VALUES OF SPECIAL INDEFINITE QUADRATIC FORMS*

GUIDO ELSNER

Abstract. For special d-dimensional hyperbolic shells E with $d \geq 5$ we show that the number of lattice points in E intersected with a d-dimensional cube C_r of edge length r, can be approximated by the volume of $E \cap C_r$, as r tends to infinity, up to an error of order $\mathcal{O}(r^{d-2})$. We generalize results and techniques, used by F. Götze (2004), to a large class of *indefinite* quadratic forms and we provide explicit bounds for the errors in terms of certain Minkowski minima related to these quadratic forms. Furthermore, we obtain, as in the positive definite case, a result for multivariate diophantine approximation and for the maximal gap between values of such indefinite forms.

1. Introduction

Let Q denote a d-dimensional quadratic form. For $a, b \in \mathbb{R}$ we consider the set E of points in the d-dimensional Euclidean space, for which Q takes values between a and b. In case that the quadratic form $Q[x]$ is positive definite, E is an elliptic shell, but in this paper we will investigate indefinite forms and hence, E is a hyperbolic shell.

For a (measurable) set $B \subset \mathbb{R}^d$ the lattice volume of B is the number of lattice points in B (formally $\mathrm{vol}_{\mathbb{Z}} B \overset{\text{def}}{=} \#(B \cap \mathbb{Z}^d)$) and $\mathrm{vol}\, B$ denotes the Lebesgue measure of B. For the hyperbolic shell E we want to approximate its lattice point volume by the Lebesgue volume. We want to investigate this approximation by estimating a *relative* lattice point rest of *large* parts of the hyperbolic shell E. Therefore we consider for $r > 0$ the d-dimensional cube C_r with edge length r and intersect the cube C_r with the hyperbolic shell E. The *relative* lattice point rest of $E \cap C_r$ is now defined by

$$\Delta \overset{\text{def}}{=} \left| \frac{\mathrm{vol}_{\mathbb{Z}}(E \cap C_r) - \mathrm{vol}(E \cap C_r)}{\mathrm{vol}(E \cap C_r)} \right|.$$

Date: February, 2007.

2000 *Mathematics Subject Classification.* 11P21.

Key words and phrases. Lattice points, ellipsoids, Minkowski's successive minima, rational and irrational indefinite quadratic forms, distribution of values of quadratic forms, Oppenheim conjecture, Davenport–Lewis conjecture.

* Research supported by the DFG, CRC 701.

We will show for special indefinite forms Q, that $\Delta = \mathcal{O}(1)$ as $r \to \infty$ (Theorem 2.1) and that even $\Delta = o(1)$ as $r \to \infty$ provided that Q is irrational (Theorem 2.2). Recall that a quadratic form $Q[x]$ and the corresponding operator Q with non-zero matrix $Q = (q_{ij})$, $1 \le i, j \le d$, is called *rational* if there exists a real number $\lambda \neq 0$ such that the matrix λQ has integer entries only; otherwise it is called *irrational*.
Similar results for forms Q of signature (p, q) satisfying $\max(p, q) \ge 3$ have been proved by Eskin, Margulis and Mozes in [EMM98]. These are quantitative versions of the well-known Oppenheim problem concerning the distribution of values of $Q[m]$, $m \in \mathbb{Z}^d$. In 1929, Oppenheim ([Opp29], [Opp31]) conjectured that if $d \ge 5$ for an irrational non-degenerate quadratic form Q the quantity $m(Q) \overset{\text{def}}{=} \inf\{|Q[m]| : m \in \mathbb{Z}^d, m \neq 0\}$ equals zero. In the rational case this was known by Meyer's Theorem (see [Cas78]). Later it was conjectured that even for $d \ge 3$ and Q irrational the equality $m(Q) = 0$ holds (for irrational diagonal forms this was suspected in [DH46] and it is not true in dimensions 3 and 4 without the assumption of irrationality). The different approaches to this and related problems involve various mathematical methods from analytic number theory, from ergodic theory, from representation theory of Lie groups, reduction theory and from the geometry of numbers. In [Mar89] Margulis established the Oppenheim conjecture in dimensions $d \ge 3$, as stated by Davenport and Heilbronn for $d \ge 5$. In his seminal work he proved that the set of values of Q at lattice points is dense in $\mathbb{R}$. Quantitative versions of this problem were later on developed by Dani and Margulis ([DM93]) and Eskin, Margulis and Moses ([EMM98]). They consist of *quantative* bounds on the ratio between the lattice point volume and the Lebesgue volume of the set of points in the cube C_r, where the quadratic form takes values in a small interval. The quantitative bounds provided in these results yield the asymptotic number of points in these regions as a polynomial in r up to a non-effective error term tending to zero in proportion to the leading term. The estimates thus obtained are implicit, since they do not provide explicit bounds in terms of diophantine approximations of irrational coefficients of the form. For a detailed discussion of results on these problems by Oppenheim, Heilbronn and Davenport and others, see [Mar97]. In [BG99] Bentkus and Götze proved explicit error bounds in the quantitative Oppenheim problem for the elliptic shell as well as for hyperbolic shells for $d \ge 9$ by a common approach. They provide more explicit bounds (in terms of diophantine approximation) for *distribution functions* of the values of the quadratic form on C_r, whereas the direct application of the previous methods seems to be restricted to the case of the concentration in compact intervals.

In [Göt04] Götze showed that in the positive definite case for $d \geq 5$ the lattice point rest is of order $\mathcal{O}(r^{d-2})$ for arbitrary forms, and of order $o(r^{d-2})$ if the form is irrational. These results refine earlier bounds of the same order for dimensions $d \geq 9$ (see also [Göt04] for the history of such estimates and further references).

In the present paper we apply techniques of [Göt04] to special *indefinite* forms and we obtain explicit bounds in terms of certain Minkowski minima of convex bodies related to these quadratic forms. Adapting these techniques, the main problem consists of the estimation of the difference between the lattice point and the Lebesgue volume by an integral of generalized theta functions. In order to achieve such an estimate, we develop tools, different from those in [Göt04], which involve adjustable smooth approximations of the indicatior functions of the hyperboloid and of the cube C_r. The bound given by an integral of theta functions does *not* use the special structure of the indefinite forms under consideration. Furthermore, a careful modification of the arguments in [Göt04] even leads to a bound in terms of the Minkowski minima mentioned above, which holds for *any* indefinite form. The special structure of the forms is only used when we estimate the appearing functions of Minkowski minima by adapting the techniques of [Göt04] to the indefinite case. As in the positive definite case we show that in the irrational case the maximal gap between successive values of the quadratic form at lattice points converges to 0 as r tends to infinity (Corollary 2.4). Furthermore, we extend the results of Bentkus and Götze ([BG99]) on distribution functions for values of quadratic forms to dimensions including 5 up to 8 (Theorem 2.7). In addition, we obtain a result for multivariate diophantine approximations for these special indefinite forms (Theorem 2.6).

This paper is organized as follows: In the second section, we state the two main results about the asymptotics of the relative lattice point rest and derive two important corollaries concerning gaps between values of the quadratic form and concerning multivariate diophantine approximations. Furthermore, we give explicit quantitative bounds for the relative lattice point rest. In the third section, we prove the results of the second section. In the fourth section, we collect auxiliary results (e.g. from geometry of numbers, metric number theory, theory of theta functions), which are used in the proofs of the theorems.

Acknowledgements
I would like to thank Prof. Dr. Friedrich Götze for drawing my attention to this topic, for various fruitful discussions and many valuable suggestions. Furthermore, I am grateful to the DFG-CRC 701 for financial support. This paper is a part of my PhD thesis [Els06].

2. Results

Let $\mathbb{R}^d$, $1 \le d < \infty$, denote the d-dimensional Euclidean space with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$ defined by $|x|^2 = \langle x, x \rangle = x_1^2 + \cdots + x_d^2$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Let $\mathbb{Z}^d$ denote the standard lattice of points with integer coordinates in $\mathbb{R}^d$.
Consider the quadratic form

$$Q[x] \stackrel{\text{def}}{=} \langle Qx, x \rangle, \quad \text{for} \quad x \in \mathbb{R}^d,$$

where $Q : \mathbb{R}^d \to \mathbb{R}^d$ denotes a symmetric linear operator in $\mathrm{GL}(d, \mathbb{R})$ with eigenvalues, say, $q_1, \dots, q_d$. We write

$$q_0 \stackrel{\text{def}}{=} \min_{1 \le j \le d} |q_j|, \qquad q \stackrel{\text{def}}{=} \max_{1 \le j \le d} |q_j|, \qquad \bar{q} \stackrel{\text{def}}{=} \max \left\{ q_0^{-1}; q \right\}. \quad (2.1)$$

In the sequel we always assume that the form is non-degenerate, that is, that $q_0 > 0$.

We say that a quadratic form Q is of *block-type*, if and only if we can write $Q = Q^+ - Q^-$, where Q^+ and Q^- are positive definite quadratic forms, $Q^+[x]$ only depends on the first d_1 coordinates of $\mathbb{R}^d$ and $Q^-[x]$ on the $d - d_1$ remaining ones only.

We define for $a, b \in \mathbb{R}$, with $a \le b$ and for $M \in \mathbb{R}^d$ the sets

$$E_{a,b;M} \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^d \ : \ a \le Q[x - M] \le b \right\}. \quad (2.2)$$

Note, that if the quadratic form $Q[x]$ is positive definite, then $E_{a,b;M}$ is an elliptic shell.
Recall that a quadratic form $Q[x]$ and the corresponding operator Q with non-zero matrix $Q = (q_{ij})$, $1 \le i, j \le d$, is called *rational* if there exists a real number $\lambda \ne 0$ such that the matrix λQ has integer entries only; otherwise it is called *irrational*.
For $r > 0$ we set $C_r \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d \ : \ |x|_\infty \le r\}$, where $|\cdot|_\infty$ denotes the maximum norm on $\mathbb{R}^d$, and

$$H_{r,M} \stackrel{\text{def}}{=} H_{r,M}^{a,b} \stackrel{\text{def}}{=} E_{a,b;M} \cap C_r. \quad (2.3)$$

For any (measurable) set $B \subset \mathbb{R}^d$ let $\operatorname{vol} B$ denote the Lebesgue measure of B and $\operatorname{vol}_{\mathbb{Z}} B$ its lattice volume, that is the number of lattice points in $B \cap \mathbb{Z}^d$. We want to investigate the approximation of the lattice volume of $H_{r,M}$ by the Lebesgue volume. Therefore we estimate the following *relative* lattice point rest of *large* parts of hyperbolic shells $H_{r,M}$, $M \in \mathbb{R}^d$, r large.

We define

$$\Delta(r, M) \stackrel{\text{def}}{=} \left| \frac{\mathrm{vol}_{\mathbb{Z}} H_{r,M} - \mathrm{vol}\, H_{r,M}}{\mathrm{vol}\, H_{r,M}} \right|. \tag{2.4}$$

The two main results of this part of the paper are the following

Theorem 2.1. *For a non-degenerate, d-dimensional, block-type form Q, $d \geq 5$ and all $M \in \mathbb{R}^d$ it holds*

$$\Delta(r, M) = \mathcal{O}(1), \qquad \textit{as } r \to \infty. \tag{2.5}$$

The estimate of Theorem 2.1 refines an explicit bound of order $\mathcal{O}(1)$ obtained for dimensions $d \geq 9$ in [BG97] for *arbitrary* ellipsoids and in [BG99] for *arbitrary* hyperbolic shells. Since this bound is optimal in the case of positive definite forms ([Göt04], p. 196), the bound in Theorem 2.1 is also optimal for block-type forms.

In case that Q is *irrational* Theorem 2.1 can be improved.

Theorem 2.2. *For a non-degenerate d-dimensional block-type form Q, $d \geq 5$ and all $M \in \mathbb{R}^d$ it holds*

$$\Delta(r, M) = o(1), \quad \textit{provided that } Q \textit{ is irrational.} \tag{2.6}$$

For *irrational* forms and dimension $d \geq 9$ the bound of Theorem 2.2 has been already proved in [BG99]. We should remark again, that the bounds of both theorems are explicit and effective.

Remark 2.3. *For $M \in \mathbb{Q}^d$ the condition $\Delta(r, M) = o(1)$ implies that Q is irrational.*

Using Theorem 2.2 we can derive easily a Corollary about gaps between values of block-type forms:

For a positive definite quadratic form, Davenport and Lewis [DL72] conjectured in 1972, that the distance between successive values v_n of the quadratic form $Q[x]$ on $\mathbb{Z}^d$ converges to zero as $n \to \infty$, provided that the dimension d is at least five and Q is irrational. This conjecture was proved by Götze [Göt04]. Now we can derive an analog result for irrational block-type forms and dimension $d \geq 5$.

For a vector $M \in \mathbb{R}^d$ let

$$V(r) \stackrel{\text{def}}{=} \{Q[x - M] \,:\, x \in \mathbb{Z}^d \cap C_r\} \tag{2.7}$$

denote the set of values of $Q[x - M]$, for lattice points $x \in \mathbb{Z}^d$ in a box of edge length r.

We define the maximal gap between successive values as

$$d(r) \stackrel{\text{def}}{=} \sup_{u\in V(r)} \inf\big\{v-u \;:\; v>u, v\in V(r)\big\}. \tag{2.8}$$

Corollary 2.4. *For a non-degenerate d-dimensional block-type form Q, $d \geq 5$, the follwing holds:*

(1) $\lim_{r\to\infty} d(r) = 0$, *provided that Q is irrational.*
(2) *For $M \in \mathbb{Q}^d$ and Q rational we get* $\lim_{r\to\infty} d(r) > 0$.

The Theorems 2.1 and 2.2 follow from Theorem 2.5 below. Furthermore, in Theorem 2.5 (combined with (3.1) in the proof of the Theorems 2.1 and 2.2), estimates of the remainder terms in (2.5) and (2.6) in terms of certain diophantine properties of Q will be given.

In order to describe the explicit bounds we need to introduce some more notations. Let $|(x,y)|_\infty$ denote the maximum norm of a vector (x,y) in $\mathbb{R}^d\times\mathbb{R}^d$. For any $t>0$ and $r\geq 2$ consider the norm F on $\mathbb{R}^d\times\mathbb{R}^d$ given by

$$F(x,y) \stackrel{\text{def}}{=} \big|\big(r\,(x+tQy)\,,\, yr^{-1}\big)\big|_\infty. \tag{2.9}$$

We introduce the so called Minkowski minima of the convex body $\{F\leq 1\}$ as

$$M_{1,t} = \inf\big\{F(m,n) \;:\; (m,n)\in(\mathbb{Z}^d\times\mathbb{Z}^d)\setminus 0\big\} \tag{2.10}$$

and we define in general $M_{k,t}$ as the infimum of $\lambda>0$ such that the set of lattice points with norm less than λ, that is

$$\big\{(m,n)\in\mathbb{Z}^d\times\mathbb{Z}^d \;:\; F(m,n)<\lambda\big\},$$

contains k linearly independent vectors. By definition we have $r\,M_{k,t}\geq 1$. For $d>4$ and $r\geq 2$ we introduce

$$\Gamma_{T,r} \stackrel{\text{def}}{=} \inf\;\big\{r^d M_{1,t}\cdots M_{d,t} \;:\; T^{-1/(d-4)}\leq |t|\leq T\big\}, \tag{2.11}$$

$$\rho(r,Q,T) \stackrel{\text{def}}{=} \bar q^{d+1}T^{-\frac12} + \bar q^{\frac{3d}{2}}\max\big\{\frac{2}{\pi r}, \frac{\pi}{2q_0 q r}, T^{-\frac{1}{d-4}}\big\} + \bar q^{d+2}\,\Gamma_{T,r}^{-\frac12+\frac2d}\log\big(\bar q T^{\frac12}\Gamma_{T,r}+1\big), \tag{2.12}$$

$$\rho(r,Q) \stackrel{\text{def}}{=} \inf_{T\geq 1}\Big\{r^{2-d} + q_0^{\frac d2} r^{2-\frac d2} + \bar q r^{2-\frac d2}(1+\log r) + \rho(r,Q,T)\Big\}. \tag{2.13}$$

For any fixed $T>1$ and irrational Q it is shown in Lemma 4.23 that

$$\lim_{r\to\infty}\Gamma_{T,r} = \infty, \tag{2.14}$$

with a speed depending on the diophantine properties of Q. This implies that

$$\lim_{r\to\infty} \rho(r, Q) = 0. \tag{2.15}$$

With these notations we may state a Theorem providing quantitative bounds for the difference between the volume and the lattice point volume of a hyperbolic shell.

Theorem 2.5. *Let Q denote a non-degenerate d-dimensional block-type form, $d \geq 5$, and $M \in \mathbb{R}^d$. Furthermore, let $c(Q, M) > 0$ be defined as in Theorem 3.1 below and $K = K(d)$ is chosen according to* (3.8)*. Then there exist constants $c_j > 0$, $j = 1, 2$, depending on d only and a constant $r_0 = r(Q, M, a, b) > 0$ such that, for any $r \geq r_0$,*

(1) $\big|\mathrm{vol}_{\mathbb{Z}} H_{r,M} - \mathrm{vol}\, H_{r,M}\big|$

$$\leq \quad c_1 \cdot r^{d-2} \cdot \Big((b-a+1)\bar{q}^d q^{-1} + c(Q, M)\bar{q}^{d+1}(\log q + 1) + 1\Big).$$

(2) $$\big|\mathrm{vol}_{\mathbb{Z}} H_{r,M} - \mathrm{vol}\, H_{r,M}\big| \quad \leq \quad c_2 \cdot r^{d-2} \cdot \Big((b-a)\,\bar{q}^d q^{-1} r^{-\frac{1}{K}}$$

$$+ \quad (b-a)\,\bar{q}^{d+1} q^{-1}\big(|M| + 2q^{-\frac{1}{2}}\frac{|a|+|b|}{r}\big)r^{-1} \; + \; c(Q, M)\cdot \rho(r, Q)\Big),$$

where $\lim_{r\to\infty} \rho(r, Q) = 0$, provided that Q is irrational.

