SECOND DERIVATIVES OF CONVEX FUNCTIONS IN THE SENSE OF A.D. ALEXANDROFF ON INFINITE DIMENSIONAL SPACES WITH MEASURES

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ABSTRACT. We consider convex functions on infinite dimensional spaces equipped with measures. Our main results give some estimates of the first and second derivatives of a convex function, where second derivatives are considered from two different points of view: as point functions and as measures.

INTRODUCTION

In recent years several works have been published on infinite dimensional extensions of the classical result of A.D. Alexandroff on the second order differentiability of convex functions and related problems (see [1], [2], [3]). Let us recall that according to Alexandroff's theorem [4] (in the two dimensional case obtained by Busemann and Feller [5]; for a proof see Ch. 6 in [6], §5.4 in [7], Appendix 2 in [8]), every finite convex function F on $\mathbb{R}^n$ has a second order derivative D^2F (the so called Alexandroff second derivative) at almost every point x in the following sense:

$$F(x+h) - F(x) - DF(x)h - \frac{1}{2}(D^2F(x)h, h) = o(|h|^2), \quad h \to 0,$$

where the first derivative DF exists almost everywhere due to the local Lipschitzness of F. The mapping D^2F with values in the space of nonnegative symmetric matrices is the density of the absolutely continuous component of the locally bounded matrix-valued Borel measure whose matrix elements are the mixed second order derivatives of F in the sense of generalized functions. In addition, for almost all x and all $h, k \in \mathbb{R}^n$ there exists a limit

$$\partial_h \partial_k F(x) := \frac{1}{2} \lim_{t\to 0} t^{-2} \Big[F(x+th+tk) + F(x-th-tk) - F(x+th) - F(x-th) - F(x+tk) - F(x-tk) + 2F(x) \Big],$$

and one has $\partial_h \partial_k F(x) = (D^2F(x)h, k)$. It may happen that DF exists not for all points in a neighborhood of x (the domain of definition of the first derivative of a convex function may fail to have inner points at all), so $D^2F(x)$ cannot be always defined as a derivative of the mapping DF, but the indicated limit enables one to define the second order derivative without reference to generalized derivatives.

It turns out that the Alexandroff theorem has no direct extension to infinite dimensions, although a number of interesting positive results have been proved. One of the negative results is that, given a nice measure μ on an infinite dimensional separable Hilbert space X, one can find a convex function that has no Alexandroff's second derivative at almost every point with respect to μ. The situation is similar to that of the Fréchet differentiability of a Lipschitzian function; moreover, a convex Lipschitzian function that fails

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to be Fréchet differentiable μ-a.e. provides a counter-example for the second order differentiability. It is known, however, that the situation with the Fréchet differentiability of Lipschitzian functions changes if one considers the differentiability along a compactly embedded subspace $E \subset X$. Then, for any sufficiently regular measure μ, one obtains the Fréchet differentiability along E almost everywhere with respect to μ (see [9], §5.11 in [10] or [11]). In this paper, we investigate along the same lines the second order differentiability of convex functions. A study of convexity along a smaller subspace has been undertaken in [12], [13], where H-convex functions have been introduced in the case of a Gaussian measure with the Cameron–Martin space H. Here we are concerned with more general measures and mostly deal with convexity on the lines parallel to a given vector along which the measure is differentiable. Our main results give some estimates of the first and second derivatives of a convex function, where second derivatives are considered from two different points of view: as point functions and as measures.

1. Terminology and auxiliary results

Throughout the term *a Radon measure* μ on a locally convex space X means a bounded (possibly signed) Borel measure which is compact inner regular, i.e., for every Borel set B and every $\varepsilon > 0$, there is a compact set $K \subset B$ such that $|\mu|(B\backslash K) < \varepsilon$, where $|\mu| = \mu^+ + \mu^-$ and $\mu = \mu^+ - \mu^-$ is the Jordan–Hahn decomposition (see [14]). Let $\|\mu\|$ denote the total variation of μ, i.e., $\|\mu\| = |\mu|(X)$. Given a μ-integrable function g, we denote by $g\mu$ the measure with density g with respect to μ. A function f on a locally convex space X is called smooth cylindrical if

$$f(x) = \varphi(l_1(x), \ldots, l_n(x)), \quad \varphi \in C_b^\infty(\mathbb{R}^n), l_i \in X^*.$$

We recall that a Radon measure μ on a locally convex space X is called differentiable along a vector $h \in X$ in the sense of Skorohod (or Skorohod differentiable) if there exists a measure $d_h\mu$ (called the Skorohod derivative of μ along h) such that for every smooth cylindrical function f one has

$$\int \partial_h f(x)\, \mu(dx) = -\int f(x)\, d_h\mu(dx),$$

where $\partial_h f(x) := \lim\limits_{t\to 0} t^{-1}(f(x+th) - f(x))$. If $d_h\mu \ll \mu$, then μ is called Fomin differentiable. In that case, the Radon–Nikodym derivative β_h^μ of $d_h\mu$ with respect to μ is called the logarithmic derivative of μ along h. The terminology is explained by the fact that in the case when $X = \mathbb{R}$ and $h = 1$, the measure μ is Fomin differentiable precisely when μ has an absolutely continuous density ϱ with $\varrho' \in L^1(\mathbb{R})$; then $\beta_1^\mu = \varrho'/\varrho$. The existence of a Skorohod derivative in the one dimensional case is equivalent to the existence of a density ϱ of bounded variation. Then $d_1\mu$ is the derivative of μ in the sense of generalized functions. The situation is similar in $\mathbb{R}^n$, e.g., μ is Fomin differentiable along n linearly independent directions if and only if μ has a density ϱ which belongs to the Sobolev class $W^{1,1}(\mathbb{R}^n)$; then $d_h\mu = \partial_h\varrho\, dx$ and $\beta_h^\mu = \partial_h\varrho/\varrho$.

The shift of a measure μ on X along a vector h, i.e., the measure $B \mapsto \mu(B-h)$, is denoted by μ_h. If μ_{th} is equivalent to μ for all real t, then μ is called quasi-invariant along h. In this case we denote by ϱ_h the density of μ_h with respect to μ. If a measure μ is quasi-invariant along h or Skorohod differentiable along h, then it is continuous along h, i.e., there holds the equality $\lim\limits_{t\to 0}\|\mu - \mu_{th}\| = 0$. A measure on $\mathbb{R}^n$ is continuous along

n linearly independent vectors precisely when it is absolutely continuous. The quasi-invariance of a measure on $\mathbb{R}^n$ along n linearly independent vectors is equivalent to the existence of a density that does not vanish almost everywhere.

