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On the Circular Law

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Abstract

We consider the joint distribution of real and imaginary parts of eigenvalues of random matrices with independent real entries with mean zero and unit variance. We prove the convergence of this distribution to the uniform distribution on the unit disc without assumptions on the existence of a density for the distribution of entries. We assume however that the entries have sub-Gaussian tails or are sparsely non-zero.

1 Introduction

Let $X_{jk}, 1 \le j,k < \infty$, be complex random variables with $\mathbb{E}X_{jk} = 0$ and $\mathbb{E}|X_{jk}|^2 = 1$. For a fixed $n \ge 1$, denote by $\lambda_1, \dots, \lambda_n$ the eigenvalues of the $n \times n$ matrix

$$\mathbf{X} = \frac{1}{\sqrt{n}}(X_n(j,k))_{j,k=1}^n, \quad X_n(j,k) = \frac{1}{\sqrt{n}}X_{jk}, \text{ for } 1 \le j,k \le n, \tag{1.1}$$

and define its empirical spectral distribution function by

$$G_n(x,y) = \frac{1}{n}\sum_{j=1}^n I_{\{\mathrm{Re}\,\{\lambda_j\} \le x,\, \mathrm{Im}\,\{\lambda_j\} \le y\}}, \tag{1.2}$$

where $I_{\{B\}}$ denotes the indicator of an event B. We investigate the convergence of the expected spectral distribution function $\mathbb{E}G_n(x,y)$ to the distribution function $G(x,y)$ of the uniform distribution over the unit disc in $\mathbb{R}^2$.

We shall assume that the random variables X_{jk} are sub-Gaussian, i. e.

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Definition 1.1. A random variable β is called sub-Gaussian (respectively β has a distribution with sub-Gaussian tails) if for any $t > 0$

$$\Pr\{|\beta| > t\} \leq C \exp\{-ct^2\}.$$

The main result of our paper is the following

Theorem 1.2. *Let X_{jk} be independent identically distributed sub-Gaussian random variables with*

$$\mathbf{E}\, X_{jk} = 0, \qquad \mathbf{E}\, |X_{jk}|^2 = 1.$$

Then $\mathbf{E}\, G_n(x, y)$ converges weakly to the distribution function $G(x, y)$ as $n \to \infty$.

We shall prove the same result for the follows class of sparse matrices. Let ε_{jk}, $j, k = 1, \ldots, n$ denote Bernoulli random variables which are independent in aggregate and independent of $(X_{jk})_{j,k=1}^n$ with $p_n := \Pr\{\varepsilon_{jk} = 1\}$. Consider the matrix $\mathbf{X}^{(\varepsilon)} = \frac{1}{\sqrt{np_n}}(\varepsilon_{jk} X_{jk})_{j,k=1}^n$. Let $\lambda_1^\varepsilon, \ldots, \lambda_n^\varepsilon$ denote the (complex) eigenvalues of the matrix $\mathbf{X}^{(\varepsilon)}$ and denote by $G_n^\varepsilon(x, y)$ the empirical spectral distribution function of the matrix $\mathbf{X}^{(\varepsilon)}$, i. e.

$$\frac{1}{n} \sum_{j=1}^n I_{\{\mathrm{Re}\,\{\lambda_j^\varepsilon\} \leq x,\, \mathrm{Im}\,\{\lambda_j^\varepsilon\} \leq y\}}. \tag{1.3}$$

Theorem 1.3. *Let X_{jk} be independent identically distributed sub-Gaussian random variables with*

$$\mathbf{E}\, X_{jk} = 0, \qquad \mathbf{E}\, |X_{jk}|^2 = 1.$$

Assume that $np_n^4 \to \infty$ as $n \to \infty$. Then $\mathbf{E}\, G_n^\varepsilon(x, y)$ converges weakly to the distribution function $G(x, y)$ as $n \to \infty$.

Remark 1.4. The assumption $np_n^4 \to \infty$ is merely technical and due to our approach to bound the minimal singular values of sparse matrices. For details see Subsection 6.2 in the Appendix.

Remark 1.5. The crucial problem of the proofs of Theorems 1.2 and 1.3 is to bound the minimal singular values of shifted matrices $\mathbf{X} - z\mathbf{I}$ and $\mathbf{X}^\varepsilon - z\mathbf{I}$. These bounds are based on the results obtained by Rudelson in [21].

The investigation of the convergence the spectral distribution functions of real or complex (non-symmetric and non-Hermitian) random matrices with independent entries has a long history. Ginibre in 1965, [10], studied the real, complex and quaternion matrices with i. i. d. Gaussian entries. He derived the joint density for the distribution of eigenvalues of matrix. Using the Ginibre results, Edelman in 1997, [4] proved the circular law for the matrices with i. i. d. Gaussian entries. Girko in 1984, [7], investigated the circular law for general matrices with independent entries assuming that the distribution of the entries have densities. As pointed out by Bai [2], Girko's proof had serious gaps. Bai in [2] gave a proof of the circular law for random matrices with independent entries assuming that the entries had bounded densities and finite sixth moments. Unfortunately this result still

does not cover the case the the Wigner ensemble and in particular ensembles of matrices with Rademacher entries. These ensembles are of some interest in various applications, see e.g. [22]. (Wigner, in his pioneering work in 1955 [23] proved the *semi-circular* law for symmetric matrices with i. i. d. Rademacher entries). A discussion of Girko's contribution to the proof of the universality of the cicular law may be found in Edelman [4] as well. Girko published several papers providing additional explanations and corrections of his arguments in his paper in 1984 [7], see, for example, [5], [8], [9]. In [5] he states the circular law for matrices with independent entries without any assumption on their densities. His proof unfortunately does not show why (assuming his conditions)

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\mathbf{E}\,\log|\det(\mathbf{X}(z)(\mathbf{X}(z))^*+\varepsilon^2\mathbf{I}|=\lim_{n\to\infty}\lim_{\varepsilon\to 0}\mathbf{E}\,\log|\det(\mathbf{X}(z)(\mathbf{X}(z))^*+\varepsilon^2\mathbf{I}|.$$

See for example Khoruzhenko's [15], remark on the "regularization of potential". Girko's [7] approach using families of spectra of Hermitian matrices for a characterisation of the circular-law based on the so-called *V-transform* was fruitful for all later work. See, for example, Girko's Lemma 1 in [2].

We shall outline his approach using logarithmic potential theory. Let ξ denote a random variable uniformly distributed over the unit disc. For any $r>0$, consider the matrix,

$$\mathbf{X}(r)=\mathbf{X}-r\xi\mathbf{I},$$

where $\mathbf{I}$ denotes the identity matrix of order n. Let $\mu_n^{(r)}$ be empirical spectral measure of matrix $\mathbf{X}(r)$ defined on the complex plane as empirical measure of the set of eigenvalues of matrix. We define a logarithmic potential of the expected spectral measure $\mathbf{E}\,\mu_n^{(r)}(ds,dt)$ as

$$U_n^{(r)}(z)=-\frac{1}{n}\mathbf{E}\,\log|\det(\mathbf{X}(r)-z\mathbf{I})|=-\frac{1}{n}\sum\mathbf{E}\,\log|\lambda_j-z-r\xi|,$$

where $\lambda_1,\dots,\lambda_n$ are the eigenvalues of the matrix $\mathbf{X}$. Note that the expected spectral measure $\mathbf{E}\,\mu_n^{(r)}$ is the convolution of the measure $\mathbf{E}\,\mu_n$ and the uniform distribution on the disc of radius r (see Lemma 6.2 in the Appendix for details).

Lemma 1.1. *Assume that the sequence $\mathbf{E}\,\mu_n^{(r)}$ converges weakly to a measure μ as $n\to\infty$ and $r\to 0$. Then*

$$\mu=\lim_{n\to\infty}\mathbf{E}\,\mu_n. \tag{1.4}$$

Proof. Let J be a random variable which is uniformly distributed on the set $\{1,\dots,n\}$ and independent of the matrix $\mathbf{X}$. We may represent the measure $\mathbf{E}\,\mu_n^{(r)}$ as distribution of a random variable $\lambda_J+r\xi$ where λ_J and ξ are independent. Computing the characteristic function of this measure and passing first to the limit with respect to $n\to\infty$ and then with respect to $r\to 0$ (see also Lemma 6.3 in the Appendix), we conclude the result. □

Now we may fix $r>0$ and consider the measures $\mathbf{E}\,\mu_n^{(r)}$. They have bounded densities. Assume that the measures $\mathbf{E}\,\mu_n$ have supports in a fixed compact set and that $\mathbf{E}\,\mu_n$ converges weakly to a measure μ. Applying Theorem 6.9 (Lower Envelope Theorem)

from [18], p. 73 (see also Subsection 6.1 in the Appendix), we obtain that under these assumptions

$$\liminf_{n\to\infty} U_n^{(r)}(z) = U^{(r)}(z), \tag{1.5}$$

for quasi-everywhere in $\mathbb{C}$ (for the definition of "*quasi-everywhere*" see for example [18], p 24 and Subsection 6.1 in the Appendix). Here $U^{(r)}(z)$ denotes the logarithmic potential of measure $\mu^{(r)}$ which is the convolution of a measure μ and of the uniform distribution on the disc of radius r. Furthermore, note that $U^{(r)}(z)$ we may represented as

$$U^{(r)}(z_0) = \frac{2}{r^2}\int_0^r vL(\mu; z_0, v)dv,$$

where

$$L(\mu; z_0, v) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U^{(\mu)}(z_0 + v\exp\{i\theta\})d\theta. \tag{1.6}$$

Applying Theorem 1.2 in [18], p. 84, (Theorem 6.2 in Subsection 6.1 in the Appendix) we get

$$\lim_{r\to 0} U_\mu^{(r)}(z) = U_\mu(z).$$

Let $s_1(\mathbf{X}) \geq \ldots \geq s_n(\mathbf{X})$ denote the singular values of matrix $\mathbf{X}$. Note that for any $M > 2$

$$\Pr\{s_1(\mathbf{X}) > M\} \leq \Pr\{s_1^2(\mathbf{X}) > 4\} \leq \sup_x |\mathbf{E}\, F_n(x) - M_1(x)| \leq Cn^{-\frac{1}{8}}, \tag{1.7}$$

where $F_n(x)$ denotes the empirical distribution function of the matrix $\frac{1}{n}\mathbf{X}\mathbf{X}^*$. Here $\mathbf{X}^*$ stands for the complex conjugate and transpose of the matrix $\mathbf{X}$, and $M_1(x)$ denotes Marchenko–Pastur distribution function with parameter 1 and density

$$m_1(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}I_{\{0<x<4\}}.$$

(See, for example, [3], Theorem 3.2). This implies that the sequence of measures $\mathbf{E}\,\mu_n$ is weakly relatively compact. These results imply that we may restrict the measures $\mathbf{E}\,\mu_n$ to some compact set K such that $\sup_n \mathbf{E}\,\mu_n(K^{(c)}) \to 0$. If we take some subsequence of the sequence of restricted measures $\mathbf{E}\,\mu_n$ which converges to some measure μ, then $\liminf_{n\to\infty} U_{\mu_n}^{(r)}(z) = U_\mu^{(r)}(z)$, $r > 0$ and $\lim_{r\to 0} U_\mu^{(r)}(z) = U_\mu(z)$. If we prove that $\liminf_{n\to\infty} U_{\mu_n}^{(r)}(z)$ exists and $U_\mu(z)$ is equal to the logarithmic potential corresponding the uniform distribution on the unit disc then the sequence of measures $\mathbf{E}\,\mu_n$ weakly converges to the uniform distribution on the unit disc. Moreover, it is enough to prove that for some sequence $r = r(n) \to 0$, $\lim_{n\to\infty} U_{\mu_n}^{(r)}(z) = U_\mu(z)$.

Furthermore, let $s_1^\varepsilon(z,r) \geq \ldots \geq s_n^\varepsilon(z,r)$ denote the singular values of matrix $\mathbf{X}^\varepsilon(z,r) = \mathbf{X}^\varepsilon(r) - z\mathbf{I}$. We shall investigate the logarithmic potential $U_{\mu_n}^{(r)}(z)$. Using elementary properties of singular values (see for instance Lemma 3.3 [11], p.35), we may represent the function $U_{\mu_n}^{(r)}(z)$ as follows

$$U_{\mu_n}^{(r)}(z) = -\frac{1}{n}\sum_{j=1}^n \mathbf{E}\,\log s_j^\varepsilon(z,r) = -\frac{1}{2}\int_0^\infty \log x \mathbf{E}\,\nu_n^\varepsilon(dx,z,r),$$

where $\nu_n^\varepsilon(\cdot, z, r)$ denotes the spectral measure of the matrix $\mathbf{H}_n^\varepsilon(z,r) = (\mathbf{X}^\varepsilon(r) - z\mathbf{I})(\mathbf{X}^\varepsilon(r) - z\mathbf{I})^*$, which is the counting measure of the set of eigenvalues of the matrix $\mathbf{H}_n^\varepsilon(z,r)$).

In Section 2) we investigate convergence of measure $\nu_n^\varepsilon(\cdot, z) = \nu^\varepsilon(\cdot, z, 0)$. In Section 3 we study the properties of the limit measures $\nu(\cdot, z)$. But the crucial problem for the proof of the circular law is the so called "regularization of potential" problem. See Khoruzhenko [15]. We solve this problem using bounds for the minimal singular values of matrices $\mathbf{X}^\varepsilon(z) := \mathbf{X}^\varepsilon - z\mathbf{I}$ based on techniques developed in Rudelson [21]. These bounds are given in Section 4 and in the Appendix, Subsection 6.2. In Section 5 we give the proof of the main Theorem. In the Appendix we combine precise statements of relevant results. from potential theory and some auxiliary inequalities for the resolvent matrices.

