On the Monge–Ampère equation on Wiener space

V.I. Bogachev, A.V. Kolesnikov

We prove that the optimal transportation mapping that takes a Gaussian measure γ on an infinite dimensional space to an equivalent probability measure $g \cdot \gamma$ satisfies the Monge–Ampère equation provided that $\log g \in L^1(\gamma)$ and $g \log g \in L^1(\gamma)$.

The Monge–Kantorovich problem and the Monge–Ampère equation have become a very popular object of research in the last decade (see [1], [2], where one can find additional references). Let μ and ν be two probability measures on $\mathbb{R}^n$ such that

$$W(\mu,\nu)^2 = \inf_{m\in\mathcal{P}(\mu,\nu)} \int_{\mathbb{R}^n\times\mathbb{R}^n} |x_1 - x_2|^2 \, m(dx_1, dx_2) < \infty, \tag{1}$$

where $\mathcal{P}(\mu,\nu)$ is the set of all Borel probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ whose projections on the factors are μ and ν. Under broad assumptions on μ and ν (see the above cited books and [3]), there exists a Borel mapping T that takes μ to ν, i.e., $\mu \circ T^{-1} = \nu$, and satisfies the equality

$$\int |T(x) - x|^2 \, \mu(dx) = W(\mu,\nu)^2.$$

Therefore, the image of μ under the mapping $x \mapsto (x, T(x))$ gives minimum to the integral on the right-hand side in (1). According to [4], if μ is absolutely continuous, the optimal transportation T is the gradient of a convex function V. As shown in [5], if $\mu = f\,dx$ and $\nu = g\,dx$ are two absolutely continuous probability measures such that μ is equivalent to Lebesgue measure and V is a convex function such that ∇V takes μ to ν, then, letting $\det(\mathrm{D}^2_{\mathrm{ac}}V)$ be the determinant of the density $D^2_{\mathrm{ac}}V$ of the absolutely continuous part of D^2V (i.e., the determinant in A.D. Alexandroff's sense), we obtain that the set M of points where the matrix $D^2_{\mathrm{ac}}V$ is defined and invertible is of full μ-measure and for almost all $x \in M$ the following Monge–Ampère equation holds:

$$f(x) = g(\nabla V(x)) \det D^2_{\mathrm{ac}} V(x).$$

The regularity of solutions of the Monge–Ampère equation has been studied in many works, see references in [6].

Our work is concerned with the infinite dimensional case and its principal result is a derivation of the Monge–Ampère equation for transformations of Gaussian measures on infinite dimensional spaces. The assertions given below extend the results proved in the recent works [7], [8], [9]. We shall use the following important existence theorem from [7]. Let X be a locally convex space and let γ be a centered Radon Gaussian measure on X with the Cameron–Martin space H equipped with the inner product $\langle\,\cdot\,,\,\cdot\,\rangle_H$; the corresponding norm is denoted by $|\cdot|_H$. One may assume without loss of generality that $X = \mathbb{R}^\infty$, the countable power of the real line, and that γ is the countable power of the standard Gaussian measure; then $H = l^2$.

Suppose that we are given a probability measure $g\cdot\gamma$ such that

$$W_H(\gamma, g\cdot\gamma)^2 = \inf_{m\in\mathcal{P}(\gamma,g\cdot\gamma)} \int_{X\times X} |x_1-x_2|_H^2\, m(dx_1,dx_2) < \infty,$$

where $\mathcal{P}(\gamma, g\cdot\gamma)$ is the set of all Radon probability measures on $X\times X$ whose projections on the first and second factors are γ and $g\cdot\gamma$. Then there exists a unique measurable mapping (called the optimal transportation) $T\colon X\to X$ that sends γ to $g\cdot\gamma$ and satisfies the equality