Higher order derivatives are defined inductively; e.g., $d_h^2\mu := d_h(d_h\mu)$. If $h, k \in X$ are such that both $d_h d_k\mu$ and $d_k d_h\mu$ exist, then one can show that $d_h d_k\mu = d_k d_h\mu$, and we say that $d_{hk}^2\mu := d_h d_k\mu = d_k d_h\mu$ exists. It is worth noting that if μ is differentiable (in Skorohod's or Fomin's sense) along h and k, then it is differentiable in the same sense along any linear combination of h and k and $d_{sh+tk}\mu = sd_h\mu + td_k\mu$. If μ is twice Fomin differentiable along h, then the density of $d_h^2\mu$ with respect to μ is denoted by $\beta_{h,h}^\mu$. We observe that if β_h^μ is in $L^2(\mu)$ and has a μ-integrable partial derivative $\partial_h\beta_h^\mu$, then μ is twice Fomin differentiable along h and $d_h^2\mu = [\partial_h\beta_h^\mu + (\beta_h^\mu)^2]\mu$.

Another useful fact that we employ below is that if μ is differentiable along h and k, then one can find differentiable (in the same sense) conditional measures on the planes parallel to the linear span L of h and k. More precisely, let Y be any closed linear subspace in X such that X is a topological sum of L and Y. Let ν be the image of $|\mu|$ under the natural projection to Y. Then one can find measures μ^y, $y \in Y$, on the subspaces $y + L$ that are differentiable along h and k in the same sense as μ and

$$\mu(B) = \int_Y \mu^y(B)\,\nu(dy), \quad B \in \mathcal{B}(X).$$

The same is true in the case of quasi-invariance or continuity along h and k. The aforementioned facts can be found, e.g., in [11], [15].

A Radon probability measure μ on X is called centered Gaussian if, for every continuous linear functional f on X, the induced measure $\mu \circ f^{-1}$ on the line is centered Gaussian. Given $h \in X$, let us set

$$|h|_H := \sup\biggl\{ f(h)\colon\ f \in X^*, \int f(x)^2\,\mu(dx) \le 1 \biggr\}.$$

The space

$$H = H(\mu) := \bigl\{ h \in X\colon\ |h|_H < \infty \bigr\}$$

is called the Cameron–Martin space of μ. It is known that H with the norm $|\cdot|_H$ is a separable Hilbert space and its natural embedding into X is compact. If X is a Hilbert space, then $H = R(X)$, where R is the Hilbert–Schmidt operator given by

$$(Rh, Rk) = \int (h,x)(k,x)\,\mu(dx).$$

To every $h \in H$, there corresponds a unique element $\widehat{h}$ in the closure of X^* in $L^2(\mu)$ specified by the equality

$$\int_X \widehat{h}(x) f(x)\,\mu(dx) = f(h), \quad f \in X^*.$$

The element $\widehat{h}$ has a linear version and is called the measurable linear functional generated by h. It is known that a centered Gaussian measure μ is infinitely differentiable along all vectors $h \in H = H(\mu)$ and $\beta_h^\mu = -\widehat{h}$. In addition, μ is quasi-invariant along h and the density of μ_h with respect to μ is given by $\varrho_h(x) = \exp\bigl[\widehat{h}(x) - |h|_H^2/2\bigr]$. The Ornstein–Uhlenbeck semigroup $(T_t)_{t\ge 0}$ on $L^1(\mu)$ is defined by the formula

$$T_t\psi(x) = \int \psi\Bigl(e^{-t}x + \sqrt{1 - e^{-2t}}y\Bigr)\,\mu(dy).$$

For these and additional facts on Gaussian measures, see, e.g., [10].

A function F on X will be called convex along a linear subspace E (or E-convex) if $h \mapsto F(x+h)$ is convex on E for every $x \in X$. If E is endowed with a norm $|\cdot|_E$, then we say that an E-convex function F has a second order derivative along E at a point x if there exist $l_x \in \partial F(x)$, where $\partial F(x)$ is the subdifferential of F at x, and a bounded linear operator $T_x\colon E \to E^*$ such that for each $h \in E$ one has

$$F(x+th) - F(x) = tl_x(h) + \frac{t^2}{2}T_x(h)(h) + o(t^2), \quad t \to 0.$$

If F is Gâteaux differentiable along E at the point x, then $\partial F(x)$ consists of a single element $D_E F(x) \in E^*$. Set $D^2F(x) := T_x$.

Now let L be the linear span of vectors h and k along which μ is continuous and let F be a real valued μ-measurable function on X such that the restriction of F to $y+L$ is convex for ν-a.e. $y \in Y$ (this is equivalent to the convexity of restrictions to $x+L$ for μ-a.e. $x \in X$). According to the Alexandroff theorem cited above, this restriction has a second order derivative at almost every point $x \in L$, where L is equipped with Lebesgue measure induced by the isomorphism of L and $\mathbb{R}^d$, $d = \dim L$. This means that the limit

$$\begin{aligned}\partial_h\partial_k F(x) :=& \frac{1}{2}\lim_{t\to 0}\ t^{-2}\Big[F(x+th+tk) + F(x-th-tk)\\ &- F(x+th) - F(x-th) - F(x+tk) - F(x-tk) + 2F(x)\Big]\end{aligned}$$

exists almost everywhere on L. Hence $\partial_h\partial_k F$ exists μ-a.e. For the same reason,

$$\partial_h^2 F(x) := \lim_{t\to 0} t^{-2}[F(x+th) + F(x-th) - F(x+th) - 2F(x)]$$

exists μ-a.e. In addition, $\partial_h^2 F(x) \geq 0$.

It should be noted that $\partial_h^2 F$ may be different from the second derivative of F in the sense of generalized functions. For example, if f is the usual Cantor function on $[0,1]$ and

$$F(x) = \int_0^x f(t)\,dt,$$

then $F'' = 0$ a.e., but F'' is not zero in the sense of distributions. As we have mentioned above, if F is a finite convex function on $\mathbb{R}^n$, then there exist locally bounded Borel measures F_{ij} on $\mathbb{R}^n$ such that the generalized derivative $\partial_{x_i}\partial_{x_j}F$ is the measure F_{ij}, and the matrix $(F_{ij}(B))_{i,j\leq n}$ is nonnegative for every Borel set B. One can write the decomposition $F_{ij} = F_{ij}^{ac}\,dx + F_{ij}^{sing}$, $F_{ij}^{ac} \in L^1_{loc}(\mathbb{R}^n)$, into the absolutely continuous and singular parts and then almost everywhere

$$\begin{aligned}F_{ij}^{ac}(x) =& \lim_{t\to 0} t^{-2}\Big[F(x+te_i+te_j) + F(x-te_i-te_j)\\ &- F(x-te_i) - F(x-te_j) + 2F(x)\Big],\end{aligned}$$