2 Convergence of $\nu_n^\varepsilon(\cdot, z)$

Denote by $F_n^\varepsilon(x, z)$ the distribution function of the measure $\nu_n^\varepsilon(\cdot, z)$,

$$F_n^\varepsilon(x,z) = \frac{1}{n}\sum_{j=1}^{n} I_{\{(s_j^\varepsilon(z))^2 < x\}},$$

where $s_1^\varepsilon(z) \geq \ldots \geq s_n^\varepsilon(z) \geq 0$ denote the singular values of the matrix $\mathbf{X}^\varepsilon(z) = \mathbf{X}^\varepsilon - z\mathbf{I}$. For a positive random variable ξ and a Rademacher random variable (r. v.) κ consider the transformed r. v. $\widetilde{\xi} = \kappa\sqrt{\xi}$. If ζ has distribution function $F_n^\varepsilon(x, z)$ the variable $\widetilde{\zeta}$ has distribution function $\widetilde{F}_n^\varepsilon(x, z)$, given by

$$\widetilde{F}_n^\varepsilon(x,z) = \frac{1}{2}(1 + \mathrm{sgn}\{x\}F_n^\varepsilon(x^2, z))$$

for all real x. Note that this induces a one-to-one corresponds between the respective measures $\nu_n^\varepsilon(\cdot, z)$ and $\widetilde{\nu}_n^\varepsilon(\cdot, z)$. The limit distribution function of $F_n^\varepsilon(x, z)$ as $n \to \infty$, is denoted by $F(\cdot, z)$ with corresponding symmetrization $\widetilde{F}(x, z)$ being the limit of $\widetilde{F}_n^\varepsilon(x, z)$ as $n \to \infty$. We have

$$\sup_x |F_n^\varepsilon(x,z) - F(x,z)| = \sup_x |\widetilde{F}_n^\varepsilon(x,z) - \widetilde{F}(x,z)|.$$

Denote by $s_n^\varepsilon(\alpha, z)$ (resp. $s(\alpha, z)$) and $S_n^\varepsilon(x, z)$ (resp. $S(x, z)$) the Stieltjes transforms of the measures $\nu_n^\varepsilon(\cdot, z)$ (resp. $\nu(\cdot, z)$) and $\widetilde{\nu}_n^\varepsilon(\cdot, z)$ (resp. $\widetilde{\nu}(\cdot, z)$) correspondingly. Then we have

$$S_n^\varepsilon(\alpha, z) = \alpha s_n^\varepsilon(\alpha^2, z), \qquad S(\alpha, z) = \alpha s(\alpha^2, z).$$

Remark 2.1. As is shown in Bai [2], the measure $\nu(\cdot, z)$ has a density $p(x, z)$ and bounded support. More precisely, $p(x, z) \leq C \max\{1, \frac{1}{\sqrt{x}}\}$. Thus the measure $\widetilde{\nu}(\cdot, z)$ has bounded support and bounded density $\widetilde{p}(x, z) = |x|p(x^2, z)$.

Theorem 2.2. *Let* $\mathbf{E}\, X_{jk} = 0$, $\mathbf{E}\,|X_{jk}|^2 = 1$, *and*

$$\varkappa_3 = \max_{1\le j,k<\infty} \mathbf{E}\,|X_{jk}|^3. \tag{2.1}$$

Then

$$\sup_x |F_n^\varepsilon(x,z) - F(x,z)| \le C\varkappa_3 (np_n)^{-\frac{1}{10}}. \tag{2.2}$$

Proof. To bound the distance between the distribution functions $\widetilde{F}_n^\varepsilon(x,z)$ and $\widetilde{F}(x,z)$ we investigate the distance between the Stieltjes transforms of these distribution functions. Introduce the Hermitian $2n\times 2n$ matrix

$$\mathbf{W} = \begin{pmatrix} \mathbf{O}_n & (\mathbf{X}^\varepsilon - z\mathbf{I}) \\ (\mathbf{X}^\varepsilon - z\mathbf{I})^* & \mathbf{O}_n \end{pmatrix},$$

where $\mathbf{O}_n$ denotes $n\times n$ matrix with all entries equal to zero. From Šur's complement formula (see for example [14], Ch. 08, p. 21) it follows that, for $\alpha = u+iv$, $v>0$,

$$(\mathbf{W}-\alpha\mathbf{I}_{2n})^{-1} = \begin{pmatrix} \alpha\left(\mathbf{X}^\varepsilon(z)(\mathbf{X}^\varepsilon(z))^* - \alpha^2\mathbf{I}_n\right)^{-1} & \mathbf{X}^\varepsilon(z)\left(\mathbf{X}^\varepsilon(z)(\mathbf{X}^\varepsilon(z))^* - \alpha^2\mathbf{I}_n\right)^{-1} \\ \left((\mathbf{X}^\varepsilon(z))^*\mathbf{X}^\varepsilon(z) - \alpha^2\mathbf{I}_n\right)^{-1}(\mathbf{X}^\varepsilon(z))^* & \alpha\left((\mathbf{X}^\varepsilon(z))^*\mathbf{X}^\varepsilon(z) - \alpha^2\mathbf{I}_n\right)^{-1} \end{pmatrix} \tag{2.3}$$

where $\mathbf{X}^\varepsilon(z) = \mathbf{X}^\varepsilon - z\mathbf{I}$ and $\mathbf{I}_{2n}$ denotes the unit matrix of order $2n$. By definition of $S_n^\varepsilon(\alpha,z)$, we have

$$S_n^\varepsilon(\alpha,z) = \frac{1}{2n}\mathbf{E}\,\mathrm{Tr}\,(\mathbf{W}-\alpha\mathbf{I}_{2n})^{-1}.$$

Set $\mathbf{R}(\alpha,z) := (R_{j,k}(\alpha,z))_{j,k=1}^{2n} = (\mathbf{W}-\alpha\mathbf{I}_{2n})^{-1}$. It is easy to check that

$$1+\alpha S_n^\varepsilon(\alpha,z) = \frac{1}{2n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{W}\mathbf{R}(\alpha,z).$$

We may rewrite this equality as

$$\begin{aligned} 1+\alpha S_n^\varepsilon(\alpha,z) = {} & \frac{1}{2n\sqrt{np_n}}\sum_{j,k=1}^n \mathbf{E}\left(\varepsilon_{jk}X_{jk}R_{k+n,j}(\alpha,z) + \varepsilon_{jk}\overline{X}_{jk}R_{j+n,k}(\alpha,z)\right) \\ & - \frac{\overline{z}}{2n}\sum_{j=1}^n \mathbf{E}\,R_{j,j+n}(\alpha,z) - \frac{z}{2n}\sum_{j=1}^n \mathbf{E}\,R_{j+n,j}(\alpha,z). \end{aligned} \tag{2.4}$$

We introduce the notations

$$\begin{aligned} \mathbf{A} &= (\mathbf{X}^\varepsilon(z)(\mathbf{X}^\varepsilon(z))^* - \alpha^2\mathbf{I})^{-1}, \quad \mathbf{B} = \mathbf{X}^\varepsilon(z)\mathbf{A}, \\ \mathbf{C} &= ((\mathbf{X}^\varepsilon(z))^*\mathbf{X}^\varepsilon(z) - \alpha^2\mathbf{I})^{-1}, \quad \mathbf{D} = \mathbf{C}(\mathbf{X}^\varepsilon(z))^*. \end{aligned}$$

With these notations we rewrite equality (2.3) as follows

$$\mathbf{R}(\alpha,z) = (\mathbf{W}-\alpha\mathbf{I}_{2n})^{-1} = \begin{pmatrix} \alpha\mathbf{A} & \mathbf{B} \\ \mathbf{D} & \alpha\mathbf{C} \end{pmatrix} \tag{2.5}$$

Equalities (2.5) and (2.4) together imply

$$1+\alpha S_n^{\varepsilon}(\alpha,z)=\frac{1}{2n\sqrt{np_n}}\sum_{j,k=1}^{n}\mathbf{E}\left(\varepsilon_{jk}X_{jk}R_{k+n,j}(\alpha,z)+\varepsilon_{jk}\overline{X}_{jk}R_{j,k+n}(\alpha,z)\right)$$
$$-\frac{z}{2n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{D}-\frac{\overline{z}}{2n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{B}. \tag{2.6}$$

In the what follows we shall use a simple resolvent equality. For two matrices $\mathbf{U}$ and $\mathbf{V}$ let $\mathbf{R}_U=(\mathbf{U}-\alpha\mathbf{I})^{-1}$, $\mathbf{R}_{U+V}=(\mathbf{U}+\mathbf{V}-\alpha\mathbf{I})^{-1}$, then

$$\mathbf{R}_{U+V}=\mathbf{R}_U-\mathbf{R}_U\mathbf{V}\mathbf{R}_{U+V}.$$

Let $\{\mathbf{e}_1,\dots\mathbf{e}_{2n}\}$ denote the canonical orthonormal basis in $\mathbb{R}^{2n}$. Let $\mathbf{W}^{(jk)}$ denote the matrix is obtained from $\mathbf{W}$ by replacing the both entries $X_{j,k}$ and $\overline{X}_{j,k}$ by 0. In our notation we may write

$$\mathbf{W}=\mathbf{W}^{(jk)}+\frac{1}{\sqrt{np_n}}\varepsilon_{jk}X_{jk}\mathbf{e}_j\mathbf{e}_{k+n}^T+\frac{1}{\sqrt{np_n}}\varepsilon_{jk}\overline{X}_{jk}\mathbf{e}_{k+n}\mathbf{e}_j^T. \tag{2.7}$$

Using this representation and the resolvent equality, we get

$$\mathbf{R}=\mathbf{R}^{(j,k)}-\frac{1}{\sqrt{np_n}}\varepsilon_{jk}X_{jk}\mathbf{R}^{(j,k)}\mathbf{e}_j\mathbf{e}_{k+n}^T\mathbf{R}-\frac{1}{\sqrt{np_n}}\varepsilon_{jk}\overline{X}_{jk}\mathbf{R}^{(j,k)}\mathbf{e}_{k+n}\mathbf{e}_j^T\mathbf{R}. \tag{2.8}$$

Here and in the what follows we omit the arguments α and z in the notation of resolvent matrices. For any vector $\mathbf{a}$, let $\mathbf{a}^T$ denote the transposed vector $\mathbf{a}$. Applying the resolvent equality again, we obtain

$$\mathbf{R}=\mathbf{R}^{(j,k)}-\frac{1}{\sqrt{np_n}}\varepsilon_{jk}X_{jk}\mathbf{R}^{(j,k)}\mathbf{e}_j\mathbf{e}_{k+n}^T\mathbf{R}^{(j,k)}-\frac{1}{\sqrt{np_n}}\varepsilon_{jk}\overline{X}_{jk}\mathbf{R}^{(j,k)}\mathbf{e}_{k+n}\mathbf{e}_j^T\mathbf{R}^{(j,k)}+\mathbf{T}^{(jk)},$$

where

$$\begin{aligned}\mathbf{T}^{(jk)}&=\frac{1}{np_n}\varepsilon_{jk}X_{jk}^2\mathbf{R}^{(j,k)}\mathbf{e}_j\mathbf{e}_{k+n}^T\mathbf{R}^{(j,k)}\mathbf{e}_j\mathbf{e}_{k+n}^T\mathbf{R}\\&+\frac{1}{np_n}\varepsilon_{jk}|X_{jk}|^2\mathbf{R}^{(j,k)}\mathbf{e}_j\mathbf{e}_{k+n}^T\mathbf{R}^{(j,k)}\mathbf{e}_{k+n}\mathbf{e}_j^T\mathbf{R}\\&+\frac{1}{np_n}\varepsilon_{jk}(\overline{X}_{jk})^2\mathbf{R}^{(j,k)}\mathbf{e}_{k+n}\mathbf{e}_j^T\mathbf{R}^{(j,k)}\mathbf{e}_j\mathbf{e}_{k+n}^T\mathbf{R}\\&+\frac{1}{np_n}\varepsilon_{jk}|X_{jk}|^2\mathbf{R}^{(j,k)}\mathbf{e}_{k+n}\mathbf{e}_j^T\mathbf{R}^{(j,k)}\mathbf{e}_{k+n}\mathbf{e}_j^T\mathbf{R}\end{aligned}$$

This implies

$$\begin{aligned}\mathbf{R}_{j,k+n}&=\mathbf{R}^{(j,k)}_{j,k+n}-\frac{1}{\sqrt{np_n}}\varepsilon_{jk}X_{jk}\mathbf{R}^{(j,k)}_{j,j}\mathbf{R}^{(j,k)}_{k+n,k+n}-\frac{1}{\sqrt{np_n}}\varepsilon_{jk}\overline{X}_{jk}(\mathbf{R}^{(j,k)}_{j,k+n})^2+\mathbf{T}^{(j,k)}_{j,k+n}\\\mathbf{R}_{k+n,j}&=\mathbf{R}^{(j,k)}_{k+n,j}-\frac{1}{\sqrt{np_n}}\varepsilon_{jk}X_{jk}\mathbf{R}^{(j,k)}_{k+n,j}\mathbf{R}^{(j,k)}_{j,k+n}-\frac{1}{\sqrt{np_n}}\varepsilon_{jk}\overline{X}_{jk}\mathbf{R}^{(j,k)}_{k+n,k+n}\mathbf{R}^{(j,k)}_{j,j}+\mathbf{T}^{(j,k)}_{k+n,j}.\end{aligned} \tag{2.9}$$

Applying these notations to the equality (2.6) and taking into account that X_{jk} and $\mathbf{R}^{(jk)}$ are independent, we get

$$1+\alpha S_n^{\varepsilon}(\alpha,z)+\frac{z}{2n}\mathrm{Tr}\,\mathbf{D}+\frac{\overline{z}}{2n}\mathrm{Tr}\,\mathbf{B}=-\frac{1}{n^2p_n}\sum_{j,k=1}^{n}\mathbf{E}\,\varepsilon_{jk}R^{(j,k)}_{j,j}R^{(j,k)}_{k+n,k+n}$$
$$-\frac{1}{2n^2p_n}\sum_{j,k=1}^{n}\mathbf{E}\,\varepsilon_{jk}|X_{jk}|^2\mathbf{E}\,(R^{(j,k)}_{j,k+n})^2$$
$$-\frac{1}{2n\sqrt{np_n}}\sum_{j,k=1}^{n}\mathbf{E}\,(\varepsilon_{jk}X_{jk}T^{(j,k)}_{k+n,j}+\varepsilon_{jk}\overline{X}_{jk}T^{(j,k)}_{j,k+n}). \tag{2.10}$$