$$W_H(\gamma, g\cdot\gamma)^2 = \int_X |T(x)-x|_H^2\, d\gamma.$$

An effective sufficient condition for $W_H(\gamma, g\cdot\gamma)$ to be finite is the finiteness of entropy of g, i.e., the integrability of $g\log g$. The optimal transportation has the form $T = I+\nabla\Phi$, where Φ belongs to the Sobolev class $W^{2,1}(\gamma)$ and is 1-convex (see the definition below). If $g>0$ γ-a.e. and $\log g\in L^1(\gamma)$, then T has an inverse mapping S, i.e., one has $T\circ S(x) = S\circ T(x) = x$ for γ-a.e. x. Moreover, S realizes the optimal transportation that takes $g\cdot\gamma$ to γ and $S = I+\nabla\Psi$, where $\Psi\in W^{2,1}(\gamma)$ is 1-convex.

Let us introduce some notation. Let $L^2(\gamma, H)$ denote the Hilbert space of H-valued γ-measurable mappings v with finite norm

$$\|v\|_{L^2(\gamma,H)} = \left(\int_X |v(x)|_H^2\, \gamma(dx)\right)^{1/2}.$$

The Hilbert–Schmidt norm of a symmetric operator A on H is defined by $\|A\|_{\mathcal{H}} = \left(\sum_{i=1}^\infty (Ae_i, Ae_i)\right)^{1/2}$, where $\{e_i\}$ is any orthonormal basis in H. Every vector $h\in H$ corresponds to a γ-measurable linear functional $\widehat{h}$ on X such that $\langle h,k\rangle_H = (\widehat{h},\widehat{k})_{L^2(\gamma)}$ for all $k\in H$ (see [10] for details). Set $x_i := \widehat{e_i}(x)$. As noted above, one may assume that we deal with the standard Gaussian product-measure on $\mathbb{R}^\infty$ and then $\widehat{e_i}$ is the usual ith coordinate function. The σ-algebra generated by $\widehat{e}_1,\ldots,\widehat{e}_n$ is denoted by $\mathcal{F}_n$. The space of smooth cylindrical functions, denoted by $\mathcal{F}C_0^\infty$, consists of all functions of the form $\zeta(x_1,\ldots,x_n)$, where $\zeta\in C_0^\infty(\mathbb{R}^n)$ for some n. The Sobolev class $W^{2,1}(\gamma)$ consists of all functions $f\in L^2(\gamma)$ that have a generalized gradient $\nabla f\in L^2(\gamma,H)$ along H such that

$$\int_X \partial_h\varphi\, f\, d\gamma = -\int_X \varphi\langle\nabla f, h\rangle_H\, d\gamma + \int_X \varphi f\widehat{h}\, d\gamma$$

for all $h\in H$ and $\varphi\in\mathcal{F}C_0^\infty$, where ∂_h is the partial derivative along h. The Sobolev class $W^{2,1}(\gamma,H)$ of H-valued mappings and the Sobolev class $W^{2,2}(\gamma)$ of functions that are twice differentiable along H with the second derivative in the space of Hilbert–Schmidt operators are defined in a similar way (see [10] for details). The

Ornstein–Uhlenbeck semigroup $\{P_t\}$ on $L^p(\gamma)$, $1 \le p < \infty$, is defined by

$$P_t f(x) := \int_X f\Big(e^{-t}x + \sqrt{1-e^{-2t}}\, y\Big)\, \gamma(dy).$$

Let $\mathcal{L}$ be the generator of $\{P_t\}$ on $L^2(\gamma)$. We recall that $\mathcal{L}$ is an extension of the operator

$$\Delta_H f - \big\langle x, \nabla_H f\big\rangle_H := \sum_{n=1}^{\infty} (\partial^2_{e_i} f - x_i \partial_{e_i} f)$$

acting on smooth cylindrical functions. Given a function $f \in L^1(\gamma)$ with $f \log f \in L^1(\gamma)$, one defines $\mathcal{L}f$ in the sense of distributions as the linear functional

$$\varphi \mapsto \int_X f \mathcal{L}\varphi \, d\gamma$$

on $\mathcal{F}C_0^\infty$ (note that $\mathcal{L}\varphi \in L^\infty(\gamma)$). If $f \in L^2(\gamma)$, then $\mathcal{L}f$ is a continuous linear functional on $W^{2,2}(\gamma)$. Convergence in the sense of distributions over (X, γ) is understood as pointwise convergence of linear functionals on $\mathcal{F}C_0^\infty$.

