

AROUND A.D. ALEXANDROV'S UNIQUENESS THEOREM FOR 3D POLYTOPES

GAIANE PANINA

ABSTRACT. Two dependent examples are presented.

1. An example of two different convex 3D polytopes such that for each pair of their parallel facets, the facets are different, and there exists a unique translation putting one facet inside the other.

2. An example of a pointed tiling of S^2 generated by a Laman-plus-one graph which can be regularly triangulated without adding new vertices.

The paper explores the relationship between the theory of pseudo triangulations and the theory of hyperbolic virtual polytopes.

1. INTRODUCTION.

In 1939, A.D.Alexandrov formulated the following uniqueness conjecture and proved it for analytic surfaces.

Uniqueness conjecture for smooth convex surfaces [2].

Let $K \subset \mathbb{R}^3$ be a smooth convex body. If for a constant C , at every point of ∂K , we have $R_1 \leq C \leq R_2$, then K is a ball. (R_1 and R_2 stand for the principal curvature radii of ∂K).

In the same paper, he claims that there is an analogous assertion for convex polytopes:

Theorem 1.1. Uniqueness theorem for convex polytopes [1]. *Let K, M be 3-dimensional convex polytopes. If for any pair of their parallel faces, none of them can be inserted strictly into another via a translation, the polytopes coincide up to a translation. \square*

Much later, it turned out that the above conjecture for smooth surfaces is wrong. The first counterexample was presented by Y. Martinez-Maure, 2001 (see [6]). Some later, the author of the paper presented a series of new counterexamples based on the theory of *hyperbolic virtual polytopes* (see [7],[8]).

No wonder that Theorem 1.1. has a natural reformulation and nice refinements in terms of hyperbolic virtual polytopes (see [9]).

In 2006, it became clear that this theory is very much related to the theory of *pointed pseudo triangulations* (see [3], [10], [11]).

Uniting the methods of both theories, we construct the following two examples.

Key words and phrases. pseudo triangulation, regular triangulation, convex polytope, virtual polytope, hyperbolic virtual polytope. MSC 52B10, 52B70.

Example 4.1

We present an example of two 3D polytopes K and L such that

- The polytopes K and L are different.
- K^ξ is a facet $\Leftrightarrow L^\xi$ is a facet (such facets are called parallel).
(K^ξ stands for the face of K with the outer normal vector ξ .)
- For any pair of parallel facets, there exists a unique translation t such that either $tK^\xi \subset L^\xi, tK^\xi \neq L^\xi$, or $tL^\xi \subset K^\xi, tL^\xi \neq K^\xi$ \square

Example 4.2

We present an example of a pointed tiling of S^2 generated by an embedded Laman-plus-one graph which can be regularly triangulated without Steiner points (i. e. without adding new vertices). \square

The two examples are dependent: the pointed tiling from Example 4.2 is the fan of the Minkowski difference $K \otimes L^{-1}$, whereas the latter is a hyperbolic polytope.

Acknowledgements. This research was inspired by my discussion with Ileana Streinu during my stay at Freie University (Berlin); the work was done at SFB 701, Bielefeld University.

2. VIRTUAL POLYTOPES. HYPERBOLIC POLYTOPES AND SPHERICAL POINTED TILINGS

Virtual polytopes were introduced by A. Pukhlikov, A. Khovanskii [4]. By definition, they are elements of the Grothendieck group of the semigroup of convex polytopes \mathcal{P} equipped with the Minkowski addition \otimes . I.e., they are formal expressions of type $K \otimes L^{-1}$, where $K, L \in \mathcal{P}$.

In the paper, we need the following two ways of their visualization.

- Virtual polytopes are *polytopal functions* ([4],[7]), i.e., finite linear combinations of indicator functions of convex polytopes. So it makes sense to speak of the *value* of a polytope K at a point $x \in \mathbb{R}^n$.
- Virtual polytopes are defined by their *support functions*, i.e., piecewise linear positively homogeneous functions defined on \mathbb{R}^n (see [4], [7], [8]).