Note that the summand $\rho(r, Q) r^{d-2}$ in the bound in Theorem 2.5 is at least of order $\mathcal{O}(r^{d/2} \log r)$. It may be indeed of this order since $r\, M_{j,t} \ll_d r$ shows that the maximal value of $\Gamma_{T,r}$ is of order $\mathcal{O}(r^d)$ and we may choose $T = \mathcal{O}(r^{\bar{\beta}})$ with $\bar{\beta} > 0$ sufficiently large.
Note that an error bound of order $r^{d/2+\varepsilon}$ has been proved by Jarnik [Jar28] for *diagonal* $Q = \mathrm{diag}(s_1, .., s_d)$, $s_j > 0$ for Lebesgue *almost all* coefficients s_j.

The proof of Theorem 2.5 is based, roughly speaking, on an 'continuous' approximation of $\big|\mathrm{vol}_{\mathbb{Z}} H_{r,M} - \mathrm{vol}\, H_{r,M}\big|$ by an integral over generalized theta functions. We will derive bounds for parts of this integral, which use the distribution of the first Minkowski minimum $M_{1,t}$. We investigate this distribution using results from metric number theory. As a consequence of this investigation, we also get a result for multivariate diophantine approximation:

For a vector $x \in \mathbb{R}^d$ let $\|x\| \stackrel{\mathrm{def}}{=} \inf_{m\in\mathbb{Z}^d} |x - m|_\infty$ denote the error of an integer approximation. For real numbers $t > 0$, $\nu > 1$ we introduce

$$D(t, \nu) = \nu \min\big\{\|tQn\| \,:\, n \in \mathbb{Z}^d,\, 0 < |n|_\infty \leq \nu\big\}, \tag{2.16}$$

and let λ denote the Lebesgue measure. Then we have

Theorem 2.6. *Assume that Q is a symmetric, non-degenerated block-type form, which is normalized such that $q_0 = 1$. Then there exists a constant $c(d) > 1$ depending on d only such that for any $r \geq 1$ and any interval $[\kappa, \xi]$ satisfying $0 < \xi - \kappa < 1$ the following inequalities hold*

$$\lambda\{t \in [\kappa, \xi] \,:\, M_{1,t} \leq \tau\} \;\leq\; c(d)\big(q\tau^2(\xi-\kappa) + \tau r^{-1}\big), \tag{2.17}$$

$$\sup_{t\in[\kappa,\xi]} M_{1,t} \;\geq\; \min\{\tau_Q, r(\xi-\kappa)\}, \tag{2.18}$$

$$\sup_{t\in[\kappa,\xi]} D(t,\nu) \;\geq\; \min\{\tau_Q, \nu(\xi-\kappa)/2\}, \tag{2.19}$$

for any $\nu \geq \tau_Q$, where $\tau_Q \overset{\text{def}}{=} \Big(\frac{c(d)+2}{2c(d)q}\Big)^{1/2}$.

Refining the proofs, we may extend Theorem 2.1 and 2.2 to include the case $a = -\infty$, i.e. the case of distribution functions. This partially extends a result obtained by Bentkus and Götze in [BG99] to the dimensions including 5 up to 8.

Theorem 2.7. *For a non-degenerate, d-dimensional, block-type form Q, $d \geq 5$ and all $M \in \mathbb{R}^d$ we set*

$$F_{r,M}(b) \overset{\text{def}}{=} \big\{x \in \mathbb{R}^d \,:\, Q[x-M] \leq b,\;\; |x|_\infty \leq r\big\}.$$

Then for the corresponding relative lattice point remainder holds

$$\Big|\frac{\operatorname{vol}_{\mathbb{Z}} F_{r,M}(b) - \operatorname{vol} F_{r,M}(b)}{\operatorname{vol} F_{r,M}(b)}\Big| = \begin{cases} o(1), & \textit{provided } Q \textit{ is irrational,} \\ \mathcal{O}(1), & \textit{otherwise,} \end{cases}$$

as $r \to \infty$.

3. Proofs

First we deduce Theorem 2.1 and 2.2 from Theorem 2.5:

Proof of Theorem 2.1 and 2.2.
By Lemma 4.1 we obtain for $M = (M_1, ..., M_d)$ and for r large

$$\begin{aligned} \text{vol } H_{r,M} \;&\gg_d\; (b-a)q^{-d/2}\left(q_0^{d/2} + r^{-1}\big|\big(|q_1|^{\frac12}M_1, ..., |q_d|^{\frac12}M_d\big)\big|\right)^{d-2} r^{d-2} \\ &\gg_d\; (b-a)q^{-d/2}q_0^{d(d-2)/2}r^{d-2}. \end{aligned} \tag{3.1}$$

Dividing the inequalities in Theorem 2.5 (1) in the general case (resp. Theorem 2.5 (2) in the irrational case) by vol $H_{r,M}$, the estimate (3.1) completes the proof of Theorem 2.1 (resp. of Theorem 2.2). ∎

Proof of Corollary 2.4.
If Q is irrational, Theorem 2.2 implies, that for any $a, b \in \mathbb{R}$

$$\left| \frac{\operatorname{vol}_{\mathbb{Z}} H_{r,M}^{a,b}}{\operatorname{vol}_{\mathbb{R}} H_{r,M}^{a,b}} - 1 \right| \to 0 \quad \text{as} \quad r \to \infty. \tag{3.2}$$

Hence, $H_{r,M}^{a,b} \cap \mathbb{Z}^d \neq \emptyset$ for all $a, b \in \mathbb{R}$ if r is sufficiently large. This implies that $\lim\limits_{r\to\infty} d(r) = 0$ and the proof of part (1) is completed.
If Q is rational, there exists a real number $\lambda > 0$, such that λQ has integer entries only. For $M \in \mathbb{Q}^d$ there exists a $\mu \in \mathbb{Z}$, $\mu \neq 0$, such that $\mu M \in \mathbb{Z}^d$. Hence, it holds that $Q[m - M] \in \lambda^{-1}\mu^{-2}\mathbb{Z}^d$ for all $m \in \mathbb{Z}^d$. Therefore $d(r) \geq \lambda^{-1}\mu^{-2} > 0$ for all $r \geq 1$, which proves part (2). □

We should remark, that by using (3.14) and (3.1) one can obtain explicit bounds for $d(r)$ in terms of r and $\rho(r, Q)$, representing diophantine properties of Q.

Proof of Remark 2.3. Analyzing the proof of Corollary 2.4 (1) we recognize that $\Delta(r, M) = o(1)$ already implies $\lim_{r\to\infty} d(r) = 0$. Under the assumption $M \in \mathbb{Q}^d$ the condition that Q is rational yields by Corollary 2.4 (2) that $\lim_{r\to\infty} d(r) > 0$. Thus, for $M \in \mathbb{Q}^d$ the irrationality of Q follows from $\Delta(r, M) = o(1)$. □

The first step in proving Theorem 2.5 is to analyze smooth approximations of the lattice volume of H_r:

For $a, b \in \mathbb{R}$ and a smoothing parameter $w > 0$ we define $g_{a,b,w} : \mathbb{R} \to [0, 1]$ by

$$g_{a,b,w}(x) \overset{\text{def}}{=} \frac{1}{w}\big((b + w - x)_+ - (b - x)_+ - (a - x)_+ + (a - w - x)_+\big). \tag{3.3}$$

This function $g_{a,b,w}$ is a linear continuous approximation of the indicator function $I_{[a,b]}$ of the interval $[a, b]$. By Lemma 4.8 we may rewrite $g_{a,b,w}$ as follows

$$\begin{aligned} g_{a,b,w}(x) &= \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{(b+w-x)z} - e^{(b-x)z} - e^{(a-x)z} + e^{(a-w-x)z} \, \frac{dz}{wz^2} \\ &= \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \exp\left[-xz\right] \cdot h_{a,b,w}(z) \frac{dz}{z}, \end{aligned} \tag{3.4}$$

where $h_{a,b,w}(z) \stackrel{\text{def}}{=} \dfrac{\exp[wz]-1}{wz} \cdot \Big(\exp[bz] - \exp[(a-w)z]\Big)$.

Using $g_{a,b,w}$ we construct a continuous approximation $V^{\mathbb{Z}}_{w,\varepsilon}(r;a,b,M)$ of the (monotone) lattice point counting function $r \mapsto \mathrm{vol}_{\mathbb{Z}}(H_{r,M})$ depending on two smoothing parameter $w > 0$ and $\varepsilon > 0$. Setting $Q_+ \stackrel{\text{def}}{=} \big(Q^T Q\big)^{\frac{1}{2}}$, we define

$$V^{\mathbb{Z}}_{w,\varepsilon}(r;a,b,M) \stackrel{\text{def}}{=} \sum_{x\in\mathbb{Z}^d} \exp\Big[-\frac{2}{r^2}\,Q_+[x]\Big]\, g_{a,b,w}\Big(Q[x-M]\Big)\, \chi_\varepsilon\left(\frac{x}{r}\right) \quad (3.5)$$

and

$$V^{\mathbb{R}}_{w,\varepsilon}(r;a,b,M) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \exp\Big[-\frac{2}{r^2}\,Q_+[x]\Big]\, g_{a,b,w}\Big(Q[x-M]\Big)\, \chi_\varepsilon\left(\frac{x}{r}\right) dx \quad (3.6)$$

where $\chi_{\pm\varepsilon}$ is a function with the following properties:

(1) For $u \in \mathbb{R}^d$ it holds

$$\chi_{\pm\varepsilon}(u) = \begin{cases} \exp\left[2 \cdot Q_+[u]\right], & \text{if } |u|_\infty \le \min\{1; 1 \pm \varepsilon\}, \\ 0, & \text{if } |u|_\infty > \max\{1; 1 \pm \varepsilon\}. \end{cases}$$

(2) There exists a constant $c_1(Q,M) > 0$ such that for

$$\overline{\chi}_{\pm\varepsilon}(x) \stackrel{\text{def}}{=} \chi_{\pm\varepsilon}(x) \cdot \exp\big[\big\langle x, 2r^{-1}QM\big\rangle\big] \quad (3.7)$$

the following estimates hold for an appropriate $K = K(d) \in \mathbb{N}$:

$$\text{(a)} \quad \int_{\mathbb{R}^d} \Big|\widehat{\overline{\chi}}_{\pm\varepsilon}(v)\Big|\, dv \quad \ll_d \quad c_1(Q,M) \cdot \varepsilon^{-K},$$

$$\text{(b)} \quad \int_{\{|v|_\infty > d^{-\frac{1}{2}} r\}} \Big|\widehat{\overline{\chi}}_{\pm\varepsilon}(v)\Big|\, dv \ \ll_d \ c_1(Q) \cdot \varepsilon^{-K} r^{-1} \ \text{ for all } \ r \ge 1. \quad (3.8)$$

The existence of such a function $\chi_{\pm\varepsilon}$ follows by standard arguments from Fourier analysis (cf. [Els06], p. 27, Lemma 2.4.5). Note, that the function

$$\psi_{r,\pm\varepsilon}(x) \stackrel{\text{def}}{=} \exp\Big[-\frac{2}{r^2}\,Q_+[x]\Big]\chi_{\pm\varepsilon}(\frac{x}{r}) \quad (3.9)$$

approximates the indicator function $I_{\{|x|_\infty \le r\}}$ and hence the equations $V^{\mathbb{Z}}_{0,0}(r;a,b,M) = \mathrm{vol}_{\mathbb{Z}}(H_{r,M})$ and $V^{\mathbb{R}}_{0,0}(r;a,b,M) = \mathrm{vol}_{\mathbb{R}}(H_{r,M})$ are suggestive.

Proof of Theorem 2.5.
For $M \in \mathbb{R}^d$, $0 < \varepsilon \le \frac{1}{4}$, there exists a constant $c = c(d) > 0$ by Lemma 4.4 such that

$$\Big|\mathrm{vol}_{\mathbb{Z}} H_{r,M} - \mathrm{vol}\, H_{r,M}\Big| \leq \max\big\{\Delta_{-\varepsilon}\ ;\ \Delta_{\varepsilon}\big\} + c\cdot(b-a)q_0^{-d/2}q^{(d-2)/2} \times \Big(\varepsilon + q_0^{-1/2}q^{1/2}|M|r^{-1} + 2q_0^{-1/2}(|a|+|b|)r^{-2}\Big)\, r^{d-2}, \quad (3.10)$$

where $\Delta_{\pm\varepsilon}$ is defined by using (3.9) as follows

$$\Delta_{\pm\varepsilon} \stackrel{\mathrm{def}}{=} \Big|\int_{\mathbb{R}^d} I_{H_{r,M}}(x)\psi_{r,\pm\varepsilon}(x)dx - \sum_{x\in\mathbb{Z}^d} I_{H_{r,M}}(x)\psi_{r,\pm\varepsilon}(x)\Big|. \quad (3.11)$$

Hence, $\Delta_{\pm\varepsilon}$ can be estimated by Lemma 4.6 by

$$\Delta_{\pm\varepsilon} \ll_d \max_{\pm} \sup_{\substack{a'\in[a-w;a+w]\\ b'\in[b-w;b+w]}} \Big|V^{\mathbb{R}}_{w,\pm\varepsilon}(r;a',b',M) - V^{\mathbb{Z}}_{w,\pm\varepsilon}(r;a',b',M)\Big| + 8wq_0^{-\frac{d}{2}}q^{\frac{d-2}{2}}\big(1+\varepsilon+q^{\frac12}\frac{|M|}{r}\big)^{d-2}r^{d-2}. \quad (3.12)$$

Collecting the estimates (3.10) and (3.12) we obtain for $w>0, 0<\varepsilon\leq\frac14$

$$\begin{aligned} r^{2-d}\cdot\Big|\mathrm{vol}_{\mathbb{Z}} H_{r,M} - \mathrm{vol}\, H_{r,M}\Big| \ \ll_d\ & (b-a)\,\bar q^d q^{-1}\varepsilon \\ +\ & (b-a)\,\bar q^{d+1}q^{-1}\big(\frac{|M|}{r} + 2q^{-\frac12}\frac{|a|+|b|}{r^2}\big) + w\bar q^d q^{-1}\big(1+\varepsilon+q^{\frac12}\frac{|M|}{r}\big)^{d-2} \\ +\ & \max_{\pm}\sup_{a',b'}\Big|V^{\mathbb{R}}_{w,\pm\varepsilon}(r;a',b',M) - V^{\mathbb{Z}}_{w,\pm\varepsilon}(r;a',b',M)\Big|\cdot r^{2-d}. \end{aligned} \quad (3.13)$$

Choosing now $w=1, \varepsilon=\frac14$ and $r>r_0$ large enough, (3.13) and the result of the following crucial Theorem 3.1 (1) below yields (note that $d\geq 5$)

$$\begin{aligned} &\Big|\mathrm{vol}_{\mathbb{Z}} H_{r,M} - \mathrm{vol}\, H_{r,M}\Big| \\ &\ll_d (b-a+1)\bar q^d q^{-1}r^{d-2} + c(Q,M)\Big(1 + q_0^{-\frac{d}{2}}r^{\frac{d}{2}} + \bar q^{d+1}(\log q+1)\,r^{d-2}\Big) \\ &\ll_d \Big((b-a+1)\bar q^d q^{-1} + c(Q,M)\bar q^{d+1}(\log q+1)+1\Big)r^{d-2}, \end{aligned}$$

for r large enough. This proves Theorem 2.5 (1).