We recall the definition of a θ-convex function introduced in [11]. Let $F\colon X \to \mathbb{R} \cup \{\infty\}$ be a measurable mapping such that $\gamma(\{F < \infty\}) > 0$ and let $\theta \in \mathbb{R}^1$. Let

$$F_\theta\colon\ H \times X \to \mathbb{R} \cup \{\infty\}, \quad F_\theta(h, w+h) = \frac{\theta}{2}|h|_H^2 + F(w+h).$$

Then F is called θ-convex if for all $h, k \in H$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, one has

$$F_\theta(\alpha h + \beta k, w + \alpha h + \beta k) \le \alpha F_\theta(h, w+h) + \beta F_\theta(k, w+k) \quad \gamma\text{-a.e.},$$

where the measure zero set on which this inequality fails may depend on h, k and α. See [11] for some equivalent definitions.

A Radon measure μ on X is called Skorohod differentiable along a vector $h \in X$ if there exists a Radon measure $d_h\mu$ such that for every smooth cylindrical function ζ one has

$$\int_X \partial_h \zeta(x)\, \mu(dx) = -\int_X \zeta(x)\, d_h\mu(dx).$$

Note that γ is differentiable along any $h \in H$ and $d_h\gamma = -\widehat{h} \cdot \gamma$. The second order derivative is defined as $d^2_{hh}\mu := d_h(d_h\mu)$. In this paper we are especially interested in the second derivatives of the 1-convex potentials Φ and Ψ. In our case Φ and Ψ possess the first Sobolev derivatives $\nabla\Phi$ and $\nabla\Psi$ along H. We define Φ_{kh}, where $k, h \in H$, as a Radon measure satisfying the relation

$$\int_X \zeta(x)\, \Phi_{kh}(dx) = -\int_X \partial_h\zeta(x) \partial_k \Phi(x)\, \gamma(dx) + \int_X \zeta(x) \partial_k\Phi(x) \widehat{h}(x)\, \gamma(dx).$$

If $\partial_k\Phi$ is differentiable in the Sobolev sense, then, according to this definition, $\Phi_{kh} = \partial_h \partial_k \Phi \cdot \gamma$.

The density of the absolutely continuous part of Φ_{kh} (with respect to γ) is denoted by Φ^{ac}_{kh} and the singular part is denoted by Φ^{s}_{kh}. Note that since Φ is 1-convex, the measure Φ^{s}_{hh} is a nonnegative (Corollary 1). In the case when there exists an $\mathcal{H}$-valued measure with matrix elements $\Phi_{e_ie_j}$, we denote this measure by the symbol $D^2\Phi$. If $\sum_{i=1}^{\infty} |\Phi^{\mathrm{ac}}_{e_ie_i}(x)|^2 < \infty$ γ-a.e., then the $\mathcal{H}$-valued mapping with matrix elements $\Phi^{\mathrm{ac}}_{e_ie_j}$ is denoted by the symbol $D^2_{\mathrm{ac}}\Phi$ (even if $D^2\Phi$ does not exist; if the measure $D^2\Phi$ exists, then $D^2_{\mathrm{ac}}\Phi$ is the density of its absolutely continuous part with respect to γ). Below we give some sufficient conditions for the existence of $D^2\Phi$.