where $e_1, \ldots, e_n$ is the standard basis in $\mathbb{R}^n$. One of our goals is to obtain a similar decomposition in the infinite dimensional case. The main point is to find a suitable analogue of second generalized derivatives. Our approach is as follows. Suppose that in the finite dimensional case we replace Lebesgue measure by some probability measure μ with a nice density ϱ. Then the generalized second derivative of F can be expressed via ϱ. Namely, we can consider the measure $F\mu$ (i.e., the measure with density F with respect to μ), look at its generalized second order derivatives $\partial_{x_i}^2(F\mu)$, which are locally finite measures. If F is twice differentiable in the usual sense, then we can recover $\partial^2_{e_i}F$ from the expression

$$(\partial^2_{e_i}F)\mu = \partial^2_{x_i}(F\mu) - 2\partial_{x_i}F\partial_{x_i}\mu - F\partial^2_{x_i}\mu,$$

where the right hand side exists as a locally bounded measure for any convex F. In general, however, the right hand side is not absolutely continuous and $\partial^2_{e_i}F$ has to be recovered from its absolutely continuous part. We shall follow this approach also in infinite dimensions.

Although the pointwise second order derivative does not completely characterize the function, it is of interest to have the integrability of the function $\partial^2_h F$. The next two lemmas give sufficient conditions for the integrability of $\partial_h F$ and $\partial^2_h F$ along with a bit stronger version of the above mentioned finite dimensional differentiability.

Lemma 1.1. *Let a nonnegative Radon measure μ on a locally convex space X be continuous along a vector h and let a function $F \in L^p(\mu)$ be convex on μ-almost all straight lines $x + \mathbb{R}^1 h$. Then $\partial_h F$ exists μ-a.e. If μ is quasi-invariant along h and the Radon–Nikodym densities ϱ_h and ϱ_{-h} of the measures μ_h and μ_{-h} with respect to μ belong to $L^s(\mu)$, where $s = p(p-r)^{-1}$ and $r \in [1, p)$, then $\partial_h F$ belongs to $L^r(\mu)$. For the inclusion $\partial_h F \in L^1(\mu)$ it suffices to have the μ-integrability of the functions $F \ln |F|$, $\varrho_h \exp \varrho_h$, $\varrho_{-h} \exp \varrho_{-h}$ or the μ-integrability of the functions $|F| \exp \sqrt{|\ln |F||/\varepsilon}$, $\varrho_h \exp(\varepsilon |\ln \varrho_h|^2)$, $\varrho_{-h} \exp(\varepsilon |\ln \varrho_{-h}|^2)$ for some $\varepsilon > 0$.*

In particular, if μ is a centered Gaussian measure and $h \in H(\mu)$, then $\partial_h F \in L^{p-\varepsilon}(\mu)$ for every $\varepsilon > 0$. If $F \exp \sqrt{|c \ln |F||} \in L^1(\mu)$, where $c > 2|h|^2_H$, then $\partial_h F \in L^1(\mu)$.

Proof. In the one dimensional case the function F is locally Lipschitzian, and by the convexity of F, almost everywhere there holds the estimate

$$|F'(x)| \le |F(x-1)| + |F(x)| + |F(x+1)|.$$

Indeed, if, say, $F'(x) > 0$, then $F(x) + F'(x) \le F(x+1)$. By using the absolute continuity of the conditional measures on the straight lines parallel to h, we obtain that $\partial_h F(x)$ exists μ-a.e. and

$$|\partial_h F(x)| \le |F(x-h)| + |F(x)| + |F(x+h)|$$

for μ-a.e. x. It remains to estimate the L^r-norms of the functions on the right-hand side by Hölder's inequality taking into account that the integral of $|F(x+h)|^r$ equals the integral of $|F(x)|^r \varrho_h(x)$. If $F \ln |F| \in L^1(\mu)$, then we use the estimate $|F|\varrho_h \le |F||\ln |F|| + \varrho_h \exp \varrho_h$ and an analogous estimate with ϱ_{-h}. The second sufficient condition for the integrability of $\partial_h F$ indicated in the lemma is considered in a similar manner. In the Gaussian case, one has $\ln \varrho_h = \widehat{h} - |h|^2_H/2$, so $\varrho_h \exp(c|\ln \varrho_h|^2) \in L^1(\mu)$ $c < (2|h|^2_H)^{-1}$. □

Lemma 1.2. *Let μ be a nonnegative Radon measure on a locally convex space X.*

(i) *Let h_n be a sequence of vectors along which μ is continuous and let L be the linear span of $\{h_n\}$. Suppose a measurable function F is convex along L. Then μ-a.e. F has the second derivative along every finite dimensional subspace in L.*

(ii) *Let $F \in L^p(\mu)$ for some $p > 1$ and $h \in X$. Assume that F is convex on μ-almost all straight lines $x + \mathbb{R}^1 h$, μ is quasi-invariant along h and that for the Radon–Nikodym density ϱ_{th} of μ_{th} with respect to μ we have*

$$\left| t^{-2}[\varrho_{th}(x) + \varrho_{-th}(x) - 2] \right| \le G(x),$$

where $G \in L^{p'}(\mu)$, $p' = p/(p-1)$. Then the limit

$$\partial^2_h F(x) = \lim_{t \to 0} \frac{F(x+th) + F(x-th) - 2F(x)}{t^2},$$

which exists μ-a.e., defines a nonnegative μ-integrable function. In particular, this assertion is true if μ is a centered Gaussian measure with the Cameron–Martin space H and $h \in H$.

Proof. (i) It is verified in the proof of the Alexandroff theorem in [6] that the second order derivative of a convex function on $\mathbb{R}^n$ exists at a point x provided that x is a Lebesgue point for the first derivative of F and for the absolutely continuous part of the second order derivative and, in addition, the singular component ζ of the second order derivative satisfies the condition $\lim_{r\to 0} |\zeta|(B(x,r))r^{-n} = 0$, where $B(x,r)$ is the closed ball of radius r centered at x. By the absolute continuity of the conditional measures, one obtains all the three conditions a.e. on every finite dimensional subspace which is a shift of the linear span of $h_1, \ldots, h_n$.

