

# PURE POINT DIFFRACTION IMPLIES ZERO ENTROPY FOR DELONE SETS WITH UNIFORM CLUSTER FREQUENCIES

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ABSTRACT. Delone sets of finite local complexity in Euclidean space are investigated. We show that such a set has patch counting and topological entropy 0 if it has uniform cluster frequencies and is pure point diffractive. We also note that the patch counting entropy vanishes whenever the repetitivity function satisfies a certain growth restriction.

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## 1. INTRODUCTION

Aperiodic order is a regime of (dis)order at the very border between order and disorder. It has attracted a lot of attention in recent years, see [2, 32, 35, 8] and references therein, though many of its facets are still rather enigmatic. On the one hand, this attention originated in intriguing consequences of aperiodic order. On the other hand, it is due to the existence of real world solids that exhibit this form of order. These substances are called quasicrystals, and were discovered (in 1982) by their diffraction properties [45]. More precisely, they exhibit pure point diffraction (indicating long range order), while typically displaying non-crystallographic symmetries at the same time.

So far, there is no axiomatic framework for aperiodic order, though various fingerprints of order and their relationships have been considered. Besides pure point diffraction, these properties include finite local complexity (FLC), the Meyer property, repetitivity, and uniform cluster frequencies (UCF). In fact, assuming finite local complexity, it is possible to study the patch counting complexity (compare [24, 25, 26]). A first (though still rather coarse) measure of order is then given by the patch counting entropy, which turns out to be finite, compare [24, 25]. Of course, in an ordered situation, it is expected to vanish. Thus, both pure point diffraction and vanishing patch counting entropy can be seen as indications of an underlying order, and it is natural to ask for their relationship. One answer is given by our main **Theorem**:

*Let  $\omega$  be a Delone set in Euclidean space that has finite local complexity and uniform cluster frequencies. If  $\omega$  is pure point diffractive, it has patch counting entropy 0.*

A precise version is given in Theorem 5 below. The theorem can also be formulated for subsets of FLC Delone sets, which can be viewed as coloured point sets, see Remark 2 for details.

In any case, the FLC assumption is necessary in order to properly define the patch counting function. The UCF assumption is also necessary in this theorem, as shown by the set of visible lattice points, which satisfies FLC (as a subset of a lattice) but not UCF (due to not being relatively dense, and hence also not being Delone itself). In fact, the set of visible lattice points possesses a natural autocorrelation with pure point diffraction [9], while having positive patch counting entropy [37].

Our proof of Theorem 5 relies on the following steps:

- (i) Pure point diffraction implies pure point dynamical spectrum (and vice versa). This has recently been shown in increasing degrees of generality [27, 17, 6, 30].
- (ii) Pure point dynamical spectrum implies metric entropy 0, as follows from a Halmos-von Neumann type representation theorem, see Theorem 4 below.
- (iii) There is a variational principle relating metric entropy and topological entropy for  $\mathbb{R}^d$ -actions, as follows from the work of Tagi-Zade [52].
- (iv) The topological entropy agrees with the patch counting entropy for FLC Delone sets. This is shown below in Theorem 1.

The proof thus shows that patch counting, topological and metric entropy coincide under the given hypotheses. Statements (ii) and (iv) are expected generalisations of the corresponding well-known statements for dynamical systems with  $\mathbb{Z}^d$ -actions (compare [1, 12, 36, 53]). They certainly belong to the mathematical “folklore” of the topic, see also the discussion in [43]. However, no proof seems to have appeared in print so far. For this reason, we include some details below. Let us also note that even the various notions of entropy for an action of  $\mathbb{R}^d$  require some care, as (unlike in the one-dimensional case) there is no first return map.

Let us put our results in perspective by considering related results. Within the regime of aperiodic order, vanishing topological entropy has been known so far for two classes of systems that are both characterised by their constructions. These systems are recognisable primitive (self-affine) substitutions [42, 19] and uniquely ergodic systems of finite type [53, 47]. Note that these systems include primitive substitutions, which do *not* have pure point diffraction in general, see [38] for examples.

In this context, it is worthwhile to point out that subshifts over finite alphabets may be strictly ergodic, but still have positive entropy, as has been known since [18] (see [16] for recent results on strictly ergodic Cantor dynamical systems with positive entropy). In fact, there even exist almost automorphic systems (e.g., Toeplitz flows) that are strictly ergodic and have positive entropy [10] (see [13] for further results in this direction and [34] for a generic counterpart).

On the other hand, let us mention that various random tilings are known that quite naturally lead to FLC Delone sets that have positive entropy (both topological and metric) as well as mixed diffraction spectrum. They are important cases where the presence of entropy is a clear indication of some form of disorder, which then manifests itself also in the spectral properties of the system.

Let us briefly return to the visible lattice points mentioned above. The natural dynamical system that arises via the closure of the lattice translations will also contain the empty set (and various others), and hence admits many invariant measures. However, if one defines natural cluster frequencies via the same averaging process as used for the natural autocorrelation, the corresponding metric entropy vanishes [37]. This interesting result indicates some weaker connection between pure point diffractiveness and vanishing metric entropy, which deserves further thought.

The paper is organised as follows. Section 2 is devoted to a brief summary of Delone sets and their associated dynamical systems. In Section 3, we discuss topological entropy and patch counting entropy, showing their equality for FLC Delone sets in Theorem 1. In Section 4, we recall basic notions concerning metric entropy and its invariance under metric isomorphism. Dynamical systems and pure point spectra are then discussed in Section 5. In particular, we relate pure point spectrum and entropy 0 in Theorem 4. Our main result, Theorem 5, is then stated and proved in Section 6. In Section 7, we briefly discuss how our results apply to model sets. Moreover, we provide a result of vanishing entropy for systems whose repetitivity function does not grow too fast. This discussion establishes vanishing entropy for all “standard” examples of aperiodic order with uniform cluster frequencies.

## 2. DELONE DYNAMICAL SYSTEMS

We are interested in combinatorial properties of Delone sets of finite local complexity. To such sets, dynamical systems are associated in a natural way. This section sets up the notation by summarising some results from [49, 46, 27].