The conditional expectation of $f \in L^1(\gamma)$ with respect to $\mathcal{F}_n$ is denoted by $\mathbb{E}(f|\mathcal{F}_n)$. Set $P_n x = \sum_{i=1}^{n} \widehat{e}_i(x) e_i$. The operator $\mathbb{E}(\cdot|\mathcal{F}_n)$ extends to bounded Radon measures as follows: $\mathbb{E}(m|\mathcal{F}_n)$ is the restriction of a measure m to the σ-algebra $\mathcal{F}_n$. It is verified directly that $P_t\mathbb{E}(f|\mathcal{F}_n) = \mathbb{E}(P_t f|\mathcal{F}_n)$.

We consider the following problem: when do the potentials Φ and Ψ satisfy an infinite dimensional analog of the Monge–Ampère equation? The heuristic formulas for the Monge–Ampère equation are

$$g = \det{}_2(I + D^2\Psi) \exp\Bigl(\mathcal{L}\Psi - \frac{1}{2}|\nabla\Psi|^2_H\Bigr),$$

$$\frac{1}{g(T)} = \det{}_2(I + D^2\Phi) \exp\Bigl(\mathcal{L}\Phi - \frac{1}{2}|\nabla\Phi|^2_H\Bigr).$$

Here $\det_2$ denotes the Carleman–Fredholm determinant which is defined for any symmetric Hilbert–Schmidt operator Λ by the formula

$$\det{}_2(I + \Lambda) = \prod_{i=1}^{\infty} (1 + \lambda_i) e^{-\lambda_i},$$

where λ_i are the eigenvalues of Λ counted with their multiplicities. As the first step one has to show that all the objects involved in equalities (2) and (3) exist indeed. It has been shown in [7] that $\mathcal{L}\Phi$ (considered as a distribution on the space (X,γ)) is a Radon measure if $g < C$. The density of its absolutely continuous part with respect to γ is denoted by $\mathcal{L}_{\mathrm{ac}}\Phi$. One can show that $\mathcal{L}_{\mathrm{ac}}\Phi := \lim_{n\to\infty} \mathcal{L}_{\mathrm{ac}}\mathbb{E}(\Phi|\mathcal{F}_n)$ almost surely. Similarly, if $g > c > 0$, then $\mathcal{L}\Psi$ is a Radon measure, and $\mathcal{L}_{\mathrm{ac}}\Psi$ is the density of its absolutely continuous part with respect to γ. Another result from [7] states that if $0 < c < g < C$, then

$$g(T)\underline{\lim}_n \det{}_2[I + D^2_{\mathrm{ac}}\mathbb{E}(\Phi|\mathcal{F}_n)] \exp\Bigl(\mathcal{L}_{\mathrm{ac}}\Phi - \frac{1}{2}|\nabla\Phi|^2_H\Bigr) \le 1,$$

$$g \ge \underline{\lim}_n \det{}_2[I + D^2_{\mathrm{ac}}\mathbb{E}(\Psi|\mathcal{F}_n)] \exp\Bigl(\mathcal{L}_{\mathrm{ac}}\Psi - \frac{1}{2}|\nabla\Psi|^2_H\Bigr).$$

We see that these results give only inequalities instead of the expected equalities. However, according to [8], if $-\log g$ is an H-convex function (which is, certainly, a very strong restriction on g), then the infinite dimensional Monge–Ampère equation holds.

The main result of this paper is the following theorem.

Theorem 1. (i) *Suppose that* $\log g \in L^1(\gamma)$ *and* $g \log g \in L^1(\gamma)$. *Then there exist* $\mathcal{H}$*-valued mappings* $D^2_{\mathrm{ac}}\Psi$ *and* $D^2_{\mathrm{ac}}\Phi$ *with matrix elements* $\Phi^{\mathrm{ac}}_{e_ie_j}$ *and* $\Psi^{\mathrm{ac}}_{e_ie_j}$ *and a subsequence* $\{n_k\}$ *such that* γ*-a.e. there exist finite limits*

$$\mathcal{L}_0\Psi = \lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n_k}\bigl(\Psi^{\mathrm{ac}}_{e_ie_i} - x_i\partial_{e_i}\Psi\bigr), \quad \mathcal{L}_0\Phi = \lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n_k}\bigl(\Phi^{\mathrm{ac}}_{e_ie_i} - x_i\partial_{e_i}\Phi\bigr).$$