(ii) The fact that $\partial_h^2 F(x) \geq 0$ whenever it exists, follows by convexity. In order to show that $\partial_h^2 F$ is integrable, it suffices, by Fatou's theorem, to obtain an upper bound on the integrals of

$$g_n(x) := n^2[F(x+n^{-1}h) + F(x-n^{-1}h) - 2F(x)].$$

We have

$$\begin{aligned}\int g_n(x)\,\mu(dx) &= \int F(y)n^2[\varrho_h(y-n^{-1}h) + \varrho_h(y+n^{-1}h) - 2]\,\mu(dy)\\ &\leq \int |F(y)|G(y)\,\mu(dy).\end{aligned}$$

In the Gaussian case, we recall that $\varrho_{th}(x) = \exp(t\widehat{h}(x) - t^2|h|_H^2/2)$, where $\widehat{h}$ is the measurable linear functional generated by h. Now it suffices to note that $\exp|\widehat{h}| \in L^s(\mu)$ for all $s < \infty$. □

Remark 1.3. It is clear from the proof of Lemma 1.1 that for every $n \geq 1$, $p > 1$ and $r < p$ there is a constant $C(n,p,r)$ such that for every convex function F from $L^p(\mu)$ with respect to the standard Gaussian measure μ on $\mathbb{R}^n$ there holds the inequality

$$\biggl(\int |\nabla F(x)|^r\,\mu(dx)\biggr)^{1/r} \leq C(n,p,q)\biggl(\int |F(x)|^p\,\mu(dx)\biggr)^{1/p}.$$

Analogous inequalities hold for many other measures.

2. Main results

We shall now see that if a function F is convex along h and is integrable in an appropriate power with respect to a sufficiently smooth measure μ, then the measure $F\mu$ is twice differentiable along h. This fact enables one to consider a generalized second derivative of F along h. Throughout X is a locally convex space.

Theorem 2.1. (i) *Suppose that a Radon probability measure μ on X is twice Skorohod differentiable along a vector* $h \in X$ *and that F is convex on μ-almost all straight lines* $x + \mathbb{R}^1 h$. *Assume also that F is integrable with respect to the measures μ and* $d_h^2\mu$ *and that* $\partial_h F$ *is integrable with respect to* $d_h\mu$. *Then the measure Fμ is twice Skorohod differentiable along h. In addition, one has*

$$\|d_h^2(F\mu)\| \leq 2\|F\|_{L^1(d_h^2\mu)} + 2\|\partial_h F\|_{L^1(d_h\mu)}. \tag{2.1}$$

(ii) *If, in addition,* $F \geq 0$ *and one has* $F^p, F|\beta_h^\mu|^p \in L^1(\mu)$ *for some* $p > 1$, *then*

$$\int |\partial_h F|^r\,d\mu < \infty \tag{2.2}$$

for some $r > 1$. Finally, if we also have $F \in L^\alpha(\mu)$ for all $\alpha \in [1, \infty)$ and $\beta_h^\mu \in L^2(\mu)$, then $\partial_h F \in L^r(\mu)$ for every $r < 2$.

Proof. Let us consider first the one dimensional case. In addition, we shall assume that the support of μ belongs to some bounded interval $[a, b]$. Clearly, F is Lipschitzian on $[a, b]$. The measure μ has an absolutely continuous density ϱ such that ϱ' has bounded variation. Hence the measure $F\mu$ is differentiable and $d_1(F\mu) = F'\mu + Fd_1\mu$. Assume, in addition, that F and ϱ are smooth. Then, certainly, the measure $d_1(F\mu)$ is Skorohod differentiable, but we need an estimate of the variation of its derivative. By convexity, $F'' \geq 0$. Therefore,

$$0 \leq \int F''(x)\varrho(x)\,dx = \int F(x)\varrho''(x)\,dx \leq \int |F(x)|\,|d_1^2\mu|(dx). \tag{2.3}$$

As we have

$$d_1^2(F\mu) = F''\mu + 2F'd_1\mu + Fd_1^2\mu,$$

we obtain from (2.3) the estimate

$$\|d_1^2(F\mu)\| \leq \|F''\mu\| + 2\|F'd_1\mu\| + \|Fd_1^2\mu\| \leq 2\|F\,d_1^2\mu\| + 2\|F'd_1\mu\|.$$

Therefore, estimate (2.1) is established in the present special case. Now, assuming that ϱ is only absolutely continuous with ϱ' of bounded variation, but still assuming that F is smooth and μ has bounded support, we can find a sequence of smooth probability densities ϱ_j with support in a fixed interval such that

$$\lim_{j\to\infty} \int \Big[|\varrho_j(x) - \varrho(x)| + |\varrho_j'(x) - \varrho'(x)|\Big]\,dx = 0$$

and $\sup_j \|d_1^2\mu_j\| < \infty$. By (2.1) we have

$$\limsup_{j\to\infty} \|d_1^2(F\mu_j)\| \leq 2\|Fd_1^2\mu\| + 2\|F'd_1\mu\|.$$

This yields (see [11], [15]) that $d_1^2(F\mu)$ exists and

$$\|d_1^2(F\mu)\| \leq \limsup_{j\to\infty} \|d_1^2(F\mu_j)\| \leq 2\|Fd_1^2\mu\| + 2\|F'd_1\mu\|.$$

Indeed, it suffices to note that for every $\psi \in C_0^\infty(\mathbb{R})$ with $|\psi| \leq 1$, one has

$$\begin{aligned}\int \psi\, d_1^2[F\mu](dx) &= \lim_{j\to\infty} \int \psi''\, F\mu_j(dx) \\ &= \lim_{j\to\infty} \int \psi\, d_1^2[F\mu_j](dx) \leq \lim_{j\to\infty} \|d_1^2(F\mu_j)\|.\end{aligned}$$

The next step is to relax the smoothness assumption on F still assuming that μ has bounded support in some $[a, b]$. To this end, it suffices to note that there exists a sequence of smooth convex functions F_j which converge uniformly to F on $[a, b]$ such that the functions F_j' converge to F' in $L^1[a, b]$. In the same manner as above, one verifies that (2.1) still holds. Now let us drop the assumption that μ has bounded support. Let ζ_j be smooth compactly supported functions such that $0 \leq \zeta_j \leq 1$, $\zeta_j(x) = 1$ if $|x| \leq j$, $\zeta_j(x) = 0$ if $|x| \geq j+1$, $\sup_j \sup_x \Big[|\zeta_j'(x)| + |\zeta_j''(x)|\Big] < \infty$. Let $\mu_j := \zeta_j\mu$. Then the measures $F\mu_j$ converge to $F\mu$ in the variation norm. In addition,

$$d_1\mu_j = \zeta_j'\mu + \zeta_j d_1\mu, \quad d_1^2\mu_j = \zeta_j''\mu + 2\zeta_j'd_1\mu + \zeta_j d_1^2\mu.$$

It is readily seen from this expression that $F'd_1\mu_j \to Fd_1\mu$ and $Fd_1^2\mu_j \to Fd_1^2\mu$ in the variation norm. Thus, we arrive at (2.1) in the general one dimensional case.

