ON THE ESTIMATION OF AN ANALYTIC SPECTRAL DENSITY OUTSIDE OF THE OBSERVATION BAND.

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Abstract. We consider a Gaussian stationary process $X(t)$ with an integer analytic spectral density $f(\lambda)$ and study a problem of its estimation. The process $X(t)$ is non-observable. Instead of it we observe a linear transformation $Y(t)$, $0 \le t \le T$, of $X(t)$ with a transfer function $a(\lambda)$, $|a(\lambda)| = 1$ if λ belongs to an interval I. We study how far from I the consistent estimation of $f(\lambda)$ is possible, $T \to \infty$.

Key words and concepts: stationary processes, band limited processes, spectral density estimation, .

1. Introduction.

The paper continues author's investigations devoted to the statistical approach to the unicity theorem for analytic functions. The unicity theorem says that if two functions f and g are holomorphic in a region G and $f(z) = g(z)$ for all z in some sequence of distinct points with limit point in G, then $f(z) = g(z)$ everywhere in G. The theorem means in particular that if an integer (or entire, " an entire function is an integral function in British usage" [3]) analytic function is observed on an interval I, it can be restored immediately in the whole complex plane. Of course, this problem of restoration is an ill posed one and small perturbations of the observations may drastically change the solution on large distance from the region of observation. We are interesting how far from the region of observation a consistent restoration is possible under small stochastic perturbations. There are a few possible statements of the problem. One of them which has been considered in [8] is the following one.

Problem 1. We are observing an integer analytic function f on an interval $[x, y]$ in the white noise of a small intensity ε, i.e. the observation $X(t)$, $x \le t \le y$, satisfies the relation

$$dX(t) = f(t) + \varepsilon dw(t), \tag{1.1}$$

$w(t)$ is a standard Wiener process.

Denote $\mathbf{F}(M, \sigma, \rho)$ the class of integer analytic functions which satisfy the following restrictions on their growth

$$\sup_{|z| \le R} |f(z)| \le M \exp\{\sigma R^\rho\}. \tag{1.2}$$

These are well known classes of integer functions of order ρ and type σ, see [3], [12]. It has been shown in [8] that if the function f in (1.1) belongs to the class

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$\mathbf{F}(M,\sigma,\rho)$, the consistent estimation of $f(z)$ when $\varepsilon \to 0$ is possible approximately in $(\ln 1/\varepsilon)^{\frac{1}{\rho}}$ vicinities of $[x,y]$ (see details in [8]).

An other scheme of observation has been considered in [9], [10]. Call it

Problem 2. Let

$$X_1, X_2, \dots X_n \tag{1.3}$$

be a sample from a distribution with an integer analytic density function $f(x)$. It is supposed that the sample (1.3) is censored by an interval $[x,y]$: only observations $X_j \in [x,y]$ can be used to construct estimators for f, all other observations disappear. It is shown in [9], [10] that if the density function f belongs to a class $\mathbf{F}(M,\sigma,\rho)$, the consistent estimation is possible approximately in $(\ln n)^{\frac{1}{\rho}}$ vicinities of $[x,y]$ (see details in [9],[10]).

In this paper we consider one more variant of the last problem. Let $X(t)$ be a Gaussian stationary process with an integer analytic spectral density $f(\lambda)$. The process $X(t)$ is the Fourier transform of a Gaussian process $Z(\lambda)$ with independent increments and $\mathbf{E}|dZ(\lambda)|^2 = f(\lambda)d\lambda$,

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda} dZ(\lambda), \tag{1.4}$$

see, for example, [5], ch. 11.

The process $X(t)$ is unobservable. Instead of it we are observing for $0 \le t \le T$ a linear transformation $Y(t)$ of $X(t)$,

$$Y(t) = \int_{-\infty}^{\infty} e^{it\lambda} a(\lambda) dZ(\lambda) \tag{1.5}$$

with a transfer function $a(\lambda)$. The process $Y(t)$ is also a stationary Gaussian process with the spectral density $g(\lambda) = |a(\lambda)|^2 f(\lambda)$. If $|a(\lambda)| = 1$ for $\lambda \in [x,y]$, then $f(\lambda) = g(\lambda)$ for $x \le \lambda \le y$. There are many ways to construct consistent when $T \to \infty$ estimators $\hat{g}(\lambda)$ for $g(\lambda)$. Any such estimator will estimate consistently $f(\lambda)$ inside $[x,y]$. How shall we set problems on the estimation $f(\lambda)$ outside $[x,y]$? On the first glance a direct analogue of the problem 2 looks as follows: the process $Y(t)$ is bandlimited to $[x,y]$, i.e. the transfer function $a(\lambda) = 0$ for $\lambda \notin [x,y]$. But this approach will not work. Indeed, in the case the observation process

$$Y(t) = \int_{x}^{y} e^{it\lambda} a(\lambda) dZ(\lambda)$$

is analytic in the whole complex plane. Thus the observation $Y(t)$ admits the analytic continuation on $\mathbb{R}^1$. By Maruyama's theorem the Gaussian process $Y(t)$ with continuous spectrum is ergodic (see [13] or [6]) and hence by ergodic theorem the correlation function

$$R_Y(t) = \mathbf{E}Y(t)\overline{Y(0)} = \lim_{S\to\infty} \frac{1}{S} \int_0^S Y(t+s)\overline{Y(s)} ds$$

can be determined by the observation $Y(t)$, $0 \le t \le T$. The correlation function $R_Y(t)$ determines the spectral density $g(\lambda), \lambda \in \mathbb{R}^1$, and hence $f(\lambda)$ for $x \le \lambda \le y$.

But then the analytic function $f(\lambda)$ is determined for all λ. (Note that linear filters with $a(\lambda)=0$, $\lambda \notin [x,y]$ are physically non-realizable.)

Therefore we take an engineering conception of bandlimited processes. Namely, we consider linear transformations $Y(t)$ with transfer functions $a(\lambda)$ such that

$$|a(\lambda)| = 1,\ x \le \lambda \le y, \qquad |a(\lambda)| < 1,\ \lambda \notin [x,y].$$

We denote the class of such transfer functions $\mathcal{A}$. Of course it is supposed that the transfer function $a(\lambda)$ is unknown, otherwise we could estimate $f(\lambda)$ as $\hat{g}(\lambda)|a(\lambda)|^{-2}$. Thus we consider below the following estimation problem.

