

INFINITE-DIMENSIONAL DIFFUSIONS AS LIMITS OF RANDOM WALKS ON PARTITIONS

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INTRODUCTION

The present paper originated from our previous study of the problem of harmonic analysis on the infinite symmetric group S_∞ . This problem leads to a family $\{P_z\}$ of probability measures depending on the complex parameter $z \in \mathbb{C}$, the z -measures, living on an infinite-dimensional compact space Ω . The space Ω , called the Thoma simplex, is a kind of dual object to the group S_∞ . The aim of this paper is to introduce “natural” stochastic dynamics related to the z -measures. Namely, we construct a family $\{\omega_z(t)\}$ of diffusion processes in Ω indexed by the same parameter $z \in \mathbb{C}$. These diffusions arise from certain Markov chains on partitions of n in a scaling limit as $n \rightarrow \infty$. Each z -measure P_z serves as a unique stationary distribution for the corresponding process $\omega_z(t)$. Moreover, the process $\omega_z(t)$ is reversible with respect to P_z and it is ergodic. We describe the spectrum of the generator of the process and compute the associated (pre)Dirichlet form.

As was shown in our previous papers, the z -measures and associated random point processes (that is, measures on infinite point configurations) are similar to random point processes coming from random matrix ensembles. A remarkable fact is that this similarity extends to related dynamical (that is, time-dependent) models. In particular, the dynamical model that we study in the present paper is similar to time-dependent random point processes (like the Dyson Brownian motion) coming from random matrices.

Now let us describe our results and their origin in more detail.

The Thoma simplex. This is the subspace $\Omega \subset [0, 1]^\infty \times [0, 1]^\infty$ consisting of couples (α, β) , where α and β are two infinite sequences of nonincreasing non-negative real numbers, $\alpha_1 \geq \alpha_2 \geq \dots \geq 0$ and $\beta_1 \geq \beta_2 \geq \dots \geq 0$, such that

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$\sum \alpha_i + \sum \beta_i \leq 1$. Define the *infinite symmetric group* S_∞ as the union of the chain $S_1 \subset S_2 \subset S_3 \subset \dots$ of finite symmetric groups. By Thoma's theorem [T], 1964, the points $\omega = (\alpha, \beta) \in \Omega$ parameterize the indecomposable normalized characters of the group S_∞ (in a hidden form, this remarkable result is contained in an earlier paper by Edrei [Ed], 1952). In this sense Ω may be regarded as a dual object to S_∞ .

Origin of the z -measures. The problem of harmonic analysis for the group S_∞ , as stated by Kerov, Olshanski, and Vershik [KOV1], consists in decomposing the so-called generalized regular representations \mathbf{T}_z . These representations depend on an arbitrary complex number z and form a deformation of the biregular representation of the group $S_\infty \times S_\infty$ in $\ell^2(S_\infty)$. As shown in [KOV2], the decomposition of \mathbf{T}_z into irreducibles is uniquely determined by an equivalence class \mathbf{P}_z of Borel measures on Ω ; moreover, except the coincidence $\mathbf{P}_z = \mathbf{P}_{\bar{z}}$, these classes are mutually singular.¹ The structure of \mathbf{P}_z substantially depends on whether z is an integral point or not. The former case was studied in [KOV2]; it turns out that for each $z \in \mathbb{Z}$, \mathbf{P}_z is supported by a countable union of finite-dimensional faces of the simplex Ω . The latter case, $z \in \mathbb{C} \setminus \mathbb{Z}$, which is of primary interest for us, is more complex. However, in this case, the construction of \mathbf{T}_z provides a distinguished measure in the class \mathbf{P}_z , which is our z -measure P_z .² The z -measures were studied in detail in a series of our papers, see [BO1–7].

Measures on Young diagrams. Let \mathbb{Y}_n denote the set of Young diagrams with n boxes. Recall that there is a bijective correspondence between the Young diagrams $\lambda \in \mathbb{Y}_n$ and the irreducible characters of S_n , so that \mathbb{Y}_n may be regarded as the dual object to S_n . The construction of the present paper relies on the fact that any probability measure P on Ω comes with a *canonical* approximation by a sequence $\{M_n\}$ of probability measures carried by finite sets \mathbb{Y}_n . Note that as n gets large, the finite sets \mathbb{Y}_n approximate, in an appropriate way, the space Ω . This intuitively agrees with the fact that the infinite group S_∞ is approximated by the finite groups S_n . Furthermore, for each n , the embedding $S_n \hookrightarrow S_{n+1}$ induces a canonical Markov transition kernel $p_{n+1,n}^\downarrow$ from \mathbb{Y}_{n+1} to \mathbb{Y}_n . It turns out that M_n and M_{n+1} are always consistent with this kernel: its application to M_{n+1} gives M_n . This can be written as $M_{n+1}p_{n+1,n}^\downarrow = M_n$; here and below we let a kernel act on a measure on the right. There exists another transition kernel, denoted as $p_{n,n+1}^\uparrow$, such that, conversely, $M_n p_{n,n+1}^\uparrow = M_{n+1}$.