We can write X as a topological sum $X = \mathbb{R}^1 h + Y$ for some closed hyperplane Y in X. Let ν denote the image of μ under the natural projection to Y. It is known (see [11, Ch. 2] or [15]) that there exist conditional measures μ^y on the straight lines $y + \mathbb{R}^1 h$, $y \in Y$, which are twice Skorohod differentiable along h. For ν-almost every $y \in Y$, the restriction of the function F to $y + \mathbb{R}^1 h$ is integrable with respect to $d_h^2 \mu^y$ and the restriction of $\partial_h F$ is integrable with respect to $d_h \mu^y$. Therefore, by using the one dimensional case, we arrive at the estimate

$$\|d_h^2(F\mu)\| \leq 2\|F\, d_h^2\mu\| + 2\|\partial_h F d_h \mu\|,$$

which is (2.1).

Now suppose $F \geq 0$ and $|F|^p, |\beta_h^\mu|^p F \in L^1(\mu)$ for some $p > 1$. According to Krugova's inequality [16], we have the estimate

$$\left(\int |\beta_h^{F\mu}(x)|^{2-\varepsilon}\, F(x)\, \mu(dx) \right)^{1/(2-\varepsilon)} \leq (1 + \varepsilon^{-1}) \|d_h(F\mu)\| + \frac{1-\varepsilon}{\varepsilon} \|d_h^2(F\mu)\| \tag{2.4}$$

for every $\varepsilon \in (0,1)$. We observe that $\beta_h^{F\mu} = \beta_h^\mu + \partial_h F/F$ a.e. with respect to the measure $F\mu$. Since by our hypothesis $\beta_h^\mu \in L^p(F\mu)$ with some $p \in (1,2)$, then also $\partial_h F/F \in L^p(F\mu)$. Now let $r \in (1,p)$. Set

$$s = \frac{p}{r}, \quad \alpha = \frac{p-1}{s} = r\frac{p-1}{p}.$$

Let $t = s(s-1)^{-1}$. Then $\alpha t = r(p-1)(p-r)^{-1}$. Since $p(p-r)(p-1)^{-1} \to p$ as $r \to 1$, there is $r > 1$ such that $r \leq p(p-r)(p-1)^{-1}$. With this r, one has $\alpha t \leq p$, hence $|F|^{\alpha t} \in L^1(\mu)$ and on the account of the equalities $rs = p$ and $\alpha s = p - 1$ we obtain by Hölder's inequality

$$\begin{aligned} \int |\partial_h F|^r\, \mu(dx) &= \int \frac{|\partial_h F|^r}{F^\alpha} F^\alpha\, \mu(dx) \\ &\leq \left(\int \frac{|\partial_h F|^{rs}}{F^{\alpha s}}\, \mu(dx) \right)^{1/s} \left(\int F^{\alpha t}\, \mu(dx) \right)^{1/t} < \infty. \end{aligned}$$

The proof is complete. □

Corollary 2.2. *Let a Radon probability measure μ on X be quasi-invariant and twice Skorohod differentiable along a vector $h \in X$ and let a μ-measurable function F be convex on μ-almost all straight lines $x + \mathbb{R}^1 h$. Suppose also that $F \in L^p(\mu)$, $F \in L^1(d_h^2\mu)$, $\beta_h^\mu \in L^{p'}(\mu)$, $\varrho_h, \varrho_{-h} \in L^s(\mu)$, where $s = p(p-r)^{-1}$. Then the measure $F\mu$ is twice Skorohod differentiable along h and there hold inequalities* (2.1) *and* (2.2).

Proof. We can apply Lemma 1.1 and the theorem proved above. □

Let us observe that if μ is twice Fomin differentiable along h, then the assumption that $F \in L^1(d_h^2\mu)$ follows from the inclusions $F \in L^p(\mu)$ and $\beta_{h,h}^\mu \in L^{p'}(\mu)$, in particular, it holds in the Gaussian case.

Corollary 2.3. *Let μ be a centered Gaussian measure, let $h \in H(\mu)$, and let a μ-measurable function F be convex on μ-almost all straight lines $x + \mathbb{R}^1 h$.*

(i) *If the functions $F(1 + |\widehat{h}|^2)$ and $\partial_h F \widehat{h}$ are in $L^1(\mu)$, then the measure $F\mu$ is twice Skorohod differentiable along h and* (2.1) *is true.*

(ii) *If $F \in L^p(\mu)$ with some $p > 1$, then assertion* (i) *is true and* (2.2) *holds.*

Proof. It suffices to recall that $\beta_h^\mu = -\widehat{h}$ and $\beta_{h,h}^\mu = |\widehat{h}|^2 - |h|_H^2$. □

As it has already been mentioned, it may happen that the pointwise second order derivative is almost everywhere zero, but the corresponding part of $d_h^2(F\mu)$ is nontrivial. Let us explain how $\partial_h^2 F$ can be interpreted in the generalized sense in analogy with the case of Lebesgue measure. Let us set

$$F_{hh} := d_h^2(F\mu) - 2\partial_h F d_h\mu - F d_h^2\mu,$$

provided that each of the three measures on the right exists separately. Heuristically, $F_{hh} = (\partial_h^2 F)\mu$, since if F is twice differentiable along h in the usual sense and $\partial_h^2 F \in L^1(\mu)$, then

$$d_h^2(F\mu) = (\partial_h^2 F)\mu + F d_h^2\mu + 2\partial_h F d_h\mu. \tag{2.5}$$

We know that $\partial_h^2 F$ exists μ-a.e. and is μ-integrable. However, (2.5) may fail (as it happens in the above mentioned one dimensional example). In general, F_{hh} can be regarded as a derivative of $\partial_h F$ along h in the sense of distributions over (X, μ).

Theorem 2.4. *Suppose that the hypotheses of Theorem* 2.1(i) *are fulfilled and that* $\partial_h F \in L^1(\mu)$ (*which holds under the assumptions of Lemma* 1.1 *or Theorem* 2.1(ii)). *Then the measure* F_{hh} *is finite and nonnegative. In addition,*

$$F_{hh} = d_h(\partial_h F\mu) - \partial_h F d_h\mu. \tag{2.6}$$

Finally, if μ *is twice Fomin differentiable along* h, *then the function* $\partial_h^2 F$ *is the Radon–Nikodym density of the absolutely continuous part of the measure*

$$F_{hh} = d_h^2(F\mu) - 2\partial_h F d_h\mu - F d_h^2\mu$$

with respect to μ.