Give now detailed explanations of the items, restricting from now on by dimension 3.

Denote by \mathcal{P} the set of all compact convex polytopes in \mathbb{R}^3 (degenerate polytopes are also included). It is a semigroup with respect to the Minkowski addition \otimes .

Denote by \mathcal{P}^* the Grothendieck group of \mathcal{P} . The element of \mathcal{P}^* that is inverse to K is denoted by K^{-1} .

A function $F : \mathbb{R}^3 \rightarrow \mathbb{Z}$ is *polytopal* if it admits a representation of the form

$$F = \sum_i a_i I_{K_i},$$

where $a_i \in \mathbb{Z}$, $K_i \in \mathcal{P}$, and I_{K_i} is the indicator function of the polytope K_i :

$$I_{K_i}(x) = \begin{cases} 1 & \text{if } x \in K_i, \\ 0 & \text{otherwise.} \end{cases}$$

The set of all polytopal functions \mathcal{M} is endowed with two ring operations. The role of addition is played by the pointwise addition, denoted by $+$. The multiplication is generated by \otimes and is denoted by the same symbol.

The unit element of the ring \mathcal{M} is obviously the function $E = I_{\{O\}}$.

Identifying convex compact polytopes with their indicator functions, we get an inclusion $\pi : \mathcal{P} \subset \mathcal{M}$. Keeping this identification in mind, we write for convenience K instead of I_K .

Due to the following theorem, all elements of the semigroup $\pi(\mathcal{P})$ are invertible in \mathcal{M} .

Theorem 2.1. (On Minkowski inversion) [4]

For any convex polytope K , we have

$$(-1)^{\dim K} I_{\text{Relint}(sK)} \otimes K = E,$$

where s is the central symmetry mapping (with respect to the origin O), $\text{Relint}(sK)$ is the relative interior of the polytope sK (i.e., the interior taken in the affine hull of K). \square

Hence the inclusion $\mathcal{P} \subset \mathcal{M}$ induces an inclusion $\mathcal{P}^* \subset \mathcal{M}$.

Definition 2.2. The image of the latter inclusion is called the group of *virtual polytopes*. For convenience we denote it by the same letter \mathcal{P}^* .

Definition 2.3. Let $K = L \otimes M^{-1}$ be a virtual polytope. The *support function* h_K of K is defined to be the pointwise difference of support functions of L and M :

$$h_K = h_L - h_M.$$

Remark.

Similarly to the convex case, the support function of a virtual polytope is piecewise linear with respect to some fan. By a *fan* we mean (as usual) a splitting of \mathbb{R}^3 in a union of polytopal cones with a common apex at O . But in the sequel, we speak of (and draw) the intersection of the fan with the unite sphere S^2 centered at O . Thus the cones correspond to the spherical polytopes (spherical cells).

Definition 2.4. [7] Let K be a virtual polytope, h be its support function. For a plane $e \subset \mathbb{R}^3$, the graph of the restriction $h|_e$ is called the *affine graph of h* and is denoted by $\Gamma(h, e)$.

Definition 2.5. [8] Let $K = \sum_i a_i K_i$ with $K_i \in \mathcal{P}$. Let $e_i(\xi)$ be the support plane to K_i with the outer normal vector ξ . The polytope $K_i^\xi = K_i \cap e_i(\xi)$ is called *the face of the polytope K_i with the normal vector ξ* , while the polytopal

function $K^\xi = \sum_i a_i K_i^\xi$ is called *the face of the polytopal function K with the normal vector ξ* .

Two-dimensional faces of a 3D polytope are called *facets*.

A face of a virtual polytope is a virtual polytope as well.