Problem 3. A stationary Gaussian process $X(t)$ with the spectral representation (1.4) has the spectral density $f \in \mathbf{F}(M,\sigma,\rho)$. Suppose also that

$$\int_{-\infty}^{\infty} f(\lambda)d\lambda \le A. \tag{1.6}$$

Denote the class of spectral densities f which belongs $\mathbf{F}(M,\sigma,\varrho)$ and satisfies (1.6) $\mathbf{G}(M,\sigma,\rho,A)=\mathbf{G}$. The observation process $Y(t)$, $0 \le t \le T$, is determined by the relation (1.4). The unknown transfer function $a \in \mathcal{A}$. The problem is to estimate the spectral density $f(\lambda)$, $f \in \mathbf{G}$, of $X(t)$ in and outside of the observation interval $[a,b]$.

Let us formulate the basic results.

Theorem 1.1. *There exists an estimate $\hat{f}(\lambda)$ of $f(\lambda)$ such that*

$$\mathbf{E}\int_{x}^{y} |f(\lambda)-\hat{f}(\lambda)|^2 d\lambda \le \frac{C}{T}\frac{\ln T}{\ln\ln T} \tag{1.7}$$

where the constant C depends on M,σ,ρ,A,x,y only.

Denote $G_T(\alpha)$ the $(\ln T)^{\alpha/\rho}$-vicinity of the interval $[x,y]$.

Theorem 1.2. *There exists an estimate $\hat{f}(\lambda)$ of $f(\lambda)$ such that for all fixed $\alpha<1$ and $\beta<\alpha$ the following inequality holds*

$$\sup_{f\in\mathbf{G},a\in\mathcal{A}} \mathbf{E}_f\{\sup_{G_T(\beta)} |f(\lambda)-\hat{f}(\lambda)|\} \le C_{\alpha,\beta}T^{\frac{1-\alpha}{2}}. \tag{1.8}$$

The constant $C_{\alpha,\beta}$ depends also on M,σ,ρ,A,x,y.

The next theorem shows that the length of consistency intervals can not be essentially increased.

Theorem 1.3. *Let L be an exterior point of the interval $G_T(\alpha)$, $\alpha>1$. Then for all sufficiently large T*

$$\inf_{\hat{f}}\ \sup_{f\in\mathbf{G},a\in\mathcal{A}} \mathbf{E}_f|\hat{f}-f(L)|^2 > c_0 > 0. \tag{1.9}$$

The constant c_0 depends on M,σ,ρ,x,y,α.

2. Construction of estimators. Proof of Theorems 1, 2.

The method of the proof does not depend on the observation's interval and for the sake of simplicity we suppose below that $[x, y] = [-1, 1]$.

We consider the following general scheme of estimator's construction. Let $\{\varphi_n\}$ be a complete orthonormal system in $L_2(-1, 1)$. Then we can expend the restriction of $f(\lambda)$ on $[-1, 1]$ into the Fourier series with respect to $\{\varphi_n\}$,

$$f(\lambda) = \sum_n a_n \varphi_n(\lambda), \quad a_n = \int_{-1}^{1} \varphi_n(\lambda) f(\lambda) d\lambda.$$

If $\hat{a}_n$ are estimators for a_n, estimate $f(\lambda)$ by a sum $\sum_{n \le N} \hat{a}_n \varphi_n(\lambda)$, the choice of N depends on the class $\mathbf{G}$. This method has been initiated by N.N.Chentsov [4].

To realize the scheme take as a complete ortonormal system in $L_2(-1, 1)$ the Legendre polynomials $P_n(\lambda)$, $n = 0, 1, \ldots$: P_n is a polynomial of degree n and $\int_{-1}^{1} P_n(\lambda) P_m(\lambda) d\lambda = \delta_{mn}$. See, for example, [14].

Lemma 2.1. *Let $f \in \mathbf{F}(M, \sigma, \rho)$. Let*

$$f(\lambda) = \sum_n a_n P_n(\lambda), \quad a_n = \int_{-1}^{1} P_n(\lambda) f(\lambda) d\lambda, \tag{2.1}$$

be its expansion into the Fourier series with respect to $\{P_n\}$. Then

$$|a_k| \le cM \exp\{-\frac{k}{\rho} \ln \frac{k}{e\rho\sigma}\} \tag{2.2}$$

where c depends on σ, ρ. If $\rho = 1$,

$$|a_k| \le M \exp\{-k \ln(k/e\sigma)\}. \tag{2.3}$$

The result should be well known. A proof of it see in [10].

Lemma 2.2 (see[2], p. 74, or [15], p. 17)**.** *Let $Q(z)$ be a polynomial of degree n. Then for all complex z*

$$|Q(z)| \le \max_{|x| \le 1} Q(x) \cdot |z + \sqrt{z^2 - 1}|^n. \tag{2.4}$$

Lemma 2.3 (see [14])**.** *The following inequalities hold*

$$\max_{|x| \le 1} |P_n(x)| \le |P_n(1)| = \sqrt{(2n+1)/2}. \tag{2.5}$$

It follows from the lemmas above that the series (2.1) converges in the whole complex plane and determines there an analytic function. Hence the equality

$$f(z) = \sum_n a_n P_n(z) \tag{2.6}$$

satisfies in the whole complex plane.

Take as a basic statistics the periodogram $I_T(\lambda)$ of the observation,

$$I_T(\lambda) = \frac{1}{2\pi T} \left| \int_0^T e^{-it\lambda} Y(t) dt \right|^2 \tag{2.7}$$

and consider as estimators of linear functionals

$$(\varphi, g) = \int_{-\infty}^{\infty} \varphi(\lambda) g(\lambda) d\lambda$$

the statistics $\int_{-\infty}^{\infty} \varphi(\lambda) I_T(\lambda) d\lambda$. In particular we estimate the coefficients

$$a_n = \int_{-1}^{1} P_n(\lambda) f(\lambda) d\lambda = \int_{-1}^{1} P_n(\lambda) g(\lambda) d\lambda$$

by the statistics

$$\hat{a}_n = \int_{-1}^{1} P_n(\lambda) I_T(\lambda) d\lambda. \tag{2.8}$$

Lemma 2.4. *The mathematical expectation*

$$\mathbf{E} I_T(\lambda) = \frac{2}{\pi T} \int_{-\infty}^{\infty} D_T^2(\lambda - \mu) g(\mu) d\mu \tag{2.9}$$

where

$$D_T(\lambda) = \frac{\sin(T\lambda/2)}{\lambda}.$$

The covariance function

$$\begin{aligned} \mathbf{E} I_T(\lambda_1) I_T(\lambda_2) = \frac{4}{\pi^2 T^2} \Big\{ & \int_{-\infty}^{\infty} f(l) D_T^2(l - \lambda_1) dl \int_{-\infty}^{\infty} f(l) D_T^2(l - \lambda_2) dl + \\ & + \Big(\int_{-\infty}^{\infty} f(l) D_T^2(l - \lambda_1) D_T(l - \lambda_2) dl \Big)^2 + \\ & + \Big(\int_{-\infty}^{\infty} f(l) D_T^2(l - \lambda_1) D_T(l - \lambda_2) dl \Big)^2 \Big\}. \end{aligned} \tag{2.10}$$

Proof for processes with the discrete time see, for example, in [1], theorems 8.2.7, 8.2.8. The formulas (2.8), (2.9) can be proved in the same way.