By definition of $\mathbf{T}^{(j,k)}$ and standard resolvent properties, we obtain the following bounds, for any $p,q=1,\dots,2n$, $j,k=1,\dots n$, and any $z=u+iv,\ v>0$,

$$|R_{p,p}-R^{(j,k)}_{pp}|\le\frac{C\varepsilon_{jk}|X_{jk}|}{\sqrt{np_n}}(|R^{jk}_{pj}||R_{k+n,p}|+|R^{jk}_{p,k+n}||R_{jp}|)$$
$$\frac{1}{n^2}\sum_{j,k=1}^{n}\mathbf{E}\,|R^{(j,k)}_{j,k+n}|^2\le\frac{C}{nv^4} \tag{2.11}$$
$$\frac{1}{n\sqrt{np_n}}\sum_{j,k=1}^{n}\mathbf{E}\,\varepsilon_{jk}|X_{jk}||T^{(j,k)}_{j,k+n}|\le\frac{C\varkappa_3}{np_nv^4} \tag{2.12}$$

For the proof of these inequalities see in the Appendix, Lemma 6.1. Using the last inequalities we obtain, that for $v>0$

$$\left|\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,R_{jj}\frac{1}{n}\sum_{k=1}^{n}R_{k+n,k+n}\ -\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbf{E}\,R^{(jk)}_{jj}R^{(jk)}_{k+n,k+n}\right|$$
$$\le\frac{C}{n^2\sqrt{np_n}v}\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbf{E}\,\varepsilon_{jk}|X_{jk}|(|R^{(jk)}_{jj}||R_{k+n,j}|+|R^{(jk)}_{j,k+n}||R_{jj}|)$$
$$\le\frac{C\varkappa_3}{nv^4}. \tag{2.13}$$

Since $\frac{1}{n}\sum_{j=1}^{n}R_{jj}=\frac{1}{n}\sum_{k=1}^{n}R_{k+n,k+n}=\frac{1}{2n}\mathrm{Tr}\,\mathbf{R}(\alpha,z)$, we obtain

$$|\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbf{E}\,R^{(jk)}_{jj}R^{(jk)}_{k+n,k+n}-\mathbf{E}\,(\frac{1}{2n}\mathrm{Tr}\,\mathbf{R}(\alpha,z))^2|\le\frac{C\varkappa_3}{nv^4} \tag{2.14}$$

Note that for any Hermitian random matrix $\mathbf{W}$ with independent entries on and above the diagonal we have

$$\mathbf{E}\,|\frac{1}{n}\mathrm{Tr}\,\mathbf{R}(\alpha,z)-\mathbf{E}\,\frac{1}{n}\mathrm{Tr}\,\mathbf{R}(\alpha,z)|^2\le\frac{C}{nv^2}. \tag{2.15}$$

The proof of this inequality is easy and due to a martingale type expansion already used by Girko. Inequalities (2.14) and (2.15) together imply that for $v > cn^{-\frac{1}{4}}$

$$|\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbf{E}\,R_{jj}^{(jk)}R_{k+n,k+n}^{(jk)} - (S_n^{\varepsilon}(\alpha,z))^2| \le \frac{C}{\sqrt{n}v^2} \tag{2.16}$$

We may now rewrite equality (2.6) as follows

$$1 + \alpha S_n^{\varepsilon}(\alpha,z) + (S_n^{\varepsilon}(\alpha,z))^2 = -\frac{z}{2n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{D} - \frac{\overline{z}}{2n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{B} + \theta\frac{1}{\sqrt{np_n}v^2}, \tag{2.17}$$

were θ is a function such that $|\theta| \le 1$ and $v > c(np_n)^{-\frac{1}{4}}$.

We now investigate the functions $T(\alpha,z) = \frac{1}{n}\mathbf{E}\,\mathrm{Tr}\,\mathbf{D}$ and $V(\alpha,z) = \frac{1}{n}\mathbf{E}\,\mathbf{B}$. Since the arguments for both functions are similar we provide it for the first one only. By definition of the matrix $\mathbf{B}$, we have

$$\mathrm{Tr}\,\mathbf{B} = \frac{1}{\sqrt{np_n}}\sum_{j,k=1}^{n}\varepsilon_{jk}X_{j,k}((\mathbf{X}^{\varepsilon}(z)(\mathbf{X}^{\varepsilon}(z))^* - \alpha^2)^{-1})_{kj}$$

According to equality (2.5), we have

$$\mathrm{Tr}\,\mathbf{B} = \frac{1}{\alpha\sqrt{np_n}}\sum_{j,k=1}^{n}\varepsilon_{jk}X_{j,k}R_{kj} - z\mathrm{Tr}\,\mathbf{A}$$

Using the resolvent equality (2.8) and Lemma 6.1, we get, for $v > c(np_n)^{-\frac{1}{4}}$

$$T(\alpha,z) = -\frac{1}{\alpha n^2}\sum_{j,k=1}^{n}\mathbf{E}\,R_{k,k+n}^{(jk)}R_{jj}^{(jk)} - \frac{z}{\alpha}S_n^{\varepsilon}(\alpha,z) + \theta\frac{C\varkappa_3}{np_nv^2}. \tag{2.18}$$

Similar to (2.16) we obtain

$$|\frac{1}{\alpha n^2}\sum_{j,k=1}^{n}\mathbf{E}\,R_{jj}^{(jk)}R_{k,k+n}^{(jk)} - V(\alpha,z)S_n^{\varepsilon}(\alpha,z)| \le \frac{C}{n\sqrt{n}v^4} \tag{2.19}$$

Inequalities (2.18) and (2.19) together imply, for $v > c(np_n)^{-\frac{1}{4}}$,

$$V(\alpha,z) = -\frac{\overline{z}S_n^{\varepsilon}(\alpha,z)}{\alpha + S_n^{\varepsilon}(\alpha,z)} + \theta\frac{C\varkappa_3}{np_nv^2|\alpha + S_n^{\varepsilon}(\alpha,z)|}. \tag{2.20}$$

Analogously we get

$$T(\alpha,z) = -\frac{zS_n^{\varepsilon}(\alpha,z)}{\alpha + S_n^{\varepsilon}(\alpha,z)} + \theta\frac{C}{np_nv^2|\alpha + S_n^{\varepsilon}(\alpha,z)|}. \tag{2.21}$$

Insecting (2.20) and (2.21) in (2.10), we get

$$(S_n^{\varepsilon}(\alpha,z))^2+\alpha S_n^{\varepsilon}(\alpha,z)+1-\frac{|z|^2S_n^{\varepsilon}(\alpha,z)}{\alpha+S_n^{\varepsilon}(\alpha,z)}=\delta_n(z), \tag{2.22}$$

where

$$|\delta_n(\alpha,z)|\leq\frac{C\varkappa_3}{np_nv^2|S_n^{\varepsilon}(\alpha,z)+\alpha|}.$$

or equivalently

$$S_n^{\varepsilon}(\alpha,z)\,(\alpha+S_n^{\varepsilon}(\alpha,z))^2+(\alpha+S_n({}^{\varepsilon}\alpha,z))-|z|^2S_n^{\varepsilon}(\alpha,z)=\widetilde{\delta}_n(\alpha,z), \tag{2.23}$$

were $\widetilde{\delta}_n(\alpha,z)=\theta\frac{C\varkappa_3}{np_nv^2}$. The last equation we may rewrite as

$$S_n^{\varepsilon}(\alpha,z)=-\frac{\alpha+S_n^{\varepsilon}(\alpha,z)}{(\alpha+S_n^{\varepsilon}(\alpha,z))^2-|z|^2}+\widehat{\delta}_n(\alpha,z), \tag{2.24}$$

were

$$\widehat{\delta}_n(\alpha,z)=\frac{\widetilde{\delta}_n(\alpha,z)}{(\alpha+S_n^{\varepsilon}(\alpha,z))^2-|z|^2}. \tag{2.25}$$

Note that

$$\left|\frac{1}{(\alpha+S_n^{\varepsilon}(\alpha,z))^2-|z|^2}\right|\leq\frac{1}{v|\alpha+S_n^{\varepsilon}(\alpha,z)|}$$

. This implies that

$$|\widehat{\delta}_n(\alpha,z)|\leq\frac{C}{np_nv^2|\alpha+S_n^{\varepsilon}(\alpha,z)|}.$$

Furthermore, we prove the following simple Lemma.

Lemma 2.1. *Let $\alpha=u+iv$, $v>0$. Let $S(\alpha,z)$ satisfy the equation*

$$S(\alpha,z)=-\frac{\alpha+S(\alpha,z)}{(\alpha+S(\alpha,z))^2-|z|^2}. \tag{2.26}$$

and $\operatorname{Im}\{S(\alpha,z)\}>0$. *Then the following inequality*

$$1-|S(\alpha,z)|^2-\frac{|z|^2|S(\alpha,z)|^2}{|\alpha+S(\alpha,z)|^2}\geq\frac{v}{v+1}.$$

holds.

Proof. The Stieltjes transform $S(\alpha,z)$ satisfies the following equation, for $\alpha=u+iv$ with $v>0$,

$$S(\alpha,z)=-\frac{\alpha+S(\alpha,z)}{(\alpha+S(\alpha,z))^2-|z|^2}. \tag{2.27}$$

Comparing the imaginary parts of both sides of this equation, we get

$$\operatorname{Im}\{\alpha+S(\alpha,z)\}=\operatorname{Im}\{\alpha+S(\alpha,z)\}\,\frac{|\alpha+S(\alpha,z)|^2+|z|^2}{|(\alpha+S(\alpha,z))^2-|z|^2|^2}+v. \tag{2.28}$$

Equations (2.26) and (2.28) together imply

$$\operatorname{Im}\{\alpha + S(\alpha, z)\}\left(1 - \frac{|\alpha + S(\alpha, z)|^2 + |z|^2}{|(\alpha + S(\alpha, z))^2 - |z|^2|^2}\right) = v. \tag{2.29}$$

Since $v > 0$ and $\operatorname{Im}\{\alpha + S(\alpha, z)\} > 0$, it follows that

$$1 - \frac{|\alpha + S(\alpha, z)|^2 + |z|^2}{|(\alpha + S(\alpha, z))^2 - |z|^2|^2} > 0.$$

In particular, we have

$$|S(\alpha, z)| \leq 1.$$

Inequality (2.29) and the last remark together imply

$$1 - \frac{|\alpha + S(\alpha, z)|^2 + |z|^2}{|(\alpha + S(\alpha, z))^2 - |z|^2|^2} = \frac{v}{\operatorname{Im}\{\alpha + S(\alpha, z)\}} \geq \frac{v}{v + 1}.$$

The proof is completed. □

To compare the function $S(\alpha, z)$ and $S_n(\alpha, z)$ we prove

Lemma 2.2. *Let*

$$|\widehat{\delta}_n(\alpha, z)| \leq \frac{v}{2}.$$

Then the following inequality holds

$$1 - \frac{|\alpha + S_n^{\varepsilon}(\alpha, z)|^2 + |z|^2}{|(\alpha + S_n^{\varepsilon}(\alpha, z))^2 - |z|^2|^2} \geq \frac{v}{4}.$$

Proof. By assumption, we have

$$\operatorname{Im}\{\widehat{\delta}_n(\alpha, z) + \alpha\} > \frac{v}{2}.$$

Repeating the arguments of Lemma 2.1 completes the proof. □

The next Lemma give as a bound for the distance between the Stieltjes transforms $S(\alpha, z)$ and $S_n^{\varepsilon}(\alpha, z)$.

Lemma 2.3. *Let*

$$|\widehat{\delta}_n(\alpha, z)| \leq \frac{v}{8}.$$

Then

$$|S_n^{\varepsilon}(\alpha, z) - S(\alpha, z)| \leq \frac{4|\widehat{\delta}_n(\alpha, z)|}{v}.$$

Proof. Note that $S(\alpha, z)$ and $S_n^{\varepsilon}(\alpha, z)$ satisfy the equations

$$S(\alpha, z) = -\frac{\alpha + S(\alpha, z)}{(\alpha + S(\alpha, z)^2 - |z|^2} \tag{2.30}$$

and

$$S_n^{\varepsilon}(\alpha, z) = -\frac{\alpha + S_n^{\varepsilon}(\alpha, z)}{(\alpha + S_n^{\varepsilon}(\alpha, z)^2 - |z|^2} + \widehat{\delta}_n(\alpha, z) \tag{2.31}$$

respectively. These equations together imply

$$\begin{aligned} S(\alpha, z) - S_n^{\varepsilon}(\alpha, z) = \frac{(\alpha + S_n^{\varepsilon}(\alpha, z))(\alpha + S(\alpha, z)) + |z|^2}{((\alpha + S(\alpha, z)^2 - |z|^2)((\alpha + S_n^{\varepsilon}(\alpha, z)^2 - |z|^2)} \\ \times (S(\alpha, z) - S_n^{\varepsilon}(\alpha, z)) + \widehat{\delta}_n(\alpha, z). \end{aligned} \tag{2.32}$$

Applying inequality $|ab| \leq \frac{1}{2}(a^2 + b^2)$, we get

$$\begin{aligned} \left| 1 - \frac{(\alpha + S_n^{\varepsilon}(\alpha, z))(\alpha + S(\alpha, z)) + |z|^2}{((\alpha + S(\alpha, z)^2 - |z|^2)((\alpha + S_n^{\varepsilon}(\alpha, z)^2 - |z|^2)} \right| \\ \geq \frac{1}{2} \left(1 - \frac{|\alpha + S_n(\alpha, z)|^2 + |z|^2}{|(\alpha + S_n^{\varepsilon}(\alpha, z))^2 - |z|^2|^2} \right) \\ + \frac{1}{2} \left(1 - \frac{|\alpha + S(\alpha, z)|^2 + |z|^2}{|(\alpha + S(\alpha, z))^2 - |z|^2|^2} \right). \end{aligned}$$

The last inequality and Lemmas 2.1 and 2.2 together imply

$$\left| 1 - \frac{(\alpha + S_n^{\varepsilon}(\alpha, z))(\alpha + S(\alpha, z)) + |z|^2}{((\alpha + S(\alpha, z)^2 - |z|^2)((\alpha + S_n^{\varepsilon}(\alpha, z)^2 - |z|^2)} \right| \geq \frac{v}{4}.$$

This completes the proof of the Lemma. □

To bound the distance between the distribution function $F_n(x, z)$ and the distribution function $F(x, z)$ corresponding the Stieltjes transform $S(\alpha, z)$ we use Corollary 2.3 from [12]. In the next lemma we give an integral bound for the distance between the Stieltjes transforms $S(\alpha, z)$ and $S_n^{\varepsilon}(\alpha, z)$.