Similarly to faces of convex polytopes, virtual faces behave linearly with respect to the Minkowski addition:

$$K_1^\xi \otimes K_2^\xi = (K_1 \otimes K_2)^\xi.$$

Definition 2.6. [9] A point X is called a *boundary point* of a virtual polytope K , if $x \in cl(supp(K^\xi))$ for some $\xi \in S^2$ such that ξ is not orthogonal to $aff(K)$. (cl denotes the closure.)

Theorem 2.7. [4] For two convex polytopes K and L , and a point $x \in \mathbb{R}^3$,

$$K \otimes L^{-1}(x) = \chi(K \cap t_x(Relint L))(-1)^{dim L},$$

where χ stands for the Euler characteristic, t_x is the translation by x . \square

Corollary 2.8. Let $M_{1,2}$ be some convex polygons. Denote by $M = M_1 \otimes M_2^{-1}$ their Minkowski difference. (M is a virtual polygone, i.e., a 2-dimensional virtual polytope).

The following two assertions are valid:

- (1) M admits a positive value at a point x if and only if $t_x M_2 \subset M_1$ or $M_1 \subset Relint(t_x M_2)$.
- (2) Assume that M admits positive values only at its boundary points. If M has no parallel edges (i.e., $dim M^\xi = 1 \Rightarrow dim M^{-\xi} = 0$), it admits positive value not more than at one point.

Proof. (1) follows easily from Theorem 2.7.

(2). Suppose $M(x) = M(y) = 1$ for $x \neq y$. Then for each point v of the segment $[xy]$, we have $M(v)=1$. Indeed, the inclusions $t_x M_2 \subset M_1$ and $t_y M_2 \subset M_1$ imply $t_v M_2 \subset M_1$. This means that the segment $[xy]$ lies on edges M^ξ and $M^{-\xi}$, where ξ is orthogonal to $[xy]$. A contradiction. \square

Definition 2.9. [7] A virtual polytope K is called *hyperbolic* if $\Gamma_{aff}(h, e(\xi))$ is a saddle surface for any $\xi \in S^2$. In the sequel, we call such virtual polytopes simply *hyperbolic polytopes*.

Recall that a piecewise linear (or any other non-smooth surface) F is called a *saddle surface* if there is no plane cutting off a bounded connected component of F .

Theorem 2.10. [9]

K is a hyperbolic polytope if and only if all non-boundary values of its facets are non-positive. \square

Pointed tilings.

All graphs are supposed to be planar. All embeddings (either in \mathbb{R}^2 , or in S^2) are supposed to be *straightened*, i.e. each edge is represented by a geodesic segment. By a *general embedding* of a graph we mean an embedding with vertices lying in a general position.

Let Γ be a graph with v vertices and e edges.

Γ is a *Laman graph* if $e = 2v - 3$ and every subset of k vertices spans at most $2k - 3$ edges.

Γ is a *Laman-plus-one graph* if there is an edge such that after its removing the graph becomes Laman.

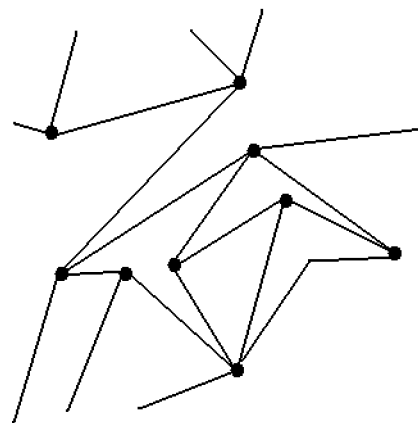
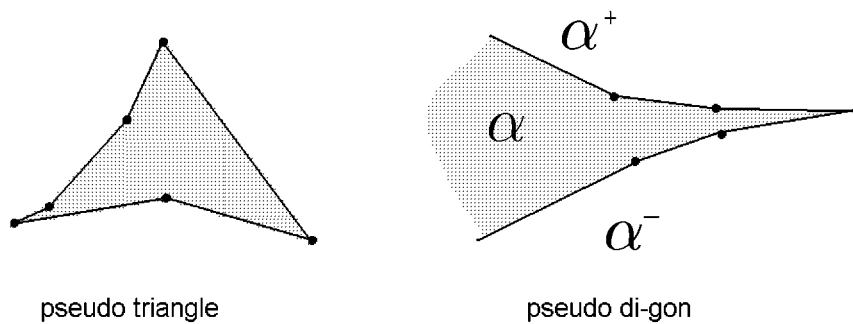
Theorem 2.11. [5] *Let a tiling T be generated by a general embedding of a Laman-plus-one graph. Then there exists a piecewise linear function which is linear on each tile and which is not globally linear.* \square