Lemma 2.5. *The bias* $\mathbf{E}\hat{a}_n - a_n$ *satisfies the inequality*

$$|\mathbf{E}\hat{a}_n - a_n| \le C T^{-1/2}. \tag{2.11}$$

The constant C depends on the class $\mathbf{G}$.

Proof. Since the equality

$$\frac{1}{\pi} \int_{-\infty}^{\infty} D_T^2(\lambda - \mu) d\mu = 1$$

and (2.8), we can write

$$
\begin{aligned}
|\mathbf{E}\hat{a}_n - a_n| &= \frac{1}{\pi}\Big|\int_{-1}^{1} P_n(\lambda) \int_{-\infty}^{\infty} D_2^2(\mu) g(\lambda + 2\mu/T) d\mu - \\
&- \frac{1}{\pi}\int_{-1}^{1} P_n(\lambda) d\lambda \int_{-\infty}^{\infty} D_2^2(\mu) g(\lambda) d\mu\Big| \le \\
&\le \frac{1}{\pi} \int_{|\mu|>T/2} \mu^{-2} d\mu \int_{-1}^{1} |P_n(\lambda)||g(\lambda) - g(\lambda+2\mu/T)| d\lambda + \\
&+ \frac{1}{\pi} \int_{|\mu|\le T/2} D_2^2(\mu) d\mu \int_{-1}^{1} |P_n(\lambda)||g(\lambda) - g(\lambda+2\mu/T)| d\lambda.
\end{aligned} \tag{2.12}
$$

The first summand on the right side of (2.11) does not exceed

$$T^{-1}\int_{-1}^{1} |P_n(\lambda) g(\lambda) d\lambda + T^{-1}\int_{-1}^{1} |P_n(\lambda) d\lambda \int_{-\infty}^{\infty} g(\mu) d\mu \le CT^{-1}.$$

If $|\lambda| \le 1$ and $|\lambda + 2\mu/T| \le 1$, then $|g(\lambda) - g(\lambda + 2\mu/T)| \le C|\mu|T^{-1}$. Hence the second summand in (2.11) does not exceed

$$\frac{C}{T}\int_{-T/2}^{T/2} |\mu| D_2^2(\mu) d\mu + C\int_{-T/2}^{T/2} D_2^2(\mu) d\mu \int_{\{1-2|\mu|/T \le \lambda \le 1\}} |P_n(\lambda)| d\lambda \le \frac{C}{\sqrt{T}}.$$

The lemma is proved.

Lemma 2.6. *The variance of* $\hat{a}_n$

$$\mathbf{E}|\hat{a}_n - \mathbf{E}\hat{a}_n|^2 \le \frac{C}{T}. \tag{2.13}$$

Proof. It follows from (2.9) that

$$
\begin{aligned}
\mathbf{E}|\hat{a}_n - \mathbf{E}\hat{a}_n|^2 &= \frac{4}{\pi^2 T^2}\int_{-1}^{1}\int_{-1}^{1} P_n(\lambda) P_n(\mu) d\lambda d\mu \Big(\int_{-\infty}^{\infty} D_T(\lambda - l) D_T(\mu - l) g(l) dl\Big)^2 + \\
&+ \frac{4}{\pi^2 T^2}\int_{-1}^{1}\int_{-1}^{1} P_n(\lambda) P_n(\mu) d\lambda d\mu \Big(\int_{-\infty}^{\infty} D_T(\lambda - l) D_T(\mu + l) g(l) dl\Big)^2 = \\
&= J_1 + J_2.
\end{aligned} \tag{2.14}
$$

We bound the first summand on the right, the second one can be bounded in the same way. We have

$$J_1 \le \frac{8}{\pi^2 T^2}\Big\{\int_{-1}^{1}\int_{-1}^{1} P_n(\lambda)P_n(\mu)d\lambda d\mu\Big(\int_{\{|l|\ge 2\}} D_T(\lambda-l)D_T(\mu-l)g(l)dl\Big)^2 +$$

$$+\Big(\int_{\{|l|\le 2\}} D_T(\lambda-l)D_T(\mu-l)g(l)dl\Big)^2\Big\} = J_{11}+J_{12}.$$

Further

$$J_{11} \le CT^{-2}\int_{-1}^{1}|P_n(\lambda)|^2 d\lambda\big(\int_{-\infty}^{\infty} g(l)dl\big)^2 \le CT^{-2}$$

and

$$J_{12} \le \frac{C}{T^2}\int_{-2}^{2}\int_{-2}^{2} g(l_1)g(l_2)\Big(\int_{-1}^{1} P_n(\lambda)P_n(\mu)D_T(\lambda-l_1)D_T(\lambda-l_2)d\lambda\Big)^2 \le$$

$$\le \frac{C(\max_{|l|\le 2} f(l))^2}{T^2}\int_{-1}^{1}\int_{-1}^{1} P_n(\lambda)P_n(\mu)d\lambda d\mu\Big(\int_{-\infty}^{\infty} D_T(\lambda-l)D_T(\mu-l)dl\Big)^2.$$

But

$$\int_{-\infty}^{\infty} D_T(x-l)D_T(y-l)dl = \pi D_T(x-y).$$

Thus

$$J_{12} \le CT^{-1}\int_{-1}^{1}|P_n(\lambda)|d\lambda\int_{-1}^{1}|P_n(\mu)|T^{-1}D_T^2(\lambda-\mu)d\mu \le CT^{-1}||P_n||^2 = CT^{-1}$$

and lemma follows.