Lemma 2.4. *For $v \geq v_0(n) = c(np_n)^{-1/4}$ the inequality*

$$\int_{-\infty}^{\infty} |S(\alpha, z) - S_n^{\varepsilon}(\alpha, z)| du \leq \frac{C(1 + |z|^2)\varkappa_3}{np_n v^6}.$$

holds.

Proof. It is enough to prove that

$$\int_{-\infty}^{\infty} |\widehat{\delta}_n(\alpha, z)| du \leq C\gamma_n,$$

where $\gamma_n = \frac{C}{np_n v^5}$. By definition of $\widehat{\delta}(\alpha, z)$, we have

$$\int_{-\infty}^{\infty} |\widehat{\delta}_n(\alpha, z)| du \le \frac{c\varkappa_3}{np_n v^2} \int_{-\infty}^{\infty} \frac{du}{|(\alpha + S_n^\varepsilon(\alpha, z))^2 - |z|^2|}. \tag{2.33}$$

Furthermore, the representation (2.24) implies that

$$\frac{1}{|(\alpha + S_n^\varepsilon(\alpha, z))^2 - |z|^2|} \le \frac{|S_n^\varepsilon(\alpha, z)|}{|\alpha + S_n^\varepsilon(\alpha, z)|} + \frac{|\widehat{\delta}_n(\alpha, z)|}{|\alpha + S_n^\varepsilon(\alpha, z)|}. \tag{2.34}$$

Note that, according to the relation (2.23),

$$\frac{1}{|\alpha + S_n^\varepsilon(\alpha, z)|} \le \frac{|z|^2 |S_n^\varepsilon(\alpha, z)|}{|\alpha + S_n^\varepsilon(\alpha, z)|^2} + |S_n^\varepsilon(\alpha, z)| + \frac{|\widetilde{\delta}_n(\alpha, z)|}{|\alpha + S_n^\varepsilon(\alpha, z)|^2} \le |S_n^\varepsilon(\alpha, z)|(1 + \frac{|z|^2}{v^2}) + \frac{|\delta_n(\alpha, z)|}{|\alpha + S_n^\varepsilon(\alpha, z)|}. \tag{2.35}$$

This inequality implies

$$\int_{-\infty}^{\infty} \frac{|S_n^\varepsilon(\alpha, z)|}{|\alpha + S_n^\varepsilon(\alpha, z)|} du \le \frac{C(1 + |z|^2)}{v^2} \int_{-\infty}^{\infty} |S_n^\varepsilon(\alpha, z)|^2 du + \int_{-\infty}^{\infty} |\delta_n(\alpha, z)| \frac{|S_n^\varepsilon(\alpha, z)|}{|\alpha + S_n^\varepsilon(\alpha, z)|} du \tag{2.36}$$

It follows from the relation (2.22), for $v > c(np_n)^{-\frac{1}{4}}$, that

$$|\delta_n(\alpha, z)| \le \frac{C\varkappa_3}{np_n v^3}. \tag{2.37}$$

The last two inequalities together imply that for sufficiently large n and $v > c(np_n)^{-\frac{1}{4}}$,

$$\int_{-\infty}^{\infty} \frac{|S_n^\varepsilon(\alpha, z)|}{|\alpha + S_n^\varepsilon(\alpha, z)|} du \le \frac{C(1 + |z|^2)}{v^2} \int_{-\infty}^{\infty} |S_n^\varepsilon(\alpha, z)|^2 du \le \frac{C(1 + |z|^2)}{v^3}. \tag{2.38}$$

The inequalities (2.35), (2.33), and the definition of $\widehat{\delta}_n(\alpha, z)$ together imply

$$\int_{-\infty}^{\infty} |\widehat{\delta}_n(\alpha, z)| du \le \frac{C(1 + |z|^2)}{np_n v^5} + \frac{C\varkappa_3}{np_n v^3} \int_{-\infty}^{\infty} |\widehat{\delta}_n(\alpha, z)| du. \tag{2.39}$$

If we choose v such that $\frac{C}{np_n v^3} < \frac{1}{2}$ we obtain

$$\int_{-\infty}^{\infty} |\widehat{\delta}_n(\alpha, z)| du \le \frac{C(1 + |z|^2)}{np_n v^5}. \tag{2.40}$$

□

In Section 3 is shown that the measure $\nu(\cdot, z)$ has bounded support and bounded density for any z. To bound the distance between the distribution functions $\mathbf{E}\, F_n(x, z)$ and $F(x, z)$ we may apply Corollary 3.2 from [12] (see also Lemma 6.4 in the Appendix). We take $V = 1$ and $v_0 = C(np_n)^{-\frac{1}{4}}$. Then Lemmas 2.1 and 2.2 together imply

$$\sup_x |\mathbf{E}\, F_n^\varepsilon(x, z) - F(x, z)| \le C(np_n)^{-\frac{1}{4}}. \tag{2.41}$$

□

3 Properties of the measure $\nu(\cdot, z)$

In this Section we investigate the properties of the measure $\nu(\cdot, z)$. At first note that there exists a solution $S(\alpha, z)$ of the equation

$$S(\alpha, z) = -\frac{S(\alpha, z) + z}{(S(\alpha, z) + z)^2 - |z|^2} \tag{3.1}$$

such that

$$\mathrm{Im}\,\{S(\alpha, z)\} \geq 0 \quad \text{for} \quad v > 0$$

and $S(\alpha, z)$ is an analytic function in the upper half-plane $\alpha = u + iv$, $v > 0$. This follows from the relative compactness of the sequence of analytic functions $S_n(\alpha, z)$, $n \in \mathbb{N}$. From (2.30) it follows immediately that

$$|S(\alpha, z)| \leq 1. \tag{3.2}$$

Set $y = S(x, z) + x$ and consider the equation (2.30) on the real line

$$y = -\frac{y}{y^2 - |z|^2} + x, \tag{3.3}$$

or

$$y^3 - xy^2 + (1 - |z|^2)y + x|z|^2 = 0. \tag{3.4}$$

Set

$$x_1^2 = \frac{5 + 2|z|^2}{2} + \frac{(1 + 8|z|^2)^{\frac{3}{2}} - 1}{8|z|^2}, \quad x_2^2 = \frac{5 + 2|z|^2}{2} - \frac{(1 + 8|z|^2)^{\frac{3}{2}} + 1}{8|z|^2}. \tag{3.5}$$

It is straightforward to check that for $|z| \leq 1$ $\sqrt{3(1 - |z|^2)} \leq |x_1|$ and $x_2^2 < 0$ for $|z| < 1$ and $x_2^2 = 0$ for $|z| = 1$, and $x_2^2 > 0$ for $|z| > 1$.

Lemma 3.1. *In the case $|z| \leq 1$ equation (3.4) has one real root for $|x| \leq |x_1|$ and three real roots for $|x| > |x_1|$. In the case $|z| > 1$ equation (3.4) has one real root for $|x_2| \leq x \leq |x_1|$ and has tree real roots for $|x| \leq |x_2|$ or for $|x| \geq |x_1|$.*

Proof. Set

$$L(y) := y^3 - xy^2 + (1 - |z|^2)y + x|z|^2.$$

We consider the roots equation

$$L'(y) = 3y^2 - 2xy + (1 - |z|^2) = 0. \tag{3.6}$$

The roots of this equation are

$$y_{1,2} = \frac{x \pm \sqrt{x^2 - 3(1 - |z|^2)}}{3}.$$

This implies that, for $|z| \leq 1$ and for

$$|x| \leq \sqrt{3(1-|z|^2)},$$

the equation (3.4) has one real root. Furthermore, direct calculations shown that

$$L(y_1)L(y_2) = \frac{1}{27}\left(-4|z|^2x^4 + (8|z|^4 + 20|z|^2 - 1)x^2 + 4(1-|z|^2)^3\right)$$

Solving the equation $L(y_1)L(y_2) = 0$ with respect to x, we get for $|z| \leq 1$ and $\sqrt{3(1-|z|^2)} \leq |x| \leq |x_1|$

$$L(y_1)L(y_2) \geq 0,$$

and for $|z| \leq 1$ and $|x| > \sqrt{\frac{20+8|z|^2}{8} + \frac{(1+8|z|^2)^{\frac{3}{2}}-1}{8|z|^2}}$

$$L(y_1)L(y_2) < 0,$$

These relations imply that for $|z| \leq 1$ the function $L(y)$ has three real roots for $|x| \geq |x_1|$ and one real root for $|x| < |x_1|$.

Consider the case $|z| > 1$ now. In this case $y_{1,2}$ are real for all x and $x_2^2 > 0$. Note that

$$L(y_1)L(y_2) \leq 0$$

for $|x| \leq |x_2|$ and for $|x| \geq |x_1|$ and

$$L(y_1)L(y_2) > 0$$

for $|x_2| < x < |x_1|$. These implies that for $|z| > 1$ and for $|x_2| < x < |x_1|$ the function $L(y)$ has one real root and for $|x| \leq |x_2|$ or for $|x| \geq |x_1|$ the function $L(y)$ has three real roots. The Lemma is proved. □

Remark 3.1. From Lemma 3.1 it follows that the measure $\nu(x,z)$ has a density $p(x,z)$ and

- $p(x,z) \leq 1$, for all x and z
- for $|z| \leq 1$, if $|x| \geq x_1$ then $p(x,z) = 0$;
- for $|z| \geq 1$, if $|x| \geq x_1$ or $|x| \leq x_2$ then $p(x,z) = 0$;
- $p(x,z) > 0$ otherwise.

The next lemma is an analogue of Lemma 4.4 in Bai [2].

Lemma 3.2. *The following equality*

$$\frac{\partial}{\partial s}\left(\int_0^\infty \log x \nu(dx,z)\right) = \frac{1}{2}\Re\{g(x,z)\} \tag{3.7}$$

holds.

Proof. Following Bai [2] Lemma 4.4, we consider

$$I(C) := \int_0^C \frac{\partial y(x)}{\partial s} dx. \tag{3.8}$$

We have

$$y^3 + 2xy^2 + x^2 y - |z|^2 y + y + x = 0. \tag{3.9}$$

Taking the derivatives with respect to x and s correspondingly, we get

$$\frac{\partial y}{\partial x}\left(3y^2 + 4xy + (1 - |z|^2 + x^2)\right) = -1 - 2y(x + y) \tag{3.10}$$

and

$$\frac{\partial y}{\partial s}\left(3y^2 + 4xy + (1 - |z|^2 + x^2)\right) = 2sy. \tag{3.11}$$

These equalities together imply

$$\frac{\partial y}{\partial s} = -\frac{2sy}{1 + 2y(x + y)} \frac{\partial y}{\partial x}. \tag{3.12}$$

From equation (3.9) it follows that

$$1 + 2y(y + x) = \pm\sqrt{1 + 4|z|^2 y^2}. \tag{3.13}$$

Using the results of Remark 3.1, it is straightforward to check that for $|z| \leq 1$

$$1 + 2y(y + x) = \sqrt{1 + 4|z|^2 y^2} \tag{3.14}$$

and for $|z| > 1$ there exists a number x_0 such that $\sqrt{1 + 4|z|^2 y^2} = 0$. Furthermore, we have for $-x_0 \leq x \leq 0$

$$1 + 2y(y + x) = \sqrt{1 + 4|z|^2 y^2} \tag{3.15}$$

and for $x < -x_0$ we obtain

$$1 + 2y(y + x) = -\sqrt{1 + 4|z|^2 y^2}. \tag{3.16}$$

Using these equalities, we get

$$\int_{-C}^0 \frac{\partial y}{\partial s} dx = -\int_{-C}^0 \frac{2sy}{1 + 2y(x + y)} \frac{\partial y}{\partial x} dx. \tag{3.17}$$

For $|z| \leq 1$, we have

$$\int_{-C}^0 \frac{\partial y}{\partial s} dx = -\int_{-C}^0 \frac{2sy}{\sqrt{1 + 4|z|^2 y^2}} \frac{\partial y}{\partial x} dx = \frac{s}{4|z|^2}\left(\sqrt{1 + 4|z|^2 y^2(-C)} + \sqrt{1 + 4|z|^2(|z|^2 - 1)}\right). \tag{3.18}$$

In the limit $C \to \infty$, we get, for $|z| \le 1$,

$$\int_{-\infty}^{0} \frac{\partial y}{\partial s} dx = \frac{s}{2}. \tag{3.19}$$

For $|z| > 1$, we have

$$\int_{-\infty}^{0} \frac{\partial y}{\partial s} dx = \int_{-x_0}^{0} \frac{2sy}{\sqrt{1+4|z|^2y^2}} \frac{\partial y}{\partial x} dx - \int_{-\infty}^{-x_0} \frac{2sy}{\sqrt{1+4|z|^2y^2}} \frac{\partial y}{\partial x} dx = \frac{s}{2|z|^2}. \tag{3.20}$$

Similar to Bai [2] (equality (4.39)) we have

$$\begin{aligned} \int_{-C}^{0} y(x)dx = \int_{-C}^{0} y(x)dx &= \int_0^C \int_0^\infty \frac{1}{u+x} \nu(du, z)dx \\ &= \ln C + \int_0^\infty \left[\ln(u+C) - \ln u\right] \nu(du, z) \\ &= \ln C + \int_0^\infty \ln(1+\frac{u}{C})\nu(du, z) - \int_0^\infty \ln u \nu(du, z) \end{aligned} \tag{3.21}$$

After differentiation we get

$$\frac{\partial}{\partial s} \int_0^\infty \ln u \nu(du, z) = \frac{\partial}{\partial s} \int_0^\infty \ln(1+\frac{u}{C})\nu(du, z) - \int_{-C}^{0} \frac{\partial}{\partial s} y(x)dx. \tag{3.22}$$

Relations (3.19)–(3.22) together imply the result. □

4 The smallest singular value

In this Section we prove a bound for the minimal singular value of the matrices $\mathbf{X} - z\mathbf{I}$. A corresponding bound for sparse matrices we shall give in the Appendix. Let $\mathbf{X} = \frac{1}{\sqrt{n}} (X_{jk})_{j,k=1}^n$ be an $n \times n$ matrix with i.i.d. entries X_{jk}, $j, k = 1, \dots, n$ and ε_{jk} $j, k = 1 \dots, n$ Bernoulli i. i. d. random variables independent on X_{jk}, $j, k = 1, \dots, n$ with $p_n = \Pr\{\varepsilon_{jk} = 1\}$. Assume that $\mathbf{E}\, X_{jk} = 0$ and $\mathbf{E}\, X_{jk}^2 = 1$. We prove the following result. Denote by $s_1(z) \ge \dots \ge s_n(z)$ the singular values of the matrix $\mathbf{X}(z) := \mathbf{X} - z\mathbf{I}$.