Proof. Let ζ be a nonnegative smooth cylindrical function. We have

$$\begin{aligned}\int \zeta(x)\, F_{hh}(dx) &= \int \zeta(x) d_h^2(F\mu)(dx) - 2\int \zeta(x)\partial_h F(x)\, d_h\mu(dx) - \int \zeta(x)F(x)\, d_h^2\mu(dx)\\ &= -\int \partial_h\zeta(x) d_h(F\mu)(dx) - 2\int \zeta(x)\partial_h F(x)\, d_h\mu(dx)\\ &\quad + \int [\partial_h\zeta(x)F(x) + \zeta(x)\partial_h F(x)]\, d_h\mu(dx)\\ &= -\int \partial_h\zeta(x)\partial_h F(x)\,\mu(dx) - \int \zeta(x)\partial_h F(x)\, d_h\mu(dx)\\ &= -\int \partial_h F(x)\, d_h(\zeta\mu)(dx).\end{aligned}$$

The right-hand side is nonnegative. This is verified by using the differentiable conditional measures for μ on the straight lines $x+\mathbb{R}^1 h$ and noting that if ϱ is an absolutely continuous probability density on the real line and G is a convex function such that $G'\varrho'$ and $G'\varrho$ are integrable, then

$$\int G'(t)\varrho'(t)\, dt \le 0.$$

Indeed, for any fixed a and b one has

$$\int_a^b G'(t)\varrho'(t)\, dt \le G'(b)\varrho(b) - G'(a)\varrho(a),$$

which follows by the integration by parts formula, since $\varrho \ge 0$ and the measure G'' is nonnegative. The integrability of $G'\varrho$ enables us to pick $a \to -\infty$ and $b \to +\infty$ in such a way that $G'(b)\varrho(b)$ and $G'(a)\varrho(a)$ tend to zero. Equality (2.6) is seen from the above chain of equalities.

In order to prove the last statement, we consider a decomposition $X = \mathbb{R}^1 h \oplus Y$, where Y is a closed hyperplane in X. Let $L_y := \{y + th\colon\ t \in \mathbb{R}^1\}$. Let us make the following observation. Suppose that μ and ν are Radon measures on X such that there exist a nonnegative Radon measure σ on Y and two families of Radon measures μ^y and ν^y on L_y, $y \in Y$, such that, for every Borel set B in X, the functions $y \mapsto \mu^y(B)$ and $y \mapsto \nu^y(B)$ are σ-measurable on Y, the functions $y \mapsto \|\mu^y\|$ and $y \mapsto \|\mu^y\|$ are σ-integrable and one has

$$\mu(B) = \int_Y \mu^y(B)\,\sigma(dy), \quad \nu(B) = \int_Y \nu^y(B)\,\sigma(dy). \tag{2.7}$$

It is known that such measures μ^y exist if we take $\sigma = |\mu|_Y$, where $|\mu|_Y$ is the image of $|\mu|$ under the projection on Y (see, e.g., [14]). Therefore, one can always take $\sigma = |\mu|_Y + |\nu|_Y$ or any other nonnegative Radon measure with respect to which both measures $|\mu|_Y$ and $|\nu|_Y$ are absolutely continuous. Clearly, $\mu^y(A) = \mu^y(A \cap L_y)$ and $\nu^y(A) = \nu^y(A \cap L_y)$. Moreover, if σ is fixed, then the measures μ^y are uniquely defined σ-a.e. Indeed, if B has the form $B = R_r \times A$, where $R_r = (-\infty, r)$ and r is rational and A is a Borel set in Y, we find from (2.7) that

$$\mu(B) = \int_A \mu^y(R_r \times \{y\} \cap L_y)\,\sigma(dy).$$

This shows that the values $\mu^y(R_r \times \{y\} \cap L_y)$, $r \in \mathbb{Q}$, are uniquely determined on a set of full σ-measure, which yields that the measures μ^y are uniquely determined for the corresponding y. Suppose now that $\mu \ll \nu$. Then it is readily verified by using the uniqueness statement that $\mu^y \ll \nu^y$ for σ-a.e. y. On the other hand, if $\mu \perp \nu$, then $\mu^y \perp \nu^y$ for σ-almost all y. Indeed, it suffices to show this in the case of nonnegative measures, because $|\mu| \perp |\nu|$. In that case, there is a Borel set B with $\mu(B) = \nu(X\backslash B) = 0$, hence $\mu^y(B^y) = \nu^y(L_y\backslash B^y) = 0$ for σ-almost all $y \in Y$. The fact we need below is that the converse is also true: if $\mu^y \perp \nu^y$ for σ-almost all y, then $\mu \perp \nu$. Again, since $|\mu^y| \perp |\nu^y|$ and $\mu \ll \int_Y |\mu^y|\,\sigma(dy)$ and similarly for ν, it suffices to consider nonnegative measures μ and ν. Letting $\nu = \mu_0 + \mu_1$, where $\mu_0 \ll \mu$ and $\mu_1 \perp \mu$, we obtain $\mu_0^y \ll \mu_y$ and $\mu_1^y \perp \mu^y$ for σ-almost all y. By the uniqueness statement, $\nu^y = \mu_0^y + \mu_1^y$, hence $\mu_0^y = 0$ for σ-almost all y, i.e., $\mu_0 = 0$.

Let us show the last assertion of the theorem in the one dimensional case. Then $\mu = \varrho\,dx$, where the functions ϱ and ϱ' are absolutely continuous. Evaluating the first and second derivatives in the sense of distributions we find $(F\varrho)' = F'\varrho + F\varrho'$,

$$(F\varrho)'' = (F'\varrho)' + F'\varrho' + F\varrho'' = \varrho F'' + 2F'\varrho' + F\varrho'' = \varrho F''_{\rm ac} + \varrho F''_{\rm sing} + 2F'\varrho' + F\varrho''.$$

It remains to recall that $F''_{\rm ac}$ coincides almost everywhere with the limit

$$\lim_{t\to 0}[F(x+t) + F(x-t) - 2F(x)]t^{-2}.$$

Let us consider the infinite dimensional case. Decomposing X as above and letting σ be the projection of μ on Y, we find the corresponding conditional measures μ^y on the straight lines L_y. It is known (see [11] or [15]) that these measures can be found twice differentiable. Moreover, for σ-almost all y, the measure $F\mu^y$ is twice Skorohod differentiable along h and

$$d_h^2(F\mu)(B) = \int_Y d_h^2(F\mu^y)(B)\,\sigma(dy)$$

for all Borel sets B in X. Since the result is true in the one dimensional case, according to the above made remark on conditional measures, we conclude that the absolutely continuous part $[d_h^2(F\mu)]_{\mathrm{ac}}$ of $d_h^2(F\mu)$ with respect to μ admits the representation

$$[d_h^2(F\mu)]_{\mathrm{ac}} = \int_Y (\partial_h^2 F)\mu^y\,\sigma(dy) = (\partial_h^2 F)\mu,$$

which completes the proof. □

Proposition 2.5. *Let μ be a nonnegative Radon measure on X that is twice Fomin differentiable along a vector h and let Ψ be a μ-a.e. finite μ-integrable nonnegative function such that the sets $\{\Psi \le c\}$ have compact closure, let $h \in X$, and let F_n, $n \in \mathbb{N}$, be μ-measurable functions such that the sequence $\{F_n\}$ is bounded in $L^p(\mu)$ for some $p > 1$. Assume that the functions F_n are convex and twice differentiable along h on almost all lines parallel to h and*

$$\partial_h F_n \beta_h^\mu,\, F_n\beta_h^\mu,\, F_n\beta_{h,h}^\mu \in L^1(\mu). \tag{2.8}$$

Suppose also that the measure $d_h^2(\Psi\mu)$ has a density $g \in L^{p'}(\mu)$ with respect to μ. Then the sequence of measures $(\partial_h^2 F_n)\mu$ is uniformly tight, i.e., for every $\varepsilon > 0$ there is a compact set K_ε such that $|(\partial_h^2 F_n)\mu|(X\backslash K_\varepsilon) < \varepsilon$ for all n.