Consider a planar embedding of a graph Γ . We say that its vertex is *pointed* if one of the adjacent angles is greater than π .

An embedding is *pointed* if all its vertices are pointed.

A *pseudo-triangle* is a simple polygon (a non-crossing broken line embedded in S^2) which has exactly 3 convex vertices.

Figure 2.12.



fragment of a pointed tiling

A *pseudo di-gon* is a spherical polygon (a non-crossing broken line embedded in the sphere with a fixed interior domain; all segments are geodesic segments) which has exactly two convex vertices.

A pseudo di-gon never fits an open hemisphere.

Definition 2.13. A *nice pseudo tiling* is a tiling of S^2 which is

- pointed
- each tile is either a (spherical) pseudo triangle or a pseudo di-gon.

Lemma 2.14. [9] *Each nice pseudotiling which contains exactly four pseudo di-gons is generated by an embedding of a Laman-plus-one graph.* \square

In the framework of the theory of hyperbolic polytopes, spherical pointed tilings appear due to the following simple observation.

Lemma 2.15. [7] *Let K be a virtual polytope.*

- *The fan of K is a pointed tiling $\Rightarrow K$ is hyperbolic.*
- *If K is simplicial, then the fan of K is a pointed tiling $\Leftrightarrow K$ is hyperbolic.* \square

Theorem 2.16. *Let H be a hyperbolic polytope, h be its support function, Σ be its fan. Then the fan Σ contains at least four cells α such that*

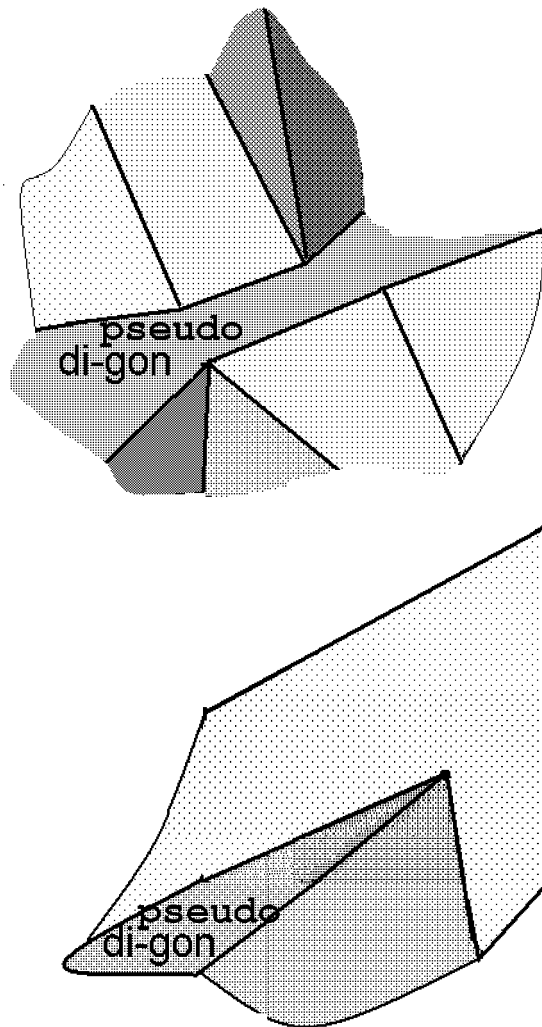
- *α is a pseudo di-gon. (It means that it is bounded by two convex broken lines α^+ and α^- , see Fig. 2.12.)*
- *The function h is concave up along one of α^\pm and concave down along the other, see Fig. 2.17.* \square

Proof. The below proof uses some extra notions for which we refer the reader to [7] and [8].