Consider the locally compact Abelian group  $\mathbb{R}^d$ . The closed ball with radius  $S \geq 0$  around  $x \in \mathbb{R}^d$  is denoted by  $B_S(x)$ . We set  $B_S = B_S(0)$ . The closed hypercube with side length  $n$  centred at the origin is denoted by  $C_n$ . Now,  $(\Omega, \alpha)$  is said to be a *topological dynamical system* over  $\mathbb{R}^d$  when  $\Omega$  is compact and  $\alpha: \mathbb{R}^d \times \Omega \rightarrow \Omega$  is a continuous action of  $\mathbb{R}^d$  on  $\Omega$ . For  $x \in \mathbb{R}^d$  and  $\omega \in \Omega$ , we write  $\alpha(x, \omega) = \alpha_x(\omega)$ . The topology of  $\Omega$  gives rise to the Borel  $\sigma$ -algebra  $\mathcal{B}$  on  $\Omega$ , i.e., the smallest  $\sigma$ -algebra on  $\Omega$  which contains all open subsets of  $\Omega$ . Below, we additionally assume that  $\Omega$  is equipped with a metric  $d$  that generates the topology.

Let  $m$  be an  $\alpha$ -invariant probability measure on  $(\Omega, \mathcal{B})$ . As usual,  $m$  is called *ergodic* if the only members  $B \in \mathcal{B}$  which are invariant up to sets of measure 0 satisfy either  $m(B) = 0$  or  $m(B) = 1$ . Of course, this is equivalent to the invariant members of the completion of  $\mathcal{B}$  all having measure 1 or 0 (compare Lemma 1 in [11, Ch. 2.1]). The

system  $(\Omega, \alpha)$  is called *uniquely ergodic* if there exists exactly one  $\alpha$ -invariant probability measure on  $(\Omega, \mathcal{B})$ . By standard reasoning, see [12, 53], such a measure is ergodic. When  $(\Omega, \alpha)$  is uniquely ergodic and minimal (meaning that each  $\mathbb{R}^d$ -orbit is dense), it is called *strictly ergodic*.

Delone sets are discrete subsets of  $\mathbb{R}^d$  whose points are distributed rather uniformly. Let positive real numbers  $r$  and  $R$  be given. A subset  $\omega$  of  $\mathbb{R}^d$  is called an  $(r, R)$ -Delone set if

- $\omega$  is *uniformly discrete* with packing radius  $r$ , i.e., any open ball of radius  $r$  in  $\mathbb{R}^d$  contains at most one point of  $\omega$ , and
- $\omega$  is *relatively dense* with covering radius  $R$ , i.e., any closed ball of radius  $R$  in  $\mathbb{R}^d$  contains at least one point of  $\omega$ .

Clearly, this is only of interest when  $r \leq R$ . The set of all  $(r, R)$ -Delone sets is denoted by  $\mathcal{D}_{r,R}$ . In the remainder of this section, we assume  $0 < r \leq R$  to be *fixed*, with  $\mathcal{D}_{r,R} \neq \emptyset$ , and usually refer to elements of  $\mathcal{D}_{r,R}$  simply as Delone sets. The group  $\mathbb{R}^d$  acts on  $\mathcal{D}_{r,R}$  in the obvious way, via  $\alpha_x(\omega) := -x + \omega := \{-x + y : y \in \omega\}$ , for each  $\omega \in \mathcal{D}_{r,R}$ . Now, let  $\omega \in \mathcal{D}_{r,R}$  be given. For  $D > 0$ , the elements of

$$p_\omega(D) := \{(-x + \omega) \cap B_D : x \in \omega\}$$

are called  $D$ -patches of  $\omega$ . By construction, each  $D$ -patch contains the point 0. The cardinality of  $p_\omega(D)$  is denoted by  $\text{card}(p_\omega(D))$ . When  $\text{card}(p_\omega(D)) < \infty$  for every  $D > 0$ , we say that  $\omega$  has *finite local complexity*, or FLC for short. If, for every  $D > 0$  and for every  $\pi \in p_\omega(D)$ , the limit

$$\lim_{n \rightarrow \infty} \frac{1}{|B_n|} \text{card}\{x \in a_n + B_n : (-x + \omega) \cap B_D = \pi\}$$

exists and is independent of the sequence  $(a_n) \subset \mathbb{R}^d$ , then  $\omega$  is said to have *uniform patch frequencies*, or UCF for short. When the corresponding property is satisfied also for all non-empty point sets of the form  $(-x + \omega) \cap K$  with  $K \subset \mathbb{R}^d$  compact and  $x \in \omega$ , which are the *clusters* of  $\omega$ , the point set  $\omega$  has *uniform cluster frequencies*. Fortunately, these two notions coincide for FLC sets. In fact, in this case, the limit can be calculated along arbitrary van Hove sequences, see [46] for details.

The notions of FLC and UCF for  $\omega$  can be interpreted in terms of an associated dynamical system. To be more precise, we need further notation. On  $\mathcal{D}_{r,R}$ , we can introduce a metric  $d$  by defining

$$d(\xi_1, \xi_2) := \min\left\{\frac{1}{\sqrt{2}}, \inf\{S > 0 : \exists u, v \in B_S : (-u + \xi_1) \cap B_{1/S} = (-v + \xi_2) \cap B_{1/S}\}\right\}.$$

Equipped with this metric,  $\mathcal{D}_{r,R}$  becomes a topological space on which the action  $\alpha$  of  $\mathbb{R}^d$  is continuous, see [49] and references therein. For  $\omega \in \mathcal{D}_{r,R}$ , the *hull*  $\Omega(\omega)$  is defined as the closure of  $\{\alpha_x(\omega) : x \in \mathbb{R}^d\}$  in  $\mathcal{D}_{r,R}$ . Then,  $\Omega(\omega)$  is compact if and only if  $\omega$  is FLC. In this case, by [46, Thm. 3.2],  $(\Omega(\omega), \alpha)$  is uniquely ergodic if and only if  $\omega$  has uniform cluster frequencies, see also [49] for an earlier statement concerning the tiling case.