Proof. According to Theorem 2.1 the measures $F_n\mu$ are twice Skorohod differentiable along h. Since the measures $\partial_h F d_h\mu$ and $F d_h^2\mu$ have bounded variations, the functions $\partial_h^2 F_n$ are μ-integrable. Therefore,

$$\begin{aligned}\int \partial_h^2 F_n(x)\,\Psi(x)\,\mu(dx) &= \int F_n(x)\,d_h^2(\Psi\mu)(dx)\\ &= \int F_n(x)g(x)\,\mu(dx) \le \|\Psi\|_{L^p(\mu)}\|g\|_{L^{p'}(\mu)}.\end{aligned}$$

As $\partial_h^2 F_n(x) \ge 0$ a.e., the integrals on the left are uniformly bounded, hence by the Chebyshev inequality the sequence of measures $(\partial_h^2 F_n)\mu$ is uniformly tight. □

Note that if $\Psi \ge 1$, then the sequence of measures $(\partial_h^2 F_n)\mu$ is uniformly bounded.

It is clear that condition (2.8) is fulfilled if the measure μ is quasi-invariant along h and $\beta_h^\mu, \beta_{h,h}^\mu, \varrho_h, \varrho_{-h} \in L^s(\mu)$ with a sufficiently large s.

Corollary 2.6. *Let μ be a centered Radon Gaussian measure on a sequentially complete locally convex space X and let H be the Cameron–Martin space of μ. Let $F \in L^p(\mu)$, where $p > 1$, be an H-convex function. Then, for any $h \in H$, the measures $(\partial_h^2 T_\varepsilon f)\mu$, $\varepsilon \in (0,1)$, are uniformly tight and converge weakly to F_{hh} as $\varepsilon \to 0$.*

Proof. We shall construct a function $\Psi \ge 1$ that is finite on a linear space of full measure such that the sets $\{\Psi \le c\}$ have compact closure and the functions $\Psi, \partial_h\Psi, \partial_h^2\Psi$ belong to all $L^p(\mu)$. It follows by our hypotheses that there exists a balanced convex compact set K of positive μ-measure (see Proposition A.1.7 in [10]). The Minkowski functional q_K of K is defined by the formula $q_K(x) = \inf\{t > 0\colon\ t^{-1}x \in K\}$ on the linear span E_K of K and $q_K(x) = +\infty$ if $x \notin E_K$. The function q_K is H-Lipschitzian and belongs to all $L^p(\mu)$ (see §4.3 in [10]). Clearly, the sets $\{q_K \le c\} = cK$ are compact. However, this function may not be sufficiently differentiable. Let us consider the function $\Psi = T_1 q_K + 1$. This function is infinitely differentiable along all directions in H and all its partial derivatives along directions from H are in all $L^p(\mu)$ (see Ch. 5 in [10]). In addition, any set $\{\Psi \le c\}$ has compact closure, because it is contained in the set $(6c + m)K$, where $m > 0$ is such that $\mu(mK) \ge 1/2$. Thus, the measures $(\partial_h^2 T_\varepsilon F)\mu$ are uniformly bounded and uniformly tight. Hence as $\varepsilon_n \to 0$ the sequence of measures $(\partial_h^2 T_{\varepsilon_n} F)\mu$ has a limit point ν in the weak topology (see Ch. 8 in [14]). This limit point coincides with the measure F_{hh}, because for every smooth cylindrical function f we have

$$\int f(x)\partial_h^2(T_\varepsilon F)(x)\,\mu(dx)$$

$$= -\int \partial_h f(x)\partial_h(T_\varepsilon F)(x)\,\mu(dx) - \int f\partial_h(T_\varepsilon F)(x)\,d_h\mu(dx)$$

$$\int \partial_h^2 f(x)T_\varepsilon F(x)\,\mu(dx) + 2\int \partial_h f(x)T_\varepsilon F(x)\,d_h\mu(dx) + \int f(x)T_\varepsilon F(x)\,d_h^2\mu(dx)$$

$$= \int T_\varepsilon F(x)[\partial_h^2 f(x) + 2\partial_h f(x)\beta_h^\mu(x) + f\beta_{h,h}^\mu(x)]\,\mu(x).$$

As $\varepsilon \to 0$, the right-hand side tends to

$$\int F(x)[\partial_h^2 f(x) + 2\partial_h f(x)\beta_h^\mu(x) + f\beta_{h,h}^\mu(x)]\,\mu(dx)$$

$$= \int f(x)\,d_h^2(F\mu)(dx) + \int 2\partial_h f(x)F(x)\,d_h\mu(dx) + \int F(x)f(x)d_h^2\mu(dx)$$

$$= \int f(x)\,d_h^2(F\mu)(dx) - 2\int f(x)\partial_h F(x)\,d_h\mu(dx) - \int F(x)f(x)d_h^2\mu(dx),$$

which coincides with the integral of f against the measure F_{hh}. □

Remark 2.7. One can also consider mixed second order partial derivatives F_{hk} by setting

$$F_{hk} := \frac{F_{h+k\,h+k} - F_{hh} - F_{kk}}{2}$$

for appropriate h and k. This notation is consistent from the point of view of generalized derivatives.