It is known that a hyperbolic polytope has at least four *horns*. Let P be a horn, and let α be the cell of the fan corresponding to the horn P . We can assume that $P = (0, 0, 0)$ and that x -coordinates of all other points of H are strictly less than 0. Therefore, $h'_x < 0$ on all cells of the fan except for α , and $h'_x = 0$ on the cell α .

Consider a plane e such that the x -axis is parallel to e and consider the restriction $h|_e$. Its graph is horizontal above α and has a negative slope above all other cells of the fan. Besides, it is a saddle surface. This easily implies the statement of the theorem (see Fig. 2.17) \square

Figure 2.17.



3. THE RELATIONSHIP BETWEEN THE DESIRED EXAMPLE AND HYPERBOLIC POLYTOPES

Lemma 3.1. *Let K be a virtual polytope, Γ be the affine graph of its support function, $\bar{\Gamma}$ be its subgraph.*

For a point $\xi \in S^2$ and a point E in $\text{aff}(K^\xi)$, we have

$$K^\xi(E) = 1 + \chi(e \cap \bar{\Gamma} \cap U(\Xi)),$$

where χ stands for the Euler characteristic; Ξ is the point on Γ such that $\pi(\Xi) = \xi$; e is the spherical graph of the point E (i.e., a 2-dimensional sphere on S^3); $U(\Xi) = \{\eta \in S^3 \mid 0 \neq |\eta, \Xi| < \epsilon\}$ is a deleted neighbourhood of Ξ .

Proof. Without loss of generality we may assume that $\dim K = 2$ and $K = K^\xi$. Indeed, the spherical graphs of K and K^ξ coincide in a neighbourhood of Ξ .

$$-\chi(e \cap \bar{\Gamma} \cap U(\Xi)) =$$

by duality,

$$\begin{aligned} & \#(\{l \mid l \subset \text{aff}(K) \text{ is an oriented line, } E \in l; l \text{ is a support line to } K\})/2 = \\ & 1 - K(E). \end{aligned}$$

The latter equality is well-known for convex polygons. Owing to linearity, it also is valid for virtual polygons. \square

Consider the following conditions for 3-dimensional convex polytopes K and L .

- (*): For each $\xi \in S^2$, $\dim(K^\xi) = 2 \Leftrightarrow \dim(L^\xi) = 2$. In this case, there exists a **unique** translation t such that either $tK^\xi \subset L^\xi, tK^\xi \neq L^\xi$ or $tK^\xi \supset L^\xi, tK^\xi \neq L^\xi$.

Definition 3.2. We say that the affine graph Γ of the support function of a hyperbolic polytope is *flat* at its vertex a , if

- locally (i. e. in a neighborhood of the point a lies in a closed semispace bounded by some plane e passing through a , and
- The set $U(a) \cap e \cap \Gamma$ has one connected component. ($U(a)$ is a deleted neighborhood of the point a .)

The conditions (*) can be reformulated in terms of hyperbolic polytopes.

Proposition 3.3. *Let K and L be 3-dimensional convex polytopes. They satisfy the condition (*) if and only if the three below statements hold.*

- The virtual polytope $H = K \otimes L^{-1}$ is hyperbolic.
- The fan of K is a refinement of the fan of H without Steiner points.
- The graph Γ_H is flat at each of its vertices.