**Remark 1.** (a) Let us mention that no metric is required for the definition of the hull  $\Omega(\omega)$ . In fact, a more natural (and automatically translation invariant) way works with a uniform structure, see [46, 6] for details. In our situation, the metric becomes useful later on for the relation between different notions of entropy, which is why we introduced the hull in this way.

(b) The condition of relative denseness is not necessary in the above construction. One may furthermore consider dynamical systems arising from suitable ensembles of uniformly discrete sets (instead of a single set), by taking the closure of the translation orbits of all sets in the ensemble, such as those emerging for lattices gases or random tilings, compare [5] and references therein for details. For clarity of presentation, we concentrate on the above setup, since we are mainly interested in properties of FLC Delone sets.

### 3. TOPOLOGICAL AND PATCH COUNTING ENTROPY

Topological entropy for  $\mathbb{R}^d$ -actions on compact metric spaces can essentially be defined in analogy to the case of  $\mathbb{Z}$ -actions. There are three equivalent approaches, via open covers,  $\varepsilon$ -separated sets, and  $\varepsilon$ -dense sets, which have been studied by Tagi-Zade [52]. We will use a definition of entropy via  $\varepsilon$ -separated sets, as it seems most adequate for our needs.

Let  $D > 0$  and  $\varepsilon > 0$  be given and consider  $\Omega = \Omega(\omega)$ . Then, a subset  $\Xi$  of  $\Omega$  is called  $(D, \varepsilon)$ -separated, if, for all  $\xi_1, \xi_2 \in \Xi$  with  $\xi_1 \neq \xi_2$ , there exists an  $x \in B_D$  with  $d(\alpha_x(\xi_1), \alpha_x(\xi_2)) > \varepsilon$ .

This definition can be put into more compact form. For  $D > 0$  and  $\xi_1, \xi_2 \in \Omega$ , define  $d_D(\xi_1, \xi_2) = \sup\{d(\alpha_x(\xi_1), \alpha_x(\xi_2)) : x \in B_D\}$ . The supremum is finite and attained since  $B_D$  is compact and  $\alpha$  is continuous by assumption. It can be shown that  $d_D$  defines a metric that is equivalent to  $d$ , but this is not needed in the sequel. Now,  $\Xi \subset \Omega$  is  $(D, \varepsilon)$ -separated if and only if, for all  $\xi_1, \xi_2 \in \Xi$  with  $\xi_1 \neq \xi_2$ , we have  $d_D(\xi_1, \xi_2) > \varepsilon$ .

By compactness of  $\Omega$ , there exists an upper bound on the number of elements of any  $(D, \varepsilon)$ -separated set. So,  $N(D, \varepsilon) := \max\{\text{card}(\Xi) : \Xi \subset \Omega \text{ is } (D, \varepsilon)\text{-separated}\}$  exists and is finite. Define now  $H_\varepsilon := H_\varepsilon(\Omega, \alpha) := \limsup_{n \rightarrow \infty} \frac{1}{|B_n|} \log N(n, \varepsilon) \geq 0$  as usual. Obviously,  $H_\varepsilon \geq H_{\varepsilon'}$  whenever  $\varepsilon \leq \varepsilon'$ . Thus, the limit

$$h_{\text{top}} := h_{\text{top}}(\Omega, \alpha) := \lim_{\varepsilon \rightarrow 0} H_\varepsilon(\Omega, \alpha)$$

exists in  $\mathbb{R}_{\geq 0} \cup \{\infty\}$ . It is called the *topological entropy* of  $(\Omega, \alpha)$ . By compactness of  $\Omega$ , equivalent metrics on  $\Omega$  lead to the same topological entropy, see also [53, Thm. 7.4]. In fact, it only depends on the topology, and can be defined independently of the metric, compare [52, 12]. It is thus clear that topological entropy is invariant under topological conjugacy. In view of our later calculations, we prefer to use a formulation with a given metric.

Of course, various normalisations are possible. For example, one can replace the balls  $B_n$  by hypercubes  $C_n$ . This may lead to some overall factor in the definition of the entropy,

as a consequence of the volume ratio of balls versus cubes of the same extension. However, this cannot influence whether the entropy is 0 or not. In the cases we deal with, the entropy will turn out to be 0, and such factors will thus not matter at all.

Recall that the cardinality of the set  $p_\omega(D)$  of all  $D$ -patches is denoted by  $\text{card}(p_\omega(D))$ . The *patch counting entropy*  $h_{\text{pc}}(\omega)$  of  $\omega$  is defined by

$$(1) \quad h_{\text{pc}}(\omega) := \limsup_{n \rightarrow \infty} \frac{1}{|B_n|} \log(\text{card}(p_\omega(n))),$$

which exists and is finite, see [24, Thm. 2.3]. The following theorem states that the patch counting entropy of an FLC Delone set coincides with the topological entropy of the associated dynamical system. This is an extension of a well-known result in symbolic dynamics, compare [36, Ch. 6.3]. In order to prepare its proof, we first derive a lower bound for the distance of two Delone sets that share the origin as a vertex.

**Lemma 1.** *Let  $\{\xi_1, \xi_2\} \subset \mathcal{D}_{r,R}$  satisfy  $0 \in \xi_1 \cap \xi_2$  and  $\xi_1 \neq \xi_2$ . Then, if  $S > 0$  is any number that satisfies  $\xi_1 \cap B_S \neq \xi_2 \cap B_S$ , one has  $d(\xi_1, \xi_2) \geq \min\{\frac{1}{\sqrt{2}}, \frac{r}{2}, \frac{1}{S}\}$ .*

*Proof.* Note that  $\xi_1 \neq \xi_2$  by assumption. Fix  $S > 0$  such that  $\xi_1 \cap B_S \neq \xi_2 \cap B_S$ . Choose  $\tilde{r} \leq \min\{r/2, 1/S\}$ . Then, for  $x, y \in B_{\tilde{r}}$  arbitrary, we have  $(\xi_1 - x) \cap B_{1/\tilde{r}} \neq (\xi_2 - y) \cap B_{1/\tilde{r}}$ , as can easily be seen by considering the cases  $x = y$  and  $x \neq y$  separately. This implies  $d(\xi_1, \xi_2) \geq \min\{\tilde{r}, 1/\sqrt{2}\}$ , and the claim follows.  $\square$

Note that the above proof does not require relative denseness of  $\xi_1$  or  $\xi_2$ .