Prove now theorem 1.1. Consider estimators

$$f_N(\lambda) = \sum_{0}^{N}\hat{a}_n P_n(\lambda), \quad N = 0, 1, \ldots. \tag{2.15}$$

The sequence $\{P_n\}$ is orthonormal in $L_2[-1,1]$ and

$$\mathbf{E}\{||f-f_N||^2\} = \sum_{0}^{N}\mathbf{E}|\hat{a}_n - a_n|^2 + \sum_{N+1}^{\infty}|a_n|^2 \le$$

$$\le 2\sum_{0}^{N}|\mathbf{E}\hat{a}_n - a_n|^2 + 2\sum_{0}^{N}\mathbf{E}|\hat{a}_n - \mathbf{E}\hat{a}_n|^2 + \sum_{N+1}^{\infty}|a_n|^2.$$

Lemma 2.2 implies that

$$\sum_{N+1}^{\infty}|a_n|^2 \le C\exp\{-\frac{2}{\rho}N\ln N\}.$$

It follows from this inequality and lemmas 2.5, 2.6 that

$$\mathbf{E}\{||f-f_N||^2\} \le C(NT^{-1} + \exp\{-\frac{N\ln N}{\rho}\}).$$

Take $N \sim \frac{\rho \ln T}{\ln\ln T}$ and put $\hat f = f_N$. We find that

$$\mathbf{E}\{||\hat f - f||^2\} \le \frac{C}{T}\frac{\ln T}{\ln\ln T}.$$

The theorem is proved.

Remark. Suppose that the observation process is not band limited.We may conjecture (again because of the unicity theorem) that then the bound can be ameliorated. Indeed this is the case. For example suppose that $\rho = 1$ and $\sigma > 0$ is known. For $|t| \le \sigma$ consider the following estimator $\hat R(t)$ for the correlation function $R(t)$ of the observation process

$$\hat R(t) = \frac{1}{T-t}\int_0^{T-t} X(t+s)X(s)ds,\ 0\le t\le\sigma, \quad \hat R(t) = \hat R(-t),\ -\sigma\le t\le 0.$$

If $|t| > \sigma$, we set $\hat R(t) = 0$. Consider as an estimator for $f(\lambda)$

$$\hat f(\lambda) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} e^{-it\lambda}\hat R(t)dt.$$

It is rather easy to show that

$$\mathbf{E}||\hat f - f||_{L_2(-1,1)} \asymp \mathbf{E}||\hat f - f||_\infty\ asympCT^{-1}.$$

Comparing the last formula with (1.7) we see how large are lossess due to band limitation.

Prove theorem 1.2. Consider at first the estimators f_n defined in (2.14). Let $L \ge 1$. Because of lemmas 2.1, 2.2 and 2.6 we will have that

$$\Delta(L) = \mathbf{E}\sup_{|\lambda\le L}|f_N(\lambda) - f(\lambda)| \le$$

(2.16)
$$\le \frac{C}{\sqrt T}\sum_0^N \sqrt n\big|L + \sqrt{|L^2-1|}\big|^n +$$
$$+\sum_{N+1}^\infty \exp\{-\frac{n}{\rho}\ln\frac{n}{e\rho\sigma}\}\big|L+\sqrt{|L^2-1|}\big|^n.$$

It follows that

$$\Delta(L) \le (CL)^N\big(T^{-1/2} + \exp\{-\frac{N}{\rho}\ln N\}\big).$$

Take $N \sim \frac{\rho}{2}\frac{\ln T}{\ln\ln T}$ and set $\hat f = f_N$. Then if $L \le C\big(\ln T\big)^{\frac{\alpha}{\rho}},\ 0<\alpha<1$,

(2.17)
$$\mathbf{E}\sup_{|\lambda\le L}\big|\hat f(\lambda) - f(\lambda)\big| \le C\exp\big\{-\frac{\ln T}{2}\big(1-\rho\frac{\ln LC}{\ln\ln T}\big)\big\} \le CT^{\frac{1-\alpha}{2}}.$$

The theorem is proved.

3. Law bounds. Proof of the theorem 1.3.

Below we restrict ourself to the observations (1.1) with the transfer functions of the following form

$$a(\lambda) = \begin{cases} 1, & \text{if } |\lambda| \leq 1, \\ \varepsilon(\frac{\sin\lambda}{\lambda}),\ 0 < \varepsilon < 1, & \text{if } |\lambda| > 1. \end{cases} \tag{3.1}$$

Denote $\mathcal{A}_0$ the set of these transfer functions.

Take a number L, $|L| > 1$. Our initial problem is the problem of estimation of the value $f(L)$ on the base of observations (1.5) with $a \in \mathcal{A}_0$. Consider together with the initial problem an additional one-parametric estimation problem of the following type. Define the following one-parametric family $\{f_\theta\}$ of spectral densities

$$f_\theta(\lambda) = \varphi_L(\lambda)(1 + \theta\psi_L(\lambda)), \quad \theta \in [0, 1], \tag{3.2}$$

where $\varphi_L \in \mathbf{G}$ and $\psi_L \in \mathbf{G}$ are two given functions such that $\varphi_L(L)\psi_L(L) \geq 1/2$. The final choice of these functions will be made later. Below we usually omit the index L and write φ, ψ instead of φ_L, ψ_L. Suppose that the initial process $X(t)$ has a spectral density f_θ and consider the problem of estimation of the parameter θ on the base of observations (1.5) with a *known* transfer function $a \in \mathcal{A}_0$.

Let $\hat{f}$ be an estimator of the value $f(L)$, $f \in \mathbf{G}$, in the initial problem. Because of

$$\theta = \frac{f_\theta(L) - \varphi(L)}{\varphi(L)\psi(L)}$$

we may consider as an estimator for θ in the additional problem

$$\hat{\theta} = \frac{\hat{f} - \varphi(L)}{\varphi(L)\psi(L)}.$$

Evidently

$$\mathbf{E}_\theta\{|\hat{\theta} - \theta|^2\} = \frac{\mathbf{F}_{f_\theta}|\hat{f} - f_\theta(L)|^2}{\varphi^2(L)\psi^2(L)} \tag{3.3}$$

and hence

$$\inf_{\hat{f}} \sup_{f \in \mathbf{G}, a \in \mathcal{A}} \mathbf{E}^\theta|\hat{f} - f_\theta(L)|^2 \geq \varphi(L)\psi(L) \inf_{\hat{\theta}} \sup_{\theta, a \in \mathcal{A}_0} \mathbf{E}|\hat{\theta} - \theta|^2. \tag{3.4}$$