Proof. This follows directly from Lemma 3.1 and Corollary 2.8. \square

4. THE MAIN EXAMPLE.

Example 4.1. We present an example of two 3D polytopes K and L such that

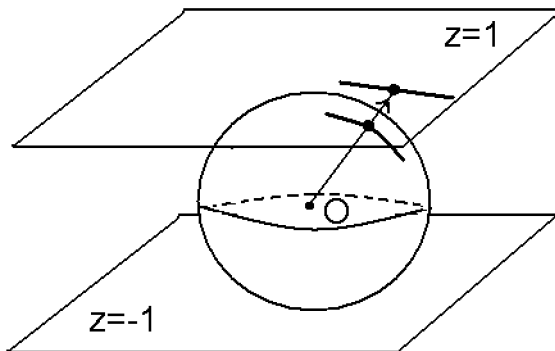
- The polytopes K and L are different.
- K^ξ is a facet $\Leftrightarrow L^\xi$ is a facet (such facets are called parallel).
- For any pair of parallel facets, there exists a unique translation t such either $tK^\xi \subset L^\xi, tK^\xi \neq L^\xi$, or $tL^\xi \subset K^\xi, tL^\xi \neq K^\xi$

Example 4.2. We present an example of a pointed tiling of S^2 generated by an embedded Laman-plus-one graph which can be regularly triangulated without Steiner points (i. e. without adding new vertices).

1. Initial regular triangulation.

Consider the collection of 12 great circles in S^2 as is depicted in Fig. 4.4. Here and in the sequel, instead of presenting spherical drawings, we draw the central projection of the upper hemisphere to the plane $z = 1$. In general, for the complete information of a spherical drawing one should consider also the central projection of the lower hemisphere to the plane $z = -1$ (Fig. 4.3) But since all the spherical pictures below are symmetric, we skip the drawings for the second plane.

Figure 4.3.



This collection of circles should be considered as an embedded graph.

Some vertices and some edges of the graph we mark bold (and call them bold).

Besides, we mark grey four spherical polygons (call them *grey di-gons*) bounded by four pairs of great semicircles.

The two vertices of such a di-gon we call the *ends of the grey di-gone*.

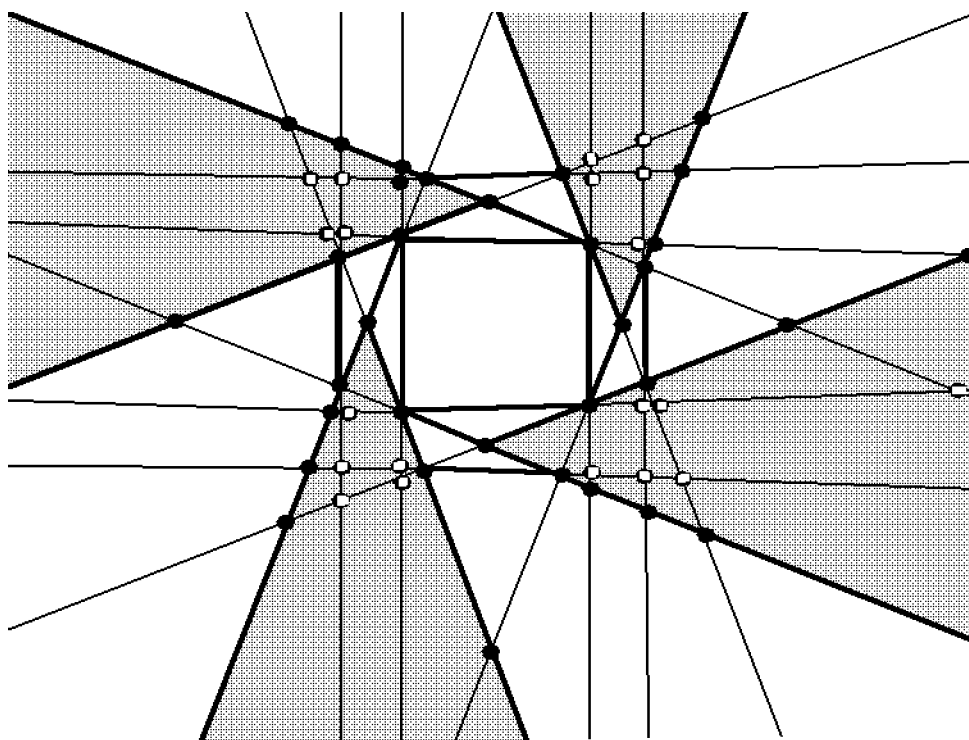
Some vertices are not depicted (for they do not fit the page). For them, the following rule holds: we mark a vertex bold whenever it lies on the boundary of a grey di-gon.