**Theorem 1.** *Let  $\omega \in \mathcal{D}_{r,R}$  be an FLC Delone set, with hull  $\Omega(\omega)$ . Then, for each  $\varepsilon < \varepsilon_0 := \min\{\frac{1}{\sqrt{2}}, \frac{r}{2}, \frac{1}{2R}\}$ , one has  $H_\varepsilon(\Omega(\omega), \alpha) = h_{\text{pc}}(\omega)$ . In particular,*

$$h_{\text{pc}}(\omega) = h_{\text{top}}(\Omega(\omega), \alpha).$$

*Proof.* For  $D > 0$ , we use the abbreviation  $n(D) := \text{card}(p_\omega(D))$ . Now, consider a subset  $\Xi^{(D)} = \{\xi_1, \dots, \xi_{n(D)}\}$  of  $\Omega(\omega)$  that represents the  $D$ -patches around the point  $0 \in \mathbb{R}^d$ , i.e.,  $\Xi^{(D)}$  has  $n(D)$  elements and  $p_\omega(D) = \{\xi \cap B_D : \xi \in \Xi^{(D)}\}$ . We show two inequalities.

“ $\leq$ ”: Assume  $n(D) > 1$  and fix some  $1 \leq i, j \leq n(D)$  with  $i \neq j$ . Choose a number  $0 < S_{ij} \leq D$  such that  $\xi_i \cap B_{S_{ij}} \neq \xi_j \cap B_{S_{ij}}$  and  $\xi_i \cap B_S = \xi_j \cap B_S$  for all  $S < S_{ij}$ . Distinguish two cases. If  $S_{ij} \leq 2R$ , Lemma 1 implies

$$d_D(\xi_i, \xi_j) \geq d(\xi_i, \xi_j) \geq \min\{\frac{1}{\sqrt{2}}, \frac{r}{2}, \frac{1}{S_{ij}}\} \geq \min\{\frac{1}{\sqrt{2}}, \frac{r}{2}, \frac{1}{2R}\} = \varepsilon_0.$$

If  $S_{ij} > 2R$ , fix  $x_0 \in \partial B_{S_{ij}}$  such that  $x_0 \in \xi_i \cup \xi_j$  but  $x_0 \notin \xi_i \cap \xi_j$ . Choose a ball  $B \subset S_{ij}$  of radius  $R$  such that  $x_0 \in B$ . Choose  $x \in B \cap \xi_i$ . Such a point exists, since  $\xi_i$  is relatively dense with radius  $R$ . By construction, we also have  $x \in \xi_j$ . Since  $x_0 \in B_{2R}(x)$ , we have  $\xi_i \cap B_{2R}(x) \neq \xi_j \cap B_{2R}(x)$ . We can now apply Lemma 1, with  $x \in B_D$  considered to be the origin. This gives

$$d_D(\xi_i, \xi_j) \geq d(\alpha_{-x}(\xi_i), \alpha_{-x}(\xi_j)) \geq \min\{\frac{1}{\sqrt{2}}, \frac{r}{2}, \frac{1}{2R}\} \geq \varepsilon_0.$$

Combining the two cases, we infer that, for any  $D > 0$  and any  $\varepsilon < \varepsilon_0$ , the set  $\Xi^{(D)}$  is indeed  $(\varepsilon, D)$ -separated. (If  $n(D) = 1$ , this is trivially the case.) This implies

$$n(D) = \text{card}(\Xi^{(D)}) \leq N(D, \varepsilon),$$

and  $h_{\text{pc}}(\omega) \leq H_\varepsilon$  follows. As  $\varepsilon$  was arbitrary with  $0 < \varepsilon < \varepsilon_0$ , we obtain “ $\leq$ ”.

“ $\geq$ ”: Let  $\varepsilon > 0$  and  $D > 0$  be arbitrary. Define  $\rho(D) := D + R + \varepsilon + 1/\varepsilon$ . Let  $\Xi$  be an arbitrary  $(D, \varepsilon)$ -separated set. For any  $\xi \in \Xi$ , we can find an  $x_\xi \in \xi \cap B_R$ , since  $\xi$  is relatively dense with radius  $R$ . Thus, we can define a mapping  $\Psi : \Xi \rightarrow \Xi^{(\rho(D))}$ , such that  $\xi \in \Xi$  is mapped to  $\eta \in \Xi^{(\rho(D))}$  whenever  $(-x_\xi + \xi) \cap B_{\rho(D)} = \eta \cap B_{\rho(D)}$ . In general, the map  $\Psi$  is not injective. A bound on the number of preimages of  $\eta$ , meaning  $\text{card}(\Psi^{-1}(\eta))$ , is obtained as follows. First, we claim that

$$(2) \quad d(x_{\xi_1}, x_{\xi_2}) \geq \varepsilon$$

whenever  $\xi_1 \neq \xi_2$  are elements of  $\Xi$  with  $\Psi(\xi_1) = \Psi(\xi_2)$ . To see this, assume the contrary. Then, a short calculation yields  $d_D(\xi_1, \xi_2) < \varepsilon$ , based on the definition of  $\rho(D)$  and the construction of  $\Psi$ . However, this is impossible, as  $\xi_1$  and  $\xi_2$  are different elements of a  $(D, \varepsilon)$ -separated set  $\Xi$ . Furthermore, by compactness of  $B_R$ , there exist  $M = M(\varepsilon) \in \mathbb{N}$  and  $x_1, \dots, x_M \in B_R$  such that

$$(3) \quad B_R \subset \bigcup_{i=1}^M B_{\varepsilon/2}(x_i).$$

Combining (2) with (3), we infer that no  $\eta \in \Xi^{(\rho(D))}$  can have more than  $M$  inverse images under  $\Psi$ . This implies  $\text{card}(\Xi) \leq M(\varepsilon) \cdot n(\rho(D))$ . As  $\Xi$  is an arbitrary  $(D, \varepsilon)$ -separated set, we infer  $N(D, \varepsilon) \leq M(\varepsilon) \cdot n(\rho(D))$ . With  $\lim_{D \rightarrow \infty} \frac{1}{D} \rho(D) = 1$ , one finds  $H_\varepsilon \leq h_{\text{pc}}(\omega)$ . As  $\varepsilon > 0$  was arbitrary, the desired inequality follows, and the claimed equality is shown.  $\square$

**Remark 2.** (a) The above proof can be adapted to coloured FLC Delone sets (with finitely many colours), compare [27, 29] for a discussion of coloured Delone sets. In this case, the FLC property is inherited, but the UCF property becomes increasingly more restrictive. Every subset of an FLC Delone set gives rise to a coloured FLC Delone set, via the obvious colouring with two colours. An example is given by the visible lattice points. Note that the number of  $D$ -patches (and possibly also the corresponding entropies) of an FLC Delone subset and its coloured counterpart may be different.