Theorem 4.1. *Let X_{jk} be independent random variables with sub-Gaussian tails, i. e.*

$$\Pr\{|X_{jk}| > t\} \le \exp\{-ct^2\}. \tag{4.1}$$

Then for any $z \in \mathbb{C}$ such that $|z| \le 4$ and for any $\gamma > \frac{c}{\sqrt{n}}$

$$\Pr\{s_n \le \gamma / Cn^2\} \le \gamma, \tag{4.2}$$

for some positive constants C and c.

The proof of this theorem is based on the arguments of Rudelson [21]. He proved the same result for $z = 0$ and for a real matrix $\mathbf{X}$. To generalize this result to complex z and complex matrices we need some modifications of his proof. To bound the smallest singular value in our case we need to consider the complex unit sphere $\mathcal{S}^{(n-1)}$ in $\mathbb{C}^n$.

By the symbols C and c with or without indices or without it we shall denote some absolute constants. We shal adapt Rudelson's enumeration of constants, i. e. the lower indices of constants correspond the number of the Theorems in Rudelson's paper.

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ denote a vector in $\mathcal{S}^{(n-1)}$ in $\mathbb{C}^n$. Then $\mathbf{a} = (|\alpha_1|, \dots, |\alpha_n|)$ is an element of the unit sphere $S^{(n-1)} \subset \mathbb{R}^n$. We shall use the arguments of Rudelson for real vectors $\mathbf{a}$. Furthermore, we need some modifications of his concentration results for complex random variables. These are Theorem 3.5 and Lemma 4.2 in [21]. We start with Theorem 3.5. We may reformulate it as follows.

Theorem 4.2. *Let β a complex random variable such that $\mathbf{E}\,\beta = 0$ and $\Pr\{|\beta| > c\} \geq c'$, for some $c, c' > 0$. Let $\beta_1, \dots, \beta_n$ be independent copies of β. Let $\Delta > 0$ and let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{C}^n$ be a vector such $a < |x_j| < \overline{C}_{3.5}a$ for a some $a > 0$ and for some positive constant $\overline{C}_{3.5}$. Let ε_j be i. i. d. Bernoulli random variables independent on β_j, $j = 1, \dots, n$. Then there exists a constant $C_{3.5}$ such that for any $\Delta < \frac{a}{2\pi}$, for any $j_0 = 1, \dots, n$ and any $u, v \in \mathbb{C}$*

$$\Pr\left\{\left|\sum_{j=1}^{n} \varepsilon_j \beta_j x_j - \sqrt{np_n}ux_{j_0} - \sqrt{np_n}v\right| < \Delta\right\} \leq \frac{C_{3.5}}{(np_n)^{\frac{5}{2}}} \sum_{k=1}^{\infty} P_k^2(x, \Delta), \tag{4.3}$$

where

$$P_k(x, \Delta) = \#\{j : \ |x_j| \in (k\Delta, (k+1)\Delta]\}.$$

Proof. The proof of this Theorem is based on Lemma 3.1 in [21]. We reformulate this result for the complex case

Lemma 4.1. *Let $c > 0$, $0 < \Delta < \frac{a}{2\pi}$, and $\beta_1, \dots, \beta_n$ be independent complex random variables such that $\mathbf{E}\,\beta_j = 0$ and $\Pr\{|\widetilde{\beta}_j| > \sqrt{2}a\} \geq c$, where $\widetilde{\beta}_j = \beta_j - \beta'_j$ and β'_j is an independent copy of β_j. Let ε_j be i. i. d. Bernoulli random variables independent on β_j, $j = 1, \dots, n$. Then, tehere exist constants c, c' such that for any $v \in \mathbb{C}$,*

$$\Pr\left\{\left|\sum_{j=1}^{n} \varepsilon_j \beta_j - v\right| < \Delta\right\} \leq \frac{C_{3.5}}{(np_n)^{\frac{5}{2}}} \int_{\frac{3a}{2}} S_\Delta^2(y)dy + ce^{-c'np_n}, \tag{4.4}$$

where

$$S_\Delta(y) = \sum_{j=1}^{n} \Pr\{|\widetilde{\beta}_j| \in [y - \pi\Delta, y + \pi\Delta]\}. \tag{4.5}$$

Proof. Let $\beta_j = \xi_j + i\ \eta_j$, and $v = c + i\ d$. In this notation we have

$$\Pr\left\{\left|\sum_{j=1}^{n} \beta_j - v\right| < \Delta\right\} \leq \min\left\{\Pr\left\{\left|\sum_{j=1}^{n} \xi_j - c\right| < \Delta\right\},\ \Pr\left\{\left|\sum_{j=1}^{n} \eta_j - d\right| < \Delta\right\}\right\} =: Q$$

Note that

$$|\xi_j|^2 + |\eta_j|^2 = |\beta_j|^2, \tag{4.6}$$

implies

$$\max\{|\xi_j|,\ |\eta_j|\} \geq \frac{|\beta_j|}{\sqrt{2}}. \tag{4.7}$$

By the Lemma of Esséen (see, for example, [20] Lemma 3, p. 38), for any $v \in \mathbb{C}$ we have

$$Q \leq C \min\left\{ \int_{[-\frac{\pi}{2},\frac{\pi}{2}]} |\phi^{\varepsilon}(t/\Delta)|dt, \int_{[-\frac{\pi}{2},\frac{\pi}{2}]} |\psi^{\varepsilon}(t/\Delta)|dt \right\}, \tag{4.8}$$

where $\phi^{\varepsilon}(t) := \mathbf{E}\exp\{it\sum_{j=1}^{n}\varepsilon_j\xi_j\}$ and $\psi^{\varepsilon}(t) := \mathbf{E}\exp\{it\sum_{j=1}^{n}\varepsilon_j\eta_j\}$. Let $\widetilde{\xi}_j = \xi_j - (\xi_j)'$ and $\widetilde{\eta}_j = \eta_j - (\eta_j)'$ where $(\xi_j)'$ and $(\eta_j)'$ denote independent copies of ξ_j and η_j respectively. Note that

$$|\widetilde{\xi}_j|^2 + |\widetilde{\eta}_j|^2 = |\widetilde{\beta}_j|^2 \tag{4.9}$$

This implies that at least $\frac{n}{2}$ of the random variables $\widetilde{\xi}_j$ or $\widetilde{\eta}_j$, $j = 1, \ldots, n$, satisfy the inequality

$$|\widetilde{\xi}_j| \geq \frac{1}{\sqrt{2}}|\widetilde{\beta}_j|, \text{ or } |\widetilde{\eta}_j| \geq \frac{1}{\sqrt{2}}|\widetilde{\beta}_j|. \tag{4.10}$$

Without loss of generality we shall assume that $m \geq \left[\frac{n}{2}\right]$ random variables $\widetilde{\xi}_j$ satisfy the inequality

$$|\widetilde{\xi}_j| \geq \frac{1}{\sqrt{2}}|\widetilde{\beta}_j|. \tag{4.11}$$

The last inequality yields

$$\Pr\{|\widetilde{\xi}_j| \geq a\} \geq \overline{c} > 0. \tag{4.12}$$

Following Rudelson, we introduce the random variable τ_j by conditioning on $|\widetilde{\xi}_j| > 2a$. We may repeat from here on his proof of Lemma 3.1 and Theorem 4.1 in [21] to obtain the result of Theorem 4.2. After simple calculations we get

$$|\phi^{\varepsilon}(t)| \leq \exp\{-p_n(1-p_n)\sum^{*}(1 - \operatorname{Re}\phi_j(t)) - \frac{1}{2}p_n^2\sum^{*}(1 - |\phi_j(t)|^2)\}, \tag{4.13}$$

where $\sum^{*}$ denote the summation over all indexes $j = 1, \ldots, n$ such that inequality (4.12) holds and $\phi_j(t) = \mathbf{E}\exp\{it\xi_j\}$. Furthermore, for all j such that (4.12) holds we have

$$1 - |\phi_j(t)|^2 \geq \overline{c}\ \mathbf{E}\,(1 - \cos\tau_j t). \tag{4.14}$$

Inequalities (4.13) and (4.14) together imply

$$|\phi(t)| \leq \exp\{-c'f(t)\}, \tag{4.15}$$

where

$$f(t) = \mathbf{E}\sum^{*}(1 - \cos\tau_j t).$$

In the what follows we repeat Rudelson arguments for the rest of proof. Let

$$T(l,r) = \{t: \ f(t/\Delta)) \le l, |t| \le r\}, \tag{4.16}$$

$$M = \max_{|t|\le \frac{\pi}{2}} f(t/\Delta). \tag{4.17}$$

To estimate M from below, notice that

$$M = \max_{|t|\le \frac{\pi}{2}} f(t/\Delta) \ge \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{E}\sum\nolimits^{*}(1-\cos(\tau_j/\Delta)t)dt \tag{4.18}$$

$$= \mathbf{E}\sum\nolimits^{*}\left(1-\frac{2}{\pi}\frac{\sin(\tau_j/\Delta)\pi/2}{\tau_j/\Delta}\right) \ge cm \ge c'n, \tag{4.19}$$

since $|\tau_j|/\Delta > 2a/\Delta > 4\pi$. We shall use the following result from Rudelson [21].

Lemma 4.2. *Let $0 < l < M/4$. Then*

$$|T(l,\pi/2)| \le c\sqrt{\frac{l}{M}}|T(M/4,\pi)|. \tag{4.20}$$

We have

$$Q \le C\int_{[-\pi/2,\pi/2]} |\phi(t/\Delta)|dt \le C\int_{[-\pi/2,\pi/2]} \exp\{-c'f(t)\}$$

$$\le C\int_0^n |T(l,\pi/2)|e^{-c'l}dl. \tag{4.21}$$

According to the last lemma we get

$$Q \le \frac{C'}{\sqrt{M}}|T(\frac{M}{4},\pi)| + ce^{-\frac{C'M}{16}} \le \frac{C'}{\sqrt{M}}|T(\frac{M}{4},\pi)| + e^{-c'm}. \tag{4.22}$$

Repeating the arguments of Rudelson in [21], we obtain

$$Q \le \frac{C}{n^{\frac{5}{2}}\Delta}\int_{\mathbb{R}\setminus[-3a/2,3a/2]} \left(\sum\nolimits^{*}\Pr\{\tau_j \in [z-\pi\Delta, z+\pi\Delta]\}\right)^2 dz + ce^{-c'n}. \tag{4.23}$$

Since τ_j are symmetric we may change the interval of integration set in the previous inequality to $(3a/2,\infty)$. Moreover, if $z \in (3a/2,\infty)$

$$\Pr\{\tau_j \in [z-\pi\Delta, z+\pi\Delta]\} \le \frac{1}{c}\Pr\{\widetilde{\xi}_j \in [z-\pi\Delta, z+\pi\Delta]\} \le \frac{1}{c}\Pr\{|\widetilde{\beta}_j| \in [z-\pi\Delta, z+\pi\Delta]\}. \tag{4.24}$$

Furthermore,

$$1 - \operatorname{Re}\{\phi_j(t)\} = \mathbf{E}\,(1-\cos\{\xi_j t\}. \tag{4.25}$$

repeating the previous arguments, we conclude the proof of Lemma 4.1. □

We continue to prove Theorem 4.2. Recall that $x_j = a_j + i\ b_j$, $u = c + i\ d$ and $v = f + i\ g$. Then the following inequality holds

$$\begin{aligned}
&\Pr\{|\sum_{j=1}^{n}\beta_j x_j - \sqrt{n}ux_{j_0} - \sqrt{n}v| \le \Delta\} \\
&\qquad \le \min\Bigg\{\Pr\{|\sum_{j=1}^{n}(\xi_j a_j - \eta_j b_j) - \sqrt{n}(c\xi_{j_0} - d\eta_{j_0}) - \sqrt{n}f| \le \Delta\}, \\
&\qquad\qquad \Pr\{|\sum_{j=1}^{n}(\eta_j a_j + \xi_j b_j) - \sqrt{n}(c\eta_{j_0} + d\xi_{j_0}) - \sqrt{n}d| \le \Delta\}\Bigg\}. \quad (4.26)
\end{aligned}$$

Note that

$$|\eta_j a_j + \xi_j b_j|^2 + |\xi_j a_j - \eta_j b_j|^2 = |x_j|^2|\beta_j|^2, \quad (4.27)$$

implies again

$$\max\{|\eta_j a_j + \xi_j b_j|, |\xi_j a_j - \eta_j b_j|\} \ge |x_j||\beta_j|/\sqrt{2}. \quad (4.28)$$

Conditioning given β_{j_0}, we may apply the result of Lemma 4.1. We obtain

$$\Pr\{|\sum_{j=1}^{n}\beta_j x_j - \sqrt{n}ux_{j_0} - \sqrt{n}v| \le \Delta\} \le \frac{C}{m^{\frac{5}{2}}\Delta}F(\mu) + ce^{-c'm}, \quad (4.29)$$

where

$$\begin{aligned}
F(\mu) &= \int_{\frac{3a}{2}}^{\infty}\widetilde{S}_{\Delta}^2(y)dy, \\
\widetilde{S}_{\Delta}(y) &= \sum_{j=1}^{n}\mu\left(\frac{1}{|x_j|}[y - \pi\Delta, y + \pi\Delta]\right), \quad (4.30)
\end{aligned}$$

and μ denotes the distribution of $|\widetilde{\beta}|$. Since

$$F(\mu) \le C\Delta\sum_{k=1}^{\infty}|\{j:\ |x_j| \in (k\Delta, (k+1)\Delta]\}|^2, \quad (4.31)$$

we obtain

$$\Pr\{|\sum_{j=1}^{n}\beta_j x_j - \sqrt{n}ux_{j_0} - \sqrt{n}v| \le \Delta\} \le \frac{C}{m^{\frac{5}{2}}}\sum_{k=1}^{\infty}|\{j:\ |x_j| \in (k\Delta, (k+1)\Delta]\}|^2. \quad (4.32)$$

This completes the proof. □

We also need the following lemma.