Denote by B the embedded graph of bold edges.

The construction has the following obvious properties:

- All bold vertices lie on the boundaries of grey polygons.
- Some vertices are not bold. Call them "white" and mark by small white circles.
- All bold vertices lie on bold edges.
- The embedding B is nearly pointed. This means that at each vertex, there is an adjacent angle which equals π .

Figure 4.4.



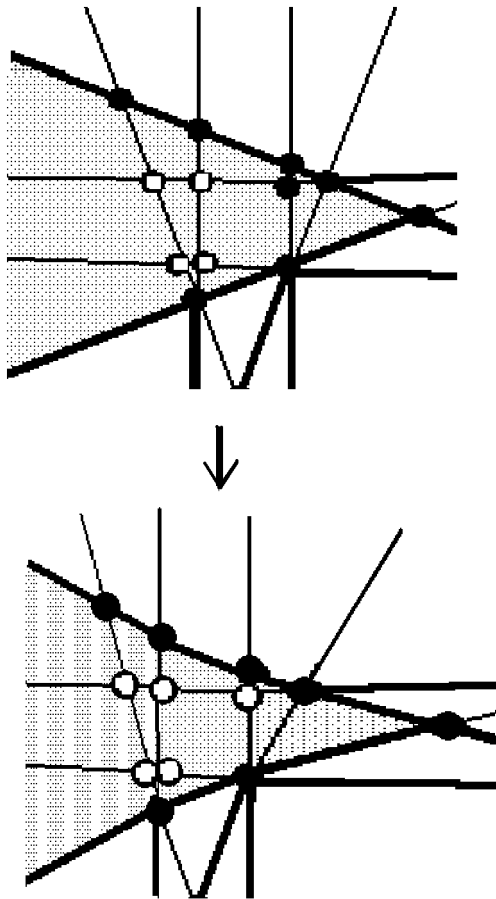
These 12 circles generate a regular tiling of S^2 (it is the fan of some zonotope). The tiling can be regularly triangulated without adding new vertices. Fix such a triangulation T .

2. Provide the pointed property of the bold graph.

Now move somewhat the bold points so that the resulted graph B' is pointed.

For this, turn grey polygons to pseudo di-gons and care about their ends as is shown in Fig. 4.5.

Figure 4.5.



3. Get rid of white vertices. The second example.

We can assume that the shift on the previous step is sufficiently small to keep the shifted triangulation T' regular. By Lemma 4.7, there is a regular triangulation of B' without new vertices at all. Denote it by T'' .

Add to the graph of B' new edges from T'' , maintaining the "pointed" property and denote the result by B'' . For this, we do not add any edges inside grey polygons and we add no new edges at the ends of the grey polygons.

We get an embedded graph. It is pointed by construction. The generated tiling consists of pseudo triangles and four pseudo di-gons. Therefore, we have a Laman-plus-one graph which gives us a hyperbolic polytope (by Lemmas 2.14, 2.15, and Theorem 2.11).

4. Pass to convex polytopes. The first example.

Let K be a convex polytope with the fan T'' , H be a hyperbolic polytope with the fan B'' .

We can assume that $K \otimes H$ is convex (if it is not, just increase K by a dilation).

The pair K and $L = H \otimes K$ of convex polytopes gives the desired example by Proposition 3.3.