(b) The above proof uses the Delone property of the underlying set in both directions. It is an open question whether the condition of relative denseness is necessary for the validity of the theorem. See Example 3 below for an FLC set which is not Delone.

(c) An alternative proof can be given, using the definition of topological entropy by open covers, in analogy to the case of  $\mathbb{Z}$ -actions. Note, however, that the dynamical system is not expansive, and entropy is computed by a sequence of open covers with diameters that shrink to 0.

(d) The above result generalises the cases of  $\mathbb{Z}$ -actions [36, Ch. 6.2] and  $\mathbb{Z}^d$ -actions [1].

**Example 1.** Quasiperiodic tilings provide, via a natural point decoration, interesting examples of aperiodic FLC Delone sets. This includes the majority of tilings used in crystallography and physics to study the properties of quasicrystals. Examples are, among various others, the Penrose tiling, the Ammann-Beenker tiling, the shield tiling, the Ammann-Kramer tiling, and the Danzer tiling. See the review [3] and references therein. The underlying point sets are regular model sets, and some tilings are substitution tilings. See Section 7 for a discussion of their entropy and diffraction properties.

**Example 2.** It is well-known that fully-packed dimer models on lattices or periodic graphs give rise to interesting random tiling examples [39, 40] and, via a natural point decoration, also to FLC Delone sets. A random tiling ensemble gives rise to a dynamical system, via the closure of the translations of the Delone sets derived from all tilings of the ensemble. The system has positive patch counting (and hence topological) entropy, and a diffraction spectrum of mixed type, see [5] for a proof of this statement. For the dimer model on the square lattice (the domino case) and on the hexagonal lattice (the lozenge case), the corresponding equilibrium (or Gibbs) measure is unique, so that also the metric entropy is positive. The topological entropy of both models is of the form

$$m(P) = \int_0^1 \int_0^1 \log |P(e(s), e(t))| ds dt,$$

for certain two-variable Laurent polynomials  $P = P(x, y)$ , where  $e(t) = \exp(2\pi it)$ . We have  $P(x, y) = 4 + x + 1/x + y + 1/y$  in the domino case [23, Eq. 8], and  $P(x, y) = 1 + x + y$  in the lozenge case [54, Eq. 5].

The quantity  $m(P)$ , the average of  $\log |P|$  over the real 2-torus, is also called the (logarithmic) *Mahler measure* of  $P$ . For  $P$  having integer coefficients only, it arises as an entropy within a certain  $\mathbb{Z}^2$ -dynamical system associated to  $P$ , see [31]. It would be interesting to investigate its relation to the  $\mathbb{R}^2$ -dynamical system discussed above, see also [51].

**Example 3.** Let us briefly discuss a situation that sheds some light on the connection between FLC sets and Delone sets<sup>1</sup>. Consider Euler's number  $e$  and define  $a_n = 1 + e + e^2 + \dots + e^{n-1}$  for  $n \in \mathbb{N}$ . The set  $\Lambda = \{\pm a_n : n \in \mathbb{N}\}$  is uniformly discrete, but not relatively dense, and hence not Delone. Since  $a_{n+1} - a_n = e^n \xrightarrow{n \rightarrow \infty} \infty$ , there are only *finitely* many patches of a given diameter, so that  $\Lambda$  is an FLC set. As the patch counting function grows only logarithmically with patch size, one can conclude that  $h_{\text{pc}} = 0$  for  $\Lambda$ .

On the other hand, as  $e$  is transcendental, the points of  $\Lambda$  are rationally independent, so that  $[\Lambda - \Lambda]$  is not a finitely generated  $\mathbb{Z}$ -module. Consequently, by a result from [24],  $\Lambda$  cannot be a subset of any FLC Delone set. Since  $\Lambda$  has vanishing density, its entropy per volume does not seem to be the right quantity to look at. However, when reconsidering it as a two-colour Delone set (which is then *not* FLC), it is not clear how to relate the various entropies, or what their values are.

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<sup>1</sup>We thank Robert V. Moody for suggesting this type of example.

## 4. METRIC ENTROPY AND VARIATIONAL PRINCIPLE

In this section, we discuss the metric entropy, its invariance under metric isomorphism and the variational principle. While we need two properties of the metric entropy, the details of its actual definition play no role below. Therefore, we omit the technicalities and concentrate on these properties instead. For details concerning its definition and the variational principle, we refer the reader to the paper of Tagi-Zade [52].

The first part of our discussion does not require a topological dynamical system. It works whenever we are given a measure dynamical system, i.e., a measurable action  $\alpha$  of  $\mathbb{R}^d$  on the measurable space  $\Omega$  together with an  $\alpha$ -invariant probability measure  $m$ . To these data, one can associate a quantity called *metric entropy* or Kolmogorov-Sinai entropy, see [11, 12] for details.