In addition, one has

$$g = \det{}_2(I + D^2_{\mathrm{ac}}\Psi)\exp\Bigl(\mathcal{L}_0\Psi - \frac{1}{2}|\nabla\Psi|^2_H\Bigr),$$

$$\frac{1}{g(T)} = \det{}_2(I + D^2_{\mathrm{ac}}\Phi)\exp\Bigl(\mathcal{L}_0\Phi - \frac{1}{2}|\nabla\Phi|^2_H\Bigr).$$

Furthermore, $(I + D^2_{\mathrm{ac}}\Psi)(I + D^2_{\mathrm{ac}}\Phi(S)) = (I + D^2_{\mathrm{ac}}\Phi(S))(I + D^2_{\mathrm{ac}}\Psi) = I$.

(ii) *Suppose that* $g > c > 0$ *and* $g\log g \in L^1(\gamma)$. *Then* $\mathcal{L}_0\Psi = \mathcal{L}_{\mathrm{ac}}\Psi$ *and*

$$g = \det{}_2(I + D^2_{\mathrm{ac}}\Psi)\exp\Bigl(\mathcal{L}_{\mathrm{ac}}\Psi - \frac{1}{2}|\nabla\Psi|^2_H\Bigr).$$

(iii) *Suppose that* $0 < g < C$ *and* $\log g \in L^1(\gamma)$. *Then* $\mathcal{L}_0\Phi = \mathcal{L}_{\mathrm{ac}}\Phi$ *and*

$$\frac{1}{g(T)} = \det{}_2(I + D^2_{\mathrm{ac}}\Phi)\exp\Bigl(\mathcal{L}_{\mathrm{ac}}\Phi - \frac{1}{2}|\nabla\Phi|^2_H\Bigr).$$

All these equalities hold almost everywhere.

The main result follows from a series of auxiliary assertions presented below. Let us consider a probability measure $g\cdot\gamma$ and approximations of g by functions $g_n \to g$ such that every g_n is measurable with respect to $\mathcal{F}_n$. We shall consider the approximations $P_{\frac{1}{n}}\mathbb{E}(g|\mathcal{F}_n)$ and $\mathbb{E}(g|\mathcal{F}_n)$. Let $\{T_n\}$ and $\{S_n\}$ be the sequences of optimal transportations sending γ to $g_n\cdot\gamma$ and $g_n\cdot\gamma$ to γ, accordingly. By the finite dimensional case

$$T_n = I + \nabla\Phi_n, \quad S_n = I + \nabla\Psi_n,$$

where Φ_n and Ψ_n are 1-convex functions. It is clear that

$$T_n \circ S_n(x) = S_n \circ T_n(x) = x \quad \gamma\text{-a.e.}$$

According to [7], one has $T_n \to T$ and $S_n \to S$ in measure γ. Passing to a subsequence we may assume that there holds a.e. convergence. The potentials Φ and Ψ are not uniquely determined, because adding constants one does not change their gradients. Throughout we use the convention that the constants are chosen in such a way that the integrals of Φ and Ψ against γ vanish (and the same for the approximations). With this convention, we obtain by the Poincaré inequality that $\Phi_n \to \Phi$ and $\Psi_n \to \Psi$ in $L^2(\gamma)$.