Lemma 4.3. *Let $x_j = a_j + i\ b_j$, $v = c + i\ d$, $\beta_j = \xi_j + i\ \eta_j$. Let β be a random variable such that $\mathbf{E}\,\beta = 0$, $\mathbf{E}\,|\beta|^2 = 1$ and let $\beta_1, \dots, \beta_n$, be independent copies of β. Let $0 < r < R$ and let $x_1, \dots, x_m \in \mathbb{C}$ such that $\frac{r}{\sqrt{m}} \le |x_j| \le \frac{R}{\sqrt{m}}$ for any j. Then there exist constants $C_{4.2}$ and $c_{4.2}$ such that for any $t > \frac{c_{4.2}}{\sqrt{m}}$ and for any $v \in \mathbb{C}$*

$$\Pr\left\{\left|\sum_{j=1}^{n} \beta_j x_j - v\right| < t\right\} \le C_{4.2} t \tag{4.33}$$

Proof. We use the simple inequality

$$\Pr\left\{\left|\sum_{j=1}^{n} \beta_j x_j - v\right| < t\right\} \le \min\{A, B\}, \tag{4.34}$$

where

$$\begin{aligned} A &= \Pr\left\{\left|\sum_{j=1}^{n} (\xi_j a_j - \eta_j b_j) - c\right| < t\right\} \\ B &= \Pr\left\{\left|\sum_{j=1}^{n} (\eta_j a_j + \xi_j b_j) - d\right| < t\right\} \end{aligned} \tag{4.35}$$

Note that random variables $\widehat{\xi}_j = \xi_j a_j - \eta_j b_j$ (resp. $\overline{\xi}_j = \xi_j a_j - \eta_j b_j$) are independent for $j = 1, \dots, n$,

$$\max\{\sum_{j=1}^{m} \mathbf{E}\,|\widehat{\xi}_j|^3,\ \sum_{j=1}^{m} \mathbf{E}\,|\overline{\xi}_j|^3\} \le \frac{CR^3}{\sqrt{m}}, \tag{4.36}$$

and

$$\max\{\sum_{j=1}^{m} \mathbf{E}\,||\widehat{\xi}_j|^2,\ \sum_{j=1}^{m} \mathbf{E}\,|\overline{\xi}_j|^2\} \ge \frac{r^2\sigma^2}{\sqrt{2}}. \tag{4.37}$$

Applying the Berry–Esséen inequality, we obtain the result. □

To conclude the proof of Theorem 4.1 we repeat the proof of Rudelson [21] in the rest.

5 Proof of the main Theorem

In this Section we give the proof of Theorem 1.2. The proof of Theorem 1.3 is similar. We have to use Theorem 6.3 instead of Theorem 4.1 and instead of Bai's results we may use the result of Section 2 for $z = 0$ only. For any $z \in \mathbb{C}$ we introduce the set $\Omega_n(z) = \{\omega \in \Omega :\ n^{-3} \ge s_n(\mathbf{X} - z\mathbf{I}),\ s_1(\mathbf{X}) \le 4\}$. From Bai [3] it follows that

$$\Pr\{s_1(X) \ge 4\} \le C n^{-\frac{1}{8}}. \tag{5.1}$$

According to Theorem 4.1,

$$\Pr\{n^{-3} \geq s_n(\mathbf{X} - z\mathbf{I})\} \leq Cn^{-\frac{1}{2}}. \tag{5.2}$$

These inequalities imply

$$\Pr\{\Omega_n(z)\} \leq Cn^{-\frac{1}{8}}. \tag{5.3}$$

Let $r = r(n)$ such that $r(n) \to 0$ as $n \to \infty$. A more specific choice will be made later. Consider the potential $U_{\mu_n}^{(r)}$. We have

$$\begin{aligned} U_{\mu_n}^{(r)} &= -\frac{1}{n}\mathbf{E}\,\log|\det(\mathbf{X} - z\mathbf{I} - r\xi\mathbf{I})| && (5.4)\\ &= -\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\log|\lambda_j - r\xi - z|I_{\Omega_n(z)} - \frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\log|\lambda_j - r\xi - z|I_{\Omega_n^{(c)}(z)} && (5.5)\\ &= \overline{U}_{\mu_n}^{(r)} + \widehat{U}_{\mu_n}^{(r)}, && (5.6) \end{aligned}$$

where I_A denotes an indicator function of an event A and $\Omega_n^{(c)}(z)$ denotes the complement of $\Omega_n(z)$.

Lemma 5.1. *Assuming the conditions of Theorem 4.1, for r such that $-n^{-1/12}\log r \to 0$ as $n \to \infty$, we have*

$$\widehat{U}_{\mu_n}^{(r)} \to 0, \ \text{as } n \to \infty. \tag{5.7}$$

Proof. By definition, we have

$$\widehat{U}_{\mu_n}^{(r)} = -\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,\log|\lambda_j - r\xi - z|I_{\Omega_n^{(c)}(z)}. \tag{5.8}$$

Applying Cauchy's inequality, we get, for any $\alpha > 0$,

$$\begin{aligned} |\widehat{U}_{\mu_n}^{(r)}| &\leq \frac{1}{n}\sum_{j=1}^{n}\mathbf{E}^{\frac{1}{1+\alpha}}|\log|\lambda_j - r\xi - z||^{1+\alpha}\,(\Pr\{\Omega_n\})^{\frac{\alpha}{1+\alpha}}\\ &\leq \left(\frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\,|\log|\lambda_j - r\xi - z||^{1+\alpha}\right)^{\frac{1}{1+\alpha}}(\Pr\{\Omega_n\})^{\frac{\alpha}{1+\alpha}}. \end{aligned} \tag{5.9}$$

Furthermore, since ξ is uniformly distributed in the unit disc and independent of λ_j, we may write

$$\mathbf{E}\,|\log|\lambda_j - r\xi - z||^{1+\alpha} = \frac{1}{2\pi}\mathbf{E}\int_{|\zeta|\leq 1}|\log|\lambda_j - r\zeta - z||^{1+\alpha}d\zeta = \ EJ_1^{(j)} + \mathbf{E}\,J_2^{(j)} + \mathbf{E}\,J_3^{(j)}, \tag{5.10}$$

where

$$J_1^{(j)} = \frac{1}{2\pi} \int_{|\zeta|\le 1,\ |\lambda_j - r\zeta - z|\le \varepsilon} |\log|\lambda_j - r\zeta - z||^{1+\alpha} d\zeta \tag{5.11}$$

$$J_2^{(j)} = \frac{1}{2\pi} \int_{|\zeta|\le 1,\ \frac{1}{\varepsilon} > |\lambda_j - r\zeta - z| > \varepsilon} |\log|\lambda_j - r\zeta - z||^{1+\alpha} d\zeta \tag{5.12}$$

$$J_3^{(j)} = \frac{1}{2\pi} \int_{|\zeta|\le 1,\ |\lambda_j - r\zeta - z| > \frac{1}{\varepsilon}} |\log|\lambda_j - r\zeta - z||^{1+\alpha} d\zeta \tag{5.13}$$

Note that

$$|J_2^{(j)}| \le \log\left(\frac{1}{\varepsilon}\right). \tag{5.14}$$

Since for any $b > 0$, the function $-u^a \log u$ is not decreasing on the interval $[0, \exp\{-\frac{1}{b}]$, we have for $0 < u \le \varepsilon < \exp\{-\frac{1}{b}\}$,

$$-\log u \le \varepsilon^b u^{-b} \log\left(\frac{1}{\varepsilon}\right). \tag{5.15}$$

Using this inequality, we obtain, for $b(1+\alpha) < 2$,

$$|J_1^{(j)}| \le \frac{1}{2\pi} \varepsilon^{b(1+\alpha)} \left(\log\left(\frac{1}{\varepsilon}\right)\right)^{1+\alpha} \int_{|\zeta|\le 1,\ |\lambda_j - r\zeta - z|\le \varepsilon} |\lambda_j - r\zeta - z|^{-b(1+\alpha)} d\zeta \tag{5.16}$$

$$\le \frac{1}{2\pi r^2} \varepsilon^b \log\left(\frac{1}{\varepsilon}\right) \int_{|\zeta|\le\varepsilon} |\zeta|^{-b(1+\alpha)} d\zeta \le C(\alpha, b) \varepsilon^2 r^{-2} \left(\log\left(\frac{1}{\varepsilon}\right)\right)^{1+\alpha} \tag{5.17}$$

If we choose $\varepsilon = r$, then we get

$$|J_1^{(j)}| \le C(\alpha, b) \left(\log\left(\frac{1}{r}\right)\right)^{1+\alpha} \tag{5.18}$$

The following bound holds for $\frac{1}{n}\sum_{j=1}^n \mathbf{E}\, J_3^{(j)}$. Note that $|\log x|^{1+\alpha} \le \varepsilon^2 |\log\varepsilon|^{1+\alpha} x^2$ for $x \ge \frac{1}{\varepsilon}$ and sufficiently small ε. Using this inequality, we obtain

$$\frac{1}{n}\sum_{j=1}^n \mathbf{E}\, J_3^{(j)} \le C(\alpha)\varepsilon^2 |\log\varepsilon| \frac{1}{n}\sum_{j=1}^n \mathbf{E}\, |\lambda_j - r\zeta - z|^2 \le C(\alpha)(1 + |z|^2 + r^2)\varepsilon^2 |\log\varepsilon|$$

$$\le C(\alpha)(2 + |z|^2) r^2 |\log r|. \tag{5.19}$$

The inequalities (5.16)–(5.19) together imply that

$$|\frac{1}{n}\sum_{j=1}^n \mathbf{E}\, |\log|\lambda_j - r\xi - z||^{1+\alpha}| \le C \left(\log\left(\frac{1}{r}\right)\right)^{1+\alpha}. \tag{5.20}$$

Furthermore, the inequalities (5.8), (5.9), and (5.20) together imply

$$|\widehat{U}_{\mu_n}^{(r)}| \leq C\left(\log\left(\frac{1}{r}\right)\right) n^{-\frac{\alpha}{3(1+\alpha)}} \tag{5.21}$$

We choose $\alpha = 3$ and rewrite the last inequality as follows

$$|\widehat{U}_{\mu_n}^{(r)}| \leq C\left(\log\left(\frac{1}{r}\right)\right) n^{-\frac{1}{4}} \tag{5.22}$$

If we choose r such that $\log(1/r)n^{-1/4} \to 0$, then (5.7) holds. Thus the Lemma is proved. □

We shall investigate $\overline{U}_{\mu_n}^{(r)}$ now. We may write

$$\overline{U}_{\mu_n}^{(r)} = -\frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\,\log|\lambda_j - z - r\xi| I_{\Omega_n} = -\frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\,\log(s_j(\mathbf{X}(z,r)) I_{\Omega_n} \tag{5.23}$$

$$= -\int_{n^{-3}}^{4+|z|} \log x d\mathbf{E}\,\overline{F}_n(x,z,r), \tag{5.24}$$

where $\overline{F}_n(\cdot,z,r)$ is the distribution function corresponding to the restriction of the measure $\nu_n(\cdot,z,r)$ on the set Ω_n. Introduce the notation

$$\overline{U}_\mu = -\int_{n^{-3}}^{4+|z|} \log x dF(x,z) \tag{5.25}$$

Integrating by parts, we get

$$\overline{U}_{\mu_n}^{(r)} - \overline{U}_\mu = -\int_{n^{-3}}^{4+|z|} \frac{\mathbf{E}\,F_n(x,z,r) - F(z,r)}{x} dx + C\theta \sup_x |\mathbf{E}\,F_n(x,z,r) - F(z,r)| \log n, \tag{5.26}$$

where θ denotes some constant such that $|\theta| \leq 1$. This implies that

$$|\overline{U}_{\mu_n}^{(r)} - \overline{U}_\mu| \leq C \log n \sup_x |\mathbf{E}\,F_n(x,z,r) - F(x,z)|. \tag{5.27}$$