It follows that we prove the theorem 1.3 if we can establish the inequality

$$\sup_{a \in \mathcal{A}_0} \sup_\theta \mathbf{E}_\theta|\hat{\theta} - \theta|^2 \geq c_0 > 0. \tag{3.5}$$

We see later that the additional estimation problem corresponds to a regular statistical experiment in the sense of [11], sect. 1.7. Hence the Fisher information $I(\theta)$ of the additional problem is well defined and by Cramér – Rao's inequality (see [11], sect. 1.7): for any estimator $\hat{\theta}$ with $\sup_\theta \mathbf{E}_\theta|\hat{\theta} - \theta|^2 < \infty$

$$\mathbf{E}_\theta|\hat{\theta} - \theta|^2 \geq \frac{(1 + d'(\theta))^2}{I(\theta)} + d^2(\theta). \tag{3.6}$$

Here $d(\theta) = \mathbf{E}_\theta\hat{\theta} - \theta$ is the bias of $\hat{\theta}$. It follows from (3.6) that if

$$\sup_\theta I(\theta) \leq 1, \tag{3.7}$$

then

$$\sup^{\theta} \mathbf{E}_\theta |\hat\theta - \theta|^2 \geq c_1 > 1/16. \tag{3.8}$$

In the next part of the Section we prove the inequality (3.7). Because of (3.4), (3.5), (3.8) it will prove the theorem.

Because the interval $G_T(\alpha)$ of theorem 1.3 depends on ρ only and to simplify the proof we will not try to construct spectral densities with given σ, ρ, M, A but satisfy ourselfes by constructing $f \in \mathbf{G}$ with given ρ and *some* σ, M, A. In other words we prove the following weakened version of Theorem 1.3: for a given ρ one can find σ, M, A such that for $f \in \mathbf{G}(M, \sigma, \rho, A)$ the inequality (1.9) will be satisfied (of course these σ, M, A will not depend on the interval $G_T(\alpha)$).

Return to the definition of the functions φ, ψ determining the family (3.2). Let $\varphi_0(\lambda) = (\sin(\lambda + i)(\lambda + i)^{-1})(\sin(\lambda - i)(\lambda - i)^{-1})$. Set

$$\varphi_L(\lambda) = \varphi(\lambda) = \varphi_0(\lambda - L) + \varphi_0(\lambda + L). \tag{3.9}$$

The function $\varphi(z)$ is an integer function. It is easy to see that

$$|\varphi_L(z)| \leq Ce^{2|z|}$$

where the constant C does not depend on L.

Let now $\psi_0(z)$ be a function from $\mathbf{F}(\sigma, M, \rho)$ with some σ, M and let the function ψ_0 satisfy the following conditions:

on the real line

$$0 \leq \psi_0(\lambda) \leq e^{-|\lambda|^\rho},\ \psi_0(\lambda) = \psi_0(-\lambda),\ \psi_0(0) = 1; \tag{3.10}$$

for all $L \in R^1$

$$|\psi_0(z - L)| \leq C \exp\{c|z|^\rho\} \tag{3.11}$$

where the constants C, c do not depend on L.

The existence of such functions will be proved at the end of the section. We define now ψ_L as

$$\psi_L(z) = \psi_0(z + L) + \psi_0(z - L). \tag{3.12}$$

Under such choice of φ, ψ all the functions f_θ will belong $\mathbf{G}$ (with a given ρ and some σ, M, A determined by φ_0, ψ_0).

The observation process $Y(t)$ of the additional problem determines on the Hilbert space $L_2(0,T) = L_T$ the family of Gaussian measures P_θ. We show that all these measures are absolutely continuous one with respect to other. To do this we apply some general results on absolute continuity of Gaussian measures in Hilbert spaces.

Lemma 3.1 (see [7], Th. 2, Sect. 4, ch. VII)**.** *Let P_1 and P_2 be two Gaussian measures with mean values zero and correlation operators $\mathbf{R}_1$ and $\mathbf{R}_2$ correspondingly defined in a Hilbert space H. In order that these measures be equivalent, it is necessary and sufficient that the operator $D = \mathbf{R}_2^{-1/2}\mathbf{R}_1\mathbf{R}_2^{-1/2} - \mathbf{I}$, $\mathbf{I}$ be the identity operator , be a Hilbert – Schmidt operator and its eigenvalues δ_k satisfy the inequality $\delta_k > -1$. If the measures P_1, P_2 are equivalent, the Radon – Nykodim derivative*

$$\frac{dP_2}{dP_1}(x) = \exp\{-\frac{1}{2}\sum_k [(\mathbf{R}_2^{-1/2}x, e_k)^2 \frac{\delta_k}{1+\delta_k} - \ln(1+\delta_k])\} \tag{3.13}$$

where e_k are the eigenfunctions of the operator $\mathbf{D}$ corresponding to the eigenvalues δ_k.

In the case when P is a Gaussian measure on $L_T = L_2(0,T)$ generated by a stationary process $X(t)$ with the correlation function $B(t)$ and the spectral density $b(\lambda)$ the correlation operator is defined on L_T by the formula

$$(\mathbf{B}u)(t) = \int_0^T B(t-s)u(s)ds.$$

To the operator $\mathbf{B}$ corresponds the bilinear form

$$(\mathbf{B}u, v)_T = \int_0^T \int_0^T B(t-s)u(t)\overline{u(s)}dtds. \tag{3.14}$$

Continue a function $u \in L_T$ on the whole real line setting $u(t) = 0$, $t \notin [0,T]$ and denote $\tilde{u}(\lambda)$ the Fourier transform of the continued function,

$$\tilde{u}(\lambda) = \int_0^T e^{it\lambda}u(t)dt.$$

We can rewrite then the form $(\mathbf{B}u, v)_T$ in the terms of $b(\lambda)$, namely

$$(\mathbf{B}u, v)_T = \int_{-\infty}^{\infty} b(\lambda)\tilde{u}(\lambda)\overline{\tilde{v}(\lambda)}d\lambda. \tag{3.15}$$

Denote E_T the class of integer analytic functions $\varphi(z)$ square summable on the real line and satisfying the inequality

$$|\varphi(z)| \le Ce^{T|z|}$$

(integer functions which satisfy the last inequality are called integer functions of exponential type T, see their theory in [3] or [12].)