For proving Theorem 2.5 (2), we choose for an arbitrary $\alpha\in(0,1)$

$$w = T^{-1/2},\ T\geq 1 \ \text{ and } \ \varepsilon=\varepsilon(r) \stackrel{\mathrm{def}}{=} r^{-\alpha K^{-1}},$$

where $K=K(d)$ is chosen according to (3.7).
Then we get by (3.13) and by Theorem 3.1 (2) below, that for r sufficiently large the following holds

$$\begin{aligned} r^{2-d}\cdot\big|\mathrm{vol}_{\mathbb Z} H_{r,M}-\mathrm{vol}\, H_{r,M}\big| \;&\ll_d\; (b-a)\,\bar q^{d} q^{-1} r^{-\alpha K^{-1}}\\ &+\; (b-a)\,\bar q^{d+1} q^{-1}\big(|M|+2q^{-\frac12}\frac{|a|+|b|}{r}\big)r^{-1} \;+\; T^{-\frac12}\bar q^{\frac d2} q^{-1}\big(2+q^{\frac12}\big)^{d-2}\\ &+\; c(Q,M)\cdot r^{\alpha}\Big(1+q_0^{-\frac d2} r^{\frac d2}+\bar q^{d} r^{\frac d2}(1+\log r)+r^{d-2}\cdot\rho(r,Q,T)\Big)r^{2-d} \end{aligned}$$

Taking the infimum over all $\alpha\in(0,1)$ we obtain

$$\begin{aligned} r^{2-d}\cdot\big|\mathrm{vol}_{\mathbb Z} H_{r,M}-\mathrm{vol}\, H_{r,M}\big| \;&\ll_d\; (b-a)\,\bar q^{d} q^{-1} r^{-\frac1K}\\ &+\; (b-a)\,\bar q^{d+1} q^{-1}\big(|M|+2q^{-\frac12}\frac{|a|+|b|}{r}\big)r^{-1} + T^{-\frac12}\bar q^{\frac d2} q^{-1}\big(2+q^{\frac12}\big)^{d-2}\\ &+\; c(Q,M)\cdot\Big(1+q_0^{-\frac d2} r^{\frac d2}+\bar q^{d} r^{\frac d2}(1+\log r)+r^{d-2}\cdot\rho(r,Q,T)\Big)r^{2-d} \end{aligned}$$

By taking the infimum over all $T\geq 1$ we get with (2.13)

$$\begin{aligned} r^{2-d}\cdot\big|\mathrm{vol}_{\mathbb Z} H_{r,M}-\mathrm{vol}\, H_{r,M}\big| \;&\ll_d\; (b-a)\,\bar q^{d} q^{-1} r^{-\frac1K}\\ &+\; (b-a)\,\bar q^{d+1} q^{-1}\big(|M|+2q^{-\frac12}\frac{|a|+|b|}{r}\big)r^{-1} \;+\; c(Q,M)\cdot\rho(r,Q), \quad (3.14) \end{aligned}$$

which proves Theorem 2.5 (2) for an appropriate choice of r_0. □

The key tool in the previous proofs is the following

Theorem 3.1. *Let Q denote a non-degenerate d-dimensional quadratic form of block-type, $d\geq 5$. Then for all $M\in\mathbb R^d$ there exist constants $c(Q,M), r_0>0$, such that for any $r\geq r_0$ and any $T\geq 1$*

(1) $$\big|V^{\mathbb Z}_{1,\pm\varepsilon}(r;a,b,M)-V^{\mathbb R}_{1,\pm\varepsilon}(r;a,b,M)\big| \ll_d\; c(Q,M)\cdot\varepsilon^{-K}\Big(1+2q_0^{-\frac d2}r^{\frac d2}+\bar q^{d+1}(\log q+1)r^{d-2}\Big).$$

(2) $$\big|V^{\mathbb Z}_{T^{-1/2},\pm\varepsilon}(r;a,b,M)-V^{\mathbb R}_{T^{-1/2},\pm\varepsilon}(r;a,b,M)\big| \;\ll_d\; c(Q,M)\cdot\varepsilon^{-K} \times\Big(1+q_0^{-\frac d2}r^{\frac d2}+\bar q^{d} r^{\frac d2}(1+\log r)+r^{d-2}\cdot\rho(r,Q,T)\Big),$$

where $\rho(r,Q,T)$ is defined as in (2.12).

Proof. We want to estimate the difference between these two approximations by integrals of theta functions. By (3.4), (3.5) and (3.6) we have

$$\big|V^{\mathbb Z}_{w,\pm\varepsilon}(r;a,b,M)-V^{\mathbb R}_{w,\pm\varepsilon}(r;a,b,M)\big| \;=$$

$$\Big|\sum_{x\in\mathbb Z^d}\exp\Big[-\frac{2}{r^2}\,Q_+[x]\Big]\,\frac{1}{2\pi i}\int\limits_{\beta-i\,\infty}^{\beta+i\,\infty}\exp\big[-z\cdot Q[x-M]\big]\cdot h_{a,b,w}(z)\frac{dz}{z}\,\chi_{\pm\varepsilon}\Big(\frac{x}{r}\Big)$$

$$-\int_{\mathbb{R}^d} \exp\Big[-\frac{2}{r^2}\,Q_+[x]\Big]\,\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \exp\big[-z\cdot Q[x-M]\big]\cdot h_{a,b,w}(z)\frac{dz}{z}\,\chi_{\pm\varepsilon}\Big(\frac{x}{r}\Big)\,dx\Big|.$$

Choosing $\beta = r^{-2}$, decomposing $Q[x-M] = Q[x] + Q[M] - 2\langle x, QM\rangle$ (Recall, that Q is self-adjoint.) and using Fubini's theorem, we get

$$\Big|V^{\mathbb{Z}}_{w,\pm\varepsilon}(r;a,b,M) - V^{\mathbb{R}}_{w,\pm\varepsilon}(r;a,b,M)\Big| \;=\Big|\int_{r^{-2}-i\infty}^{r^{-2}+i\infty} \exp\big[-zQ[M]\big]\,h_{a,b,w}(z)$$

$$\times\Big\{\sum_{x\in\mathbb{Z}^d}\exp\Big[-\frac{2}{r^2}\,Q_+[x] - zQ[x] + i\langle x, 2t\,\mathrm{Im}(z)QM\rangle\Big]\overline{\chi}_{\pm\varepsilon}\Big(\frac{x}{r}\Big)$$

$$-\int_{\mathbb{R}^d}\exp\Big[-\frac{2}{r^2}\,Q_+[x] - zQ[x] + i\langle x, 2\,\mathrm{Im}(z)QM\rangle\Big]\overline{\chi}_{\pm\varepsilon}\Big(\frac{x}{r}\Big)\,dx\Big\}\frac{dz}{z}\Big|,$$

where $\overline{\chi}_{\pm\varepsilon}$ is defined as in (3.7).
Since $\overline{\chi}_{\pm\varepsilon}(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{\overline{\chi}}_{\pm\varepsilon}(v)\exp[-i\langle x,v\rangle]dv$ holds by the Fourier inversion theorem, we obtain

$$\Big|V^{\mathbb{Z}}_{w,\pm\varepsilon}(r;a,b,M) - V^{\mathbb{R}}_{w,\pm\varepsilon}(r;a,b,M)\Big| \;=$$

$$\Big|\int_{r^{-2}-i\infty}^{r^{-2}+i\infty} \exp\big[-zQ[M]\big]\,h_{a,b,w}(z)\tfrac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{\overline{\chi}}_{\pm\varepsilon}(v)$$

$$\times\quad\Big\{\sum_{x\in\mathbb{Z}^d}\exp\Big[-\frac{2}{r^2}\,Q_+[x] - zQ[x] + i\langle x, 2\,\mathrm{Im}(z)\,QM - \frac{v}{r}\rangle\Big]$$

$$-\int_{\mathbb{R}^d}\exp\Big[-\frac{2}{r^2}\,Q_+[x] - zQ[x] + i\langle x, 2\,\mathrm{Im}(z)QM - \frac{v}{r}\rangle\Big]dx\Big\}dv\,\frac{dz}{z}\Big|. \tag{3.15}$$

For $v\in\mathbb{C}^d$ we introduce the following theta sum and theta integral

$$\theta_v(z) \;\stackrel{\text{def}}{=}\; \exp\big[-zQ[M]\big]\sum_{x\in\mathbb{Z}^d}\exp\left[-\Theta_{Q,M,r,v}(z,x)\right], \tag{3.16}$$

$$\theta_{0,v}(z) \;\stackrel{\text{def}}{=}\; \exp\big[-zQ[M]\big]\int_{R^d}\exp\left[-\Theta_{Q,M,r,v}(z,x)\right]dx \tag{3.17}$$

where $\Theta_{Q,M,r,v}(z,x) \stackrel{\text{def}}{=} \frac{2}{r^2}Q_+[x] - z\cdot Q[x] - i\cdot\langle x, \frac{v}{r} - 2\,\mathrm{Im}(z)QM\rangle$.
Then we can rewrite (3.15) as follows

$$\Big|V^{\mathbb{Z}}_{w,\pm\varepsilon}(r;a,b,M) - V^{\mathbb{R}}_{w,\pm\varepsilon}(r;a,b,M)\Big|$$

$$= \Big| \int\limits_{r^{-2}-i\,\infty}^{r^{-2}+i\,\infty} h_{a,b,w}(z) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\chi}_{\pm\varepsilon}(v) \cdot \big\{\theta_v(z) - \theta_{0,v}(z)\big\} dv \, \frac{dz}{z} \Big|.$$

Consider the segments $J_0 \overset{\text{def}}{=} [r^{-2} - i \cdot \frac{1}{r}; r^{-2} + i \cdot \frac{1}{r}]$ and $J_1 \overset{\text{def}}{=} \big(r^{-2} + i\,\mathbb{R}\big) \setminus J_0$. Then we may split

$$\Big|V^{\mathbb{Z}}_{w,\pm\varepsilon}(r;a,b,M) - V^{\mathbb{R}}_{w,\pm\varepsilon}(r;a,b,M)\Big|$$

$$\begin{aligned}
\ll_d \quad & \Big| \int\limits_{J_0} h_{a,b,w}(z) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\chi}_{\pm\varepsilon}(v) \cdot \big\{\theta_v(z) - \theta_{0,v}(z)\big\} dv \, \frac{dz}{z} \\
& - \int\limits_{J_1} h_{a,b,w}(z) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\chi}_{\pm\varepsilon}(v) \cdot \theta_{0,v}(z) dv \, \frac{dz}{z} \\
& + \int\limits_{J_1} h_{a,b,w}(z) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\chi}_{\pm\varepsilon}(v) \cdot \theta_v(z) dv \, \frac{dz}{z} \Big| \\
= \quad & \Big| I_0 - I_1 + I_2 \Big|, \quad \text{say.}
\end{aligned} \tag{3.18}$$

Before estimating these integrals we derive a bound for $h_{a,b,w}(r^{-2}+it), t \in \mathbb{R}$. Using

$$\Big| \frac{\exp\{w(r^{-2}+it)\} - 1}{w} \Big| \le \min\Big\{ e\big|r^{-2}+it\big|, \frac{e+1}{w} \Big\}, \tag{3.19}$$

for $r^2 \ge \max(w,b) > 0, \; r \ge 1,$ we obtain

$$\left| \frac{h_{a,b,w}(r^{-2}+it)}{r^{-2}+it} \right| \ll \Big(e^{br^{-2}} + e^{ar^{-2}}\Big) \cdot \frac{1}{w|r^{-2}+it|^2} \ll \frac{1}{w|r^{-2}+it|^2}, \tag{3.20}$$

as well as

$$\left| \frac{h_{a,b,w}(r^{-2}+it)}{r^{-2}+it} \right| \ll \Big(e^{br^{-2}} + e^{ar^{-2}}\Big) \cdot |r^{-2}+it|^{-1} \ll |r^{-2}+it|^{-1}. \tag{3.21}$$

Estimation of I_0: Inequality (3.21) and Lemma 4.12 for $t \in J_0$ yields

$$\begin{aligned}
\Theta_t \quad &\stackrel{\text{def}}{=} \quad \Big|\big(r^{-2}+i\,t\big)^{-1} h_{a,b,w}(r^{-2}+i\,t)\Big| \\
&\qquad \times \quad \Big|\int_{\mathbb{R}^d} \widehat{\overline{\chi}}_{\pm\varepsilon}(v)\cdot\big\{\theta_v(r^{-2}+i\,t)-\theta_{0,v}(r^{-2}+i\,t)\big\}dv\Big| \\
&\ll_d \quad q_0^{-\frac{d}{2}}|r^{-2}+it|^{-\frac{d+2}{2}}\exp\big[-c(Q)\cdot\mathrm{Re}\big((r^{-2}+it)^{-1}\big)\big]\cdot\int_{\mathbb{R}^d}\Big|\widehat{\overline{\chi}}_{\pm\varepsilon}(v)\Big|\,dv \\
&\qquad +2\cdot|r^{-2}+it|^{-1}\int_{\mathbb{R}^d}\Big|\widehat{\overline{\chi}}_{\pm\varepsilon}(v)\Big|\,I_{(r,\infty)}(|v|)dv,
\end{aligned}$$

where $c(Q)$ is chosen according to Lemma 4.12. Writing $|r^{-2}+it| = r^{-2}(1+r^4t^2)^{1/2}$ and $\mathrm{Re}\big((r^{-2}+it)^{-1}\big) = \frac{r^2}{1+r^4t^2}$, we may introduce the variable $s=(1+r^4t^2)^{-1}$ and the function $h(s) \stackrel{\text{def}}{=} s^{(d+2)/4}\exp\{-c(Q)\,s\,r^2\}$. The maximal value of h on $[0,\infty)$ is attained at $s_0 = \dfrac{d+2}{4\,c(Q)\,r^2}$ and it is bounded by $(c(Q)\,r^2)^{-(d+2)/4}$ up to a constant depending on d only.

Using the properties of $\chi_{\pm\varepsilon}$ (see p. 10) and the fact that $|v|\geq r$ implies $|v|_\infty \geq d^{-1/2}r$ we now obtain

$$\begin{aligned}
\sup_{t\in J_0}\Theta_t &\ll_d q_0^{-\frac{d}{2}}\,r^{d+2}\sup_{s\geq 0}h(s)\int_{\mathbb{R}^d}\Big|\widehat{\overline{\chi}}_{\pm\varepsilon}(v)\Big|\,dv + 2r^2\int_{\mathbb{R}^d}\Big|\widehat{\overline{\chi}}_{\pm\varepsilon}(v)\Big|\,I_{(r,\infty)}(|v|)dv \\
&\ll_d q_0^{-\frac{d}{2}}\,r^{d+2}\,(c(Q)\,r^2)^{-\frac{d+2}{4}}\cdot\int_{\mathbb{R}^d}\Big|\widehat{\overline{\chi}}_{\pm\varepsilon}(v)\Big|\,dv \\
&\qquad\qquad + \quad 2r^2\int_{\mathbb{R}^d}\Big|\widehat{\overline{\chi}}_{\pm\varepsilon}(v)\Big|\,I_{(d^{-1/2}r,\infty)}(|v|_\infty)dv \\
&\ll_d q_0^{-d/2}\,r^{d+2}\,(c(Q)\,r^2)^{-(d+2)/4}\cdot c_1(Q,M)\cdot\varepsilon^{-K} + c_1(Q,M)\cdot\varepsilon^{-K}\cdot r.
\end{aligned}$$

Integrating this bound over J_0, we get for an appropriately chosen constant $c_2(Q,M)>0$

$$|I_0| \leq \int_{-\frac{1}{r}}^{\frac{1}{r}}\Theta_t dt \ll_d c_2(Q,M)\cdot\varepsilon^{-K}q_0^{-\frac{d}{2}}r^{\frac{d}{2}} + c_1(Q,M)\cdot\varepsilon^{-K}. \tag{3.22}$$

Estimation of I_1: Using Lemma 4.11, (4.13) and (4.15), we have

$$\Big|\theta_{0,v}(z)\Big| \ll_d q_0^{-\frac{d}{2}}|z|^{-\frac{d}{2}}. \tag{3.23}$$

Therefore, we get by the properties of $\chi_{\pm\varepsilon}$ (see p. 10) and (3.21) for $r^2 \geq \max\{w, b, 1\}$

$$\begin{aligned} |I_1| &\ll_d q_0^{-\frac{d}{2}} c_1(Q,M) \cdot \varepsilon^{-K} \int_{J_1} \Big| \big(r^{-2} + i\,t\big)^{-\left(1+\frac{d}{2}\right)} \Big|\, dt \\ &\ll_d q_0^{-\frac{d}{2}} c_1(Q,M) \cdot \varepsilon^{-K} \int_{\frac{1}{r}}^{\infty} t^{-\left(1+\frac{d}{2}\right)}\, dt \ll_d q_0^{-\frac{d}{2}} c_1(Q,M) \cdot \varepsilon^{-K} r^{\frac{d}{2}}, \end{aligned} \tag{3.24}$$

using the symmetry in t around 0.