Note that, for any $r > 0$, $|s_j(z) - s_j(z,r)| \leq r$. This implies that

$$\mathbf{E}\,F_n(x-r,z) \leq \mathbf{E}\,F_n(x,z,r) \leq \mathbf{E}\,F_n(x+r,z). \tag{5.28}$$

Hence, we get

$$\sup_x |\mathbf{E}\,F_n(x,z,r) - F(x,z)| \leq \sup_x |\mathbf{E}\,F_n(x,z) - F(x,z)| + \sup_x |F(x+r,z) - F(x,z)|. \tag{5.29}$$

Since the distribution function $F(x,z)$ has a density $p(x,z)$ which is bounded (see Remark 3.1) we obtain

$$\sup_x |\mathbf{E}\,F_n(x,z,r) - F(x,z)| \leq \sup_x |\mathbf{E}\,F_n(x,z) - F(x,z)| + Cr. \tag{5.30}$$

Choose $r = cn^{-\frac{1}{8}}$. Inequalities 5.30 and (2.41) together imply

$$\sup_x |\mathbf{E}\,\overline{F}_n(x,z,r) - \overline{F}(x,z)| \le Cn^{-\frac{1}{8}}. \tag{5.31}$$

From inequalities (5.31) and (5.27) it follows that

$$|\overline{U}^{(r)}_{\mu_n} - \overline{U}_{\mu}| \le Cn^{-\frac{1}{8}} \log n. \tag{5.32}$$

Note that

$$|\overline{U}^{(r)}_{\mu_n} - U_{\mu}| \le |\int_0^{n^{-3}} \log x dF(x,z)| \le Cn^{-\frac{1}{8}} \log n. \tag{5.33}$$

Let $K = \{z \in \mathbb{C} : \ |z| \le 4\}$ and let $K^{(c)}$ denote $\mathbb{C} \setminus K$. According to inequality (1.2), we have

$$1 - p_n := \mathbf{E}\,\mu_n^{(r)}(K^{(c)}) \le \Pr\{s_1(\mathbf{X}) > 3\} \le \sup_x |F_n(x) - M_1(x)| \le Cn^{-\frac{1}{8}}. \tag{5.34}$$

Furthermore, let $\overline{\mu}_n^{(r)}$ and $\widehat{\mu}_n^{(r)}$ be probability measures supported on the compact set K and $K^{(c)}$ respectively, such that

$$\mathbf{E}\,\mu_n^{(r)} = p_n \overline{\mu}_n^{(r)} + (1-p_n)\widehat{\mu}_n^{(r)}. \tag{5.35}$$

Introduce the logarithmic potential of the measure $\overline{\mu}_n^{(r)}$,

$$U_{\overline{\mu}_n^{(r)}} = -\int \log|z-\zeta| d\mu_n \overline{\mu}_n^{(r)}(\zeta). \tag{5.36}$$

Similar to the proof of Lemma 5.1 we show that

$$\lim_{n\to\infty} |U^{(r)}_{\mu_n} - U_{\overline{\mu}_n^{(r)}}| \le Cn^{-\frac{1}{8}} \log n. \tag{5.37}$$

This implies that

$$\lim_{n\to\infty} U_{\overline{\mu}_n^{(r)}}(z) = U_{\mu}(z) \tag{5.38}$$

for all $z \in \mathbb{C}$. Since the measures $\overline{\mu}_n^{(r)}$ are compactly supported, Theorem 6.9 from [18] and Corollary 2.2 from [18] (see also the Appendix, Theorem 6.1 and Corollary 6.5), together imply that

$$\lim_{n\to\infty} \overline{\mu}_n^{(r)} = \mu \tag{5.39}$$

in the weak topology. Inequality (5.34) and relations (5.35) and (5.35) together imply that

$$\lim_{n\to\infty} \mathbf{E}\,\mu_n^{(r)} = \mu \tag{5.40}$$

in weak topology. Finally, by Lemma 1.1 we get

$$\lim_{n\to\infty} \mathbf{E}\,\mu_n = \mu \tag{5.41}$$

in the weak topology. Thus Theorem 1.2 is proved.

6 Appendix

In this Section we collect some technical results.

Lemma 6.1. *Let* $\varkappa_3 = \max_{j,k} \mathbf{E}\,|X_{jk}|^3$. *The following inequality holds*

$$\frac{1}{n\sqrt{n}} \sum_{j,k=1}^{n} \mathbf{E}\,|X_{jk}|(|T^{(jk)}_{k+n,j}| + |T^{(jk)}_{j,k+n}|) \le \frac{C\varkappa_3}{\sqrt{n}v^3} \tag{6.1}$$

Proof. Introduce the notations

$$B := \frac{1}{n\sqrt{n}} \sum_{j,k=1}^{n} \mathbf{E}\,|X_{jk}|(|T^{(jk)}_{k+n,j}| + |T^{(jk)}_{j,k+n}|) \tag{6.2}$$

and

$$\begin{aligned}
B_1 &:= \frac{1}{n^2\sqrt{n}} \sum_{j,k=1}^{n} \mathbf{E}\,|X_{jk}|^3 |R^{(jk)}_{k+n,j}|^2 |R_{k+n,j}| \\
B_2 &:= \frac{1}{n^2\sqrt{n}} \sum_{j,k=1}^{n} \mathbf{E}\,|X_{jk}|^3 |R^{(jk)}_{k+n,j}||R^{(jk)}_{k+n,k+n}||R_{j,j}| \\
B_3 &:= \frac{1}{n^2\sqrt{n}} \sum_{j,k=1}^{n} \mathbf{E}\,|X_{jk}|^3 |R^{(jk)}_{k+n,k+n}||R^{(jk)}_{j,j}||R_{k+n,j}| \\
B_4 &:= \frac{1}{n^2\sqrt{n}} \sum_{j,k=1}^{n} \mathbf{E}\,|X_{jk}|^3 |R^{(jk)}_{k+n,k+n}||R^{(jk)}_{k+n,j}||R_{j,j}| \\
B_5 &:= \frac{1}{n^2\sqrt{n}} \sum_{j,k=1}^{n} \mathbf{E}\,|X_{jk}|^3 |R^{(jk)}_{j,k+n}||R^{(jk)}_{k+n,j}||R_{j,k+n}| \\
B_6 &:= \frac{1}{n^2\sqrt{n}} \sum_{j,k=1}^{n} \mathbf{E}\,|X_{jk}|^3 |R^{(jk)}_{j,j}||R^{(jk)}_{k+n,k+n}||R_{j,k+n}| \\
B_7 &:= \frac{1}{n^2\sqrt{n}} \sum_{j,k=1}^{n} \mathbf{E}\,|X_{jk}|^3 |R^{(jk)}_{j,k+n}||R^{(jk)}_{j,j}||R_{k+n,j}| \\
B_8 &:= \frac{1}{n^2\sqrt{n}} \sum_{j,k=1}^{n} \mathbf{E}\,|X_{jk}|^3 |R^{(jk)}_{j,k+n}|^2 |R_{j,k+n}|
\end{aligned} \tag{6.3}$$

It is easy to check that

$$\max\{B_k,\ k = 1, \dots, 8\} \le \frac{C\varkappa_3}{\sqrt{n}v^3}. \tag{6.4}$$

This implies that

$$B \le \frac{C\varkappa_3}{\sqrt{n}v^3}. \tag{6.5}$$

□

Lemma 6.2. *Let μ_n be the empirical spectral measure of the matrix $\mathbf{X}$ and ν_r be the uniform distribution on the disc of radius r. Let $\mu_n^{(r)}$ be the empirical spectral measure of the matrix $\mathbf{X}(r) = \mathbf{X} - r\xi\mathbf{I}$, where ξ is a random variable which is uniformly distributed on the unit disc. Then the measure $\mathbf{E}\,\mu_n^{(r)}$ is the convolution of the measures $\mathbf{E}\,\mu_n$ and ν_r, i. e.*

$$\mathbf{E}\,\mu_n^{(r)} = (\mathbf{E}\,\mu_n) * (\nu_r). \tag{6.6}$$

Proof. Let J be a random variable which is uniformly distributed on the set $\{1, \dots, n\}$. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the matrix $\mathbf{X}$. Then $\lambda_1 + r\xi, \dots, \lambda_n + r\xi$ are eigenvalues of the matrix $\mathbf{X}(r)$. Let δ_x be denote the Dirac measure. Then

$$\mu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j} \tag{6.7}$$

and

$$\mu_n^{(r)} = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j + r\xi}. \tag{6.8}$$

Denote by μ_{nj} the distribution of λ_j. Then

$$\mathbf{E}\,\mu_n = \frac{1}{n}\sum_{j=1}^{n} \mu_{nj} \tag{6.9}$$

and

$$\mathbf{E}\,\mu_n^r = \frac{1}{n}\sum_{j=1}^{n} \mu_{nj} * \nu_r = \left(\frac{1}{n}\sum_{j=1}^{n} \mu_{nj}\right) * (\nu_r) = (\mathbf{E}\,\mu_n) * (\nu_r). \tag{6.10}$$

The Lemma is proved. □

Let

$$f_n^{(r)}(t, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{itx + ivy\} dG_n^{(r)}(x, y) \tag{6.11}$$

and

$$f_n(t, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{itx + ivy\} dG_n(x, y), \tag{6.12}$$

where

$$G_n^{(r)}(x, y) = \frac{1}{n}\sum_{j=1}^{n} \Pr\{\operatorname{Re}\lambda_j + r\xi \le x, \operatorname{Im}\lambda_j + r\xi \le y\}, \tag{6.13}$$

and

$$G_n(x,y) = \frac{1}{n}\sum_{j=1}^{n} \Pr\{\mathrm{Re}\,\lambda_j \le x, \mathrm{Im}\,\lambda_j \le y\}. \tag{6.14}$$

Denote by $h(t,v)$ the characteristic function of the joint distribution of the real and imaginary parts of ξ,

$$h(t,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{iux + ivy\} dG(x,y). \tag{6.15}$$

Lemma 6.3. *The following relations hold*

$$f_n^{(r)}(t,v) = f_n(t,v)h(rt,rv). \tag{6.16}$$

If for any t, v there exists $\lim_{n\to\infty} f_n(t,v)$, then

$$\lim_{r\to 0}\lim_{n\to\infty} f_n^{(r)}(t,v) = \lim_{n\to\infty}\lim_{r\to 0} f_n^{(r)}(t,v) = \lim_{n\to\infty} f_n(t,v). \tag{6.17}$$

Proof. The first equality follows immediately from the independence of the random variable ξ and the matrix $\mathbf{X}$. Since $\lim_{r\to 0} h(rt,rv) = h(0,0) = 1$ the first equality implies the second one. □

Lemma 6.4. *Let F and G be distribution functions with Stieltjes transforms $S_F(z)$ and $S_G(z)$ respectively. Assume that $\int_{-\infty}^{\infty} |F(x) - G(x)|dx < \infty$. Let $G(x)$ have a bounded support J and density bounded by some constant K. Let $V > v_0 > 0$ and a be positive numbers such that*

$$\gamma = \frac{1}{\pi}\int_{|y|\le a} \frac{1}{u^2+1}\, du > \frac{3}{4}.$$

Then there exist some constants C_1, C_2, C_3 depending on J and K only such that

$$\begin{aligned}\sup_x |F(x) - G(x)| \le\ & C_1 \sup_{x\in J}\int_{-\infty}^{x} |S_F(u+iV) - S_G(u+iV)|\, du \\ & + \sup_{u\in J}\int_{v_0}^{V} |S_F(u+iv) - S_G(u+iv)|dv + C_3\ v_0\end{aligned} \tag{6.18}$$

6.1 Some facts from logarithmic potential theory

We cite here some definitions and Theorems about logarithmic potentials, see [18]. Let $\Sigma \subset \mathbb{C}$ be a compact set of the complex plane and $\mathcal{M}(\Sigma)$ the collection of all positive Borel probability measures with support in Σ. The *logarithmic energy* of $\mu \in \mathcal{M}(\Sigma)$ is defined as

$$I(\mu) := \int\int \log\frac{1}{|z-t|} d\mu(z) d\mu(t), \tag{6.19}$$

and the energy of Σ by

$$V := \inf\{I(\mu) | \mu \in \mathcal{M}(\Sigma)\}. \tag{6.20}$$

The quantity

$$cap(\Sigma) := e^{-V} \tag{6.21}$$

is called the *logarithmic capacity* of Σ.

The *capacity* of an arbitrary Borel set E is defined as

$$\cap\,(E) := sup\{cap(K)|K \subset E, K \text{ compact}\}. \tag{6.22}$$

Note that every Borel set of capacity zero has zero two-dimensional Lebesgue measure. A property is said to hold *quasi-everywhere* (q. e.) on a set E if the set of exceptional points is of capacity zero. The next Theorem is called *Lower Envelope Theorem*

Theorem 6.1. *Let μ_n, $n = 1, 2\ldots$, be a sequence of positive Borel probability measures having support in a fixed compact set. If $\mu_n \to \mu$ weakly, then*

$$\liminf_{n\to\infty} U^{\mu_n}(z) = U^{\mu}(z) \tag{6.23}$$

for quasi-every $z \in \mathbb{C}$.

The following fact is Corollary 2.2 from the Unicity Theorem of logarithmic potential theory (see [18], p. 98).

Corollary 6.5. *If μ and ν are compactly supported measures and the potentials U^{μ} and U^{ν} coincides almost everywhere with respect to two-dimensional Lebesgue measure, then $\mu = \nu$.*

For reader convenience we give here the statement of Theorem 1.2 from [18].