The following important identity was proved in [9]:

$$\int_X \log\frac{g_m}{g_n} g_m\, d\gamma = \frac{1}{2}\int_X |S_n - S_m|_H^2 g_m\, d\gamma + \int_X \Big[\mathrm{Tr}\big(DS_n(DS_m)^{-1} - I\big) - \log\det\big(DS_n(DS_m)^{-1}\big)\Big] g_m\, d\gamma,$$

where DF denotes the derivative of a mapping F. Note that both integrands are nonnegative. Letting $g_n = 1$ or $g_m = 1$ we obtain the following relations:

$$\int_X \log\frac{1}{g_n}\, d\gamma = \frac{1}{2}\int_X |S_n(x) - x|_H^2\, \gamma(dx) - \int_X \log\det{}_2 DS_n\, d\gamma,$$

$$\int_X g_m \log g_m\, d\gamma = \frac{1}{2}\int_X |S_m(x) - x|_H^2 g_m\, \gamma(dx) - \int_X \log\det{}_2\big[(DS_m)^{-1}\big] g_m\, d\gamma.$$

These formulas give the following estimates of the transportation cost:

$$\frac{1}{2}\int_X |\nabla\Psi_n|_H^2\, d\gamma \le \int_X \log\frac{1}{g_n}\, d\gamma, \quad \frac{1}{2}\int_X |\nabla\Phi_n|_H^2\, d\gamma \le \int_X g_n \log g_n\, d\gamma.$$

The latter estimate is Talagrand's inequality [12].

Theorem 2. *Let $\log g, g\log g \in L^1(\gamma)$ and $g_n = P_{1/n}\mathbb{E}(g|\mathcal{F}_n)$. Then there exists measurable mappings K and L with values in the space of symmetric Hilbert–Schmidt operators such that, for some subsequence $\{n_k\}$, the mappings $DS_{n_k} - I = D^2\Psi_{n_k}$ and $(DS_{n_k})^{-1} - I$ converge γ-a.e. to $(I+K)^2 - I$ and $(I+L)^2 - I$ in the Hilbert–Schmidt norm. Moreover,*

$$(I+K)(I+L) = (I+L)(I+K) = I$$

and the following inequalities hold:

$$\int_X \log\frac{1}{g}\, d\gamma \ge \frac{1}{2}\int_X |\nabla\Psi|_H^2\, d\gamma - \int_X \log\det{}_2\big[(I+K)^2\big]\, d\gamma,$$

$$\int_X g\log g\, d\gamma \ge \frac{1}{2}\int_X |\nabla\Phi|_H^2\, d\gamma - \int_X \log\det{}_2\big[(I+L)^2\big] g\, d\gamma.$$

In particular,

$$\int_X \Big(\|K\|_{\mathcal{H}}^2 + \|L\|_{\mathcal{H}}^2\Big) \min(1, g)\, d\gamma < \infty.$$

An important step in the proof of Theorem 1 is to show that the operators $(I + K(x))^2 - I$ and $\big[I + L(T(x))\big]^2 - I$ coincide a.e. with $D^2_{\mathrm{ac}}\Psi(x)$ and $D^2_{\mathrm{ac}}\Phi(x)$, respectively. This step, however, employs Theorem 2 in the foregoing formulation.

Corollary 1. *Let $\log g \in L^1(\gamma)$ and $g\log g \in L^1(\gamma)$. Then, for any $h, k \in H$, there exist bounded Radon measures Ψ_{hk} and Φ_{hk} and one has*

$$\Psi_{hk} = \frac{1}{2}\big[\Psi_{(h+k)(h+k)} - \Psi_{hh} - \Psi_{kk}\big], \quad \Phi_{hk} = \frac{1}{2}\big[\Phi_{(h+k)(h+k)} - \Phi_{hh} - \Phi_{kk}\big].$$

In addition, the measures Φ^s_{hh} and Ψ^s_{hh} are nonnegative and one has

$$\Phi^{\mathrm{ac}}_{hh} \geq l_h(T) \quad \gamma\text{-a.e.}, \quad \Psi^{\mathrm{ac}}_{hh} \geq k_h \quad \gamma\text{-a.e.},$$

where

$$k_h := \big\langle ((I+K)^2 - I)h, h \big\rangle_H, \quad l_h := \big\langle ((I+L)^2 - I)h, h \big\rangle_H.$$

Corollary 2. (i) *Suppose that $g \geq c > 0$ for some constant c and $g \log g \in L^1(\gamma)$. Then there exists an $\mathcal{H}$-valued measure $D^2\Psi$ of bounded variation.*

(ii) *Suppose that $0 < g \leq C$ for some constant C and $\log g \in L^1(\gamma)$. Then there exists an $\mathcal{H}$-valued measure $D^2\Phi$ of bounded variation.*

The following lemma is a generalization of Lemma 7.2 in [7].