The famous Paley – Wiener theorem (see, for example, [3], Th. 6.8.1) asserts that the class E_T coinsides with the class of functions u which can be represented as

$$u(\lambda) = \int_{-T}^{T} e^{it\lambda}v(t)dt, \quad v \in L_2(-T,T).$$

In particular, the unit ball determined by the form $(\mathbf{B}u,u)_T)$: $\{u \in L_T : (\mathbf{B}u,u) \le 1\}$ corresponds to the set $\{u \in E_T : \int_{-\infty}^{\infty} b(\lambda)|u(\lambda)|^2 d\lambda \le 1\}$.

Lemma 3.2 (see [7], Th. 3, Sect. 5, ch. VII)**.** *Let P_1 and P_2 be two Gaussian measures generated by two stationary Gaussian processes ξ_1, ξ_2 possessing spectral densities $f_1(\lambda), f_2(\lambda)$ correspondingly. Suppose that the spectral densities f_1, f_2 satisfy the following conditions:*

1. there exists a function $\varphi_0 \in E_\sigma$ and positive constants c_1, c_2 such that

$$c_1|\varphi_0(\lambda)|^2 \le f_1(\lambda) \le c_2|\varphi_0(\lambda)|^2, \ \lambda \in R^1;$$

2.

$$\int_{-\infty}^{\infty} \left| \frac{f_2(\lambda) - f_1(\lambda)}{f_1(\lambda)} \right| d\lambda < \infty.$$

Then the restrictions P_{1T}, P_{2T} of the measures P_1, P_2 on L_T are equivalent for any positive T.

The observation process of the additional problem has as the possible spectral densities the functions

$$g_\theta(\lambda) = |a(\lambda)|^2\varphi(\lambda)(1+\theta\psi(\lambda))$$

and it follows from the last lemma that for all θ the measures P_0, P_θ are equivalent for all $T>0$.

Denote $R_\theta(t)$ the correlation function corresponding to the spectral density $g_\theta(\lambda)$ and let $\mathbf{R}_\theta$ denote the corresponding correlation operator. Let $h(\lambda) = \varphi(\lambda)|a(\lambda)|^2\psi(\lambda)$. The Fourier transform of this function

$$H(t) = \int_{-\infty}^{\infty} e^{it\lambda}h(\lambda)d\lambda$$

is a correlation function with the corresponding correlation operator $\mathbf{H}$ on L_T,

$$\mathbf{H}u(t) = \int_0^T H(t-s)u(s)ds.$$

Thus the operators

$$\mathbf{R}_\theta = \mathbf{R}_0 + \theta\mathbf{H}$$

and the operators $\mathbf{D}_\theta$ corresponding to the operator $\mathbf{D}$ of lemma 6 are

$$\mathbf{D}_\theta = \mathbf{R}_0^{-1/2}\mathbf{R}_\theta\mathbf{R}_0^{-1/2} - \mathbf{I} = \theta\mathbf{R}_0^{-1/2}\mathbf{H}\mathbf{R}_0^{-1/2} = \theta\mathbf{D}.$$

In particular, the eigenvalues of $\mathbf{D}_\theta$ are equal to $\theta\delta_k$ where δ_k are the eigenvalues of $\mathbf{D}$. It follows then from lemma 3.1 that the density function

$$p_\theta(x) = \frac{dP_\theta}{dP_0}(x) = \exp\Big\{\frac{1}{2}\sum\big\{(\mathbf{R}_0^{-1/2}x, e_k)_T^2\frac{\theta\delta_k}{1+\theta\delta_k} - \ln(1+\theta\delta_k)\big\}\Big\} \tag{3.16}$$

where e_k are the eigenfunctions of the operator $\mathbf{D}$.

Lemma 3.3. *The Fisher information*

$$I(\theta) = \mathbf{E}_\theta\big\{\big(\frac{d}{d\theta}\ln p_\theta(Y)\big)^2\big\} = \frac{1}{2}\sum_k\frac{\delta_k^2}{(1+\theta\delta_k)^2}. \tag{3.17}$$

Proof. Apply (3.16). We get that

$$\frac{d}{d\theta}\ln p_\theta(x) = \frac{1}{2}\sum_k\frac{\delta_k}{1+\theta\delta_k}\Big\{\frac{(\mathbf{R}_0^{-1/2}x, e_k)^2}{1+\theta\delta_k} - 1\Big\}$$

and hence

$$I(\theta) = \mathbf{E}_\theta\Big|\frac{d}{d\theta}\ln p_\theta(Y)\Big|^2 = \frac{1}{4}\mathbf{E}_\theta\Big|\sum_k\frac{\delta_k}{1+\theta\delta_k}\Big[\frac{(\mathbf{R}_0^{-1/2}Y, e_k)^2}{1+\theta\delta_k} - 1\Big]\Big|^2. \tag{3.18}$$

To compute the last expectation we apply the following evident result:

if Z is a Gaussian stationary process with the correlation operator $\mathbf{B}$, then for all $v\in L_T$ random variables $(Z,v)_T$ are Gaussian and

$$\mathbf{E}(Z,v)_T = \mathbf{E}Z(0)\cdot(1,v)_T,\ \mathbf{E}(Z,u)_T(Z,v)_T = (\mathbf{B}u,v)_T.$$

It follows that under P_θ the random variables $\xi_k = (\mathbf{R}_0^{-1/2}Y, e_k)_T = (Y, \mathbf{R}_0^{-1/2}e_k)_T$ are Gaussian with means zero and correlations

$$\mathbf{E}_\theta(\xi_k\xi_l) = \big((\mathbf{R}_0+\theta\mathbf{H})\mathbf{R}_0^{-1/2}e_k, \mathbf{R}_0^{-1/2}e_l\big) = \big((\mathbf{I}+\mathbf{D}_\theta)e_k, e_l\big) = (1+\theta\delta_k)\delta_{kl}$$

where the last δ_{kl} is the Kroneker symbol. Thus the Gaussian random variables ξ_k are independent with means zero and variances $\mathbf{E}\xi_k^2 = 1+\theta\delta_k$. The equality (3.18) gives then that

$$I(\theta) = \frac{1}{4}\sum_k \frac{\delta_k}{(1+\theta\delta_k)^2}\mathbf{E}_\theta\left|\frac{(\mathbf{R}_0^{-1/2}Y, e_k)^2}{1+\theta\delta_k} - 1\right|^2 = \frac{1}{2}\sum_k \frac{\delta_k}{(1+\theta\delta_k)^2}.$$

The lemma is proved. It follows from the lemma that

$$I(\theta) \le \frac{1}{2}\sum_k \delta_k^2 \le \frac{1}{2}\max_k \delta_k \sum_k \delta_k.$$

Below we study two last factors.

