0}$, $\pi_m\in\mathbb{Q}_p^{\mathrm{alg}}$, such that $P(\pi_0)=0$, $\pi_0\neq0$ and $P(\pi_{m+1})=\pi_m$, for all $m\geq0$. The element $(\pi_m)_{m\geq0}$ is a generator of the Tate module of $\mathfrak{G}_P$ which is a free rank one $\mathbb{Z}_p-$module. For example, one can choose $\mathfrak{G}_P=\mathbb{G}_m$, hence $P(X)=(X+1)^p-1$, and $\pi_m=\xi_m-1$, where ξ_m is a compatible sequence of primitive $p^{m+1}-$th root of 1, i.e. $\xi_0^p=1$ and $\xi_m^p=\xi_{m-1}$. One has the following facts:

(1) Every rank one *solvable* differential module over $\mathcal{R}_K$ has a basis in which the associated operator is

$$\mathrm{L}(a_0,\boldsymbol{f}^-(T)):=\delta_1-\Big(a_0-\sum_{j=0}^s\pi_{s-j}\sum_{i=0}^j f_i^-(T)^{p^{j-i}}\partial_{T,\log}(f_i^-(T))\Big)\;, \tag{6.14.1}$$

where, $a_0\in\mathbb{Z}_p$, and $\boldsymbol{f}^-(T):=(f_0^-(T),\ldots,f_s^-(T))$ is a Witt vector in $\mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$, with $K_s:=K(\pi_s)$.

(2) Note that, even if π_j does not belong to K, the resulting polynomial $\sum_{j=0}^s\pi_{s-j}\sum_{i=0}^j f_i^-(T)^{p^{j-i}}\partial_{T,\log}(f_i^-(T))$ has coefficients in K.

(3) The Taylor solution at ∞ of the differential module in this basis is given by the so called $\boldsymbol{\pi}$-exponential attached to $\boldsymbol{f}^-(T)$:

$$T^{a_0}\cdot\mathrm{e}_{p^m}(\boldsymbol{f}^-(T),1):=T^{a_0}\cdot\exp(\sum_{j=0}^s\pi_{s-j}\frac{\phi_j^-(T)}{p^j})\;, \tag{6.14.2}$$

where $\langle\phi_0^-(T),\ldots,\phi_s^-(T)\rangle \in (T^{-1}\mathcal{O}_K[T^{-1}])^s$ is the phantom vector of $\boldsymbol{f}^-(T)$, namely one has $\phi_j^-(T) = \sum_{i=0}^j p^i f_i^-(T)^{p^{j-i}}$.

(4) The correspondence $\boldsymbol{f}^-(T) \mapsto \mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ is a group morphism

$$\mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}]) \xrightarrow{\mathrm{e}_{p^s}(-,1)} 1+\pi_s T^{-1}\mathcal{O}_{K_s}[[T^{-1}]] \;. \tag{6.14.3}$$

(5) Reciprocally, $\mathrm{L}(a_0,\boldsymbol{f}^-(T))$ is solvable for all pairs $(a_0,\boldsymbol{f}^-(T))$.

(6) The operator $\mathrm{L}(a_0,\boldsymbol{f}^-(T))$ has a Frobenius structure (cf. Def. 7.5) if and only if $a_0 \in \mathbb{Z}_{(p)} := \mathbb{Z}_p \cap \mathbb{Q}$.

(7) The operators $\mathrm{L}(a_0,\boldsymbol{f}_1^-(T))$ and $\mathrm{L}(b_0,\boldsymbol{f}_2^-(T))$ define isomorphic differential modules if and only if $a_0 - b_0 \in \mathbb{Z}$ and the Artin-Schreier equation

$$\bar{\mathrm{F}}(\overline{\boldsymbol{g}^-(T)}) - \overline{\boldsymbol{g}^-(T)} = \overline{\boldsymbol{f}_1^-(T) - \boldsymbol{f}_2^-(T)} \tag{6.14.4}$$

has a solution $\overline{\boldsymbol{g}^-(T)}$ in $\mathbf{W}_s(k_\infty(\!(t)\!))$, where t is the reduction of T, and $\bar{\mathrm{F}}$ is the Frobenius of $\mathbf{W}_s(k_\infty(\!(t)\!))$ (sending $(\bar{g}_0,\ldots,\bar{g}_s)$ into $(\bar{g}_0^p,\ldots,\bar{g}_s^p)$).

(8) In particular the most important fact concerning $\boldsymbol{\pi}-$exponentials is that the series $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ is over-convergent (i.e. belongs to $\mathcal{R}_K$) if and only if the Artin-Schreier equation $\bar{\mathrm{F}}(\overline{\boldsymbol{g}^-(T)}) - \overline{\boldsymbol{g}^-(T)} = \overline{\boldsymbol{f}^-(T)}$ has a solution $\overline{\boldsymbol{g}^-(T)}$ in $\mathbf{W}_s(k_\infty(\!(t)\!))$.

Remark 6.15. By deformation, every solvable q-difference equation, with $|q-1|<1$, has a solution at ∞ of the form $T^{a_0}\cdot \mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$. Its matrix in this basis is then

$$A(q,T) = \mathrm{e}_{p^s}(\boldsymbol{f}^-(qT),1)/\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1) = \mathrm{e}_{p^s}(\boldsymbol{f}^-(qT)-\boldsymbol{f}^-(T),1)\;.$$

By deformation $A(q,T)\in\mathcal{R}_K$. This is confirmed by the fact that $\boldsymbol{f}^-(qT)$ and $\boldsymbol{f}^-(T)$ have the same reduction in $\mathbf{W}_s(k_\infty(\!(t)\!))$, and hence $\mathrm{e}_{p^s}(\boldsymbol{f}^-(qT)-\boldsymbol{f}^-(T),1)\in\mathcal{R}_K$ by the point (8) of the previous classification.

7. Quasi unipotence and $p-$adic local monodromy theorem

In this section we show how to deduce the q-analogue of the p-adic local monodromy theorem (cf. [**And02**], [**Ked04**], [**Meb02**]) by deformation.

Let K be a complete discrete valued field with perfect residue field (this hypothesis is necessary to have the p-adic local monodromy theorem).

Let $\mathcal{E}_K^\dagger \subset \mathcal{R}_K$ be the so called *bounded Robba ring*, $\mathcal{E}_K^\dagger := \{\sum_{i\in\mathbb{Z}} a_iT^i \in \mathcal{R}_K \mid \sup|a_i| < +\infty\}$. Then, since K is discrete valued, $(\mathcal{E}_K^\dagger,|\cdot|_{(0,1)})$ is an *henselian* valued field, with residue field $k(\!(t)\!)$. It has two topologies arising from $|\cdot|_{(0,1)}$, and from the inclusion in $\mathcal{R}_K$. It is not complete with respect to these two topologies, but $\mathcal{E}_K^\dagger$ is dense into $\mathcal{R}_K$. One has

$$\mathcal{H}_K^\dagger \;\subset\; \mathcal{E}_K^\dagger \;\subset\; \mathcal{R}_K\;. \tag{7.0.1}$$

7.1. Frobenius Structure. Let $\varphi : K \to K$ be an absolute Frobenius (i.e. a ring morphism lifting of the $p-$th power map of k). Since $\mathcal{R}_K$ is not a local ring, and does not have a residual ring, we need a particular definition:

Definition 7.1. An absolute *Frobenius* on $\mathcal{R}_K$ (resp. $\mathcal{H}_K^\dagger$, $\mathcal{E}_K^\dagger$) is a continuous ring morphism, again denoted by $\varphi:\mathcal{R}_K\to\mathcal{R}_K$, extending φ on K and such that $\varphi(\sum a_iT^i) = \sum\varphi(a_i)\varphi(T)^i$, where $\varphi(T) = \sum_{i\in\mathbb{Z}} b_iT^i \in \mathcal{R}_K$ (resp. $\varphi(T)\in\mathcal{H}_K^\dagger$, $\varphi(T)\in\mathcal{E}_K^\dagger$) verifies $|b_i|<1$, for all $i\neq p$, and $|b_p-1|<1$.