Let us continue by explaining the notion of metric isomorphism of measure preserving transformations. Let two measure spaces  $(\Omega_i, \mathcal{B}_i, m_i)$ ,  $i \in \{1, 2\}$ , be given. Let  $\alpha_i$  be measure preserving actions of  $\mathbb{R}^d$  on  $\Omega_i$ . Recall that the measure algebras  $(\tilde{\mathcal{B}}_i, \tilde{m}_i)$  arise from the corresponding  $\sigma$ -algebras by identifying elements whose symmetric difference has measure 0. We then say that  $(\Omega_1, \alpha_1)$  and  $(\Omega_2, \alpha_2)$  are *metrically isomorphic* (or conjugate modulo 0) if there is a measure algebra isomorphism

$$\Phi: (\tilde{\mathcal{B}}_2, \tilde{m}_2) \longrightarrow (\tilde{\mathcal{B}}_1, \tilde{m}_1), \quad \text{with } \Phi(\alpha_{1,x}(V)) = \alpha_{2,x}(\Phi(V)),$$

for all  $x \in \mathbb{R}^d$ . A direct consequence of the definitions of metric entropy and metric isomorphism is the following well-known fact, compare [11, 12, 53].

**Lemma 2.** *Metric entropy is invariant under metric isomorphisms.* □

Let us now turn to the variational principle, which relates metric and topological entropy.

**Theorem 2.** *Let  $\Omega$  be a compact metric space, equipped with the Borel  $\sigma$ -algebra and a continuous action  $\alpha$  of  $\mathbb{R}^d$ . Then, the topological entropy is the supremum of the metric entropies taken over all  $\alpha$ -invariant probability measures on  $\Omega$ .*

This is proved in [52] by first relating the  $\mathbb{R}^d$ -entropies to corresponding  $\mathbb{Z}^d$ -entropies via restricting the underlying translation group. Since these entropies are shown to coincide, the statement follows from the variational principle for  $\mathbb{Z}^d$ -actions, which is discussed in [44, Sec. 6]. It is an extension of the case  $d = 1$  from [12, 53], see also [15] and references given there for further details.

## 5. PURE POINT SPECTRUM AND THE REPRESENTATION THEOREM

First, we recall basic notions concerning pure point spectra and formulate the representation theorem. This circle of ideas is sometimes discussed under the name Halmos-von Neumann theorem, see [53, 11, 48] for details, and [43] for its recent appearance in a related context. We then present an abstract result relating pure point spectrum and vanishing metric entropy for ergodic dynamical systems on compact metric spaces.

A measurable dynamical system  $(\Omega, \alpha, m)$  gives rise to a unitary representation  $T = T^{(m)}$  of  $\mathbb{R}^d$  on  $L^2(\Omega, m)$ , which is given by

$$T_x f = f \circ \alpha_x \quad \text{for } f \in L^2(\Omega, m) \text{ and } x \in \mathbb{R}^d.$$

An  $f \in L^2(\Omega, m)$  is called an *eigenfunction* of  $T$  if  $f \neq 0$  and if there exists a  $\lambda$  in the dual group  $\widehat{\mathbb{R}^d}$  of  $\mathbb{R}^d$  with  $T_x f = (\lambda, x)f$  for all  $x \in \mathbb{R}^d$ . If  $L^2(\Omega, m)$  has a basis that purely consists of eigenfunctions of  $T$ ,  $T$  is said to have *pure point dynamical spectrum*. Alternatively, one then also says that  $(\Omega, \alpha, m)$  has pure point spectrum.

There is a representation theorem for ergodic measure preserving  $\mathbb{R}^d$ -actions with pure point spectrum, as for ergodic  $\mathbb{Z}$ -actions. It can be proved by adapting the arguments for  $\mathbb{Z}$ -actions, as given in [53, Thms. 3.4 and 3.6], to the case of  $\mathbb{R}^d$ -actions, see also the detailed exposition in [48, Ch. I.5] for the case  $d = 1$ . For the convenience of the reader, we provide some details for the general case.

An action  $\beta$  of  $\mathbb{R}^d$  on a compact group  $\mathbb{T}$  is called a *rotation* if there exists a group homomorphism  $i: \mathbb{R}^d \rightarrow \mathbb{T}$  such that  $\beta_x(\xi) = i(x)\xi$ , for all  $x \in \mathbb{R}^d$ . If  $i$  has dense range, then  $(\mathbb{T}, \beta)$  is uniquely ergodic, as the normalised Haar measure is the unique translation invariant probability measure on  $\mathbb{T}$ . From now on,  $\alpha$  denotes an  $\mathbb{R}^d$ -action.

**Theorem 3.** *Let  $(\Omega, \alpha, m)$  be an ergodic measurable dynamical system with pure point dynamical spectrum. Then,  $\alpha$  is metrically isomorphic to a uniquely ergodic rotation  $\beta$  on some compact Abelian group  $\mathbb{T}$  (equipped with its unique normalised Haar measure). If  $L^2(\Omega, m)$  admits a countable orthonormal basis,  $\mathbb{T}$  is metrisable with a  $\beta$ -invariant metric.*

*Proof.* Denote the  $\alpha$ -invariant probability measure on  $\Omega$  by  $m$ . By assumption, there exists  $P \subset \widehat{\mathbb{R}^d}$  and a basis  $\{f_\lambda\}_{\lambda \in P}$  on  $L^2(\Omega, m)$  such that  $f_\lambda$  is an eigenfunction of  $T$  to  $\lambda \in P$ . (Here, we use that every eigenvalue has multiplicity 1 due to ergodicity.)

Now, equip  $P$  with the discrete topology and denote its dual group by  $\mathbb{T}$ . Then,  $\mathbb{T}$  is a compact Abelian group whose dual is  $P$ . As  $P \subset \widehat{\mathbb{R}^d}$ , the natural map  $i: \mathbb{R}^d \rightarrow \mathbb{T}$ , defined by  $i(t)(\lambda) := (\lambda, t)$ , has dense range. It induces a continuous action

$$\beta: \mathbb{R}^d \times \mathbb{T} \rightarrow \mathbb{T}, \quad (x, \sigma) \mapsto i(x)\sigma.$$

As  $i$  has dense range and the Haar measure is unique, this action is uniquely ergodic. We now show that  $(\Omega, \alpha, m)$  is metrically isomorphic to  $(\mathbb{T}, \beta)$ . This proceeds in three steps, see also [53, Thm. 3.4]:

Step 1: Without loss of generality, we may assume that  $f_\lambda f_\mu = f_{\lambda+\mu}$ ,  $\forall \lambda, \mu \in P$ .

*Argument:* Follow the proof of [53, Thm. 3.4] or [48, Ch. I.5].