Lemma 1. *Let $\log g \in L^2(\gamma)$. Then $\mathcal{L}\Psi$ is a bounded Radon measure and γ-a.e. one has $\mathcal{L}_{\mathrm{ac}}\mathbb{E}(\Psi|\mathcal{F}_n) \to \mathcal{L}_{\mathrm{ac}}\Psi$. If $|\log g|^2 g \in L^1(\gamma)$, then $\mathcal{L}\Phi$ is a bounded Radon measure and γ-a.e. one has $\mathcal{L}_{\mathrm{ac}}\mathbb{E}(\Phi|\mathcal{F}_n) \to \mathcal{L}_{\mathrm{ac}}\Phi$.*

Lemma 2. *Let $F\colon \mathbb{R}^d \to \mathbb{R}^d$ be a locally integrable mapping such that its generalized derivative DF is a locally bounded measure with values in the space of nonnegative symmetric matrices. Let $D_{\mathrm{ac}}F$ be the operator-valued density of the absolutely continuous component of DF and let $\Omega := \{x\colon \det D_{\mathrm{ac}}F(x) > 0\}$. Then the measure $\lambda|_\Omega \circ F^{-1}$ is absolutely continuous.*

Lemma 3. (i) *Let $g > c > 0$ and $g \log g \in L^1(\gamma)$. Then the series $\sum_{i=1}^\infty \Psi^s_{e_ie_i}$ converges in variation to a bounded nonnegative Borel measure on X.*

(ii) *If $0 < g \leq C$ and $\log g \in L^1(\gamma)$, then the series $\sum_{i=1}^\infty \Phi^s_{e_ie_i}$ converges in variation to a bounded nonnegative Borel measure on X.*

Proposition 1. *Let $X = \mathbb{R}^d$ and let γ be the standard Gaussian measure.*

(i) *Let $g = e^\Phi$ and $D^2\Phi \leq M$, where $M < 1$. Then $I + D^2\Phi \leq \frac{1}{\sqrt{1-M}} I$. In addition,*

$$\int_{\mathbb{R}^d} |\nabla\Phi|^2\, d\gamma + (1-M)\int_{\mathbb{R}^d} \|D^2\Phi\|^2_{\mathcal{H}}\, d\gamma \leq 2\int_{\mathbb{R}^d} g\log g\, d\gamma.$$

(ii) *Let $g = e^\Psi$ and $D^2\Psi \geq -M$, where $M > -1$. Then $I + D^2\Psi \leq \sqrt{1+M}\cdot I$. In addition,*

$$\int_{\mathbb{R}^d} |\nabla\Psi|^2\, d\gamma + \frac{1}{1+M}\int_X \|D^2\Psi\|^2_{\mathcal{H}}\, d\gamma \leq 2\int_{\mathbb{R}^d} \log\frac{1}{g}\, d\gamma.$$

Corollary 3. *Let g be a probability density with respect to γ. Suppose that $g = e^{-V}$, where the function V is $(1-\varepsilon)$-convex, $\varepsilon > 0$. Then $\Phi \in W^{2,2}(\gamma)$. If $g = e^W$, where W is an M-convex function for some $M > -1$, then $\Psi \in W^{2,2}(\gamma)$.*

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Department of Mechanics and Mathematics, Moscow State University, 119992 Moscow, Russia

Scuola Normale Superiore, Centro di Ricerca Matematica Ennio De Giorgi, I-56100 Pisa, Italy, E-mail: sascha77@mail.ru