Definition 7.2. We denote by ϕ the particular absolute Frobenius on $\mathcal{R}_K$ given by the choice

$$\phi(T) := T^p \;, \qquad \phi(f(T)) := f^{\varphi}(T^p) \;. \tag{7.2.1}$$

where $f^{\varphi}(T)$ is the series obtained from $f(T)$ by applying $\varphi : K \to K$ on the coefficients.

Let B be one of the rings $\mathcal{H}_K^{\dagger}$, $\mathcal{E}_K^{\dagger}$, or $\mathcal{R}_K$. For all $q \in \mathrm{D}^-(1,1)$, the following diagrams are commutative

$$\begin{array}{ccc} \mathrm{B} & \xrightarrow{\phi} & \mathrm{B} \\ {\scriptstyle\sigma_{q^p}}\downarrow & \odot & \downarrow{\scriptstyle\sigma_q} \\ \mathrm{B} & \xrightarrow[\phi]{} & \mathrm{B} \end{array} \quad ; \quad \begin{array}{ccc} \mathrm{B} & \xrightarrow{\phi} & \mathrm{B} \\ {\scriptstyle p\cdot\delta_1}\downarrow & \odot & \downarrow{\scriptstyle\delta_1} \\ \mathrm{B} & \xrightarrow[\phi]{} & \mathrm{B} \end{array} \quad . \tag{7.2.2}$$

Definition 7.3 (Frobenius functor). Let $S \subseteq \mathrm{D}^-(1,r)$, $r > 0$. Let

$$r' := \min(\; r^{1/p} \;,\; r \cdot |p|^{-1} \;) \;. \tag{7.3.1}$$

The Frobenius functor (cf. def. 5.46)

$$\phi^* : \quad (\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{[r]} \longrightarrow (\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{[r']} \;, \tag{7.3.2}$$

$$(\text{resp.} \quad \phi^* : \qquad \sigma - \mathrm{Mod}(\mathrm{B})_S^{[r]} \longrightarrow \qquad \sigma - \mathrm{Mod}(\mathrm{B})_S^{[r']} \;) \tag{7.3.3}$$

is defined as $\phi^*(\mathrm{M},\sigma^{\mathrm{M}},\delta_1^{\mathrm{M}}) = (\;\phi^*(\mathrm{M})\;,\;\sigma^{\phi^*(\mathrm{M})}\;,\;\delta_1^{\phi^*(\mathrm{M})}\;)$, where

1. $\phi^*(\mathrm{M}) := \mathrm{M} \otimes_{\mathrm{B},\phi} \mathrm{B}$ is the scalar extension of M via ϕ,
2. the morphism $\sigma^{\phi^*(\mathrm{M})}$ is given by $\sigma_q^{\phi^*(\mathrm{M})} = \sigma_{q^p}^{\mathrm{M}} \otimes \sigma_q^{\mathrm{B}}$:

$$q \longmapsto \sigma_{q^p}^{\mathrm{M}} \otimes \sigma_q \quad : \quad S \xrightarrow{\sigma^{\phi^*(\mathrm{M})}} \mathrm{Aut}_K^{\mathrm{cont}}(\phi^*(\mathrm{M})) \;, \tag{7.3.4}$$

3. the derivation is given by

$$\delta_1^{\phi^*(\mathrm{M})} = (p \cdot \delta_1^{\mathrm{M}}) \otimes \mathrm{Id}_{\mathrm{B}} + \mathrm{Id}_{\mathrm{M}} \otimes \delta_1^{\mathrm{B}} \;, \tag{7.3.5}$$

4. a morphism $\alpha : \mathrm{M} \to \mathrm{N}$ is sent into $\alpha \otimes 1 : \phi^*(\mathrm{M}) \to \phi^*(\mathrm{N})$.

Remark 7.4. 1.– Let $\mathrm{M} \in (\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K^{\dagger})_S^{[r]}$. Let $Y(T,1) = \sum_{i\geq 0} Y_i(T-1)^i$, $Y_i \in M_n(K)$, be its Taylor solution at 1. Then the Taylor solution of $\phi^*(\mathrm{M})$ is

$$Y^{\phi}(T^p,1) \;\;:=\;\; \sum_{i\geq 0} \varphi(Y_i)(T^p-1)^i \;. \tag{7.4.1}$$

2.– Let $\mathbf{e} = \{\mathrm{e}_1,\ldots,\mathrm{e}_n\}$ be a basis of M. Let $\sigma_q - A(q,T)$ and $\delta_1 - G(1,T)$ be the operators associated to σ_q^{M} and δ_1^{M} in this basis. Then the operators associated to $\phi^*(\mathrm{M})$ in the basis $\mathbf{e} \otimes 1$ are

$$\sigma_q - A^{\varphi}(q^p,T^p) \;, \qquad \delta_1 - p \cdot G^{\varphi}(1,T^p) \;, \tag{7.4.2}$$

where, according with (2.6.2), one has $A(q^p,T) = A(q,q^{p-1}T)\cdots A(q,qT)A(q,T)$.

Definition 7.5 (Frobenius structure). Let B be one of the rings $\mathcal{H}_K^{\dagger}$, $\mathcal{E}_K^{\dagger}$, or $\mathcal{R}_K$. Let $S \subseteq \mathrm{D}^-(1,1)$ be a subset. Let M be a discrete $\sigma-$module (resp. $(\sigma,\delta)-$module) over S. We will say that M has a *Frobenius structure of order* $h \geq 1$, if there exists an isomorphism $(\phi^*)^h(\mathrm{M}) \xrightarrow{\sim} \mathrm{M}$, over B where $(\phi^*)^h := \phi^* \circ \cdots \circ \phi^*$, $h-$times. We denote by

$$\sigma - \mathrm{Mod}(\mathrm{B})_S^{(\phi)} , \qquad (\text{resp.} \quad (\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{(\phi)} \ ) \tag{7.5.1}$$

the full subcategory of $\sigma - \mathrm{Mod}(\mathrm{B})_S^{[1]}$ (resp. $(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{[1]}$) whose objects have a Frobenius structure of some order.

Remark 7.6. If M has a Frobenius structure, then $r = r'$ (cf. (7.3.1)) and hence M is solvable.

Remark 7.7. We observe that objects in $\sigma - \mathrm{Mod}(\mathrm{B})_S^{(\phi)}$ and $(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{(\phi)}$ are, by Definition 6.11, *admissible*.