Step 2: There exists a map  $M: L^2(\Omega, m) \rightarrow L^2(\mathbb{T})$  which commutes with the action of  $\mathbb{R}^d$  and satisfies  $M(fg) = M(f)M(g)$  and  $M(\bar{g}) = \overline{M(g)}$ ,  $\forall f, g \in L^\infty(\Omega, m)$ .

*Argument:* Define  $j: \mathbb{R}^d \rightarrow \widehat{\mathbb{T}}$  by  $j(x)(\rho) := (\rho, i(x))$ . Thus, each  $j(x)$  is a character and in fact a bounded eigenfunction on  $\mathbb{T}$ . Then, the map  $\sum c_\lambda f_\lambda \mapsto \sum c_\lambda j(\lambda)$  has the desired properties by Step 1.

Step 3: There exists an isomorphism of measure algebras, which commutes with the action of  $\mathbb{R}^d$ .

*Argument:* Apply  $M$  to characteristic functions.

Finally, we discuss metrisability: If  $L^2(\Omega, m)$  permits a countable orthonormal basis, then so does  $\mathbb{T}$ . Thus, the set of characters of  $\mathbb{T}$  (which gives an orthonormal basis) is countable. Let  $\chi_n, n \in \mathbb{N}$ , be an enumeration of the characters and define a metric by

$$d_{\mathbb{T}}(\xi, \eta) := \sum_{n \in \mathbb{N}} 2^{-n} |(\chi_n, \xi - \eta)|.$$

Then,  $d_{\mathbb{T}}$  is invariant as each character is an eigenfunction of  $\beta$ . Obviously  $d_{\mathbb{T}}$  is also continuous, and it separates points by Pontryagin duality. Thus, it generates the topology of the compact group  $\mathbb{T}$ .  $\square$

**Remark 3.** When the Delone set  $\omega$  is a model set, see [33] for details, it emerges from a cut and project scheme, which provides a natural candidate for the compact group  $\mathbb{T}$  as the factor group of the embedding space by the embedding lattice. This can be made explicit by means of the *torus parametrisation*, see [4, 20, 46, 7] for details.

**Lemma 3.** *Let  $(\Omega, \alpha)$  be a topological dynamical system with an  $\mathbb{R}^d$ -invariant metric  $d$  on  $\Omega$ . Then, one has  $h_{\text{top}}(\Omega, \alpha) = 0$ .*

*Proof.* As the metric  $d$  is  $\mathbb{R}^d$ -invariant,  $N(D, \varepsilon)$  does not depend on  $D$ . Thus,  $H_\varepsilon = 0$  for every  $\varepsilon > 0$ , and the statement follows.  $\square$

**Theorem 4.** *Let  $(\Omega, \alpha, m)$  be a measurable dynamical system that is ergodic and has a pure point dynamical spectrum, with a countable basis of eigenfunctions. Then,  $(\Omega, \alpha, m)$  has metric entropy 0.*

*Proof.* By Theorem 3,  $(\Omega, \alpha, m)$  is metrically isomorphic to a rotation  $(\mathbb{T}, \beta)$ , which is uniquely ergodic by Theorem 3, with Haar measure  $\mu$  as the invariant measure. As we have a countable basis,  $\mathbb{T}$  admits a  $\beta$ -invariant metric. Lemma 3 then gives that  $(\mathbb{T}, \beta)$  has topological entropy 0. Since topological entropy is an upper bound for measure theoretic entropy (where we actually get equality here due to unique ergodicity),  $(\mathbb{T}, \beta, \mu)$  has metric entropy 0. Since  $(\mathbb{T}, \beta, \mu)$  and  $(\Omega, \alpha, m)$  are metrically isomorphic, Lemma 2 implies that  $(\Omega, \alpha, m)$  has metric entropy 0 as well.  $\square$

## 6. PURE POINT DIFFRACTION AND ENTROPY 0

In this section, we finally study the connection between pure point diffraction and entropy. We start by discussing diffraction, as initiated in a mathematical setting by Hof [21], compare [46, 17, 6, 30] and references given there for recent results. Let  $\omega \in \mathcal{D}_{r,R}$  with hull  $\Omega = \Omega(\omega)$  be given. Let  $m$  be a translation invariant measure on the hull. Then,  $m$  induces a measure, called *autocorrelation* of  $\Omega$  and denoted by  $\gamma_{\Omega, m} = \gamma_{\Omega}$ , on  $\mathbb{R}^d$  as

follows: Choose a continuous function  $\sigma$  with compact support and  $\int_{\mathbb{R}^d} \sigma(t) dt = 1$ . For  $\varphi$  continuous with compact support on  $\mathbb{R}^d$ , we then set

$$\gamma_\Omega(\varphi) = \int_\Omega \left( \sum_{s,t \in \omega} \sigma(s) \varphi(s-t) \right) dm(\omega).$$

It turns out that the measure  $\gamma_\Omega$  does not depend on the choice of  $\sigma$ . It is not hard to see that  $\gamma_\Omega$  is positive definite. Thus, its Fourier transform  $\widehat{\gamma}_\Omega$  exists and is a positive measure. It is called the *diffraction measure* of  $\Omega$ . When the Fourier transform  $\widehat{\gamma}_\Omega$  is a pure point measure, we say that  $\Omega$  is *pure point diffractive*.

In the case of finite local complexity and uniform cluster frequencies, the autocorrelation measure  $\gamma_\omega$  of any fixed element  $\omega \in \Omega$  can be computed as

$$\gamma_\omega = \lim_{n \rightarrow \infty} \frac{1}{|B_n|} \sum_{x,y \in \omega \cap B_n} \delta_{x-y},$$

where  $\delta_z$  denotes the unit point mass at  $z \in \mathbb{R}^d$ , and the limit is taken in the vague topology. In fact, the limit can be computed along arbitrary van Hove sequences, and does not depend on the choice of  $\omega$ , due to unique ergodicity. So,  $\gamma_\omega = \gamma_\Omega$  for all  $\omega \in \Omega$  in this case. In general, whenever  $\gamma_\omega$  is well-defined (possibly with respect to a specified averaging sequence), its Fourier transform exists and is a positive measure. One calls the single Delone set  $\omega$  *pure point diffractive* when  $\widehat{\gamma}_\omega$  is a pure point measure.