Estimation of I_2: The estimate $\big|h_{a,b,w}(r^{-2}+i\,t)\big| \ll_d \min\big\{1, \big(|r^{-2}+i\,t|w\big)^{-1}\big\}$ given by (3.20) and (3.21) implies

$$\begin{aligned} |I_2| &\ll_d \int_{\mathbb{R}^d} \int_{|t|>\frac{1}{r}} \Big|\theta_v\big(\frac{1}{r^2} + it\big)\Big| \min\Big\{1, \frac{1}{w\,|r^{-2}+it|}\Big\} \frac{dt}{|r^{-2}+it|} \Big|\widehat{\widehat{\chi}}_{\pm\varepsilon}(v)\Big|\, dv \\ &\ll_d \int_{\mathbb{R}^d} \int_{|u|>\frac{2}{\pi r}} \Big|\theta_v\big(r^{-2} + i\pi\frac{u}{2}\big)\Big|\, g(u)\, du \Big|\widehat{\widehat{\chi}}_{\pm\varepsilon}(v)\Big|\, dv, \end{aligned} \tag{3.25}$$

where

$$g(u) = \min\big\{1, (w\,|u|)^{-1}\big\}\, |u|^{-1}. \tag{3.26}$$

Using Lemma 4.15 and the properties of $\chi_{\pm\varepsilon}$ (see p. 10), we have

$$\begin{aligned} |I_2| &\ll_d \bar{q}^d r^{d/2} \int_{\mathbb{R}^d} \int_{|u|>\frac{2}{\pi r}} (M_{1,t} \cdots M_{d,t})^{-1/2} \cdot g(u)\, du \Big|\widehat{\widehat{\chi}}_{\pm\varepsilon}(v)\Big|\, dv \\ &\ll_d \bar{q}^d r^{d/2} \cdot c_1(Q,M) \cdot \varepsilon^{-K} \int_{|u|>\frac{2}{\pi r}} (M_{1,t} \cdots M_{d,t})^{-1/2} \cdot g(u)\, du, \end{aligned} \tag{3.27}$$

where $M_{j,t}$ denote Minkowski's successive minima for the norm on $\mathbb{R}^{2d}$ related to Q, defined by (4.27) and (4.30) and $c_1(Q,M) > 0$ is a constant chosen according to (3.7). Denote

$$G(\kappa,\xi) \stackrel{\text{def}}{=} \int_{\kappa}^{\xi} g(t)\, dt, \qquad \text{for } 0 < \kappa < \xi \leq \infty. \tag{3.28}$$

For $\kappa \geq \xi > 0$ we define $G(\kappa,\xi) = 0$. Note that

$$G(\kappa,\xi) = \begin{cases} \log(\xi/\kappa), & \text{for } \kappa \leq \xi \leq w^{-1}, \\ -\log(w\kappa) + 1 - (w\xi)^{-1}, & \text{for } \kappa \leq w^{-1} \leq \xi, \\ (w\kappa)^{-1} - (w\xi)^{-1}, & \text{for } w^{-1} \leq \kappa \leq \xi. \end{cases} \tag{3.29}$$

The equality (3.29) and the definition of the function G imply the bound

$$G(\kappa,\xi) \le \min\bigl\{\bigl|\log(w\kappa)\bigr| + 1,\ \bigl|\log(\xi/\kappa)\bigr|,\ (w\kappa)^{-1}\bigr\} \quad \text{for} \quad \kappa,\xi > 0. \tag{3.30}$$

Writing

$$M(t) = M_{1,t}\cdots M_{d,t},$$

the upper bound for $|I_2|$ in (3.27) in terms of Minkowski's successive minima now yields

$$\begin{aligned} |I_2| \quad &\ll_d \quad \overline{q}^d r^{d/2}\cdot c_1(Q,M)\cdot \varepsilon^{-K}\int_{|t|>\frac{2}{\pi r}} \frac{g(t)}{M(t)^{1/2}}\,dt \\ &= \quad 2\overline{q}^d r^{d/2}\cdot c_1(Q,M)\cdot\varepsilon^{-K} I_3, \end{aligned} \tag{3.31}$$

where

$$I_3 = \int_{\frac{2}{\pi r}}^{\infty} \frac{g(t)}{M(t)^{1/2}}\,dt. \tag{3.32}$$

The last equality in (3.31) follows from the fact that the functions $g(\cdot)$ and $M(\cdot)$ are even (see (4.33)).

After this preparations, we may now complete the proof of Theorem 3.1:

Proof of Theorem 3.1 (1).
Let

$$\gamma(\kappa,\xi) = r^d \inf_{\kappa\le t\le\xi} M(t), \qquad \text{for } \kappa,\xi\in\mathbb{R}. \tag{3.33}$$

Applying Lemma 4.22 for the interval with endpoints $\kappa = \frac{2}{\pi r}$ and $\xi = \infty$, we get

$$\begin{aligned} I_3 \quad \ll_d \quad q_0^{-1} r^{d/2-2}\int_{\gamma_0}^{D_0} v^{-1/2+1/d}\bigl(qv^{1/d}G\bigl(\kappa_0(v^{1/d}),\infty\bigr) + g\bigl(\kappa_0(v^{1/d})\bigr)\bigr)\,\frac{dv}{v} \\ +G(\frac{2}{\pi r},\infty) \end{aligned} \tag{3.34}$$

with

$$\gamma_0 = \gamma(\frac{2}{\pi r},\infty),\ D_0 = \max\bigl\{\Bigl(\frac{r}{2d}\Bigr)^d,\gamma_0\bigr\},\ \kappa_0(v) = \max\bigl\{\frac{2}{\pi r},\frac{1}{2qvd^{1/2}}\bigr\}. \tag{3.35}$$

Note that $\gamma_0 \ge 1$ by (4.29). In the sequel we choose $w = 1$. Using (3.26), (3.30), (3.31), (3.34), (3.35) and hence $g\bigl(\kappa_0(v^{1/d})\bigr) \ll_d q\,v^{1/d}$, we obtain for

$d > 4$ and $r \geq \max\left\{\frac{2}{\pi\, q}; \frac{2}{\pi}\right\}$:

$$\begin{aligned}
|I_2| &\ll_d c_1(Q,M)\cdot \varepsilon^{-K}\bar{q}^{d+1} r^{d/2} r^{d/2-2}\int_1^{D_0} v^{-1/2+2/d}\big(\log(q v^{1/d})+2\big)\frac{dv}{v}\\
&\qquad + c_1(Q,M)\cdot\varepsilon^{-K}\bar{q}^{d} r^{d/2}(\log r+1)\\
&\ll_d c_1(Q,M)\cdot\varepsilon^{-K}\bar{q}^{d+1}(\log q+1)\, r^{d-2}. \qquad (3.36)
\end{aligned}$$

For $r \geq r_0 \stackrel{\text{def}}{=} \max\left\{\frac{2}{\pi\, q}; \frac{2}{\pi}, r_0(Q,M)\right\}$, where $r_0(Q,M)$ is a constant chosen as in Lemma 4.12 and 4.13, this bound for I_2 yields in view of (3.18), (3.22) and (3.24), that

$$\begin{aligned}
\Big|V^{\mathbb{Z}}_{1,\pm\varepsilon}(r;a,b,M) - V^{\mathbb{R}}_{1,\pm\varepsilon}(r;a,b,M)\Big| &\ll_d c_2(Q,M)\cdot\varepsilon^{-K} q_0^{-\frac{d}{2}} r^{\frac{d}{2}}\\
&\quad + c_1(Q,M)\cdot\varepsilon^{-K}\big(1+q_0^{-\frac{d}{2}} r^{\frac{d}{2}} + \bar{q}^{d+1}(\log q+1)\, r^{d-2}\big),
\end{aligned}$$

where the constants $c_1(Q,M)$ and $c_2(Q,M)$ are chosen according to Lemma 3.7 and (3.22). Setting $c(Q,M) \stackrel{\text{def}}{=} \max\{c_1(Q,M), c_2(Q,M)\}$, this proves Theorem 3.1 (1).

Proof of Theorem 3.1 (2).
In order to use nontrivial bounds for $\gamma(\kappa,\xi)$ in the irrational case we introduce further auxiliary parameters η, T such that $\frac{2}{\pi r} \leq \eta \leq T$ with $T \geq 1$ which will be determined and optimized later. Thus we may split the integral I_3 in (3.32) which bounds $|I_2|$ in (3.31) into the parts

$$\begin{aligned}
I_3 &= \int_{\frac{2}{\pi r}}^{\eta}\frac{g(t)}{M(t)^{1/2}}\,dt + \int_{\eta}^{T}\frac{g(t)}{M(t)^{1/2}}\,dt + \int_{T}^{\infty}\frac{g(t)}{M(t)^{1/2}}\,dt\\
&= I_4 + I_5 + I_6, \qquad \text{say.} \qquad (3.37)
\end{aligned}$$

We define similarly to (3.35)

$$\gamma_1 = \gamma(\frac{2}{\pi r},\eta),\ \gamma_2 = \gamma(\eta,T),\ \gamma_3 = \gamma(T,\infty), \qquad (3.38)$$

$$D_j = \max\big\{(2d)^{-d} r^d,\, \gamma_j\big\}, \qquad j = 1,2,3, \qquad (3.39)$$

$$\kappa_1(v) = \max\big\{\frac{2}{\pi r}, f(v)\big\},\ \kappa_2(v) = \max\big\{\eta, f(v)\big\},\ \kappa_3(v) = \max\big\{T, f(v)\big\}, \qquad (3.40)$$

where $f(v) = (2qvd^{1/2})^{-1},\ v > 0$. By (4.29) we have again

$$\gamma_j \geq 1, \qquad j = 1,2,3. \qquad (3.41)$$

Using (3.26) and (3.40), we see that

$$g(\kappa_j(v)) \leq 2qvd^{1/2}, \qquad j = 1,2,3. \qquad (3.42)$$

First, we apply Lemma 4.22 as above to the interval with endpoints $\kappa = \frac{2}{\pi r}$ and $\xi = \eta$. Corollary 4.17 implies that, if $\eta \geq \dfrac{\pi}{2\, q_0\, q\, r}$ the quantity γ_1 (defined by (3.33) and (3.38)) satisfies

$$\gamma_1 \geq \delta \stackrel{\text{def}}{=} (d\,q\,\eta)^{-d}, \tag{3.43}$$

since $d \geq 5$ and $\inf_{t \in [\frac{2}{\pi\, r}, \eta]} \left\{ \frac{q_0 |t| r}{2}; \frac{1}{q|t|r} \right\} = \dfrac{1}{q\,\eta\, r}$, whenever $\eta \geq \dfrac{\pi}{2\, q_0\, q\, r}$. Lemma 4.22 yields in view of (3.29), (3.30), (3.42) and (3.43) the estimate

$$\begin{aligned} I_4 \;&\ll_d\; q_0^{-1} r^{d/2-2} \int_{\gamma_1}^{D_1} v^{-1/2+1/d} \big(v^{1/d} q\, G\big(\kappa_1(v^{1/d}), \eta\big) + g\big(\kappa_1(v^{1/d})\big)\big) \frac{dv}{v} \\ &\qquad + G(\frac{2}{\pi r}, \eta) \\ &\ll_d\; q_0^{-1} q\, r^{d/2-2} \int_{\delta}^{D_1} v^{-1/2+2/d} \left(\left|\log(q\, v^{1/d} \eta)\right| + 1 \right) \frac{dv}{v} + G(\frac{2}{\pi r}, \eta) \\ &\ll_d\; q_0^{-1} q^{d/2-1} r^{d/2-2} \eta^{d/2-2} + G(\frac{2}{\pi r}, \eta), \end{aligned} \tag{3.44}$$

provided that $d > 4$, using the change of variables $v = \delta u$ in the last inequality.

In order to estimate I_5 we choose $\kappa = \eta$, and $\xi = T$. By Lemma 4.22 we obtain as above

$$\begin{aligned} I_5 \;&\ll_d\; q_0^{-1} r^{d/2-2} \int_{\gamma_2}^{D_2} v^{-1/2+1/d} \big(v^{1/d} q\, G\big(\kappa_2(v^{1/d}), T\big) + g\big(\kappa_2(v^{1/d})\big)\big) \frac{dv}{v} \\ &\qquad + G(\eta, T) \\ &\ll_d\; q_0^{-1} q\, r^{d/2-2} \int_{\gamma_2}^{D_2} v^{-1/2+2/d} \left(\left|\log(q\, v^{1/d}/w)\right| + 1 \right) \frac{dv}{v} + G(\eta, T) \\ &\ll_d\; q_0^{-1} q\, r^{d/2-2} {\gamma_2}^{-1/2+2/d} \left(\left|\log(q\gamma_2)\right| + \left|\log w\right| + 1 \right) + G(\eta, T). \end{aligned} \tag{3.45}$$

Finally for the term I_6 choose $\kappa = T$ and $\xi = \infty$ and use (3.41) for $j = 3$. Recall that we choose $T \geq 1$. Thus, similarly as above, using Lemma 4.22 and the fact, that $G(\kappa_3(v^{1/d}), \infty) \leq G(T, \infty) \leq T^{-1} w^{-1}$ and $g\big(\kappa_3(v^{1/d})\big) \leq T^{-2} w^{-1}$, we obtain (see (3.26), (3.30) and (3.40))

$$\begin{aligned} I_6 \;&\ll_d\; q_0^{-1} r^{d/2-2} \int_{1}^{D_3} v^{-1/2+1/d} \big(v^{1/d} q\, G\big(\kappa_3(v^{1/d}), \infty\big) + g\big(\kappa_3(v^{1/d})\big)\big) \frac{dv}{v} \\ &\qquad + G(T, \infty) \\ &\ll_d\; q_0^{-1} q\, r^{d/2-2} T^{-1} w^{-1} + G(T, \infty). \end{aligned} \tag{3.46}$$

Collecting (3.44)–(3.46), we get by combining the terms $G(\kappa,\xi)$ and using (3.37) and the estimates (3.41)

$$I_3 \ll_d q_0^{-1} r^{\frac{d}{2}-2}\Big\{q^{\frac{d}{2}-1}\eta^{\frac{d}{2}-2} + q\gamma_2^{-\frac{1}{2}+\frac{2}{d}}\big(\log(q\gamma_2)+|\log w|+1\big)+\frac{q}{Tw}\Big\} + G(\frac{2}{\pi r},\infty). \tag{3.47}$$

In view of (3.31) this bound for I_3 yields

$$\begin{aligned}|I_2| \ll_d\ & c_1(Q,M)\cdot\varepsilon^{-K}\cdot q^d r^{\frac{d}{2}}\big(1+\log r\big) + c_1(Q,M)\cdot\varepsilon^{-K}\cdot q_0^{-1}\bar{q}^d\cdot r^{d-2}\\ &\times\Big\{(Tw)^{-1}+q^{\frac{d}{2}-1}\eta^{\frac{d}{2}-2}+q\gamma_2^{-\frac{1}{2}+\frac{2}{d}}\big(\log(q\gamma_2)+|\log w|+1\big)\Big\}\\ \ll_d\ & c(Q,M)\cdot\varepsilon^{-K}\cdot\bar{q}^d r^{\frac{d}{2}}\big(1+\log r\big) + c(Q,M)\cdot\varepsilon^{-K}\cdot r^{d-2}\\ &\times\Big\{\frac{\bar{q}^{d+1}}{Tw}+\bar{q}^{\frac{3d}{2}}\eta^{\frac{d}{2}-2}+\bar{q}^{d+2}\gamma_2^{-\frac{1}{2}+\frac{2}{d}}\big(\log(\bar{q}\gamma_2)+|\log w|+1\big)\Big\},\end{aligned} \tag{3.48}$$

where $c(Q,M) \overset{\text{def}}{=} \max\{c_1(Q,M),c_2(Q,M)\}$. By Lemma 4.23 for η,T fixed, we have $\gamma_2\to\infty$ for $r\to\infty$ and we may now choose the auxiliary parameters η, w and T to minimize the right hand side of (3.48) as follows. Let

$$T\ge 1,\ w=T^{-1/2},\ \eta=\max\Big\{\frac{2}{\pi r},\frac{\pi}{2q_0\,q\,r},T^{-\frac{1}{d-4}}\Big\}, \tag{3.49}$$

provided that $d\ge 5$.
For $r\ge r_0 \overset{\text{def}}{=} \max\Big\{\frac{2}{\pi},\frac{\pi}{2q_0\,q},r_0(Q,M)\Big\}$, where $r_0(Q,M)$ is a constant chosen as in Lemma 4.12 and 4.13, we obtain in view of (3.18), (3.22), (3.24), (3.38), (3.41), (3.48) and (3.49) the following bound:

$$\begin{aligned}&\Big|V^{\mathbb{Z}}_{T^{-1/2},\varepsilon}(r;a,b,M)-V^{\mathbb{R}}_{T^{-1/2},\varepsilon}(r;a,b,M)\Big|\\ &\qquad\ll_d c(Q,M)\cdot\varepsilon^{-K}\Big(1+q_0^{-\frac{d}{2}}r^{\frac{d}{2}}+\bar{q}^d r^{\frac{d}{2}}(1+\log r)+r^{d-2}\cdot\rho(r,Q,T)\Big),\end{aligned}$$

where $\rho(r,Q,T)$ is defined as in (2.12). This completes the proof of Theorem 3.1 (2). □

Proof of Theorem 2.6.
The estimate (2.17) immediately follows from Corollary 4.21. This inequality ensures that there exists a $t\in[\kappa,\xi]$ such that $M_{1,t}>\tau$ whenever $c(d)\big(q\tau^2(\xi-\kappa)+\tau r^{-1}\big)<\xi-\kappa$. This condition is equivalent to

$$\tau<\left(\frac{1}{c(d)}-q\tau^2\right)(\xi-\kappa)r. \tag{3.50}$$

Due to the fact, that $\tau \le \tau_Q$, where $\tau_Q \overset{\text{def}}{=} \left(\frac{c(d)+2}{2c(d)q}\right)^{\frac{1}{2}}$, implies $\frac{1}{c(d)} - q\tau^2 \ge \frac{1}{2}$, we may conclude, that the condition (3.50) (and hence $M_{1,t} > \tau$) follows from the inequality $\tau \le \min\{\tau_Q, r(\xi - \kappa)/2\}$, which proves (2.18).