Remark 7.8. Suppose that $\mathrm{M} \in (\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{[1]}$ has a Frobenius structure of order $h \geq 1$. If $Y(T,1)$ is the Taylor solution of M at 1, this means that there exists a matrix $H(T) \in GL_n(\mathcal{H}_K^\dagger)$ such that

$$Y^{\varphi^h}(T^{p^h},1) = H(T)\cdot Y(T,1) \ . \tag{7.8.1}$$

Indeed $\mathcal{A}_K(1,1)$ is a $\mathcal{H}_K^\dagger$-discrete σ-algebra over $\mathrm{D}^-(1,1)$ trivializing M (cf. Def.3.5).

Proposition 7.9. *Let ξ be a p^n-th root of unity, and let $q \in \mathcal{Q}_1 - \boldsymbol{\mu}(\mathcal{Q}_1)$. Let $\mathrm{M} \in \sigma_q - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)}$. Then $\mathrm{Def}_{q,\xi}(\mathrm{M})$ is trivial (i.e. isomorphic to a direct sum of the unit object).*

Proof : Let $Y(T,1) \in GL_n(\mathcal{H}_K^\dagger)$ be the Taylor solution at 1 of M in some basis $\mathbf{e}$. Then, by remark 7.8, there exists $H(T)$ such that $Y^{\varphi^h}(T^{p^h},1) = H(T)\cdot Y(T,1)$. Hence, one has also $Y^{\varphi^{nh}}(T^{p^{nh}},1) = H_n(T)\cdot Y(T,1)$, for some $H_n(T) \in GL_n(\mathcal{H}_K^\dagger)$. Since $\sigma_\xi(Y^{\varphi^{nh}}(T^{p^{nh}})) = Y^{\varphi^{nh}}(T^{p^{nh}})$, then in the basis $H_n(T)\cdot\mathbf{e}$ the matrix of σ_ξ is trivial: $A(\xi,T) = \mathrm{Id}$ (cf. Section 3.2.1). □

7.2. Special coverings of $\mathcal{H}_K^\dagger$, canonical extension. We recall briefly the notions of *special coverings*. The residue field of $\mathcal{E}_K^\dagger$ is $k(\!(t)\!)$ (with respect to the norm $|.|_{(0,1)}$). On the other hand, the residue ring of $\mathcal{H}_K^\dagger$ (with respect to the Gauss norm $|.|_{(0,1)}$) is $k[t,t^{-1}]$. One has

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{H}_K^\dagger} & \subseteq & \mathcal{O}_{\mathcal{E}_K^\dagger} \\ \big\downarrow & \odot & \big\downarrow \\ k[t,t^{-1}] & \subseteq & k(\!(t)\!). \end{array} \tag{7.9.1}$$

We denote by $\mathcal{O}_K[T,T^{-1}]^\dagger$ the weak completion of $\mathcal{O}_K[T,T^{-1}]$, in the sense of Monsky and Washnitzer (cf. [MW68]). One has

$$\mathcal{H}_K^\dagger = \mathcal{O}_K[T,T^{-1}]^\dagger \otimes_{\mathcal{O}_K} K \ . \tag{7.9.2}$$

7.2.1. *Special coverings of $\mathbb{G}_{m,k}$* . Let us look at the residual situation. The morphism

$$\widehat{\eta} := \mathrm{Spec}(k(\!(t)\!)) \quad \hookrightarrow \quad \mathbb{G}_{m,k} = \mathrm{Spec}(k[t,t^{-1}]) \tag{7.9.3}$$

gives rise, by pull-back, to a map

$$\left\{ \begin{array}{l} \text{Finite Étale} \\ \text{coverings of } \widehat{\eta} \end{array} \right\} \xleftarrow{\text{Pull-back}} \left\{ \begin{array}{l} \text{Finite Étales} \\ \text{coverings of } \mathbb{G}_{m,k} \end{array} \right\} . \tag{7.9.4}$$

It is known (cf. [**Kat86**, 2.4.9]) that this map is surjective, and moreover that there exists a full sub-category of the right hand category, called *special coverings of* $\mathbb{G}_{m,k}$, which is equivalent, via pull-back, to the category on the left hand side. Special coverings are defined by the property that they are tamely ramified at ∞, and that their geometric Galois group has a unique $p-$Sylow subgroup (cf. [**Kat86**, 1.3.1]).

On the other hand, if $\pi \in \mathcal{O}_K$ is an uniformizer element, then both $(\mathcal{O}_{\mathcal{E}_K^\dagger}, (\pi))$ and $(\mathcal{O}_K[T,T^{-1}]^\dagger, (\pi))$ are Henselian couples in the sense of [**Ray70**, Ch.II] (cf. [**Mat02**, 5.1]). One can show that the precedent situation lifts in characteristic 0: (7.9.5)

$$\begin{array}{ccccc}
\left\{\begin{array}{l}\text{Special}\\ \text{extensions of } \mathcal{H}_K^\dagger\end{array}\right\} & \xrightarrow[\sim]{-\otimes\mathcal{E}_K^\dagger} & \left\{\begin{array}{l}\text{Finite unramified}\\ \text{extensions of } \mathcal{E}_K^\dagger\end{array}\right\} & \xrightarrow[\sim]{-\otimes\mathcal{R}_K} & \left\{\begin{array}{l}\text{Special}\\ \text{ext. of } \mathcal{R}_K\end{array}\right\} \\
{\scriptstyle -\otimes K}\uparrow\wr & \odot & \wr\uparrow{\scriptstyle -\otimes K} & & \\
\left\{\begin{array}{l}\text{Special extensions}\\ \text{of } \mathcal{O}_K[T,T^{-1}]^\dagger\end{array}\right\} & \xrightarrow[\sim]{-\otimes\mathcal{O}_{\mathcal{E}_K^\dagger}} & \left\{\begin{array}{l}\text{Finite onramified}\\ \text{extensions of } \mathcal{O}_{\mathcal{E}_K^\dagger}\end{array}\right\} & & \\
{\scriptstyle -\otimes k}\downarrow\wr & \odot & \wr\downarrow{\scriptstyle -\otimes k} & & \\
\left\{\begin{array}{l}\text{Special}\\ \text{coverings of } \mathbb{G}_{m,k}\end{array}\right\} & \xrightarrow[\sim]{\text{Pull-back}} & \left\{\begin{array}{l}\text{Finite étale}\\ \text{coverings of } \hat{\eta}\end{array}\right\} & &
\end{array}$$

where, by special extension of $\mathcal{O}_K[T,T^{-1}]^\dagger$ (resp. $\mathcal{H}_K^\dagger$, $\mathcal{R}_K$) we mean a finite étale Galois extension of $\mathcal{O}_K[T,T^{-1}]^\dagger$ (resp. $\mathcal{H}_K^\dagger$, $\mathcal{R}_K$) coming, by henselianity, from a special cover of $\mathbb{G}_{m,k}$.