Lemma 3.4. *The eigenvalues δ_k of the operator $\mathbf{D}$ satisfy the inequality*

$$\max_k \delta_k \le C\big(e^{-(L/2)^\rho} + \varepsilon^2 e^{36LT\ln 9L} + e^{-tL}\big) \tag{3.19}$$

where ε is defined by (3.1). The constant C does not depend on L and T.

Proof. We have

$$\max \delta_k = \sup_{||u||_T=1} (\mathbf{D}u, u)_T = \sup_{||u||_T=1} (\mathbf{R}_0^{-1/2}\mathbf{H}\mathbf{R}_0^{-1/2}u, u)_T = \sup_v (\mathbf{H}v, v)_T$$

where the last upper bound is taken over all $v \in L_T$ such that $(\mathbf{R}_0 v, v)_T = 1$. Taking into account the relation (3.15) we find that

$$\max \delta_k = \sup_v \int_{-\infty}^{\infty} h(\lambda)|v(\lambda)|^2 d\lambda = \sup_v \int_{-\infty}^{\infty} |a(\lambda)|^2\psi(\lambda)\varphi(\lambda)|v(\lambda)|^2 d\lambda \tag{3.20}$$

where the upper bound is taken over all $v \in E_T$ such that

$$\int_{-\infty}^{\infty} |a(\lambda)|^2\varphi(\lambda)|v(\lambda)|^2 = 1. \tag{3.21}$$

Further a non-negative function $u \in E_\sigma$ can be represented as $|v(\lambda)|^2$, $v \in E_{\sigma/2}$ (see [12]). Thus

$$\max \delta_k \le \sup_v \int_{-\infty}^{\infty} \psi(\lambda)|a(\lambda)|^2|v(\lambda)|^2 d\lambda$$

and the upper bound is taken over all $v \in E_{T+1}$ such that

$$\int_{-\infty}^{\infty} |a(\lambda)|^2|v(\lambda)|^2 d\lambda = \int_{-1}^{1} |v(\lambda)|^2 d\lambda + \varepsilon^2 \int_{|\lambda|>1} \left|\frac{\sin\lambda}{\lambda}v(\lambda)\right|^2 d\lambda \le 1. \tag{3.22}$$

The properties (3.10) (3.11) of the function ψ imply that

$$\max \delta_k \le C\big(\exp\{-(L/2)^\rho\} + \varepsilon^2 \sup_v \max_{3L/2\ge|\lambda|\ge L/2} |\frac{\sin\lambda}{\lambda}v(\lambda)|^2. \tag{3.23}$$

To estimate the last summond on the right hand side take a function v satisfying (3.22) and set $v_1(z) = v(z)\frac{\sin z}{z}$. Then $v_1 \in E_{T+2}$ and satisfies the inequalities

$$\int_{-1}^{1} |v_1(\lambda)|^2 d\lambda \leq 1, \int_{-\infty}^{\infty} |v_1(\lambda)|^2 d\lambda \leq \varepsilon^{-2}. \tag{3.24}$$

The second inequality together with Paley – Wiener theorem implies that

$$|v_1(z)| \leq \varepsilon^{-1} e^{|\Im z|(T+2)} \leq \varepsilon^{-1} e^{(T+2)|z|}. \tag{3.25}$$

Represent the function $v_1(\lambda)$ by its expansion with respect to the Legendre polynomials (see Sect. 2). We get that $v_1(\lambda) = \sum_n a_n P_n(\lambda)$ and (see lemma 2.1)

$$\sum_n |a_n|^2 \leq 1, \quad |a_n| \leq C\varepsilon^{-1} \exp\{-n \ln n/eT\}.$$

Hence (see lemma 2.2) for $L/2 \leq |\lambda| \leq 3L/2$

$$|v(\lambda)| \leq \big(\sum_0^N |P_n(\lambda)|^2\big)^{1/2} + C\varepsilon^{-1} \sum_{N+1}^{\infty} |a_n||P_n(\lambda)|^2 \leq (9L)^N + C\varepsilon^{-1}(9L)^N e^{-N \ln N/eT}.$$

Take here $N \asymp 18TL$. We find that

$$\varepsilon \max_{L/2 \leq |\lambda| \leq 3L/2} |v(\lambda)(\sin \lambda)\lambda^{-1}| \leq C\big(\varepsilon \exp\{18TL \ln 9L\} + e^{-TL}\big). \tag{3.26}$$

Lemma 3.5. *The following inequality holds*

$$tr\mathbf{D} = \sum_k \delta_k \leq C(T + \ln^2 \varepsilon^{-1}). \tag{3.27}$$

Proof. The trace of the operator $\mathbf{D} = \mathbf{R}_0^{-1/2}\mathbf{H}\mathbf{R}_0^{-1/2}$ is

$$tr\mathbf{D} = \sum_k (\mathbf{D}u_k, u_k)_T$$

where $\{u_k\}$ is an orthonormal basis in L_T. If we consider such basis for which $v_k = \mathbf{R}_0^{1/2}u_k$ are defined, we find that

$$tr\mathbf{D} = \sum_k (\mathbf{H}v_k, v_k)_T$$

where $\{v_k\}$ are orthonormal with respect to the bilinear form corresponding to the operator $\mathbf{R}_0$, $(\mathbf{R}_0 v_k, v_l)_T = \delta_{kl}$. Arguing as above rewrite the forms $(\mathbf{H}v, v)_T, (\mathbf{R}_0 v, v)_T$ in the terms of the corresponding spectral densities. We find that

$$tr\mathbf{D} = \sum_k \int_{-\infty}^{\infty} \psi(\lambda)|a(\lambda)|^2|v(\lambda)|^2 d\lambda \tag{3.28}$$

where the functions $v_k \in E_{T+1}$ and satisfy the conditions

$$\int_{-\infty}^{\infty} |a(\lambda)|^2 v_k(\lambda)v_l(\lambda)d\lambda = \delta_{kl}. \tag{3.29}$$

Setting $a(\lambda)v_k(\lambda) = r_k(\lambda)$ we find that

$$tr\mathbf{D} = \int_{-\infty}^{\infty} \psi(\lambda) \sum_k r_k^2(\lambda) d\lambda \tag{3.30}$$

and the set of the functions $\{r_k\}$ is an orthonormal system in the subspace $a\dot{E}_{T+1}$ of $L_2(-\infty,\infty)$