The following result from [27] (see [17, 6, 30] for further generalisations) gives a crucial characterisation of pure point diffractiveness.

**Lemma 4.** *Let  $\omega \in \mathcal{D}_{r,R}$  be given, with finite local complexity, hull  $\Omega = \Omega(\omega)$  and invariant probability measure  $m$ . Then, the following assertions are equivalent:*

- (i)  $\Omega$  is pure point diffractive, i.e.,  $\widehat{\gamma}_\Omega$  is a pure point measure.
- (ii)  $(\Omega, \alpha, m)$  has pure point dynamical spectrum. □

This lemma and the considerations of the preceding sections imply the following.

**Corollary 1.** *Let  $\omega \in \mathcal{D}_{r,R}$  be given, with finite local complexity, hull  $\Omega = \Omega(\omega)$  and ergodic measure  $m$ . If  $\widehat{\gamma}_\Omega$  is pure point,  $(\Omega, \alpha, m)$  has metric entropy 0.*

*Proof.* By Lemma 4,  $(\Omega, \alpha, m)$  has pure point dynamical spectrum. Consequently, the metric entropy vanishes by Theorem 4. □

We can now state and prove our main result.

**Theorem 5.** *Let  $\omega \in \mathcal{D}_{r,R}$  be given, with finite local complexity and uniform cluster frequencies. If  $\omega$  is pure point diffractive, one has  $h_{pc}(\omega) = 0$ .*

*Proof.* As  $\omega$  has uniform cluster frequencies,  $(\Omega(\omega), \alpha)$  is uniquely ergodic by [46, Thm. 3.2]. Denote the unique  $\alpha$ -invariant ergodic measure by  $m$ . Corollary 1 and the variational

principle from Theorem 2 then imply that  $h_{\text{top}}(\Omega(\omega), \alpha) = 0$ . By Theorem 1, we obtain that the patch counting entropy satisfies

$$h_{\text{pc}}(\omega) = h_{\text{top}}(\Omega(\omega), \alpha) = 0,$$

which completes the argument.  $\square$

**Remark 4.** As we have not assumed repetitivity of  $\omega$ , the hull  $\Omega(\omega)$  may be bigger than the LI-class of  $\omega$  (so we need not have strict ergodicity). Consequently, different elements of  $\Omega(\omega)$  may possess different patch counting functions. However, by the structure of the hull, they are all majorised by the patch counting function of  $\omega$  itself (note that one can only “lose” patches in the limiting process). Consequently, with  $h_{\text{pc}}(\omega) = 0$  from Theorem 5, one also has  $h_{\text{pc}}(\omega') = 0$  for all  $\omega' \in \Omega(\omega)$ .

## 7. APERIODIC ORDER AND ENTROPY 0

There are two main classes of examples of aperiodic order. One class consists of model sets, and the other is given by primitive substitutions. In this section, we discuss the vanishing of the entropy for these classes.

As regular model sets have uniform cluster frequencies and are pure point diffractive [46], our main result immediately implies:

**Theorem 6.** *Regular model sets have patch counting and topological entropy 0.*  $\square$

For primitive self-affine substitutions, patch counting entropy 0 is shown in [19]. Here, we discuss a simple criterion for entropy 0 whenever the repetitivity function does not grow too fast. In particular, this criterion applies to linearly repetitive systems as introduced in [14, 25], which include primitive self-similar substitutions [25, 50]. We thus recover a part of the result from [19] by a different method.

**Definition 1.** A Delone set  $\omega$  has *repetitivity function*  $F: [1, \infty) \rightarrow [1, \infty)$  if any patch of size  $D \geq 1$  is contained in any patch of size  $F(D)$ , i.e., if for any  $x \in \omega$  the inclusion

$$p_\omega(D) \subset \{(\omega - y) \cap B_D : y \in \omega \cap B_{F(D)}(x)\}$$

holds. If  $F$  can be chosen to be  $F(D) = cD$  for some  $c > 1$ ,  $\omega$  is called *linearly repetitive*.

**Theorem 7.** *If  $\omega$  is a Delone set in  $\mathbb{R}^d$  with repetitivity function  $F$ , there exists some  $\kappa > 0$  with  $\text{card}(p_\omega(D)) \leq \kappa |B_{F(D)}|$  for all  $D \geq 1$ . Thus,  $h_{\text{pc}}(\omega) = 0$  whenever one has  $F(D) = o(\exp(D))$ . In particular,  $h_{\text{pc}}(\omega) = 0$  when  $\omega$  is linearly repetitive.*

*Proof.* By the Delone property, there exists a constant  $\kappa_1 > 0$  such that

$$\text{card}(\omega \cap B_D(p)) \leq \kappa_1 |B_D|$$

for all  $D \geq 1$  and  $p \in \mathbb{R}^d$ . Let  $D \geq 1$  be arbitrary and choose some  $x \in \omega$ . By definition of the repetitivity function, every patch of size  $D$  can be found in  $B_{F(D)}(x)$ . Obviously, there

cannot be more patches of size  $D$  in  $B_{F(D)}(x)$  than there are points of  $\omega$  in  $B_{F(D)}$ . This gives  $\text{card}(p_\omega(D)) \leq \text{card}(\omega \cap B_{F(D)}(x)) \leq \kappa_1 |B_{F(D)}|$ , which proves the statement.  $\square$

**Remark 5.** (a) Note that the proof of Theorem 7 does not use relative denseness.  
 (b) For aperiodic linearly repetitive sets, there is also a corresponding lower bound. This had been conjectured by Lagarias and Pleasants [25] and was later proved in [28]. In particular, for primitive self-similar substitutions, the patch counting function grows asymptotically proportional to  $|B_D|$ , see also the discussion in [41].  
 (c) Linear repetitivity (and thus  $h_{\text{pc}}(\omega) = 0$ ) alone does *not* imply the absence of continuous diffraction components, as can be seen from the Thue-Morse chain (singular continuous parts) or the Rudin-Shapiro chain (absolutely continuous parts), see [38, 22] for details.

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