Remark 7.10. One can show that every unramified extension $(\mathcal{E}_K^\dagger)'$ of $\mathcal{E}_K^\dagger$ (resp. special extension $\mathcal{R}'_K$ of $\mathcal{R}_K$) is *non canonically* isomorphic to $\mathcal{E}_{K'}^\dagger$ (resp. $\mathcal{R}_{K'}$), for some finite Galois unramified extension K'/K. This situation is analogue to the classical one, in which every extension of $\mathbb{C}(\!(T)\!)$ is of the form $\mathbb{C}(\!(T^{m/n})\!)$, for some integers $m, n \geq 0$, and hence it is isomorphic to $\mathbb{C}(\!(Z)\!)$, with $T = f(Z)$, $f(Z) := Z^{n/m}$. In the case of special extensions of $\mathcal{E}_K^\dagger$ or $\mathcal{R}_K$, the relation between the new variable and the old one is highly non trivial, and essentially unknown. If k'/k is the residue field of K', and if $t = \overline{f}(z) \in k'(\!(z)\!)$ is the relation between t and z in characteristic p, then the relation between T and Z in characteristic 0 is given by $T = f(Z) \in \mathcal{O}_{\mathcal{E}_K^\dagger}$, where $f(Z)$ is an arbitrary Laurent series, obtained from $\overline{f}(z)$ by lifting coefficient by coefficient.

Let us write $\mathcal{E}_{K,T}^\dagger := \mathcal{E}_K^\dagger$, and $\mathcal{E}_{K',Z}^\dagger := (\mathcal{E}_K^\dagger)'$. Let $q \in \mathrm{D}^-(1,1)$, the automorphism σ_q of $\mathcal{H}_{K,T}^\dagger$ extends uniquely to a continuous $K'-$linear automorphism of rings $\sigma'_q : \mathcal{E}_{K',Z}^\dagger \to \mathcal{E}_{K',Z}^\dagger$ inducing the identity on $k'(\!(z)\!)$. Indeed let $P(Z) = 0$, $P(X) \in \mathcal{O}_{\mathcal{E}_{K,T}^\dagger}[X]$, be the minimal polynomial of Z. Let $P^{\sigma_q}(X)$ be the polynomial obtained from P by applying σ_q to the coefficients. Then, by henselianity, there is bijection between roots of $P(X)$ and of $P^{\sigma_q}(X)$. Hence σ'_q is the unique continuous $K'-$linear automorphism given by $\sigma'_q(Z) := Z'$, where Z' is the unique root of P^{σ_q} whose reduction in $k(\!(z)\!)$ is equal to z. The same considerations show that σ extends uniquely to an automorphism of every special extension of $\mathcal{H}_K^\dagger$ and $\mathcal{R}_K$, and commutes with the action of $\mathrm{Gal}(k(\!(t)\!)^{\mathrm{sep}}/k(\!(t)\!))$.

The great problem of the theory is that the extended automorphism does not send Z into qZ. The general "Confluence" theory introduced in section 4 will be crucial in solving this problem.

7.3. Quasi unipotence of differential equations.

Definition 7.11. We denote by $\widetilde{\mathcal{H}_K^\dagger}$ (resp. $\widetilde{\mathcal{E}_K^\dagger}$, $\widetilde{\mathcal{R}_K}$) the union of all special extensions of $\mathcal{H}_K^\dagger$ (resp. unramified extension of $\mathcal{E}_K^\dagger$; special extensions of $\mathcal{R}_K$)

Definition 7.12. Let $S \subseteq \mathrm{D}^-(1,1)$ be a subset (resp. $S \subseteq \mathrm{D}^-(1,1)$, with $S^\circ \neq \emptyset$). A discrete $(\sigma,\delta)-$module on S (resp. discrete $\sigma-$module on S) is called *quasi-unipotent* if it is trivialized by the discrete $(\sigma,\delta)-$algebra

$$\widetilde{\mathcal{H}_K^\dagger}[\log(T)] \qquad (\text{resp. } \widetilde{\mathcal{E}_K^\dagger}[\log(T)]\ ,\ \widetilde{\mathcal{R}_K}[\log(T)]\ )\ . \tag{7.12.1}$$

Remark 7.13. Let $\mathrm{B} := \mathcal{H}_K^\dagger$, or $\mathcal{E}_K^\dagger$, $\mathcal{R}_K$. We observe that M is trivialized by $\widetilde{\mathrm{B}}[\log(T)]$, if and only if M is trivialized by $\mathrm{B}'[\log(T)]$, where B' is a (*finite*) special extension of B. Indeed the entries of a fundamental matrix of solutions of M in $\widetilde{\mathrm{B}}[\log(T)]$ lie all in a finite extension.

Theorem 7.14 ($p-$adic local monodromy theorem [And02],[Ked04],[Meb02]). *Objects in $\delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$ become quasi-unipotent after an eventual extension of the field of constants. In other words, if $\mathrm{M} \in \delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$, then there exists a finite extension K'/K such that $\mathrm{M} \otimes_K K'$ is quasi unipotent (i.e. trivialized by $\widetilde{\mathcal{H}_{K'}^\dagger}[\log(T)]$).*□

Theorem 7.15 (cf. [Mat02, 7.10,7.15]). *If a differential equation $\mathrm{M} \in \delta_1 - \mathrm{Mod}(\mathcal{R}_K)$ is quasi-unipotent, then it has a Frobenius structure. Moreover, the scalar extension functor*

$$\delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)} \xrightarrow{-\otimes \mathcal{R}_K} \delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)} \tag{7.15.1}$$

is essentially surjective. □

Theorem 7.16 ([Mat02, 7.15]). *There exists a full sub-category of $\delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)}$, denoted by $\delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}$, which is equivalent to $\delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$ via the scalar extension functor (7.15.1). Objects in that category are trivialized by $\widetilde{\mathcal{H}_K^\dagger}[\log(T)]$.*□

Definition 7.17 (Canonical extension). Objects in $\delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}$ will be called *special objects*. We will denote by

$$\delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)} \xrightarrow[\sim]{\mathrm{Can}} \delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Special}} \ \subset\ \delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)} \tag{7.17.1}$$

the section of the functor (7.15.1), whose image is the category of special objects (cf. Theorem 7.16). We will call it the *canonical extension functor*.

Corollary 7.18 ([And02, 7.1.6]). *Let $\mathrm{M} \in \delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$, then, up to replace K by a finite extension K'/K, M decomposes in a direct sum of submodules of the form $\mathrm{N} \otimes U_m$, where N is a module trivialized by a special extension of $\mathcal{R}_K$, and U_m is the $m-$dimensional object defined by the operator (cf. Ex. 6.14)*

$$\delta_1 - \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} .\square \tag{7.18.1}$$

REMARK 7.19. The $\log(T)$ appearing in 7.12.1, is used uniquely to trivialize the module U_m, $m \geq 2$ (cf. Ex. 6.14).

LEMMA 7.20. *Let* $\mathrm{N} \in \delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)}$ *be a (special) object trivialized by* $\widetilde{\mathcal{H}_K^\dagger}$. *Let* $\widetilde{Y} = (\widetilde{y}_{i,j}) \in GL_n(\widetilde{\mathcal{H}_K^\dagger})$ *be a fundamental matrix solution of* N. *Let* $(\mathcal{E}^\dagger)'$ *(resp.* $\mathcal{R}'$*) be the smallest special extension of* $\mathcal{E}_K^\dagger$ *(resp.* $\mathcal{R}_K$*), such that* $\mathrm{N} \otimes \mathcal{E}_K^\dagger$ *is trivialized by* $(\mathcal{E}^\dagger)'$ *(*$\mathrm{N} \otimes \mathcal{R}_K$ *is trivialized by* $\mathcal{R}'$*). Then one has*

$$(\mathcal{E}^\dagger)' = \mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}] \,, \qquad \mathcal{R}' = \mathcal{R}_K[\{\widetilde{y}_{i,j}\}_{i,j}] \,. \tag{7.20.1}$$

In other words, the smallest special extension of $\mathcal{E}_K^\dagger$ *(resp.* $\mathcal{R}_K$*) trivializing* M *is generated by the solutions of* M.