We estimate now the sum $\sum_k r_k^2(\lambda)$. Let

$$r_j(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x}\tilde{r}_j(x)dx.$$

Then

$$\sum_k r_k^2(\lambda) = \frac{1}{2\pi}\sum_k \Big(\int_{-\infty}^{\infty} e^{i\lambda x}\sqrt{2\pi}\tilde{r}_k(x)dx\Big)^2$$

and the functions $\{\sqrt{2\pi}\tilde{r}_k\}$ constitute an orthonormal system in $L_2(-\infty,\infty)$. It follows that

$$\sum_k r_k^2(\lambda) \leq \frac{1}{2\pi}\sup_r \Big|\int_{-\infty}^{\infty} e^{i\lambda x} r(x)dx\Big|^2 \tag{3.31}$$

and the upper bound is taken over all $r, ||r|| = 1$ which are the Fourier transform of functions represented as av, $v \in E_{T+1}$. Hence

$$\sum_k r_k^2(\lambda) \leq |a(\lambda)|^2 \sup_v |v(\lambda)|^2 \tag{3.32}$$

where upper bound is taken over all $v \in E_{T+1}$ under the condition

$$\int_{-\infty}^{\infty} |a(\lambda)|^2 |v(\lambda)|^2 d\lambda = 1. \tag{3.33}$$

Consider now separatly two cases: $|\lambda| > 1$ and $|\lambda| \leq 1$.

Let $|\lambda| > 1$. It follows from (3.33) that

$$\int_{-\infty}^{\infty} \frac{\sin^2\lambda}{\lambda^2}|v(\lambda)|^2 d\lambda \leq \varepsilon^{-2}.$$

Hence

$$\Big|\frac{\sin\lambda}{\lambda}v(\lambda)\Big| \leq \frac{\sqrt{T+2}}{\varepsilon}$$

and

$$\sum_k r_k^2(\lambda) \leq (|a(\lambda)| \sup_v |v(\lambda)|)^2 \leq T+2.$$

Let now $|\lambda| \leq 1$. Functions $v_1(\lambda)\frac{\sin\lambda}{\lambda}$ satisfy the inequalities

$$\int_{-1}^{1} |v_1(\lambda)|^2 d\lambda \leq 1, \quad \int_{-\infty}^{\infty} |v_1(\lambda)|^2 d\lambda \leq \varepsilon^{-2}.$$

Hence expanding the function $v_1(\lambda)$ with respect to the Legendre polynomials (see Sect.2) we find that for $N > eT$

$$(3.34) \qquad |v_1(\lambda)| \le \big(\sum_0^N |P_n(1)|^2\big)^{1/2} + \frac{C}{\varepsilon}\sum_{N+1} e^{-n\ln n/eT} \le C(N + \varepsilon^{-1}e^{-N}).$$

Take here $N \sim \ln 1/\varepsilon$. We get that

$$(3.35) \qquad |v(\lambda)| \le C\ln\varepsilon^{-1}.$$

Now we are ready to prove the inequality (3.7).

Lemma 3.6. *Let the numbers ε, L in the additional problem satisfy the relation*

$$(3.36) \qquad \ln\varepsilon^{-1} = 18LT\ln 9L - LT.$$

Then for sufficiently large T and $L > (\ln T)^{\alpha/\rho}$, $\alpha > 1$, the Fisher information of the additional problem satisfies the inequality

$$(3.37) \qquad I(\theta) \le C\big(Te^{(-L/4)^{\rho}} + e^{-LT/2}\big) < 1.$$

The lemma is an immediate corollary of the inequalities (3.19) and (3.27). We have noticed above that the inequality (3.7) proves the theorem. Thus to finish the proof we have only to construct the function ψ_0 satisfying the conditions (3.10), (3.11). The construction coincides in principle with analogous constructins in papers [9], [10] and we omit the details. We will distingushe three cases.

The first and the simplest case is when ρ is an even integer, $\rho = 2k$. We set in this case $\psi_0(z) = \exp\{-z^{2k}\}$. Evidently this function satisfies (3.10). The function $\psi_L(z) = \psi_0(z - L)$ is an integral function and for $|z| < L/4k$ $\Re(z-L)^2k > 0$ and hence $|\psi_L(z)| < 1$. If $|z| > L/4k$, then $|z - L|^{2k} \le 2^{2k-1}(|z|^{2k} + L^{2k}) \le (C|z|)^{2k}$.

If $\rho = 2k+1$, $k \ge 0$, is an odd integer, we set

$$\psi_0(z) = \left(\Pi_{n=1}^{\infty}\frac{\sin(z^{\rho}n^{-(1+\delta)})}{z^{\rho}n^{-(1+\delta)}}\right)^2, \quad \delta > 0.$$

Applying the Stirling formula we get that

$$|\psi_0(\lambda)| \le \left(\Pi_{n<|\lambda|^{\rho/(1+\varepsilon)}}\big(|\lambda|n^{-(1+\delta)}\big)^{\rho}\right)^{-2} \le Ce^{-c|\lambda|^{\rho}}.$$

Further

$$\Big|\frac{\sin z}{z}\Big| \le |z|^{-1}e^{|\Im z|}$$

and hence for $|z| \le \alpha|L|, /, \alpha$ be a small positive number,

$$|\psi_L(z)| \le C\exp\{-c_1|z-L|^{\rho} + c_2|\Im(z-L)^{\rho}|\} \le C$$

For $|z| > \alpha|L|$ evidently $|\psi_L(z)| \le Ce^{c|z|^{\rho}}$.

The case of non integer ρ is more complicated. Roughly speaking the construction is the following one. Define the integer p from the relations $p < \rho < p+1$. Consider at first the function

$$V(z) = \Pi_{r=1}^{\infty}G\big(\frac{z}{r^{1/\rho}}, p\big)$$

where

$$G(u,p) = (1-u)\exp\{u + \frac{u^2}{2} + \cdots + \frac{u^p}{p}\}.$$

Take $\alpha = \frac{3\pi}{2\rho}$ and set $V_\alpha(z) = V(ze^{-i\alpha})$. The function ψ_0 is then defined as follows

$$\psi_0(z) = V_\alpha(z)V_{-\alpha}(z)V_\alpha(-z)V_{-\alpha}(-z).$$

The proof that this function satisfies the all necessary demands see in [10].

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