By definition of $M_{1,t}$ the inequality $M_{1,t} > \overline{\tau} \overset{\text{def}}{=} \min\{\tau_Q, r(\xi-\kappa)/2\}$ implies that if $0 < |n|_\infty < \overline{\tau} r$ then $\overline{\tau} r \, \|tQn\| > \overline{\tau}^2$. For $\nu > \tau_Q$ exists a $r \ge 1$ such that $\nu = \overline{\tau} r$. Therefore, we get by (2.16) that $D(t, v) \ge \overline{\tau}^2$. Furthermore, we have $\overline{\tau}^2 = \min\{\tau_Q^2, \nu(\xi - \kappa)/2\}$, since either $r(\xi - \kappa)/2 > \tau_Q$ and $\overline{\tau} = \tau_Q$ or $\overline{\tau} = r(\xi - \kappa)/2$ otherwise. This proves (2.19). $\square$

Proof of Theorem 2.7.
Since the cube C_r is compact the quantity

$$a_r \overset{\text{def}}{=} \min\big\{Q[x - M] : x \in C_r\big\} \tag{3.51}$$

is a well-defined real number and we obviously get

$$F_{r,M}(b) = H_{r,M}^{a_r,b}, \tag{3.52}$$

where $H_{r,M}^{a_r,b}$ is defined as in (2.3).
A careful analysis of the proof shows, that Theorem 3.1 also holds for $a = a_r, r \ge r_0$. This, together with Lemma 4.7 yields that for $K = K(d)$ chosen according to (3.8) there exist constants $c_j > 0,\ j = 1, ..., 5,$ depending on Q and d only and a constant $r_0 = r(Q, M, b) > 0$ such that, for any $r \ge r_0$, it holds (cf. proof of Theorem 2.5):

(1) $\big|\mathrm{vol}_{\mathbb{Z}} F_{r,M}(b) - \mathrm{vol}\, F_{r,M}(b)\big| \;\le\; r^{d-2} \cdot \big(c_1 \cdot (b - a_r + 1) + c_2\big).$

(2) $\big|\mathrm{vol}_{\mathbb{Z}} F_{r,M}(b) - \mathrm{vol}\, F_{r,M}(b)\big|$

$$\le\; r^{d-2} \cdot \big(c_3 \cdot (b - a_r)\, r^{-\frac{1}{K}} + c_4 \cdot (b - a_r)\, r^{-1} \,+\, c_5 \cdot \rho(r, Q)\big),$$

where $\lim_{r\to\infty} \rho(r, Q) = 0$, provided that Q is irrational.

Dividing these inequalites by the inequality in Lemma 4.3 (2) for $\xi = 1$ completes the proof of Theorem 2.7. $\square$

4. Lemmas

In the sequel, let $I = [a, b], a, b \in \mathbb{R}$ and I_0 denote finite intervals. For $M \in \mathbb{R}^d$ we consider

$$H(r) \overset{\text{def}}{=} H(r, I_0, I, M) \overset{\text{def}}{=} \big\{x \in \mathbb{R}^d \,:\, r^{-1}|x|_\infty \in I_0,\ Q[x - M] \in I\big\}. \tag{4.1}$$

The diagonal matrix $D(Q)$ is defined by

$$\big(D(Q)\big)_{i,j} \stackrel{\text{def}}{=} \begin{cases} \sqrt{|q_i|}, & \text{if } j = i, \\ 0, & \text{otherwise,} \end{cases} \qquad 1 \le i,j \le d.$$

Lemma 4.1. *Let* $I_0 = [0,\xi]$ *and* $\tau = \xi + \frac{|D(Q)M|}{r}, \sigma = q_0^{d/2}\xi - \frac{|D(Q)M|}{r}$. *For the volume of* $H(r)$ *defined in* (4.1) *it holds*

$$\text{vol } H(r) \ll_d (b-a) q_0^{-d/2} q^{(d-2)/2} \tau^{d-2} r^{d-2}.$$

If $\sigma > 0$ *and* $|a| + |b| \le \sigma^2 r^2/5$ *then*

$$\text{vol } H(r) \gg_d (b-a) q^{-d/2} \sigma^{d-2} r^{d-2}.$$

Proof. [BG99], p. 1023, Lemma 8.2 or [Els06], p. 24, Lemma 2.4.3 □

Lemma 4.2. *Let* $I_0 = [1-\delta, 1+\delta]$, $0 \le \delta \le 1/4$. *Assume that* r *is large enough, that*

$$\varepsilon_1 \stackrel{\text{def}}{=} r^{-1}|D(Q)M| \le q_0^{1/2}/4 \quad \textit{and} \quad \varepsilon_2 \stackrel{\text{def}}{=} r^{-2}\left(|a| + |b|\right) \le \frac{1}{8} q_0 \qquad (4.2)$$

holds. Then for the volume of $H(r)$ *defined in* (4.1) *it holds*

$$\text{vol } H(r) \ll_d (b-a)\left(\delta + q_0^{-1/2}\varepsilon_1 + 2q_0^{-1/2}\varepsilon_2\right) r^{d-2} q_0^{-d/2} q^{(d-2)/2}.$$

Proof. [BG99], p. 1025, Lemma 8.3 or [Els06], p. 26, Lemma 2.4.4 □

Due to the fact, that for a_r defined as in 3.51 the inequality

$$\frac{|a_r|}{r^2} \le q \qquad (4.3)$$

holds for r large enough, we obtain in the case $a = a_r$ the following lemma by slightly modifying the proof of Lemma 4.1 given in [BG99] resp. [Els06]. Using these modifications we also get an analog result as in Lemma 4.2.

Lemma 4.3. *Let* $I_r \stackrel{\text{def}}{=} [a_r, b]$. *There exist constants* $C_{Q,1}, C_{Q,2} \ge 1$ *depending on* d *and* Q *only and a constant* $r_0 = r_0(Q,M,b) \ge 1$ *such that for* $r \ge r_0$ *the volume of* $F(r) \stackrel{\text{def}}{=} H(r, I_0, I, M)$ *defined as in* (4.1) *can be estimated as follows:*

(1) *For* $I_0 = [0,\xi]$ *it holds:* $\text{vol } F(r) \le (b - a_r) \cdot C_{Q,1} \cdot \xi^{d-2} r^{d-2}$.

(2) *For* $I_0 = [0,\xi]$ *it holds:* $\text{vol } F(r) \ge (b - a_r) C_{Q,1} \cdot \xi^{d-2} r^{d-2}$

(3) *For* $I_0 = [(1-\delta), 1+\delta)], 0 \le \delta \le 1/4$ *it holds:*

$$\text{vol } F(r) \le (b - a_r) C_{Q,2} \cdot \delta \cdot r^{d-2}.$$

The constants $C_{Q,1}, C_{Q,2}$ *can be computed explicitly.*

In the sequel we want to estimate the error terms caused by the approximations of the (lattice point) volumes of the hyperbolic shell $H_{r,M}$:

In the notation of (3.5)-(3.6), considering for $\varepsilon > 0$

$$\psi_{r,\pm\varepsilon}(x) = \exp\bigl[-\frac{2}{r^2}Q_+[x]\bigr]\chi_{\pm\varepsilon}\bigl(\frac{x}{r}\bigr)$$

and

$$\Delta_{\pm\varepsilon} = \Bigl|\int_{\mathbb{R}^d} I_{H_{r,M}}(x)\psi_{r,\pm\varepsilon}(x)dx - \sum_{x\in\mathbb{Z}^d} I_{H_{r,M}}(x)\psi_{r,\pm\varepsilon}(x)\Bigr|,$$

defined as in (3.9) and (3.11), respectively, we define additionally

$$v_\varepsilon \quad \stackrel{\text{def}}{=} \quad \mathrm{vol}\Bigl(H_{r,M} \cap \bigl\{x \in \mathbb{R}^d \bigl| r(1-\varepsilon) \le |x|_\infty \le r(1+\varepsilon)\bigr\}\Bigr) \qquad (4.4)$$

and get the following estimate

Lemma 4.4. *For $0 < \varepsilon \le \frac{1}{4}$ there exists a constant $c = c(d) > 0$ such that*

$$\begin{aligned}\bigl|\mathrm{vol}_{\mathbb{Z}}\, H_{r,M} - \mathrm{vol}\, H_{r,M}\bigr| \;\; &\le \;\; \max\bigl\{\Delta_{-\varepsilon}\,;\,\Delta_\varepsilon\bigr\} \; + \; c\cdot(b-a)q_0^{-d/2}q^{(d-2)/2} \\ &\times \Bigl(\varepsilon + q_0^{-1/2}q^{1/2}|M|r^{-1} + 2q_0^{-1/2}(|a|+|b|)r^{-2}\Bigr)\, r^{d-2}. \qquad (4.5)\end{aligned}$$

Proof. Obviously, we can estimate

$$\mathrm{vol}_{\mathbb{Z}}\, H_{r,M} \;\le\; \sum_{x\in\mathbb{Z}^d} I_{H_{r,M}}(x)\psi_{r,\varepsilon}(x), \qquad \mathrm{vol}\, H_{r,M} \;\le\; \int_{\mathbb{R}^d} I_{H_{r,M}}(x)\psi_{r,-\varepsilon}(x)dx + v_\varepsilon,$$

$$\mathrm{vol}_{\mathbb{Z}}\, H_{r,M} \;\ge\; \sum_{x\in\mathbb{Z}^d} I_{H_{r,M}}(x)\psi_{r,-\varepsilon}(x), \qquad \mathrm{vol}\, H_{r,M} \ge \int_{\mathbb{R}^d} I_{H_{r,M}}(x)\psi_{r,\varepsilon}(x)dx - v_\varepsilon.$$

If $\mathrm{vol}_{\mathbb{Z}}\, H_{r,M} - \mathrm{vol}\, H_{r,M} \ge 0$ these estimates imply

$$\bigl|\mathrm{vol}_{\mathbb{Z}}\, H_{r,M} - \mathrm{vol}\, H_{r,M}\bigr| \le \Delta_{+\varepsilon} + v_\varepsilon,$$

and otherwise we obtain

$$\bigl|\mathrm{vol}_{\mathbb{Z}}\, H_{r,M} - \mathrm{vol}\, H_{r,M}\bigr| \le \Delta_{-\varepsilon} + v_\varepsilon.$$

Using Lemma 4.2 for $I_0 = [1-\varepsilon, 1+\varepsilon]$ we get since $|D(Q)M| \le q^{1/2}|M|$ that

$$v_\varepsilon \ll_d (b-a)\Bigl(\varepsilon + q_0^{-1/2}q^{1/2}|M|r^{-1} + 2q_0^{-1/2}(|a|+|b|)r^{-2}\Bigr)\, r^{d-2}q_0^{-d/2}q^{(d-2)/2},$$

which proves (4.5). ☐

Lemma 4.5. *For fixed $a, b \in \mathbb{R}, w > 0$ and the functions g defined in* (3.3) *the following holds*

(1) *There exist* $a' \in [a-w; a+w]$ *and* $b' \in [b-w; b+w]$ *such that*

$$\sum_{x\in\mathbb{Z}^d} \big(I_{[a,b]} - g_{a',b',w}\big)\big(Q[x-M]\big)\psi_{r,\pm\varepsilon}(x) = 0.$$

(2) $$\sup_{\substack{a'\in[a-w;a+w]\\ b'\in[b-w;b+w]}} \Big|\int_{\mathbb{R}^d} \big(I_{[a,b]} - g_{a',b',w}\big)\big(Q[x-M]\big)\psi_{r,\pm\varepsilon}(x)dx\Big| \ll_d 8wq_0^{-\frac{d}{2}}q^{\frac{d-2}{2}}\Big(1+\varepsilon+q^{\frac{1}{2}}\frac{|M|}{r}\Big)^{d-2}r^{d-2}.$$

Proof. The sum in (1) is finite, since $\psi_{r,\pm\varepsilon}$ has bounded support. Hence, the map

$$G : (a', b') \longmapsto \sum_{x\in\mathbb{Z}^d} \big(I_{[a,b]} - g_{a',b',w}\big)\big(Q[x-M]\big)\psi_{r,\pm\varepsilon}(x)$$

is continuous and (1) follows by the intermediate value theorem.
For all $a' \in [a-w; a+w]$ and all $b' \in [b-w; b+w]$ we can estimate

$$\Big|\big(I_{[a,b]} - g_{a',b',w}\big)\big(Q[x-M]\big)\Big| \le I_{\big([a-2w;a+2w]\,\cup\,[b-2w;b+2w]\big)}\big(Q[x-M]\big). \quad (4.6)$$

This implies

$$\begin{aligned}
&\sup_{a',b'}\Big|\int_{\mathbb{R}^d} \big(I_{[a,b]} - g_{a',b',w}\big)\big(Q[x-M]\big)\psi_{r,\pm\varepsilon}(x)dx\Big| \\
&\quad\le \int I_{\big([a-2w;a+2w]\,\cup\,[b-2w;b+2w]\big)}\big(Q[x-M]\big)\psi_{r,\pm\varepsilon}(x)dx \\
&\quad\le \int \big(I_{[a-2w;a+2w]} + I_{[b-2w;b+2w]}\big)\big(Q[x-M]\big)I_{\big[0;r(1+\varepsilon)\big]}(|x|_\infty)dx, \qquad (4.7)
\end{aligned}$$

since $\psi_{r,\pm\varepsilon}(x) \le I_{\big[0;r(1+\varepsilon)\big]}(|x|_\infty)$.
Using Lemma 4.1 with $I_0 = [0, 1+\varepsilon]$, we get by (4.7)

$$\begin{aligned}
&\sup_{a',b'}\Big|\int_{\mathbb{R}^d} \big(I_{[a,b]} - g_{a',b',w}\big)\big(Q[x-M]\big)\psi_{r,\pm\varepsilon}(x)dx\Big| \\
&\quad\ll_d 8wq_0^{-d/2}q^{(d-2)/2}\big(1+\varepsilon+r^{-1}|D(Q)M|\big)^{d-2}r^{d-2} \\
&\quad\le 8wq_0^{-d/2}q^{(d-2)/2}\big(1+\varepsilon+r^{-1}q^{1/2}|M|\big)^{d-2}r^{d-2}, \qquad (4.8)
\end{aligned}$$

which proves (2). □

Lemma 4.6. *Consider* $\Delta_{\pm\varepsilon}, \varepsilon > 0$ *defined in* (3.11)*. Then:*

$$\Delta_{\pm\varepsilon} \ll_d \sup_{\substack{a'\in[a-w;a+w]\\ b'\in[b-w;b+w]}} \Big|V^{\mathbb{R}}_{w,\pm\varepsilon}(r; a', b', M) - V^{\mathbb{Z}}_{w,\pm\varepsilon}(r; a', b', M)\Big| + 8wq_0^{-\frac{d}{2}}q^{\frac{d-2}{2}}\Big(1+\varepsilon+q^{\frac{1}{2}}\tfrac{|M|}{r}\Big)^{d-2}r^{d-2},$$

where $V^{\mathbb{R}}_{w,\pm\varepsilon}(r;a',b',M)$ *and* $V^{\mathbb{Z}}_{w,\pm\varepsilon}(r;a',b',M)$ *are defined as in* (3.6) *and* (3.5) *respectively.*

Proof. Using approximations in virtue of functions g defined in (3.3) we obtain by triangle inequality (Recall the definition of $\psi_{r,\pm\varepsilon}$ in (3.9).)

$$\begin{aligned}
\Delta_{\pm\varepsilon} &= \Big|\int_{\mathbb{R}^d} I_{H_{r,M}}(x)\psi_{r,\pm\varepsilon}(x)dx - \sum_{x\in\mathbb{Z}^d} I_{H_{r,M}}(x)\psi_{r,\pm\varepsilon}(x)\Big| \\
&\le \Big|\int_{\mathbb{R}^d}\big(I_{[a,b]} - g_{a',b',w}\big)\big(Q[x-M]\big)\psi_{r,\pm\varepsilon}(x)dx\Big| \\
&\qquad + \Big|V^{\mathbb{R}}_{w,\pm\varepsilon}(r;a',b',M) - V^{\mathbb{Z}}_{w,\pm\varepsilon}(r;a',b',M)\Big| \\
&\qquad\qquad + \Big|\sum_{x\in\mathbb{Z}^d}\big(I_{[a,b]} - g_{a',b',w}\big)\big(Q[x-M]\big)\psi_{r,\pm\varepsilon}(x)\Big|. \qquad (4.9)
\end{aligned}$$

Choosing a', b' according to Lemma 4.5 (1) and estimating the first summand by taking the supremum, we obtain

$$\begin{aligned}
\Delta_{\pm\varepsilon} &\le \sup_{\substack{a'\in[a-w;a+w]\\ b'\in[b-w;b+w]}}\Big|\int_{\mathbb{R}^d}\big(I_{[a,b]} - g_{a',b',w}\big)\big(Q[x-M]\big)\psi_{r,\pm\varepsilon}(x)dx\Big| \\
&\qquad + \sup_{\substack{a'\in[a-w;a+w]\\ b'\in[b-w;b+w]}}\Big|V^{\mathbb{R}}_{w,\pm\varepsilon}(r;a',b',M) - V^{\mathbb{Z}}_{w,\pm\varepsilon}(r;a',b',M)\Big|. \qquad (4.10)
\end{aligned}$$

The application of Lemma 4.5 (2) completes the proof. □

Repeating the proofs of Lemma 4.4, 4.5 and 4.6 in the case $a = a_r$ using Lemma 4.3 instead of Lemma 4.1 and 4.2 we get immediately

Lemma 4.7. *For* $F_{r,M}(b)$ *defined as in* (3.52) *there exist constants* $r_0 = r_0(Q,M,b) \ge 1$ *and* $c_{Q,1}, c_{Q,2} \ge 1$ *depending on* Q *and* d *only, such that for* $w > 0, 0 < \varepsilon < \frac{1}{4}$ *the following estimate holds:*

$$\begin{aligned}
\big|\operatorname{vol}_{\mathbb{Z}} F_{r,M}(b) - \operatorname{vol} F_{r,M}(b)\big| &\le \big(c_{Q,1}(b-a_r)\varepsilon + c_{Q,2}w(1+\varepsilon)^{d-2}\big)r^{d-2} \\
&\quad + \sup_{\substack{a'\in[a_r-w;a_r+w]\\ b'\in[b-w;b+w]}}\Big|V^{\mathbb{R}}_{w,\pm\varepsilon}(r;a',b',M) - V^{\mathbb{Z}}_{w,\pm\varepsilon}(r;a',b',M)\Big|,
\end{aligned}$$

where $V^{\mathbb{R}}_{w,\pm\varepsilon}(r;a',b',M)$ *and* $V^{\mathbb{Z}}_{w,\pm\varepsilon}(r;a',b',M)$ *are defined as in* (3.6) *and* (3.5) *respectively.*

Lemma 4.8. *For any* $\beta > 0, T \in \mathbb{R}$ *it holds*

$$\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\exp\{zT\}\,\frac{dz}{z^2} = \max\{T,0\} = T_+. \qquad (4.11)$$

Proof. Complement the interval $(\beta - i\,\infty, \beta + i\,\infty)$ by an infinite half circle in $\mathrm{Re}\, z \geq 0$ (resp. $\mathrm{Re}\, z \leq 0$) for $T < 0$ (resp. $T \geq 0$) and apply standard residue calculus. □

Lemma 4.9. *For a symmetric, $d \times d$ complex matrix Ω, whose imaginary part is positive definite the following holds:*

$$\sum_{m \in \mathbb{Z}^d} \exp\Big[\pi i \cdot \Omega[m] + 2\pi i \langle m, v\rangle\Big] = \Big(\det\Big(\frac{\Omega}{i}\Big)\Big)^{-\frac{1}{2}} \cdot \exp\Big[-\pi i \cdot \Omega^{-1}[v]\Big] \times \sum_{n \in \mathbb{Z}^d} \exp\Big[-\pi i \cdot \Omega^{-1}[n] + 2\pi i \langle n, \Omega^{-1} v\rangle\Big]$$

and

$$\int_{\mathbb{R}^d} \exp\big[\pi i \cdot \Omega[x] + 2\pi i \langle x, v\rangle\big] dx = \left(\det\left(\frac{\Omega}{i}\right)\right)^{-\frac{1}{2}} \cdot \exp\left[-\pi i \cdot \Omega^{-1}[v]\right],$$

where $\Omega^{-1}\big[x\big]$ denotes the quadratic form $\langle \Omega^{-1} x, x\rangle$, defined by the inverse operator $\Omega^{-1} : \mathbb{C}^d \to \mathbb{C}^d$ (which exists since Ω is an element of Siegel's upper half plane).