Proof : Since M is trivialized by $(\mathcal{E}^\dagger)'$, then $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}] \subseteq (\mathcal{E}^\dagger)'$. Hence the differential field $\mathcal{E}_{K'}^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$ is an unramified extension, and is then a special extension. Since $(\mathcal{E}^\dagger)'$ is minimal, then $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}] = (\mathcal{E}^\dagger)'$. The case over $\mathcal{R}_K$ follows from the case over $\mathcal{E}_K^\dagger$. □

COROLLARY 7.21. *We preserve the notations of lemma 7.20. There exists a unique* $K-$*linear ring automorphism* σ_q *of* $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$, *which induces the identity on the residual field.*

Proof : By Lemma 7.20, $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$ is a special extension (i.e. Henselian). Hence, by Remark 7.10, the extension of σ_q to $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$ is unique.□

7.4. Quasi unipotence of $\sigma-$modules and $(\sigma,\delta)-$modules with Frob. structure.

COROLLARY 7.22. *Let* $S \subseteq \mathrm{D}_K^-(1,1)$ *(resp.* $S^\circ \neq \emptyset$*). The scalar extension functors*

$$(\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{(\phi)} \xrightarrow{-\otimes \mathcal{R}_K} (\sigma,\delta) - \mathrm{Mod}(\mathcal{R}_K)_S^{(\phi)} \tag{7.22.1}$$

$$\sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{(\phi)} \xrightarrow{-\otimes \mathcal{R}_K} \sigma - \mathrm{Mod}(\mathcal{R}_K)_S^{(\phi)} \tag{7.22.2}$$

are essentially surjective.

Proof : By Proposition 6.12 and Remark 7.6, it is enough to prove that the functors

$$(\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K^\dagger)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)} \xrightarrow{-\otimes \mathcal{R}_K} (\sigma,\delta) - \mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)} \tag{7.22.3}$$

$$\sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)} \xrightarrow{-\otimes \mathcal{R}_K} \sigma - \mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)} \tag{7.22.4}$$

are essentially surjective. By Theorem 7.15 there exists a basis of M in which the matrix $G(1,T)$ of δ_1^{M} lies in $M_n(\mathcal{H}_K^\dagger)$. Moreover, $\mathrm{Can}(\mathrm{M},\delta_1^{\mathrm{M}})$ is Taylor admissible, since all solvable differential equations are Taylor admissible. By Proposition 6.12, for all $q \in \mathrm{D}^-(1,1)$, the matrix $A(Q,T) := Y_G(QT,T)$ belongs to $GL_n(\mathcal{H}_K^\dagger)$. □

LEMMA 7.23. *Let* $\mathrm{M} \in \delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)}$. *Assume that* K *is sufficiently large in order that* M *is quasi unipotent. Let* $(\mathcal{E}^\dagger)'$ *be the smallest special extension of* $\mathcal{E}_K^\dagger$ *such that* M *is trivialized by* $(\mathcal{E}^\dagger)'[\log(T)]$. *Let* $\widetilde{Y} \in GL_n(\widetilde{\mathcal{E}_K^\dagger}[\log(T)])$ *be a fundamental matrix solution of* M. *Then there exists a* K'/K *such that the matrix*

$$\widetilde{A}(q,T) := \sigma_q(\widetilde{Y}) \cdot \widetilde{Y}^{-1} \tag{7.23.1}$$

belongs to $GL_n(\mathcal{E}_{K'}^\dagger)$, *for all* $q \in \mathrm{D}_{K'}^-(1,1)$.

Proof : By Corollary 7.18, one can assume that $\mathrm{M} = N$, or $\mathrm{M} = U_m$, where N and U_m was defined in the Corollary 7.18. The case "$\mathrm{M} = U_m$" is trivial, since both the matrices of $\delta_1^{U_m}$ and of $\sigma_q^{U_m}$ are constant (cf. Ex. 6.14). Let now $\mathrm{M} = \mathrm{N}$ (i.e. M is trivialized by $\widetilde{\mathcal{E}_K^\dagger}$). In this case the solution matrix $\widetilde{Y}$ lies in $GL_n((\mathcal{E}^\dagger)')$ (cf. Remark 7.13). Up to enlarge K, for all $\gamma \in \mathrm{Gal}((\mathcal{E}^\dagger)'/\mathcal{E}_K^\dagger)$, one has

$$\gamma(\widetilde{Y}) = \widetilde{Y} \cdot H_\gamma \ , \quad H_\gamma \in GL_n(K) \ . \tag{7.23.2}$$

Since σ_q commutes with every $\gamma \in \mathrm{Gal}((\mathcal{E}^\dagger)'/\mathcal{E}_K^\dagger)$ (cf. Remark 7.10), one finds

$$\gamma(\widetilde{A}(q,T)) = \gamma(\sigma_q(\widetilde{Y}) \cdot \widetilde{Y}^{-1}) = \sigma_q(\widetilde{Y}) \cdot H_\gamma \cdot (\widetilde{Y} \cdot H_\gamma)^{-1} = \widetilde{A}(q,T) \ . \tag{7.23.3}$$

Hence $\widetilde{A}(q,T)$ belongs to $\mathcal{E}_K^\dagger$, for all $|q-1| < 1$. □

Theorem 7.24 ($p-$adic local monodromy theorem (generalized form)). *Let $S \subset \mathrm{D}^-(1,1)$, be a subset (resp. $S^\circ \neq \emptyset$). Then every object $\mathrm{M} \in (\sigma,\delta) - \mathrm{Mod}(\mathcal{R}_K)_S^{(\phi)}$ (resp. $\mathrm{M} \in \sigma - \mathrm{Mod}(\mathcal{R}_K)_S^{(\phi)}$) is quasi unipotent, up to replace K by a finite extension K'/K depending on* M.

Proof : By Proposition 6.12 and Remark 7.6, one has $(\sigma,\delta) - \mathrm{Mod}(\mathcal{R}_K)_S^{(\phi)} = (\sigma,\delta) - \mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)}$, (resp. $\sigma - \mathrm{Mod}(\mathcal{R}_K)_S^{(\phi)} = \sigma - \mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)}$). On the other hand, $(\sigma,\delta) - \mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)} = \sigma - \mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{\mathrm{an},(\phi)}$ (cf. (2.17.1)). Hence, without loss of generality, we can suppose that M is a Taylor admissible (in particular analytic) $(\sigma,\delta)-$module on the disk $\mathrm{D}^-(1,1)$, with a Frobenius structure. Up to enlarge K, we can also assume that $(\mathrm{M},\delta_1^{\mathrm{M}})$ is quasi unipotent. Let us consider a basis of M in which the matrices of δ_1^{M} and σ_q^{M} have coefficients in $\mathcal{H}_K^\dagger$ (cf. Corollary 7.22).