We recall that a countably additive measure m on a measurable space $(X, \mathcal{B})$ with values in a normed space E is said to have bounded semivariation if

$$\|m\|_E := \sup\{\|l(m)\|\colon\ l \in E^*, \|l\| \le 1\} < \infty,$$

where $\|l(m)\|$ is the usual total variation of the scalar measure $l(m)$. It is easily seen that

$$\sup_{B\in\mathcal{B}} |m(B)|_E \le \|m\|_E \le 2\sup_{B\in\mathcal{B}} |m(B)|_E.$$

The total variation of m is defined as

$$v(m) := \sup\Big\{\sum_{i=1}^n |m(B_i)|_E\Big\},$$

where the supremum is taken over all finite partitions of X into disjoint sets $B_i \in \mathcal{B}$. If E is infinite dimensional, then m may have finite semivariation, but infinite total variation. We recall that if m is a countably additive measure of bounded total variation with values in a Hilbert space E, then there exist a probability measure μ and a μ-integrable mapping $f\colon\ X \to E$ such that $m = f\mu$, i.e.

$$m(B) = \int_B f\,d\mu, \quad \forall\, B \in \mathcal{B}.$$

Corollary 2.8. *Let μ be a centered Gaussian measure on X with the Cameron–Martin space H and let $F \in L^2(\mu)$ be H-convex. Then the formula*

$$(T_B h, h)_H := F_{hh}(B), \quad B \in \mathcal{B}(X),$$

defines a countably additive measure $B \mapsto T_B$ with values in the space HS of all Hilbert–Schmidt operators on H equipped with the Hilbert–Schmidt norm $\|\cdot\|_{HS}$. In addition, this measure has bounded semivariation such that

$$\|T_B\|_{HS} \le \sqrt{2}\|F\|_{L^2(\mu)}.$$

Proof. Suppose first that $F \in W^{2,2}(\mu)$. Then the second derivative $D_H^2 F(x)$ is a nonnegative Hilbert–Schmidt operator on H. Let $B \in \mathcal{B}(X)$. Then the formula

$$(T_B h, h)_H = \int_B D_H^2 F(x)(h, h)\, \mu(dx)$$

defines a nonnegative Hilbert–Schmidt operator whose Hilbert–Schmidt norm is majorized by $\int \|D_H^2 F(x)\|_{HS}^2\, \mu(dx)$. Let $\{e_j\}$ be an orthonormal basis such that $T_B e_j = t_j e_j$. We observe that the functions

$$\xi_j := \partial_{e_j}\beta_{e_j}^{\mu} + |\beta_{e_j}^{\mu}|^2 = -1 + |\widehat{e_j}|^2$$

are mutually orthogonal in $L^2(\mu)$ and have equal norms $\sqrt{2}$ in $L^2(\mu)$. Since the functions $\partial_{e_j}^2 F$ are nonnegative, the Hilbert–Schmidt norm of T_B can be estimated as follows:

$$\begin{aligned}\sum_{j=1}^{\infty} t_j^2 &= \sum_{j=1}^{\infty}\left|\int_B \partial_{e_j}^2 F(x)\, \mu(dx)\right|^2 \\ &\le \sum_{j=1}^{\infty}\left|\int_X \partial_{e_j}^2 F(x)\, \mu(dx)\right|^2 = \sum_{j=1}^{\infty}\left|\int_X F(x)\xi_j(x)\, \mu(dx)\right|^2 \\ &\le 2\int_X |F(x)|^2\, \mu(dx).\end{aligned}$$

Therefore, the claim is true for the functions $T_{1/k}F$. Letting $k \to \infty$ and noting that $T_{1/k}F \to F$ in $L^2(\mu)$, we obtain the claim by the previous corollary. □

It is worth noting that analogous results hold for a broader class of a-convex functions considered in [12]. To be more specific, recall that given a centered Radon Gaussian measure μ, a μ-measurable function f with values in the extended real line is said to be a-convex along a vector $h \in H = H(\mu)$, where $a \in \mathbb{R}^1$, if f is finite a.e. and the function $f + \frac{a}{2}\widehat{h}^2$ is convex on the straight lines $x + \mathbb{R}^1 h$. It is clear that some of the above results extend also to such functions: it suffices to apply them to $f + \frac{a}{2}\widehat{h}^2$ and use the obvious differentiability properties of $\widehat{h}$. Applications of some results in this paper to the Monge–Ampère equation on the Wiener space can be found in [17].

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References

[1] Borwein J.M., Noll D. Second order differentiability of convex functions in Banach spaces// Trans. Amer. Math. Soc. 1994. V. 342. P. 43–82.

[2] Matoušek J., Matouškovà E. A highly non-smooth notm on Hilbert space// Israel J. Math. 1999. V. 112. P. 1–27.

[3] Matoušková E., Zajiček L. Second order differentiability and Lipschitz smooth points of convex functionals// Czech. Math. J. 1998. V. 48. P. 617–640.

[4] Alexandroff A.D. The existence almost everywhere of the second differential of a convex function and certain related properties of convex surfaces. Uchen. Zap. Leningrad. Univ. 1939. T. 37. N 6. P. 3–35 (in Russian).

[5] Busemann H., Feller W. Krümmungseigenschaften konvexer Flächen. Acta Math. 1935. V. 66. P. 1–47.

[6] Evans L.C., Gariepy R.F. Measure theory and fine properties of functions. CRC Press, Boca Raton, Florida, 1992.

[7] Gol'dshteĭn V.M., Reshetnyak Yu.G. Quasiconformal mappings and Sobolev spaces. Translated and revised from the 1983 Russian original. Kluwer Academic Publ., Dordrecht, 1990 (Russian ed.: Introduction to the theory of functions with generalized derivatives and quasiconformal mappings. Moscow, 1983).

[8] Krylov N.V. Nonlinear elliptic and parabolic equations of second order. Reidel, Dordrecht, 1987. Translated from Russian (Russian ed.: Moscow, 1985).

[9] Bogachev V.I., Mayer-Wolf E. Some remarks on Rademacher's theorem in infinite dimensions// Potential Anal. 1996. V. 5. N 1. P. 23–30.

[10] Bogachev V.I. Gaussian measures. Amer. Math. Soc., Rhode Island, 1998.

[11] Bogachev V.I. Differentiable measures and the Malliavin calculus// J. Math. Sci. (New York). 1997. V. 87. P. 3577–3731.

[12] Feyel D., Üstünel A.S. The notion of convexity and concavity on Wiener space// J. Funct. Anal. 2000. V. 176. P. 400–428.

[13] Üstünel A.S., Zakai M. Transformation of measure on Wiener space. Springer-Verlag, Berlin, 2000.

[14] Bogachev V.I. Foundations of measure theory. V. 1,2, RCD, Moscow – Izhevsk, 2003 (in Russian).

[15] Bogachev V.I., Smolyanov O.G. Analytic properties of infinite dimensional distributions. Uspehi Mat. Nauk. 1990. V. 45. N 3. P. 3–83 (in Russian); English transl.: Russian Math. Surveys. 1990. V. 45. N 3. P. 1–104.

[16] Krugova E.P. On the integrability of logarithmic derivatives of measures// Matem. Zametki. 1993. V. 53. N 5. P. 76–86 (in Russian); English transl.: Math. Notes. 1993. V. 53. N 5. P. 506–512.

[17] Bogachev V.I., Kolesnikov A.V. On the Monge–Ampère equation in infinite dimensions. BiBoS Preprint N 05–01–175, Universität Bielefeld, 2005.

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