Proof. See [Mum83], p. 195 (5.6) and Lemma 5.8. □

Corollary 4.10. *For $z \in \mathbb{C}^d, \mathrm{Re}\, z > 0, v \in \mathbb{C}^d$ and a positive definite, symmetric $d \times d$ matrix Ω it holds*

$$\sum_{m \in \mathbb{Z}^d} \exp\big[-z\Omega[m] + 2\pi i \langle m, v\rangle\big] = \left(\det\left(z \cdot \frac{\Omega}{\pi}\right)\right)^{-\frac{1}{2}} \cdot \sum_{n \in \mathbb{Z}^d} \exp\left[-\frac{\pi^2}{z} \Omega^{-1}[n + v]\right].$$

Proof. Apply Lemma 4.9 to the matrix $\frac{i}{\pi} z \Omega$. □

Lemma 4.11. *For $z = \frac{1}{r^2} + it, r > 0, t \in \mathbb{R}$ and all $v \in \mathbb{C}^d$ it holds*

$$\begin{aligned} &\sum_{m \in \mathbb{Z}^d} \exp\Big[-\tfrac{2}{r^2} Q_+[m] - zQ[m] + 2\pi i \langle m, v\rangle\Big] \\ &= \det\Big(\frac{1}{\pi}\big(\frac{2}{r^2} Q_+ + zQ\big)\Big)^{-\frac{1}{2}} \cdot \exp\Big[-\pi^2 \big(\frac{2}{r^2} Q_+ + zQ\big)^{-1}\big[v\big]\Big] \\ &\qquad \times \sum_{n \in \mathbb{Z}^d} \exp\Big[-\pi^2 \big(\frac{2}{r^2} Q_+ + zQ\big)^{-1}\big[n\big] - 2\pi^2 \langle \big(\frac{2}{r^2} Q_+ + zQ\big)^{-1} n, v\rangle\Big], \end{aligned} \tag{4.12}$$

and

$$\int_{\mathbb{R}^d} \exp\Big[-\tfrac{2}{r^2}Q_+[x] - zQ[x] + 2\pi i\langle x, v\rangle\Big]dx$$
$$= \det\Big(\frac{1}{\pi}\big(\frac{2}{r^2}Q_+ + zQ\big)\Big)^{-\frac{1}{2}} \cdot \exp\Big[-\pi^2\big(\frac{2}{r^2}Q_+ + zQ\big)^{-1}\big[v\big]\Big], \quad (4.13)$$

where $\big(\frac{2}{r^2}Q_+ + zQ\big)^{-1}\big[x\big]$ *denotes the quadratic form* $\langle\big(\frac{2}{r^2}Q_+ + zQ\big)^{-1}x, x\rangle$, *defined by means of the positive definite operator* $\big(\frac{2}{r^2}Q_+ + zQ\big)^{-1} : \mathbb{R}^d \to \mathbb{R}^d$.

Proof. For $\Omega \overset{\mathrm{def}}{=} \frac{i}{\pi}\big(\frac{2}{r^2}Q_+ + zQ\big)$ and $z = \frac{1}{r^2} + it, t \in \mathbb{R}$ the imaginary part $\operatorname{Im}\Omega$ is positive definite. The application of Lemma 4.9 to Ω completes the proof. □

Lemma 4.12. *Let* $\theta_v(z)$ *and* $\theta_{0,v}(z)$ *denote the theta sum and theta integral in* (3.16) *and* (3.17) *respectively. Then there is a constant* $c = c(Q) > 0$, *such that for* $r \geq r_0 = r_0(Q, M) \geq 1$ *and* $t \in \mathbb{R}$, $|t| < \frac{1}{r}$, *the following bound holds*

$$\big|\big(\theta_v - \theta_{0,v}\big)(r^{-2} + it)\big| \ll_d q_0^{-\frac{d}{2}}|r^{-2} + it|^{-\frac{d}{2}} \exp\big[-c \cdot \operatorname{Re}\big((r^{-2} + it)^{-1}\big)\big] + 2I_{(r,\infty)}(|v|).$$

Proof. Using Lemma 4.11 we obtain by (3.16), (3.17) and the self-adjointness of the matrix $\big(\frac{2}{r^2}Q_+ zQ\big)^{-1}$, that

$$\big(\theta_v - \theta_{0,v}\big)(z)$$
$$= \exp\big[-zQ[M]\big]\det\Big(\frac{1}{\pi}\Omega\Big)^{-\frac{1}{2}} \cdot \exp\Big[-\pi^2\Omega^{-1}\big[-\frac{\tilde{v}}{2\pi r}\big]\Big]$$
$$\times \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \exp\Big[-\pi^2\Omega^{-1}\big[n\big] - 2\pi^2\langle\Omega^{-1}n, -\frac{\tilde{v}}{2\pi r}\rangle\Big]$$
$$= \exp\big[-zQ[M]\big]\det\Big(\frac{1}{\pi}\Omega\Big)^{-\frac{1}{2}} \cdot \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \exp\Big[-\Omega^{-1}\big[\pi n - \frac{\tilde{v}}{2r}\big]\Big], \quad (4.14)$$

where $\Omega \overset{\mathrm{def}}{=} \big(\frac{2}{r^2}Q_+ + zQ\big)$ and $\tilde{v} \overset{\mathrm{def}}{=} 2r\operatorname{Im}(z)QM - v$. Note that for $z = r^{-2} + it$ and $t \leq \frac{1}{r}$ there exists a constant $c_0 = c_0(Q, M) > 0$ such that $|2r\operatorname{Im}(z)QM| \leq c_0$ uniformly in r.
Using $\det\big(\frac{1}{\pi}\Omega\big) = \frac{1}{\pi^d}\prod_{1\leq j\leq d}\big(\frac{2}{r^2}|q_j| + zq_j\big)$ and $\Big|\frac{2}{r^2}|q_j| + zq_j\Big| \geq |q_j| \cdot \big|r^{-2} + it\big|$

for $z = r^{-2} + it$ and all $1 \le j \le d$, we have

$$\left|\det\left(\frac{1}{\pi}\Omega\right)^{-\frac{1}{2}}\right| \le \pi^{\frac{d}{2}} \cdot q_0^{-\frac{d}{2}} \cdot |z|^{-\frac{d}{2}}. \tag{4.15}$$

Since Ω can be orthogonal diagonalized, the matrix $\mathrm{Re}\,(\Omega^{-1})$ has eigenvalues $\mathrm{Re}\left(\left(\frac{2}{r^2}|q_j| + zq_j\right)^{-1}\right)$, $1 \le j \le d$. For $t \le \frac{1}{r}$ we have

$$\mathrm{Re}\left(\left(\frac{2}{r^2}|q_j| + zq_j\right)^{-1}\right) \ge \frac{1}{|q_j|}\,\mathrm{Re}(z^{-1}) \ge \frac{1}{q}\,\mathrm{Re}(z^{-1}), \quad 1 \le j \le d.$$

Hence,

$$\begin{aligned}\left|\exp\left[-\pi^2\Omega^{-1}\left[n - \frac{\tilde{v}}{2\pi r}\right]\right]\right| &= \exp\left[-\mathrm{Re}\left(\Omega^{-1}\left[\pi n - \frac{\tilde{v}}{2r}\right]\right)\right] \\ &= \exp\left[-\mathrm{Re}\left(\Omega^{-1}\right)\left[\pi n - \frac{\tilde{v}}{2r}\right]\right] \\ &\le \exp\left[-\frac{1}{q}\,\mathrm{Re}(z^{-1}) \cdot \left|\pi n - \frac{\tilde{v}}{2r}\right|^2\right]. \end{aligned} \tag{4.16}$$

Using (4.14), (4.15) and (4.16) we get

$$\begin{aligned}\left|\left(\theta_v - \theta_{0,v}\right)(r^{-2} + it)\right| \quad &\ll_d \quad \exp\left[-\tfrac{1}{r^2}Q[M]\right]\, q_0^{-\frac{d}{2}}|r^{-2} + it|^{-\frac{d}{2}} \\ &\times \sum_{n \in \mathbb{Z}^d\setminus\{0\}} \exp\left[-\frac{1}{q}\,\mathrm{Re}\left((r^{-2} + it)^{-1}\right) \cdot \left|\pi n - \frac{\tilde{v}}{2r}\right|^2\right]. \end{aligned} \tag{4.17}$$

For $|\tilde{v}| \le \pi r$ we obtain

$$\exp\left[-\frac{1}{q}\,\mathrm{Re}\left((r^{-2} + it)^{-1}\right) \cdot \left|\pi n - \frac{\tilde{v}}{2r}\right|^2\right] \le \exp\left[-\frac{1}{q}\,\mathrm{Re}\left((r^{-2} + it)^{-1}\right) \cdot \frac{|\pi n|^2}{2}\right]$$

and hence, for an appropriate constant $c = c(Q) > 0$

$$\begin{aligned}&\sum_{n \,\in\, \mathbb{Z}^d\setminus\{0\}} \exp\left[-\frac{1}{q}\,\mathrm{Re}\left((r^{-2} + it)^{-1}\right) \cdot \left|\pi n - \frac{\tilde{v}}{2r}\right|^2\right] \\ &\qquad \le \sum_{n \in \mathbb{Z}^d\setminus\{0\}} \exp\left[-\frac{1}{q}\,\mathrm{Re}\left((r^{-2} + it)^{-1}\right) \cdot \frac{|\pi n|^2}{2}\right] \\ &\qquad \ll \exp\left[-c \cdot \mathrm{Re}\left((r^{-2} + it)^{-1}\right)\right]. \end{aligned} \tag{4.18}$$

For $|\tilde{v}| > \pi r$ set $\tilde{v} = L\pi r + w$, with $L \in \mathbb{Z}, |w| \le \pi r$, then $w = \widetilde{v'}$ for $v' \stackrel{\mathrm{def}}{=} v + L\pi r$. By (4.14) we have obviously $\theta_v = \theta_{v'}$ and therefore we get by (4.18) and (4.17) the inequality

$$
\begin{aligned}
&\bigl|\bigl(\theta_v - \theta_{0,v}\bigr)(r^{-2}+it)\bigr| \\
&\quad\le\ \bigl|\bigl(\theta_{v'} - \theta_{0,v'}\bigr)(r^{-2}+it)\bigr| + \bigl|\bigl(\theta_{0,v'} - \theta_{0,v}\bigr)(r^{-2}+it)\bigr| \\
&\quad\ll_d\ \exp\bigl[-\frac{1}{r^2}Q[M]\bigr]\ q_0^{-\frac{d}{2}}|r^{-2}+it|^{-\frac{d}{2}}\cdot\exp\Bigl[-c\cdot\mathrm{Re}\bigl((r^{-2}+it)^{-1}\bigr)\Bigr] \\
&\qquad\qquad\qquad + \ \bigl|\theta_{0,v'}(r^{-2}+it)\bigr| + \bigl|\theta_{0,v}(r^{-2}+it)\bigr| \\
&\quad\le\ \exp\bigl[-\frac{1}{r^2}Q[M]\bigr]\ q_0^{-\frac{d}{2}}|r^{-2}+it|^{-\frac{d}{2}}\cdot\exp\Bigl[-c\cdot\mathrm{Re}\bigl((r^{-2}+it)^{-1}\bigr)\Bigr] + 2.
\end{aligned}
\tag{4.19}
$$

The result now follows by (4.18), (4.17) and (4.19) for $r \ge r_0$, $r_0 \ge 1$ large enough, since $|\tilde v| > \pi r$ implies $|v| \ge \pi r - c_0(Q,M) \ge r$ for r large enough. □

Lemma 4.13. *Let $\theta_v(z)$ denote the theta function in* (3.16) *depending on Q and $v \in \mathbb{C}^d$. For $r \ge r_0 = r_0(Q,M) \ge 1$, $t \in \mathbb{R}$, the following bound holds*

$$
\begin{aligned}
\bigl|\theta_v(r^{-2}+it)\bigr| &\ll_d (\det\Omega)^{-1/4} r^{d/2}\,\psi(r,t)^{1/2}, \quad \textit{where} && (4.20)\\
\psi(r,t) &= \sum_{m,n\in\mathbb{Z}^d} \exp\Bigl\{-\frac{r^2}{2}\Omega^{-1}[\pi m - 2t\,Qn] - \frac{2}{r^2}\Omega[n]\Bigr\},
\end{aligned}
$$

with $\Omega \overset{\text{def}}{=} 2\cdot Q_+ + Q$.

Note that the right hand side of this inequality is independent of $v \in \mathbb{C}^d$.

Proof. For any $x, y \in \mathbb{R}^d$ the equalities

$$ 2\,(\Omega[x] + \Omega[y]) = \Omega[x+y] + \Omega[x-y], \tag{4.21} $$

$$ \langle \Omega(x+y),\, x-y\rangle = \Omega[x] - Q[y] \tag{4.22} $$

hold. Rearranging $\theta_v(z)\,\overline{\theta_v(z)}$ and using (4.22), we would like to use $m+n$ and $m-n$ as new summation variables on a lattice. But both vectors have the same parity, i.e., $m+n \equiv m-n \mod 2$. Since they are dependent one has to consider the 2^d sublattices indexed by $\alpha = (\alpha_1,\dots,\alpha_d)$ with $\alpha_j = 0,1$, for $1 \le j \le d$:

$$ \mathbb{Z}^d_\alpha \overset{\text{def}}{=} \{m \in \mathbb{Z}^d : m \equiv \alpha \mod 2\}, $$

where, for $m = (m_1, \dots, m_d)$, $m \equiv \alpha \mod 2$ means $m_j \equiv \alpha_j \mod 2$, $1 \le j \le d$. Thus writing

$$\theta_{v,\alpha}(z) \stackrel{\text{def}}{=} \sum_{m \in \mathbb{Z}^d_\alpha} \exp\left[-\frac{2}{r^2} Q_+[m] - z \cdot Q[m] - i \cdot \langle m, \frac{v}{r} - 2tQM\rangle\right],$$

we obtain $\theta_v(z) = \exp\bigl[-zQ[M]\bigr] \sum_\alpha \theta_{v,\alpha}(z)$ and hence by the Cauchy–Schwarz inequality

$$\bigl|\theta_v(z)\bigr|^2 \le 2^d \exp\bigl[-\frac{2}{r^2} Q[M]\bigr] \sum_\alpha \bigl|\theta_{v,\alpha}(z)\bigr|^2. \tag{4.23}$$

Using (4.22) and the absolute convergence of $\theta_\alpha(z)$, we may rewrite the quantity $\theta_{v,\alpha}(z)\,\overline{\theta_{v,\alpha}(z)}$ for $z = \frac{1}{r^2} + it$ and $\tilde{v} \stackrel{\text{def}}{=} v - 2trQM$ as

$$\begin{aligned}
&\theta_{v,\alpha}(z)\,\overline{\theta_{v,\alpha}(z)} \\
&\quad = \sum_{m,n \in \mathbb{Z}^d_\alpha} \exp\left[-\frac{1}{r^2}\bigl(\Omega[m] + \Omega[n]\bigr) - it \cdot \bigl(Q[m] - Q[n]\bigr) - i \cdot \langle m - n, \frac{\tilde{v}}{r}\rangle\right] \\
&\quad = \sum_{m,n \in \mathbb{Z}^d_\alpha} \exp\left[-\frac{2}{r^2}\bigl(\Omega[\overline{m}] + \Omega[\overline{n}]\bigr) - 2i \cdot \bigl\langle 2t \cdot Q\overline{m} + \frac{\tilde{v}}{r}, \overline{n}\bigr\rangle\right]
\end{aligned} \tag{4.24}$$

where $\overline{m} = \dfrac{m+n}{2}$, $\overline{n} = \dfrac{m-n}{2}$.