We forget σ^{M}, and consider only the differential equation $(\mathrm{M},\delta_1^{\mathrm{M}})$. The lemma 7.23 provides a second structure of discrete $\sigma-$module on M on $\mathrm{D}^-(1,1)$, arising from the fact that M is trivialized by $\widetilde{\mathcal{H}_K^\dagger}[\log(T)]$. Let us call $(\mathrm{M},\delta_1^{\mathrm{M}},\widetilde{\sigma}^{\mathrm{M}})$ this second structure. By construction $(\mathrm{M},\delta_1^{\mathrm{M}},\widetilde{\sigma}^{\mathrm{M}})$ is quasi unipotent (cf. Lemma 7.23), hence it remains to show that

$$\sigma^{\mathrm{M}} = \widetilde{\sigma}^{\mathrm{M}} \ . \tag{7.24.1}$$

This will follow by a typical argument of uniqueness in difference Galois Theory: we consider the Taylor solution of M at the point 1:

$$Y(T,1) = (y_{i,j})_{i,j} \in GL_n(\mathcal{A}_K(1,1)) \ . \tag{7.24.2}$$

On the other hand, in the same basis, we consider the solution $\widetilde{Y}$ of M in $\widetilde{\mathcal{H}_K^\dagger}$:

$$\widetilde{Y} = (\widetilde{y}_{i,j})_{i,j} \in GL_n(\widetilde{\mathcal{H}_K^\dagger}[\log(T)]) \ . \tag{7.24.3}$$

We shall verify that, for all $q \in \mathrm{D}^-(1,1)$, the matrix $A(q,T) \in GL_n(\mathcal{H}_K^\dagger)$ defined by the Propagation Theorem 6.3:

$$A(q,T) := Y(qT,1) \cdot Y(T,1)^{-1} \ , \tag{7.24.4}$$

is actually equal to the matrix $\widetilde{A}(q,T) \in GL_n(\mathcal{E}_K^\dagger)$ defined by the Lemma 7.23:

$$\widetilde{A}(q,T) := \sigma_q(\widetilde{Y}) \cdot \widetilde{Y}^{-1} \ . \tag{7.24.5}$$

We need now the following lemma:

LEMMA 7.25. *Let* $(\mathrm{M},\delta_1^{\mathrm{M}})\in\delta_1-\mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$. *Then, after an eventual finite extension* K'/K, *the decomposition given at Corollary 7.18 extends to a decomposition of the analytic* $(\sigma,\delta)-$*module* $(\mathrm{M},\delta_1^{\mathrm{M}},\sigma^{\mathrm{M}})$ *on* $\mathrm{D}^-(1,1)$, *attached to* M *via the Proposition 6.12, and also to a decomposition of the discrete* $(\sigma,\delta)-$*module* $(\mathrm{M},\delta_1^{\mathrm{M}},\widetilde{\sigma}^{\mathrm{M}})$ *on* $\mathrm{D}^-(1,1)$, *attached to* M *via the Lemma 7.23.*

Proof of Lemma 7.25 : By Proposition 6.12, Lemma 7.23 and Lemma 4.4, the forget functors

$$(\sigma,\delta)-\mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}},\mathrm{C})^{\mathrm{an,const},(\phi)}_{\mathrm{D}^-(1,1)}\xrightarrow[\sim]{\mathrm{Res}_1^{\mathrm{D}^-(1,1)}}\delta_1-\mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$$

$$\text{and } (\sigma,\delta)-\mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}},\mathrm{C})^{\mathrm{disc,const},(\phi)}_{\mathrm{D}^-(1,1)}\xrightarrow[\sim]{\mathrm{Res}_1^{\mathrm{D}^-(1,1)}}\delta_1-\mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$$

are equivalences by the Theorem 7.14, where $\mathcal{R}_{K^{\mathrm{alg}}}:=\mathcal{R}_K\otimes_K K^{\mathrm{alg}}$ and $\mathrm{C}:=\widetilde{\mathcal{R}_{K^{\mathrm{alg}}}}[\log(T)]$. Indeed every object on these categories comes by scalar extension from a module over $\mathcal{R}_{K'}$, for some finite extension K'/K. □

Continuation of proof of 7.24 : By Lemma 7.25, it is sufficient to discuss the case $\mathrm{M}=U_m$ and $\mathrm{M}=\mathrm{N}$ in the notations of Corollary 7.18. The case $\mathrm{M}=U_m$ is trivial (cf. 6.14): one has that $A(q,T)=\widetilde{A}(q,T)$ has constant coefficients.

Let us suppose that $\mathrm{M}=\mathrm{N}$ is trivialized by $\widetilde{\mathcal{H}_K^\dagger}$. We consider then the discrete $(\sigma,\delta)-$algebras $\mathcal{E}_K^\dagger[\{y_{i,j}\}_{i,j}]$ and $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$. By differential Galois theory, one has an isomorphism commuting with δ_1^{M}

$$\mathcal{E}_K^\dagger[\{y_{i,j}\}_{i,j}]\xrightarrow[y_{i,j}\longmapsto\widetilde{y}_{i,j}]{\sim}\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]\,. \tag{7.25.1}$$

We must show that this isomorphism commutes also with σ_q. This follows immediately from the fact that there is a unique $K-$linear automorphism σ_q on $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$, inducing the identity on $k(\!(t)\!)$ (cf. Corollary 7.21). □

7.5. The confluence of André-Di Vizio. In [ADV04] authors consider σ_q-modules over $\mathcal{R}_K$ with a Frobenius structure (i.e. $\sigma_q-\mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$). Moreover they restrict themselves to a fixed $q\in\mathrm{D}^-(1,1)$ satisfying $|q-1|<|p|^{\frac{1}{p-1}}$, hence they have no troubles with roots of unity. Their strategy is to find a "theory of slopes" (*filtration de type Hasse-Arf*) for this kind of $q-$difference equations in order to apply the main theorem of [And02] and deduce the Tannakian equivalence T_q (see Section 0.1). The proof of this fact needs a remarkable effort and is the central result of the paper. By composition with the Tannakian equivalence T_1 given by the p-adic Local monodromy theorem, they obtain an equivalence between q-difference equations with Frobenius structure and differential equations with Frobenius structure. They call "confluence" the composite functor $T_1^{-1}\circ T_q$.

In a second time they try to describe explicitly this equivalence. For this reason they introduce the notion of "*confluent weak Frobenius structure*" using the fact that the sequence q^{p^n} goes to 1 ([ADV04, 12.3]). In this last section we show that their "confluence" coincides with our more general "constant confluence".

REMARK 7.26. We recall that an *antecedent* of a σ_q-module M over $\mathcal{R}_K$ is a σ_q-module M_{-s} isomorphic to M: $\Phi:(\phi^*)^s(\mathrm{M}_{-s})\xrightarrow{\sim}\mathrm{M}$.