Lemma 4.6. *Let H be a hyperbolic polytope with the fan B'' , let e be a plane. Then each vertex of affine graph $\Gamma(e)$ of its support function is flat.*

Proof. The grey polygons are the only four pseudo di-gons of the tiling B'' , therefore these are the pseudo di-gons from Theorem 2.16.

Take a vertex of the graph. it corresponds to a bold point lying on one of the pseudo di-gons.

Two cases are possible.

1. The vertex in question is not an endpoint of the pseudo di-gon. Then the flatness follows from Theorem 2.16.

2. The vertex in question is an endpoint of the pseudo di-gon. Then there are exactly three adjacent edges of the graph, and the flatness follows from Fig. 2.17. \square

Lemma 4.7. *Let D be a tiling of the sphere S^2 such that the following two conditions hold.*

1. *D has a regular subtriangulation T without new vertices on the edges of D .*

2. *Each tile of D fits some open hemisphere.*

Then D has a regular subtriangulation without new vertices at all.

Proof. Let h be a convex function which is piecewise linear with respect to T .

Consider a collection of affine graphs of h : take a plane in \mathbb{R}^3 , take the restriction of h on the plane, then take the graph Γ of the restriction. (Remind that the construction preserves convexity property: if all such graphs are concave down, then the function is convex.)

Now treat each tile t of D separately.

Choose a plane e such that the central projection maps the whole t on e . Take the corresponding graph Γ . It is a piecewise linear surface. Take its skeleton (the edges and the vertices), remove the vertices coming from the vertices of T , lying strictly inside D , and take the convex hull of the rest. The old edges do not disappear. We get some triangulation of the t . The construction described doesn't depend on the choice of e .

Treating this way all the tiles one by one, we get a convex function which is piecewise linear with respect to some subtriangulation of D . \square

REFERENCES

- [1] A.D.Alexandrov, A general uniqueness theorem for closed surfaces. Doklady Akad. Nauk SSSR 19, No. 4(1937), pp.227-229.
- [2] A.D.Alexandrov, On uniqueness theorems for closed surfaces. (Russian) Doklady Akad. Nauk SSSR 22 No. 3(1939), pp.99-102.
- [3] R.Haas, D. Orden, G. Rote, F Santos, B. Servatius, H. Servatius, D. Souvaine, I. Streinu, W. Whiteley, Planar minimally rigid graphs and pseudo-triangulations. (English) Comput. Geom. 31, No.1-2, 31-61 (2005).
- [4] A. Khovanskii, A. Pukhlikov, Finitely additive measures of virtual polytopes. St. Petersburg Math. J. 4, No. 2 (1993), pp. 337-356.
- [5] G. Laman, On graphs and rigidity of plane skeletal structures, J. Eng. Math. 4 (1970).
- [6] Y. Martinez-Maure, Contre-exemple à une caractérisation conjecturée de la sphère. C.R. Acad. Sci. Paris 332, No. 1 (2001), pp.41-44.
- [7] G.Panina, New counterexamples to A.D. Alexandrov's uniqueness hypothesis. Advances in Geometry, No. 5 (2005), pp. 301-317.
- [8] G.Panina, On hyperbolic virtual polytopes and hyperbolic fans. CESJM, 4(2006), No. 2, pp. 270-293.
- [9] G. Panina, A.D. Alexandrov's uniqueness theorem for convex polytopes and its refinements, to appear.
- [10] G. Panina, Planar pseudo-triangulations, spherical pseudo-tilings and hyperbolic virtual polytopes, www.arxiv.org, math.MG/0607171.
- [11] I. Streinu, Pseudo-triangulations, rigidity and motion planning. Discrete Comput. Geom. 34 (2005), no. 4, 587-635.

GAIANE PANINA, INSTITUTE FOR INFORMATICS AND AUTOMATION, V.O. 14 LINE 39, ST.PETERSBURG, 199178, RUSSIA

E-mail address: panina@iiias.spb.su