Note that the map $H : \bigcup_\alpha \mathbb{Z}^d_\alpha \times \mathbb{Z}^d_\alpha \to \mathbb{Z}^d \times \mathbb{Z}^d, (m,n) \longmapsto \Bigl(\dfrac{m+n}{2}, \dfrac{m-n}{2}\Bigr)$ is a bijection. Therefore we get by (4.23)

$$\begin{aligned}
&\exp\bigl[\tfrac{2}{r^2} Q[M]\bigr] \cdot \bigl|\theta_v(z)\bigr|^2 \\
&\quad \ll_d \sum_{\alpha \in \{0,1\}^d} \sum_{\overline{m}, \overline{n} \in \mathbb{Z}^d_\alpha} \exp\left[-\frac{2}{r^2}\bigl(\Omega[\overline{m}] + \Omega[\overline{n}]\bigr) - 2i \cdot \bigl\langle 2t \cdot Q\overline{m} + \frac{\tilde{v}}{r}, \overline{n}\bigr\rangle\right] \\
&\quad = \sum_{\overline{m}, \overline{n} \in \mathbb{Z}^d} \exp\left[-\frac{2}{r^2}\bigl(\Omega[\overline{m}] + \Omega[\overline{n}]\bigr) - 2i \cdot \bigl\langle 2t \cdot Q\overline{m} + \frac{\tilde{v}}{r}, \overline{n}\bigr\rangle\right].
\end{aligned} \tag{4.25}$$

In this double sum fix $\overline{n}$ and sum over $\overline{m} \in \mathbb{Z}^d$ first. Using Corollary 4.10 for $z = \frac{2}{r^2}$, we get for $\delta \stackrel{\text{def}}{=} \bigl(\det\bigl(\frac{2}{\pi r^2} \cdot \Omega\bigr)\bigr)^{-\frac{1}{2}}$ by the symmetry of Q

$$\begin{aligned}
\theta_v(z, \overline{n}) &\stackrel{\text{def}}{=} \sum_{\overline{m} \in \mathbb{Z}^d} \exp\left[-\frac{2}{r^2}\bigl(\Omega[\overline{m}] + \Omega[\overline{n}]\bigr) - 2i \cdot \bigl\langle 2t \cdot Q\overline{m} + \frac{\tilde{v}}{r}, \overline{n}\bigr\rangle\right] \\
&= \delta \sum_{m \in \mathbb{Z}^d} \exp\left[-\frac{r^2}{2} \Omega^{-1}[\pi m - 2t\,Q\overline{n}] - \frac{2}{r^2}\Omega[\overline{n}] - 2i\langle \frac{\tilde{v}}{r}, \overline{n}\rangle\right].
\end{aligned}$$

Thus,

$$\big|\theta_v(z,\overline{n})\big| \le \delta \sum_{m\in\mathbb{Z}^d} \exp\Big\{-\frac{r^2}{2}\,\Omega^{-1}[\pi m - 2t\,Q\overline{n}] - \frac{2}{r^2}\Omega[\overline{n}]\Big\}. \tag{4.26}$$

Hence, we obtain by (4.25) and (4.26)

$$\big|\theta_v(z)\big|^2 \ll_d \exp\big[-\tfrac{2}{r^2}Q[M]\big]\ \ (\det\Omega)^{-1/2}\, r^d$$

$$\times \sum_{m,\overline{n}\in\mathbb{Z}^d} \exp\Big\{-\frac{r^2}{2}\,\Omega^{-1}[\pi m - 2t\,Q\overline{n}] - \frac{2}{r^2}\,\Omega[\overline{n}]\Big\},$$

which proves Lemma 4.13 for $r > r_0 = r_0(Q,M) \ \stackrel{\mathrm{def}}{=}\ |Q[M]|^{1/2} + 1$. □

In the following we shall use some facts in the geometry of numbers (see [Dav58]).

Let $F : \mathbb{R}^d \to [0,\infty)$ denote a norm on $\mathbb{R}^d$, that is $F(\alpha x) = |\alpha|\, F(x)$, for $\alpha \in \mathbb{R}$, and $F(x+y) \le F(x) + F(y)$. The successive minima $M_1 \le \cdots \le M_d$ of F with respect to the lattice $\mathbb{Z}^d$ are defined as follows: Let $M_1 = \inf\big\{F(m) : m \ne 0, m\in\mathbb{Z}^d\big\}$ and define M_k as the infimum of $\lambda > 0$ such that the set $\big\{m\in\mathbb{Z}^d : F(m) < \lambda\big\}$ contains k linearly independent vectors. It is easy to see that these infima are attained, that is there exist linearly independent vectors $a_1,\dots,a_d \in \mathbb{Z}^d$ such that $F(a_j) = M_j$.

Lemma 4.14. *Let $L_j(x) = \sum_{k=1}^d q_{jk}\, x_k$, $1 \le j \le d$, denote linear forms on $\mathbb{R}^d$ such that $q_{jk} = q_{kj}$, $j,k = 1,\dots,d$. Assume that $r \ge 1$ and let $\|v\|$ denote the distance of the number v to the nearest integer. Then the number of $m = (m_1,\dots,m_d) \in \mathbb{Z}^d$ such that*

$$\|L_j(m)\| < r^{-1},\ |m_j| < r, \qquad \textit{for all}\ \ 1 \le j \le d,$$

is bounded from above by $c_d\,(M_1\cdots M_d)^{-1}$, where $c_d > 0$ denotes a constant depending on d only, $M_1 \le \cdots \le M_d$ are the first d of the $2d$ successive minima $M_1 \le \cdots \le M_{2d}$ of the norm $F : \mathbb{R}^{2d} \to [0,\infty)$ defined for vectors $y = (x,\overline{x}) \in \mathbb{R}^{2d}$, $x,\overline{x}\in\mathbb{R}^d$, $\overline{x} = (\overline{x}_1,\dots,\overline{x}_d)$, as

$$F(y) \ \stackrel{\mathrm{def}}{=}\ \max\big\{r\,|\,L_1(x) - \overline{x}_1\,|,\dots, r\,|\,L_d(x) - \overline{x}_d\,|,\ r^{-1}\,|x|_\infty\big\}. \tag{4.27}$$

Moreover,

$$\frac{1}{2d} \le M_k\, M_{2d+1-k} \le (2d)^{2d-1}, \qquad 1 \le k \le 2d. \tag{4.28}$$

Proof. [Dav58], (20), p. 113, Lemma 3. □

Note that for some constant, say $c(d) > 0$, depending on d only

$$r^{-1} \le M_1 \le \cdots \le M_d \le c(d), \tag{4.29}$$

where the first inequality is obvious by $F(m, \overline{m}) \ge r^{-1}|m|_\infty$. If here $m = 0$ then $\overline{m} \ne 0$ and $F(m, \overline{m}) = r\,|\overline{m}|_\infty \ge r^{-1}|\overline{m}|_\infty \ge r^{-1}$. Finally, $M_d \ll_d 1$ follows from (4.28) for $k = d$.

In the following we shall consider linear forms

$$L_j(x) = \sum_{k=1}^{d} t\, q_{jk}\, x_k, \qquad 1 \le j \le d, \tag{4.30}$$

where $Q = (q_{ij})$, $i, j = 1, \dots, d$, denotes the components of the matrix Q and where $t \in \mathbb{R}$ is arbitrary. We denote the corresponding successive minima of the norm $F(\cdot)$ defined by (4.27) and (4.30) for fixed t by $M_{j,t}$, $j = 1, \dots, d$. Thus, we can write

$$M_{j,t} = \big|L(m, n, t)\big|_\infty, \tag{4.31}$$

for some $m, n \in \mathbb{Z}^d$, where

$$L(m, n, t) = \big(r\,(m_1 - t(Qn)_1), \dots, r\,(m_d - t(Qn)_d),\, r^{-1}n_1, \dots, r^{-1}n_d\big). \tag{4.32}$$

It is easy to see from the definition that

$$M_{j,t} = M_{j,-t}, \qquad j = 1, \dots, d,\ t \in \mathbb{R}. \tag{4.33}$$

Lemma 4.15. *Let* $r \ge 1$. *Then*

$$\Big|\theta(r^{-2} + i\,t\frac{\pi}{2})\Big| \ll_d q_0^{-\frac{3d}{4}}\, r^{d/2}\, (M_{1,t} \cdots M_{d,t})^{-1/2}.$$

Proof. By Lemma 4.13 we need to estimate the theta series $\psi(r, t\pi/2)$. Since the matrix $\Omega = 2Q_+ + Q$ is positive definite we may use the inequalities $\Omega^{-1}[x] \ge \frac{1}{3q}|x|_\infty^2$ and $\Omega[x] \ge q_0|x|_\infty^2$, and we get with $c_Q = \min\Big\{\frac{\pi^2}{6\,q}, 2q_0\Big\}$

$$\psi(r, t\,\frac{\pi}{2}) \ll_d \sum_{m,n \in \mathbb{Z}^d} \exp\big\{-c_Q\, |L(m, n, t)|_\infty^2\big\}, \tag{4.34}$$

where $L(m, n, t)$ is defined in (4.32). Let

$$H \stackrel{\text{def}}{=} \big\{(m, n) \in \mathbb{Z}^{2d} \,:\, |L(m, n, t)|_\infty < 1\big\}.$$

Now, Lemma 4.14 may be restated for the forms (4.30) as

$$\#H \ll_d (M_{1,t} \cdots M_{d,t})^{-1}. \tag{4.35}$$

In order to bound $\psi(r, t\pi/2)$, we introduce for $k \overset{\text{def}}{=} (k_1, \dots, k_{2d}) \in \mathbb{Z}^{2d}$ the sets

$$B_k \overset{\text{def}}{=} \Big[k_1 - \frac{1}{2}, k_1 + \frac{1}{2}\Big) \times \cdots \times \Big[k_{2d} - \frac{1}{2}, k_{2d} + \frac{1}{2}\Big) \quad \text{and}$$

$$H_k \overset{\text{def}}{=} \big\{(m, n) \in \mathbb{Z}^{2d} : L(m, n, t) \in B_k\big\}$$

such that $\mathbb{R}^{2d} = \bigcup_k B_k$. For any fixed $(m^*, n^*) \in H_k$ we have

$$(m - m^*, n - n^*) \in H \qquad \text{for any } (m, n) \in H_k.$$

Hence, we conclude for any $k \in \mathbb{Z}^{2d}$

$$\#H_k \le \#H \ll_d (M_{1,t} \cdots M_{d,t})^{-1}. \tag{4.36}$$

Since $x \in B_k$ implies $|x|_\infty \ge |k|_\infty/2$, we obtain by (4.34) and (4.36)

$$\begin{aligned} \psi(r, t\pi/2) &\ll_d \#H_0 + \sum_{k \in \mathbb{Z}^{2d} \setminus 0} \sum_{m,n \in \mathbb{Z}^{2d}} \mathbb{I}\big\{L(m, n, t) \in B_k\big\} \exp\big\{-c_Q |k|_\infty^2/4\big\} \\ &\ll_d (M_{1,t} \cdots M_{d,t})^{-1} \sum_{k \in \mathbb{Z}^{2d}} \exp\big\{-c_Q |k|_\infty^2/4\big\} \\ &\ll_d (M_{1,t} \cdots M_{d,t})^{-1} (c_Q^{-1/2} + 1)^{2d}, \end{aligned}$$

using similar bounds as in (4.18). Some simple bounds together with Lemma 4.13 finally conclude the proof of Lemma 4.15. □

In the following we consider an arbitrary, real, symmetric, non-degenerate $d^* \times d^*$ - matrix Q^*. The norm on $\mathbb{R}^{d^*}$, associated by (4.32), and the associated successive minima are denoted by L^* and $M^*_{j,t}$, $1 \le j \le d^*$, respectively.

Lemma 4.16. *Let $(m, n), (m', n') \in \mathbb{Z}^{2d^*} \setminus 0$; $t, t' \in \mathbb{R}$ and $r \ge 1$. Let $M \overset{\text{def}}{=} |L^*(m, n, t)|_\infty$ and $M' \overset{\text{def}}{=} |L^*(m', n', t')|_\infty$. Assume that $\langle Q^* n, n'\rangle > 0$ and*

$$\max\{M, M'\} \le (4\, d^*)^{-1/2}. \tag{4.37}$$

Then for

$$\Delta = \Delta(m, n; m', n') \overset{\text{def}}{=} \big|\langle n', m\rangle - \langle m', n\rangle\big| \tag{4.38}$$

the following holds:

$$\begin{aligned} &\text{i)} \quad \Delta = 0 \;\Rightarrow\; |t - t'| \le \frac{(d^*)^{1/2} \max\{M, M'\}\, (|n| + |n'|)}{r\, \langle Q^* n, n'\rangle}, \\ &\text{ii)} \quad \Delta \ne 0 \;\Rightarrow\; |t - t'| \ge \langle Q^* n, n'\rangle^{-1}/2. \end{aligned} \tag{4.39}$$

In particular, assuming $n = n'$ *and* (4.37) *the alternative* i) *in* (4.39) *holds. Furthermore, assuming* $(m, n) \in \mathbb{Z}^{2d^*} \backslash 0$ *and* $M = |L^*(m, n, t)|_\infty \le (4\, d^*)^{-1/2}$ *we have either*

$$\text{i)} \quad |t| \le \frac{2\, d^*\, M\, |n|}{r\, |Q^* n|} \quad \textit{or} \quad \text{ii)} \quad |t| \ge \frac{1}{2\, |Q^* n|}. \tag{4.40}$$

This means t, t' *resp.* $t, 0$ *have to be either 'near' to each other or 'far' apart.*

Proof. [Göt04], p. 217, Lemma 3.6 or [Els06], p. 38, Lemma 2.4.17 □

The application of Lemma 4.16 to $Q^* = Q$ yields the following

Corollary 4.17. *Let* $r \ge 1$ *and* $d \ge 4$. *Then*

$$M_{1,t} \cdots M_{d,t} \ge d^{-d} \Big(\min\Big\{ \frac{q_0 |t|\, r}{2}, \, \frac{1}{q\, |t|\, r} \Big\} \Big)^d. \tag{4.41}$$

Proof. Since $|Q n| = |Q_+ n|$ we have $|Q n| \ge q_0 |n|$, and $|n| \ge q^{-1} |Q n|$. In the case, where $M_{j,t} \le (4\, d)^{-1/2}$ we obtain by (4.40), $|n|_\infty \le r\, M_{j,t}$ and $2\, d^{1/2} \le d$:

$$\begin{aligned} &\text{i)} \qquad |t|\, r\, d^{-1} q_0 \le |t|\, r\, d^{-1} \frac{|Q n|}{|n|} \le 2\, M_{j,t} \\ \text{or} & \\ &\text{ii)} \qquad \frac{1}{|t|} \le 2\, |Q n| \le 2 q\, |n| \le 2\, d^{1/2} q\, |n|_\infty \le q\, d\, r\, M_{j,t}, \end{aligned} \tag{4.42}$$

for appropriate $(m, n) \in \mathbb{Z}^{2d}$ depending on j such that $M_{j,t} = |L(m, n, t)|_\infty$. Note that if $M_{j,t} \ge (4\, d)^{-1/2}$, then $M_{j,t} \ge d^{-1}$ since $d \ge 4$. Combined with (4.42), this proves Corollary 4.17 since

$$\min\Big\{ \frac{q_0 |t|\, r}{2}, \, \frac{1}{q\, |t|\, r} \Big\} \le 1$$

(recall that $q_0 \le q$). □

In the following two Lemmas we will additionally assume that the matrix Q^* is positive definite. The smallest and the largest eigenvalue of Q^* is denoted by q_0^* and q^* respectively.

Lemma 4.18. *Let* $[\kappa, \xi] \subset \mathbb{R}$, $0 < \kappa < \xi < \infty$. *Define for* $g \in C^1[\kappa, \xi]$ *such that* $g \ge 0$, $g' \le 0$ *on* $[\kappa, \xi]$,

$$H_{\kappa,\xi}(\tau) \stackrel{\text{def}}{=} H_{\kappa,\xi,Q^*}(\tau) \stackrel{\text{def}}{=} \int_\kappa^\xi \mathbb{I}\{ M^*_{1,t} \le \tau \}\, g(t)\, dt. \tag{4.43}$$

Then, for all

$$\kappa > (q_0^* r)^{-1}, \qquad r^{-1} \le \tau \le (2d^*)^{-1}, \tag{4.44}$$

we have

$$H_{\kappa,\xi}(\tau) \quad \ll_{d^*} \quad \overline{H}_{\kappa,\xi}(\tau) \overset{\text{def}}{=} \frac{q^*}{q_0^*}\,\tau^2 \int_{\kappa(\tau r)}^{\xi} g(t)\,dt + \frac{1}{q_0^*}\,\frac{\tau}{r}\,g(\kappa(\tau r)), \tag{4.45}$$

where $\kappa(v) = \max\{\kappa, (2q^* v\, d^{1/2})^{-1}\}$, *provided that* $\kappa(\tau r) \le \xi$. *In the case where* $\kappa(\tau r) > \xi$, *we have* $H_{\kappa,\xi}(\tau) = 0$.