DEFINITION 7.27 ([**ADV04**, 12.11]). Let $|q-1| < |p|^{\frac{1}{p-1}}$ and let $s \geq 1$ be a natural number. A *confluent weak Frobenius structure* on a σ_q-module $(\mathrm{M}, \sigma_q^{\mathrm{M}}) \in \sigma_q - \mathrm{Mod}(\mathcal{R}_K)$ is a sequence $\{\mathrm{M}_m = (\mathrm{M}, \sigma_{q^{p^{sm}}}^{\mathrm{M}_m})\}_{m\geq 0}$ of $q^{p^{sm}}$-difference structures on M, with $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0}) = (\mathrm{M}, \sigma_q^{\mathrm{M}})$, together with a family of automorphisms

$$\Phi_m : (\ (\phi^*)^s(\mathrm{M}_{m+1})\ ,\ (\phi^*)^s(\sigma_{q^{p^{s(m+1)}}}^{\mathrm{M}_{m+1}})\ ) \xrightarrow{\sim} (\mathrm{M}_m, \sigma_{q^{p^{sm}}}^{\mathrm{M}_m})\ , \tag{7.27.1}$$

of $q^{p^{sm}}-$difference modules, such that

(1) The operators $\Delta_{q^{p^{sm}}}^{\mathrm{M}_m} := \frac{\sigma_{q^{p^{sm}}}^{\mathrm{M}_m} - 1}{q^{p^{sm}} - 1}$ converge to a derivation $\Delta^{\mathrm{M}_\infty}$ on M.
(2) If $\mathrm{M}_\infty := (\mathrm{M}, \Delta^{\mathrm{M}_\infty})$ is this differential module, then the sequence of isomorphisms (7.27.1) converges to a Frobenius isomorphism

$$\Phi_\infty : \phi^*(\mathrm{M}_\infty) \xrightarrow{\sim} \mathrm{M}_\infty\ . \tag{7.27.2}$$

Actually, if M has a Frobenius structure, then the notion of "Confluent weak Frobenius structure" is nothing else that the constant confluence:

LEMMA 7.28. *Let $|q-1| < |p|^{\frac{1}{p-1}}$. If $\mathrm{M} \in \sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$, then M has a "Confluent weak Frobenius structure" and moreover*

$$\mathrm{M}_m \xrightarrow{\sim} \mathrm{Def}_{q,q^{p^{sm}}}(\mathrm{M})\ , \quad \mathrm{M}_\infty \xrightarrow{\sim} \mathrm{Def}_{q,1}(\mathrm{M})\ . \tag{7.28.1}$$

Proof : In term of matrix solution the previous condition can be written as follows: if $Y_m(T,1)$ is the Taylor solution of M_m in the basis $\mathbf{e}_m$, then the existence of Φ_m is equivalent to the relation $Y_{m+1}^{\varphi^s}(T^{p^s}, 1) = H_m(T)Y_m(T,1)$, where $H_m(T) \in GL_n(\mathcal{R}_K)$ is the matrix of Φ_m in the basis $(\mathbf{e}_{m+1}, \mathbf{e}_m)$. Suppose then that M has a Frobenius structure of order $s \geq 1$, that is $Y_0^{\varphi^s}(T^{p^s}, 1) = H(T)Y_0(T,1)$, for some $H(T) \in GL_n(\mathcal{R}_K)$. Then, for all $m \geq 1$, we can put

$$\mathrm{M}_m := \mathrm{M}_0\ , \qquad \mathbf{e}_m := \mathbf{e}_0\ , \qquad Y_m(T,1) := Y_0(T,1)\ . \tag{7.28.2}$$

With this choice of M_m one has $H_m(T) = H(T)$, for all $m \geq 0$, hence the sequence Φ_m is constant, and $\Phi_m = \Phi_\infty$. On the other hand it is clear that the operator $\Delta_{q^{p^{sm}}}^{\mathrm{M}_m}$ is equal to $\Delta_{q^{p^{sm}}}^{\mathrm{M}}$, the fact that the sequence $\Delta_{q^{p^{sm}}}^{\mathrm{M}}$ converges to a derivation on M is proved by the Propagation Theorem and is actually independent on the existence of a Frobenius Structure on M (cf. Remark 6.6). □

REMARK 7.29. A.— Lemma 7.28 gives a straightforward and explicit construction of the functor $D^{\mathrm{conf}(\phi)} \circ V_{\sigma_q}^{(\phi)}$ of [**ADV04**].
B.— By our result it is evident that the "slopes" of a σ_q-module are the "slopes" of the attached differential module, by constant confluence.
C.— The quasi-unipotence of $q-$difference equations is a straightforward consequence of the *constant confluence* and of the $p-$adic local monodromy theorem for differential equations. Reciprocally one can prove the $p-$adic local monodromy theorem for differential equations by confluence by using the results of [**ADV04**].
D.— The equivalence provided by the Propagation Theorem requires only the definition and the formal properties of the Taylor solution $Y(x,y)$. For this reason this equivalence is not a consequence of the theory developed until today, but conversely the Confluence implies the main results of [**ADV04**] and also of [**DV04**].

Index

$\mathcal{A}_K(c,R)$ =Analytic funct. on $\mathrm{D}^-(c,R)$, 6
$\mathcal{A}_K(I)$ =analytic functions on $\mathcal{C}_K(I)$, 6
$K-$affinoïd, 7
Analytic $(\sigma,\delta)-$modules on U, 11
Analytic $\sigma-$modules on U, 10
Antecedent, 39

Bounded Robba ring, 32

$\mathrm{C}_S^{(\sigma,\delta)}$, 12
C_S^{σ}, 12
Can, 36
$\mathrm{Conf}_q := \mathrm{Def}_{q,1}$, 15
Confluent weak Frobenius structure, 40
Constant $\sigma-$modules
  Constant $(\sigma,\delta)-$modules, 13
Constant confluence functor, 15
constant deformation functor, 15
Constant solutions on S, 12

d_q, 17
$\mathrm{Def}_{q,q'}$, 15
$\delta_1^{\mathrm{M}} : \mathrm{M} \to \mathrm{M}$, 9
$\delta_q^{\mathrm{Hom(M,N)}}$, 10
$\delta_q^{\mathrm{M}\otimes\mathrm{N}}$, 10
Discrete $(\sigma,\delta)-$modules on S, 9
Discrete $\sigma-$algebra over S, 12
Discrete $\sigma-$modules on S, 8

Frobenius functor, 33
Frobenius structure, 33
Fundamental matrix of solutions, 13

Generic radius of convergence, 24

$\mathcal{H}_K$, $\mathcal{H}_K^\dagger$, $\mathcal{H}_K^\dagger(X) := \bigcup_{\varepsilon>0}\mathcal{H}_K(X_\varepsilon)$, 7
$\mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{N})$, 9
$\mathrm{Hom}_S^{\sigma}(\mathrm{M},\mathrm{N})$, 8

I_ε, 26

Logarithmic properties, 7

$\boldsymbol{\mu}(\mathcal{Q})$, 8

$\mathcal{O}_K[T,T^{-1}]^\dagger$, 34
Ω, 24
$\Omega[[T-c]]_q$, $\Omega\{T-c\}_{q,R}$, 18

$p-$adic local monodromy theorem, 36
$p-$adic local monodromy theorem
  (generalized form), 38
$\phi(T) := T^p$, 33
ϕ^*, 33

$\mathcal{Q}(\mathrm{B})$, $\mathcal{Q}_1(\mathrm{B})$, 8
Quasi unipotence, 36

$Ray(\mathrm{M},\rho) := Ray_0(\mathrm{M},\rho)$, 25
$Ray_c(f(T))$, 8
$\mathrm{Res}_{S'}^S$, 14
$\rho_{c,X}$, 16