Theorem 6.2. *Let μ be a finite positive measure of compact support on the plane. Then for any z_0 and $r > 0$ the mean value*

$$L(U^{\mu}; z_0, r) := \frac{1}{2\pi}\int_{-\pi}^{\pi} U^{\mu}(z_0 + r\exp\{i\theta\}d\theta \tag{6.24}$$

exists as a finite number, and $L(U^{\mu}; z_0, r)$ is a non-increasing function of r that is absolutely continuous on any closed subinterval of $(0, \infty)$. Furthermore,

$$\lim_{r\to 0} L(U^{\mu}; z_0, r) = U^{\mu}(z_0). \tag{6.25}$$

6.2 Minimal singular values

of sparse matrices In this Section we reformulate some statements from the paper of Rudelson [21] to adapt his proof to sparse matrices. Let ε_{jk} be independent Bernoulli random variables with $\Pr\{\varepsilon_{jk} = 1\} = p_n$. Assume that ε_{jk}, $j, k = 1, \ldots, n$ are independent on X_{jk}, $j, k = 1, \ldots, n$. Consider the matrix

$$\mathbf{X}^{\varepsilon} = (\frac{1}{\sqrt{p_n}}\varepsilon_{jk}X_{jk})_{j,k=1}^{n}. \tag{6.26}$$

Theorem 6.3. *Let X_{jk}, $j,k = 1,\dots n$ be centered sub-Gaussian random variables of variance 1. Then for any $\gamma > c_{1,1}/\sqrt{np_n}$*

$$\Pr\left\{ \textit{there exists } \mathbf{x} \in S^{n-1} \;\middle|\; \|\mathbf{X}^\varepsilon \mathbf{x}\| \le \frac{\gamma\sqrt{p_n}}{C_{11}(np_n)^{\frac{3}{2}}} \right\} \le c\gamma p_n^{-\frac{5}{2}} \tag{6.27}$$

if n is large enough.

The generalization of this result to the complex case is based on similar arguments as in Section 4 for the case $p_n = 1$.

Proof. We adapt Rudelson's proof for sparse matrices giving only the necccessary new statements of some Lemmas and Theorems in Rudelson's proof. To prove these results is enough to repeat Rudelson's proof of the corresponding Theorems and Lemmas.

Lemma 6.6. *(Lemma 3.1 in [21]) Let $c > 0$, $0 < \Delta < a/2\pi$ and let $\xi_1, \dots, \xi_n$ be independent random variables such that $\mathbf{E}\,\xi_i = 0$, $\Pr\{\xi_i > 2a\} \ge c$ and $\Pr\{-\xi_i > 2a\} \ge c$. For $y \in \mathbb{R}$ set*

$$S_\Delta(y) = \sum_{j=1}^{n} \frac{1}{2}(\Pr\{\xi_j \in [y - \pi\Delta, y + \pi\Delta]\} + \Pr\{-\xi_j \in [y - \pi\Delta, y + \pi\Delta]\}), \tag{6.28}$$

Let $\varepsilon_1, \dots, \varepsilon_n$ be identically distributed Bernoulli random variables independent on $\xi_1, \dots, \xi_n$ and independent in aggregate, with $\Pr\{\varepsilon_j = 1\} = p_n$. Then for any $v \in \mathbb{R}$

$$\Pr\left\{ \Big| \sum \varepsilon_j \xi_j - v \Big| < \Delta \right\} \le \frac{C}{n^{\frac{5}{2}} p_n^{\frac{3}{2}} \Delta} \int_{\frac{3a}{2}}^{\infty} S_\Delta^2(y) dy + c \exp\{-c' n p_n\}. \tag{6.29}$$

Theorem 6.4. *(Theorem 3.5 in [21]) Let $\xi_1, \dots, \xi_n$ i. i. d. be sub-Gaussian random variables such that $\mathbf{E}\,\xi_i = 0$ and $\Pr\{\xi_i > c\} \ge c'$, $\Pr\{-\xi_i > c\} \ge c'$ for some $c, c' > 0$. Let $\Delta > 0$ and let $(x_1, \dots, x_m) \in \mathbb{R}^m$ be a vector such $a < |x_j| < \overline{C}_{3.5} a/\sqrt{p_n}$. Let $\varepsilon_1, \dots, \varepsilon_n$ be independent on $\xi_1, \dots, \xi_n$ and independent in aggregate identically distributed Bernoulli random variables with $\Pr\{\varepsilon_j = 1\} = p_n$. Then for any $\Delta < a/(2\pi)$ and for any $v \in \mathbb{R}$*

$$\Pr\left\{ \left| \sum_{j=1}^{m} \xi_j \varepsilon_j x_j - v \right| < \Delta \right\} \le \frac{C_{3.5}}{(mp_n)^{\frac{5}{2}}} \sum_{k=1}^{\infty} P_k^2(x, \Delta), \tag{6.30}$$

where

$$P_k^2(x, \Delta) = |\{j \;\big|\; |x_j| \in (k\Delta, (k+1)\Delta]\}|. \tag{6.31}$$

Lemma 6.7. *(Lemma 4.1 in [21]) Assuming the conditions of Theorem 6.3, for the matrix $\mathbf{X}^\varepsilon$ and for every $v \in \mathbb{R}$, we have*

$$\Pr\{\|\mathbf{X}^\varepsilon\| \le C_{4.1}\sqrt{np_n}\} \le \exp\{-c_{4.1} n p_n\}. \tag{6.32}$$

Lemma 6.8. *(Lemma 4.2 in [21]) Let $\xi_1, \dots, \xi_n$ be i. i. d. sub-Gaussian random variables such that $\mathbf{E}\,\xi_i = 0$ and $\mathbf{E}\,\xi_i^2 = 1$. Let $0 < r < R$ and let $x_1, \dots, x_m \in \mathbb{R}$ be such that $\frac{r}{\sqrt{m}} < |x_j| < \frac{R}{\sqrt{mp_n}}$ for any j. Then for $t \geq \frac{c_{4.2}}{\sqrt{mp_n}}$ and for any $v \in \mathbb{R}$*

$$\Pr\left\{ \Big| \sum_{j=1}^{m} \xi_j \varepsilon_j x_j - v \Big| < t \right\} \leq C_{4.2} t/\sqrt{p_n}. \tag{6.33}$$

Lemma 6.9. *(Lemma 4.4 in [21])Let $\Delta > 0$ and let Y be a random variable such that for any $t \geq \Delta$, $\Pr\{|Y - v| < t\} \leq Lt$. Let $\mathbf{y} = (Y_1, \dots, Y_n)$ be a random vector, whose coordinates are independent copies of Y. Then for any $\mathbf{z} \in \mathbb{R}^n$*

$$\Pr\left\{ \|\mathbf{y} - \mathbf{z}\| \leq \Delta\sqrt{n} \right\} \leq (C_{4.4} L \Delta)^n. \tag{6.34}$$

We define the set $\sigma(\mathbf{x})$ for any $\mathbf{x} \in \mathcal{S}^{(n-1)}$ as

$$\sigma(\mathbf{x}) = \{ i \mid |x_i| \leq R/\sqrt{np_n} \}. \tag{6.35}$$

Let P_I be the coordinate projection on the set $I \subset \{1, \dots, n\}$. Set

$$V_P = \{ \mathbf{x} \in \mathcal{S}^{(n-1} \mid \|P_{\sigma(\mathbf{x})}\mathbf{x}\| < r \}$$
$$V_S = \{ \mathbf{x} \in \mathcal{S}^{(n-1} \mid \|P_{\sigma(\mathbf{x})}\mathbf{x}\| \geq r \}.$$

Lemma 6.10. *(Lemma 5.1 in [21])For any $r < 1/2$*

$$\log N(V_P, \mathcal{B}_2^n, 2r) \leq \frac{np_n}{R^2} \log \frac{3R^2}{rp_n}. \tag{6.36}$$

Lemma 6.11. *(Lemma 5.2 in [21])*

$$\Pr\left\{ \textit{there exists } \mathbf{x} \in V_P \;\Big|\; \|\mathbf{X}^\varepsilon \mathbf{x}\| \leq C_{4.1}\sqrt{np_n}/2 \right\} \leq \exp\{-c_{4.1} np_n\}. \tag{6.37}$$

For $\mathbf{x} = (x_1, \dots, x_n) \in V_S$ denote

$$J(\mathbf{x}) = \{ j \mid \frac{r}{2\sqrt{n}} \leq |x_j| \leq \frac{R}{\sqrt{np_n}} \}. \tag{6.38}$$

Note that

$$|J(\mathbf{x})| \geq (r^2/2R^2 p_n) n =: m. \tag{6.39}$$

Let $0 < \Delta < r/2\sqrt{n}$ be a number to be chosen later. We shall cover the interval $[\frac{r}{2\sqrt{n}}, \frac{R}{\sqrt{np_n}}]$ by

$$k = \left[\frac{R/\sqrt{p_n} - r/2}{\sqrt{n}\Delta} \right]. \tag{6.40}$$

consecutive intervals $(j\Delta, (j+1)\Delta]$, where $j = k_0, (k_0+1), \dots, (k_0+k)$ and k_0 is the largest number such that $k_0\Delta < r/2\sqrt{n}$.

(Definition 5.3 in [21]).Let $\Delta > 0$ and $Q > 1$. We say that a vector $\mathbf{x} \in V_S$ has a (Δ, Q)-regular profile if there exists a set $J \subset J(\mathbf{x})$ such that $|J| \geq m/2$ and

$$\sum_{i=1}^{\infty} P_i^2(\mathbf{x}|_J, \Delta) \leq Q m^{\frac{5}{2}} \Delta =: C_{5.3} Q \frac{m^2}{k}. \tag{6.41}$$

Lemma 6.12. *(Lemma 6.1 in [21]) Let $\Delta \leq \frac{r}{4\pi\sqrt{n}}$. Let $\mathbf{x} \in V_S$ be a vector of (Δ, Q)-regular profile. Then for any $t \geq \Delta$*

$$\Pr\left\{ \left| \sum_{j=1}^{n} \xi_j \varepsilon_j x_j - v \right| \leq t \right\} \leq C_{6.1} Q t / p_n. \tag{6.42}$$

Theorem 6.5. *(Theorem 6.2 in [21]) Let $\frac{r}{4\pi\sqrt{np_n}} > \Delta > 0$ and let U be the set of vectors of (Δ, Q)-regular profile. Then*

$$\Pr\{\textit{there exists } \mathbf{x} \in U \mid \|\mathbf{X}^{\varepsilon}\mathbf{x}\| \leq \frac{\Delta\sqrt{p_n}}{2\sqrt{n}}\} \leq C_{6.1} Q \Delta n / \sqrt{p_n} \tag{6.43}$$

Lemma 6.13. *(Lemma 7.1 in [21]) Let $\overline{C}_{7.1}(np_n)^{-\frac{3}{2}} \leq \Delta \leq (np_n)^{-\frac{1}{2}}$ where $C_{7.1} = \frac{2R^3}{r^2}$ and let W_S be the set of vectors of (Δ, Q)-singular profile. Let $\eta > 0$ be such that*

$$C(\eta) < C_{5.3} Q, \tag{6.44}$$

where $C(\eta)$ is the function defined in Lemma 2.1 in [21]. Then there exists a Δ-net in W_S in the l_∞-metric such that

$$|\mathcal{N}| \leq \left(\frac{C_{7.1}}{\Delta\sqrt{np_n}} \right)^n \eta^{c_{7.1} n p_n}. \tag{6.45}$$

Theorem 6.6. *(Theorem 7.3 in [21]) There exists an absolute constant Q_0 with the following property. Let $\Delta > C_{7.3}(np_n)^{-\frac{3}{2}}$, where $C_{7.3} = \max\{c_{4.2}, \overline{C}_{7.1}\}$. Denote by Ω_Δ the event that there exists a vector $\mathbf{x} \in V_S$ of (Δ, Q_0)-singular profile such that $\|\mathbf{X}^{\varepsilon}\mathbf{x}\| \leq \frac{\Delta\sqrt{p_n}}{2} n$. Then*

$$\Pr\{\Omega_\Delta\} \leq 3\exp\{-np_n\}. \tag{6.46}$$

To prove Theorem 6.3 we combine the probability estimates of the previous sections. Let $\gamma > \frac{c_{1.1}}{\sqrt{np_n}}$ where the constant $c_{1.1}$ will be chosen later. Define the exceptional sets:

$$\Omega_0 = \{\omega \mid \|\mathbf{X}^{\varepsilon}\| > C_{2.3}\sqrt{np_n}\},$$
$$\Omega_P = \{\omega \mid \text{there exists } \mathbf{x} \in V_P \|\mathbf{X}^{\varepsilon}\mathbf{x}\| \leq C_{4.1}\sqrt{np_n}\}.$$

Let Q_0 be the number defined in Theorem 6.6. Set

$$\Delta = \frac{\gamma}{2C_{6.1} Q_0 n p_n}. \tag{6.47}$$

The assumption on γ implies $\Delta \geq C_{7.3}(np_n)^{-\frac{3}{2}}$ if we set $c_{1.1} = 2C_{6.1}Q_0C_{7.3}$. Denote by W_S the set of vectors of the (Δ, Q_0)-singular profile and by W_R the set of vectors of the (Δ, Q_0)-regular profile. Set

$$\Omega_S = \left\{ \omega \;\middle|\; \text{there exists } \mathbf{x} \in W_S \|\mathbf{X}^\varepsilon \mathbf{x}\| \leq \frac{\Delta\sqrt{p}_n}{2} n = \frac{1}{4C_{6.1Q_0}\sqrt{p_n}}\gamma \right\}$$
$$\Omega_R = \left\{ \omega \;\middle|\; \text{there exists } \mathbf{x} \in W_R \|\mathbf{X}^\varepsilon \mathbf{x}\| \leq \frac{\Delta\sqrt{p_n}}{2\sqrt{n}} = \frac{1}{4C_{6.1Q_0}} \gamma n^{-\frac{3}{2}} p_n^{-1} \right\}. \tag{6.48}$$

By Theorem 6.6 $\Pr\{\Omega_S\} \leq 3\exp\{-np_n\}$, and by Theorem 6.5 $\Pr\{\Omega_R\} \leq Cn\Delta/p_n^{\frac{1}{2}} \leq C\gamma p_n^{-\frac{3}{2}}$. Choosing $\gamma = \frac{c}{\sqrt{np_n}}$, we conclude the proof. □

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