Proof. [Göt04], p. 219, Lemma 3.8 □

For indicator functions g Lemma 4.18 reads as follows.

Lemma 4.19. *Let* λ *denote the Lebesgue measure. There exists a constant* $c(d^*)$ *depending on* d^* *only such that for any* $r \ge 1$, $\tau > 0$ *and any interval* $[\kappa, \xi]$ *with* $\xi > \kappa$ *the following holds:*

$$I(\tau) \overset{\text{def}}{=} \lambda\{t \in [\kappa,\xi] \,:\, M_{1,t}^* \le \tau\} \le c(d^*) \left(\frac{q^*}{q_0^*}\,\tau^2\,(\xi - \kappa) + \frac{1}{q_0^*}\,\tau\, r^{-1} \right).$$

Proof. [Göt04], p. 222, Lemma 3.9 □

We now return to general (not necessary positive definite) non-degenerate, symmetric, real $d \times d$ - matrix Q, to the corresponding norm L (see (4.32)) and the associated successive minima $M_{j,t}$(see (4.31)).

In the sequel we will assume, that Q is a *block-type* matrix, that is, that there exist positive definite matrices $Q^+ \in \mathrm{GL}(\mathbb{R}^{d^+})$, $Q^- \in \mathrm{GL}(\mathbb{R}^{d^-})$, $d^+ + d^- \ge 5$ with

$$Q = \begin{pmatrix} Q^+ & 0 \\ 0 & -Q^- \end{pmatrix}.$$

We denote the corresponding successive minima of the norm $F^{\pm}(\cdot)$, defined by the analogon of (4.27) and (4.30) for $Q^{\pm}$, for a fixed t by $M_{j,t}^{\pm}$, $j = 1, \dots, d^{\pm}$. Thus, we can write

$$M_{j,t}^{\pm} = \left| L^{\pm}(m,n,t) \right|_{\infty}, \tag{4.46}$$

for some $m, n \in \mathbb{Z}^{d^{\pm}}$, where

$$L^{\pm}(m,n,t) = \left(r\,(m_1 - t\,(Q^{\pm} n)_1), \dots, r\,(m_{d^{\pm}} - t\,(Q^{\pm} n)_{d^{\pm}}), \frac{1}{r}\,n_1, \dots, \frac{1}{r}\,n_{d^{\pm}} \right).$$

As in (4.33) we have

$$M_{j,t}^{\pm} = M_{j,-t}^{\pm}, \qquad j = 1, \dots, d^{\pm},\ t \in \mathbb{R}. \tag{4.47}$$

In this special case there is a simple relation between the first successive minimum of Q and those of Q^+ and Q^-.

Lemma 4.20. *For $t \in \mathbb{R}$ holds*

$$M_{1,t} \geq \min\left\{M^+_{1,t}, M^-_{1,t}\right\}. \tag{4.48}$$

In particular, for $\tau \in \mathbb{R}$,

$$\mathbb{I}\{M_{1,t} \leq \tau\} \;\leq\; \mathbb{I}\{M^+_{1,t} \leq \tau\} + \mathbb{I}\{M^-_{1,t} \leq \tau\}.$$

Proof. Choose $(m, n) = \left(\binom{m_+}{m_-}, \binom{n_+}{n_-}\right) \in \mathbb{Z}^d \setminus 0$ such that $M_{1,t} = |L(m, n, t)|_\infty$. It is easy to see, that

$$M_{1,t} = |L(m, n, t)|_\infty = \max\left\{\left|L^+(m_+, n_+, t)\right|_\infty, \left|L^-(m_-, n_-, -t)\right|_\infty\right\}.$$

Since $(m, n) \neq 0$, it follows $(m_+, n_+) \neq 0$ or $(m_-, n_-) \neq 0$ and hence by (4.47),

$$\left|L^+(m_+, n_+, t)\right|_\infty \geq M^+_{1,t} \quad \text{or} \quad \left|L^-(m_-, n_-, -t)\right|_\infty \geq M^-_{1,-t} = M^-_{1,t}.$$

This proves (4.48). □

Corollary 4.21. *Again, λ denotes the Lebesgue measure. Then there exists a constant $c = c(d) > 1$ depending on d only, such that for any $r \geq 1$, $\tau > 0$ and any interval $[\kappa, \xi]$ with $\xi > \kappa$ the following holds:*

$$I(\tau) \overset{\text{def}}{=} \lambda\{t \in [\kappa, \xi] : M_{1,t} \leq \tau\} \leq c \cdot \left(\frac{q}{q_0}\,\tau^2\,(\xi - \kappa) + \frac{1}{q_0}\,\tau\, r^{-1}\right).$$

Proof. Using Lemma 4.19 and Lemma 4.20 we obtain

$$\begin{aligned} I(\tau) \;&\leq\; \int \mathbb{I}\{M^+_{1,t} \leq \tau\} + \mathbb{I}\{M^-_{1,t} \leq \tau\}\, \lambda(dt) \\ &\leq\; \big(c(d^+) + c(d^-)\big)\left(\frac{q}{q_0}\,\tau^2\,(\xi - \kappa) + \frac{1}{q_0}\,\tau\, r^{-1}\right), \end{aligned}$$

where we have used, that q (resp. q_0) is larger (resp. smaller) than the corresponding largest (resp. smallest) eigenvalue of Q^+ and Q^-. Taking $c \overset{\text{def}}{=} \max_{\substack{d^+, d^- \in \mathbb{N} \\ d^+ + d^- = d}} \big(c(d^+) + c(d^-)\big)$ completes the proof. □

Lemma 4.22. *Let $M(t) = M_{1,t} \cdots M_{d,t}$, $\gamma = \gamma(\kappa, \xi) = r^d \inf_{\kappa \leq t \leq \xi} M(t)$ and introduce*

$$D = \max\big\{(2d)^{-d} r^d,\, \gamma\big\} \quad \textit{and} \quad G(\kappa, \xi) = \int_\kappa^\xi g(t)\, dt,$$

for $0 < \kappa < \xi \le \infty$ and let $g(t)$ and $\kappa(v)$ be as in Lemma 4.18. For $\kappa > \xi$ we define $G(\kappa, \xi) = 0$. Then

$$\begin{aligned} I_{\kappa,\xi} &\stackrel{\text{def}}{=} \int_\kappa^\xi \frac{g(t)}{M(t)^{1/2}}\, dt \\ &\ll_d q_0^{-1} r^{d/2-2} \int_\gamma^D v^{-1/2+1/d} \left(q v^{1/d} G(\kappa(v^{1/d}), \xi) \,+\, g(\kappa(v^{1/d})) \right) \frac{dv}{v} \\ &\quad + \; G(\kappa, \xi). \end{aligned} \tag{4.49}$$

Proof. We generalize the proof in [Göt04], p. 222, Lemma 3.10:
Write $\overline{\gamma} \stackrel{\text{def}}{=} \inf_{\kappa \le t \le \xi} M(t)$ and $c_d = (2d)^{-d}$. If $\overline{\gamma} \ge c_d$, then $I_{\kappa,\xi} \ll_d G(\kappa, \xi)$ and (4.49) is obvious. In the case

$$\overline{\gamma} < c_d \tag{4.50}$$

we define

$$J_{\kappa,\xi}(v) \stackrel{\text{def}}{=} \int_\kappa^\xi g(t)\, I_{\{M(t) \le v\}}\, dt \tag{4.51}$$

for $0 < \kappa < \xi$. Since $M_{j,t} \le M_{d,t} \ll_d 1$, for $j = 1, \ldots, d$, by Lemma 4.14, there exists a constant $\bar{M}$ depending on d only such that $M(t) \le \bar{M}$ for all t. Therefore we have for all $t \in [\kappa, \xi]$

$$M(t)^{-1/2} = \int_{\overline{\gamma}}^{\bar{M}} \varepsilon^{-1/2}\, d I_{\{M(t) \le \varepsilon\}}.$$

Hence, Fubini's Theorem implies

$$I_{\kappa,\xi} = \int_{\overline{\gamma}}^{\bar{M}} \varepsilon^{-1/2}\, d J_{\kappa,\xi}(\varepsilon).$$

Splitting the integral $I_{\kappa,\xi}$ into the part where $\varepsilon \le c_d$ and its complement, we obtain

$$I_{\kappa,\xi} \le \int_{\overline{\gamma}}^{c_d} \varepsilon^{-1/2}\, d J_{\kappa,\xi}(\varepsilon) + c_d^{-1/2} \int_\kappa^\xi g(t)\, dt.$$

Using partial integration we have by (4.50) and the definition of $\bar{\gamma}$,

$$\begin{aligned} I_{\kappa,\xi} &\le c_d^{-1/2} \underbrace{J_{\kappa,\xi}(c_d)}_{= G(\kappa,\xi)} - \overline{\gamma}^{-1/2} \underbrace{J_{\kappa,\xi}(\bar{\gamma})}_{=0} + \frac{1}{2} \int_{\overline{\gamma}}^{c_d} \varepsilon^{-3/2} J_{a,b}(\varepsilon)\, d\varepsilon + c_d^{-1/2} G(\kappa, \xi) \\ &= \frac{1}{2} \int_{\overline{\gamma}}^{c_d} \varepsilon^{-3/2} J_{a,b}(\varepsilon)\, d\varepsilon + 2 c_d^{-1/2} G(\kappa, \xi). \end{aligned} \tag{4.52}$$

Furthermore, $M(t) \geq (M_{1,t})^d \geq r^{-d}$ (see (4.29)) implies together with Lemma 4.20

$$\begin{aligned} J_{\kappa,\xi}(\varepsilon) &\leq \int_\kappa^\xi g(t) I_{\{(M_{1,t})^d \leq \varepsilon\}}\, dt = \int_\kappa^\xi g(t) I_{\{M_{1,t} \leq \varepsilon^{1/d}\}}\, dt \\ &\leq \int_\kappa^\xi g(t) I_{\{M_{1,t}^+ \leq \varepsilon^{1/d}\}}\, dt + \int_\kappa^\xi g(t) I_{\{M_{1,t}^- \leq \varepsilon^{1/d}\}}\, dt \\ &= H_{\kappa,\xi,Q^+}(\varepsilon^{1/d}) + H_{\kappa,\xi,Q^-}(\varepsilon^{1/d}), \end{aligned} \tag{4.53}$$

where $H_{\kappa,\xi,Q^\pm}$ is defined as in (4.43) in Lemma 4.18. The smallest and the largest eigenvalue of $Q^\pm$ is denoted by $q_0^\pm$ and $q^\pm$, respectively.
Since $r^{-d} \leq \varepsilon \leq c_d$ and hence $r^{-1} \leq \varepsilon^{1/d} \leq (2d)^{-1} \leq (2d^\pm)^{-1}$ Lemma 4.18 can be applied and by changing the variable $v = r^d \varepsilon$ we obtain

$$\begin{aligned} &\textstyle\int_{\bar\gamma}^{c_d} \varepsilon^{-3/2} H_{\kappa,\xi,Q^\pm}\left(\varepsilon^{1/d}\right) d\varepsilon \\ &\quad\leq c(d^\pm) \int_{\bar\gamma}^{c_d} \varepsilon^{-3/2} \left(\frac{q^\pm}{q_0^\pm}\, \varepsilon^{2/d} G(\kappa(\varepsilon^{1/d} r), \xi) + \frac{1}{q_0^\pm}\, \frac{\varepsilon^{1/d}}{r}\, g(\kappa(\varepsilon^{1/d} r)) \right) d\varepsilon \\ &\quad\leq c(d^\pm) \int_\gamma^D r^{\frac{3d}{2}-2} v^{-\frac{3}{2}+\frac{1}{d}} \left(\frac{q^\pm}{q_0^\pm}\, v^{1/d} G(\kappa(v^{1/d}), \xi) + \frac{1}{q_0^\pm}\, g(\kappa(v^{1/d})) \right) r^{-d} dv \\ &\quad= r^{\frac{d}{2}-2} \cdot c(d^\pm) \int_\gamma^D v^{-\frac{1}{2}+\frac{1}{d}} \left(\frac{q^\pm}{q_0^\pm}\, v^{1/d} G(\kappa(v^{1/d}), \xi) + \frac{1}{q_0^\pm} g(\kappa(v^{1/d})) \right) \frac{dv}{v}. \end{aligned}$$

Analyzing the proof of Lemma 4.18 we may assume w.l.o.g that the constant $c(d)$ is monotone increasing in d. Since $q_0 = \min\left\{q_0^+; q_0^-\right\}$ and $q = \max\{q^+; q^-\}$, we have

$$\begin{aligned} &\textstyle\int_{\bar\gamma}^{c_d} \varepsilon^{-3/2} H_{\kappa,\xi,Q^\pm}\left(\varepsilon^{1/d}\right) d\varepsilon \\ &\quad\leq r^{d/2-2} \cdot c(d) \int_\gamma^D v^{-\frac{1}{2}+\frac{1}{d}} \left(\frac{q}{q_0}\, v^{1/d} G(\kappa(v^{1/d}), \xi) + \frac{1}{q_0} g(\kappa(v^{1/d})) \right) \frac{dv}{v}. \end{aligned} \tag{4.54}$$

Thus we conclude by using (4.52), (4.53), and (4.54)

$$\begin{aligned} I_{\kappa,\xi} \ll_d\ & r^{d/2-2} \int_\gamma^D v^{-1/2+1/d} \left(\frac{q}{q_0}\, v^{1/d} G(\kappa(v^{1/d}), \xi) + \frac{1}{q_0} g(\kappa(v^{1/d})) \right) \frac{dv}{v} \\ &+ G(\kappa, \xi), \end{aligned}$$

which proves (4.49). This completes the proof of Lemma 4.22. □

Lemma 4.23. *Let* $0 < \kappa < \xi < \infty$*. Then*

$$\lim_{r\to\infty} \inf_{t\in[\kappa,\xi]} \big(r\, M_{1,t}\big) \cdots \big(r\, M_{d,t}\big) = \infty$$

provided that Q *is irrational.*

Proof. [Göt04], p. 224, Lemma 3.11 or [Els06], p. 47, Lemma 2.4.24 □

References

[BG97] V. Bentkus and F. Götze. On the lattice point problem for ellipsoids. *Acta Arith.*, 80(2):101–125, 1997.

[BG99] V. Bentkus and F. Götze. Lattice point problems and distribution of values of quadratic forms. *Ann. of Math. (2)*, 150(3):977–1027, 1999.

[Cas78] J. W. S. Cassels. *Rational quadratic forms*, volume 13 of *London Mathematical Society Monographs*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1978.

[Dav58] H. Davenport. Indefinite quadratic forms in many variables. II. *Proc. London Math. Soc. (3)*, 8:109–126, 1958.

[DH46] H. Davenport and H. Heilbronn. On indefinite quadratic forms in five variables. *J. London Math. Soc.*, 21:185–193, 1946.

[DL72] H. Davenport and D. J. Lewis. Gaps between values of positive definite quadratic forms. *Acta Arith.*, 22:87–105, 1972.

[DM93] S. G. Dani and G. A. Margulis. Limit distributions of orbits of unipotent flows and values of quadratic forms. In *I. M. Gel′fand Seminar*, volume 16 of *Adv. Soviet Math.*, pages 91–137. Amer. Math. Soc., Providence, RI, 1993.

[Els06] G. Elsner. Distributions of values of indefinite forms and higher-order spectral estimates for finite markov chains. *PhD thesis*, 2006.

[EMM98] A. Eskin, G. Margulis, and S. Mozes. Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture. *Ann. of Math. (2)*, 147(1):93–141, 1998.

[Göt04] F. Götze. Lattice point problems and values of quadratic forms. *Invent. Math.*, 157(1):195–226, 2004.

[Jar28] V. Jarník. Über Gitterpunkte in mehrdimensionalen Ellipsoiden. *Math. Ann.*, 100(1):699–721, 1928.

[Mar89] G. A. Margulis. Discrete subgroups and ergodic theory. In *Number theory, trace formulas and discrete groups (Oslo, 1987)*, pages 377–398. Academic Press, Boston, MA, 1989.

[Mar97] G. A. Margulis. Oppenheim conjecture. In *Fields Medallists' lectures*, volume 5 of *World Sci. Ser. 20th Century Math.*, pages 272–327. World Sci. Publishing, River Edge, NJ, 1997.

[Mum83] D. Mumford. *Tata lectures on theta. I*, volume 28 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1983.

[Opp29] A. Oppenheim. The minima of indefinite quaternary quadratic forms. *Proc. Nat. Acad. Sci. USA*, 15:724–727, 1929.

[Opp31] A. Oppenheim. The minima of indefinite quaternary quadratic forms. *Ann. of Math. (2)*, 32(2):271–298, 1931.

Fac. of Mathematics, Univ. Bielefeld, P.O.Box 100131, 33501 Bielefeld, Germany

E-mail address: GUIDE@math.uni-bielefeld.de