S, 8
$S^\circ := S - \boldsymbol{\mu}(Q)$, 8
$\langle S\rangle$, 8
$S_\tau := S \cap \mathrm{D}^-(1,\tau)$, 26
$(\sigma,\delta)-$constants, 12
$(\sigma,\delta)-\mathrm{Mod(B)}_S^{\mathrm{disc}}$, 9
$(\sigma,\delta)-\mathrm{Mod(B)}_U^{\mathrm{an}}$, 11
$(\sigma,\delta)-\mathrm{Mod(B,C)}_S^{\mathrm{const}}$, 14
$(\sigma,\delta)-\mathrm{Mod(B,C)}_U^{\mathrm{an,const}}$, 14
$(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K(X))_S^{\mathrm{adm}}$,
  $\sigma-\mathrm{Mod}(\mathcal{H}_K(X))_S^{\mathrm{adm}}$, 27
$(\sigma,\delta)-\mathrm{Mod}(\mathcal{R}_K)_S^{[r]}$, $(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{[r]}$,
  27
$\sigma-\mathrm{Mod(B)}_S^{\mathrm{disc}}$, 8
$\sigma-\mathrm{Mod(B)}_S^{(\phi)}$, $(\sigma,\delta)-\mathrm{Mod(B)}_S^{(\phi)}$, 34
$\sigma-\mathrm{Mod(B)}_U^{\mathrm{an}}$, 10
$\sigma-\mathrm{Mod(B,C)}_S^{\mathrm{const}}$, 14
$\sigma-\mathrm{Mod}(\mathcal{R}_K)_S^{[r]}$, $\sigma-\mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{[r]}$, 26
$\sigma-\mathrm{Mod}(\mathcal{R}_K)_U^{\mathrm{an}}$, $\sigma-\mathrm{Mod}(\mathcal{R}_K)_S^{\mathrm{disc}}$,
  $\sigma-\mathrm{Mod}(\mathcal{H}_K^\dagger)_U^{\mathrm{an}}$, $\sigma-\mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{\mathrm{disc}}$, 11
$\sigma^{\mathrm{M}}:\langle S\rangle \to \mathrm{Aut}_K^{\mathrm{cont}}(\mathrm{M})$, 8
$(\sigma_q,\delta_q)-\mathrm{Mod(B,C)}_U^{\mathrm{an}}$, 15
$(\sigma_q,\delta_q)-\mathrm{Mod(B,C)}_S$, 15
$(\sigma_q,\delta_q)-\mathrm{Mod}(\mathcal{R}_K)^{[r]}$,
  $(\sigma_q,\delta_q)-\mathrm{Mod}(\mathcal{H}_K^\dagger)^{[r]}$,
  $(\sigma_q,\delta_q)-\mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{sol}(\rho)}$, 26
σ_q-constants, 12
$\sigma_q-\mathrm{Mod}(\mathcal{A}_K(I))^{\mathrm{sol}(\rho)}$, 25
$\sigma_q-\mathrm{Mod(B)}$, 9
$\sigma_q-\mathrm{Mod}(B,\mathrm{C})_U^{\mathrm{an}}$, 15
$\sigma_q-\mathrm{Mod(B,C)}_S$, 15
$\sigma_q-\mathrm{Mod}(\mathcal{H}_K^\dagger)^{[r]}$, 25
$\sigma_q-\mathrm{Mod}(\mathcal{R}_K)^{[r]}$, 26
$\sigma_q^{\mathrm{Hom(M,N)}}$, 9
$\sigma_q^{\mathrm{M}\otimes\mathrm{N}}$, 9
Special coverings, 35

$(T-c)_{q,n}$, 17
$t_{c,\rho}$ =generic point of $\mathrm{D}^+(\mathrm{c},\rho)$, 24
$t_\rho := t_{0,\rho}$, 25
Taylor admissible discrete $\sigma-$modules, and
  discrete $(\sigma,\delta)-$modules, 27
Transfer Principle, 20

$X := \mathrm{D}^+(c_0,R_0) - \bigcup_{i=1}^n \mathrm{D}^-(c_i,R_i)$, 7
X_ε, 7

$\mathrm{Y}_{A(q,T)}(x,y)$, 20
$\mathrm{Y}_{G(1,T)}(x,y)$, 19

References

[ADV04] Yves André and Lucia Di Vizio. q-difference equations and p-adic local monodromy. *Astérisque*, (296):55–111, 2004. Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes. I.

[And02] Y. André. Filtrations de type Hasse-Arf et monodromie p-adique. *Invent. Math.*, 148(2):285–317, 2002.

[And04] Yves André. Sur la conjecture des p-courbures de Grothendieck-Katz et un problème de Dwork. In *Geometric aspects of Dwork theory. Vol. I, II*, pages 55–112. Walter de Gruyter GmbH & Co. KG, Berlin, 2004.

[Ber90] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[Chr83] Gilles Christol. *Modules différentiels et équations différentielles p-adiques*, volume 66 of *Queen's Papers in Pure and Applied Mathematics*. Queen's University, Kingston, ON, 1983.

[CM02] G. Christol and Z. Mebkhout. Équations différentielles p-adiques et coefficients p-adiques sur les courbes. *Astérisque*, 279:125–183, 2002. Cohomologies p-adiques et applications arithmétiques, II.

[CR94] G. Christol and P. Robba. *Équations différentielles p-adiques*. Actualités Mathématiques. [Current Mathematical Topics]. Hermann, Paris, 1994. Applications aux sommes exponentielles. [Applications to exponential sums].

[DM] P. Deligne and James S. Milne. *Tannakian Categories*, volume 900 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

[DV02] Lucia Di Vizio. Arithmetic theory of q-difference equations: the q-analogue of Grothendieck-Katz's conjecture on p-curvatures. *Invent. Math.*, 150(3):517–578, 2002.

[DV04] Lucia Di Vizio. Introduction to p-adic q-difference equations (weak Frobenius structure and transfer theorems). In *Geometric aspects of Dwork theory. Vol. I, II*, pages 615–675. Walter de Gruyter GmbH & Co. KG, Berlin, 2004.

[Kat82] Nicholas M. Katz. A conjecture in the arithmetic theory of differential equations. *Bull. Soc. Math. France*, 110(2):203–239, 1982.

[Kat86] Nicholas M. Katz. Local-to-global extensions of representations of fundamental groups. *Ann. Inst. Fourier (Grenoble)*, 36(4):69–106, 1986.

[Ked04] Kiran S. Kedlaya. A p-adic local monodromy theorem. *Ann. of Math. (2)*, 160(1):93–184, 2004.

[Mat02] Shigeki Matsuda. Katz correspondence for quasi-unipotent overconvergent isocrystals. *Compositio Math.*, 134(1):1–34, 2002.

[Meb02] Z. Mebkhout. Analogue p-adique du théorème de Turrittin et le théorème de la monodromie p-adique. *Invent. Math.*, 148(2):319–351, 2002.

[MW68] P. Monsky and G. Washnitzer. Formal cohomology. I. *Ann. of Math. (2)*, 88:181–217, 1968.

[Pul] A. Pulita. Rank one solvable $p-$adic differential equations and finite abelian characters via lubin-tate groups. ————, ——–:—-, ——.

[Ray70] M. Raynaud. *Anneaux locaux henséliens*. Lecture Notes in Mathematics, Vol. 169. Springer-Verlag, Berlin, 1970.

[vdPS03] M. van der Put and M. F. Singer. *Galois theory of linear differential equations*, volume 328 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2003.

???????????, Universität Bielefeld, Bielefeld, Germany

E-mail address: pulita@math.jussieu.fr