This picture can also be described as follows: Any sequence $\{M_n\}$ of probability measures consistent with the “down” transition kernels can be viewed as the law of a Markov sequence $\{\lambda^{(n)}\}$ of random variables such that, for each $n = 1, 2, \dots$, the n th variable $\lambda^{(n)}$ takes values in \mathbb{Y}_n and is distributed according to M_n , while the “down” transition kernel $p_{n+1,n}^\downarrow$ describes the conditional distribution of $\lambda^{(n)}$ given $\lambda^{(n+1)}$. Then the “up” transition kernel $p_{n,n+1}^\uparrow$ determines the conditional distribution of $\lambda^{(n+1)}$ given $\lambda^{(n)}$.³

¹The latter fact is not too surprising because the carrying space Ω is infinite dimensional.

²Actually, the definition of the z -measures can be extended to a larger set of values of the parameter, see §3 below. According to this, starting from §3 we use, instead of the single subscript z , two subscripts z, z' .

³In this context, the Thoma simplex appears as the *entrance boundary* for the sequence $\{\lambda^{(n)}\}$ viewed as a Markov chain with discrete time n ranging in reverse direction, from $+\infty$ to 1, see

It should be noted that although the “down” and “up” transition kernels play symmetric roles, there is a substantial difference between them, for the “down” kernel is a canonical object associated with the inductive limit group $S_\infty = \varinjlim S_n$, while the “up” kernel varies depending on the concrete sequence $\{M_n\}$.

The “up–down” Markov chains. The starting point of our construction is the observation that for any n one can build a “natural” Markov chain in \mathbb{Y}_n with stationary distribution M_n . We call it the *n*th *up–down chain*: By definition, its transition operator T_n is defined as the superposition $T_n = p_{n,n+1}^\uparrow \circ p_{n+1,n}^\downarrow$ (the order of reading is from left to right when T_n is applied to measures, and from right to left when applied to functions). Note that if $(\dots, \lambda(t-1), \lambda(t), \lambda(t+1), \dots)$ stands for a trajectory of our Markov chain in \mathbb{Y}_n then for any time moment t , the Young diagrams $\lambda(t)$ and $\lambda(t+1)$ either coincide or differ from each other by a minimal possible transformation preserving the total number of boxes: displacement of a single boundary box to a new position.

One can introduce the graph with the vertex set \mathbb{Y}_n and the edges formed by couples of Young diagrams such that their symmetric difference consists of two boxes. Our chain jumps along the edges and so it is a random walk on this graph.

Now the idea is to look at the limit behavior of the up–down Markov chains as $n \rightarrow \infty$. We do not know what can be said in the abstract context, i.e., when $\{M_n\}$ comes from an arbitrary probability measure P on Ω , but in the concrete case of the z -measures $P = P_z$ we are able to prove the convergence of the chains to continuous time Markov processes in Ω which turn out to be diffusion processes. Here the limit transition assumes appropriate scalings both of space (from \mathbb{Y}_n to Ω) and of time (from discrete to continuous). The space scaling is the same as that leading from M_n to P : roughly speaking, it consists in shrinking a diagram $\lambda \in \mathbb{Y}_n$ by the factor of n^{-1} . The time scaling makes one step of the n th chain with large number n equivalent to a small time increment $\Delta t \approx n^{-2}$.

Note that, instead of the up–down chains, we could equally well deal with the down–up chains corresponding to the transition operators of the form $p_{n,n-1}^\downarrow \circ p_{n-1,n}^\uparrow$. At the finite level one gets slightly different chains but the limit process does not change.

The above definition of the up–down and down–up Markov chains does not pretend to originality. The second named author learnt it long ago from Sergei Kerov (but Kerov never published it). In a different context, such Markov chains appeared in Fulman’s work [F]: he was interested in the Plancherel distributions M_n which correspond to the case when P is the delta measure at the distinguished point $(\alpha \equiv 0, \beta \equiv 0)$ of Ω . Quite possibly, the trick is well known, for it is very simple and can be applied to any Markov sequence $(\dots, \xi^{(n-1)}, \xi^{(n)}, \xi^{(n+1)}, \dots)$ of random variables with values in possibly varying spaces.

Limit transition. Our analysis of the large n asymptotics of the up–down Markov chains uses the technique explained in Ethier–Kurtz [EK2] and goes as follows. Let $C(\Omega)$ be the Banach space of continuous functions with the supremum norm on the compact space Ω . We embed \mathbb{Y}_n into Ω by means of a map first introduced by Vershik and Kerov, which gives us a projection of $C(\Omega)$ onto the (finite–dimensional)

[KOO]. This Markov chain is a little bit unusual and highly non–homogeneous: its state space varies with time. Nevertheless, the theory of boundaries (see, e.g., [KSK]) can be readily adapted to such a situation.

Banach space $C(\mathbb{Y}_n)$. The Markov operator T_n can be viewed as an operator in the latter space. We show that, in an appropriate sense,

$$\lim_{n \rightarrow \infty} n^2(T_n - 1) = A, \quad (0.1)$$

where A is a closable, dissipative operator A in $C(\Omega)$ with dense range. By the Hille–Yosida theorem and a general theorem due to Kurtz, the closure \bar{A} of the operator A generates a contractive semigroup $\{T(t)\}$ in $C(\Omega)$, and moreover, we have convergence of semigroups:

$$\lim_{n \rightarrow \infty} T_n^{[n^2 t]} = T(t).$$

This implies that the limit semigroup $\{T(t)\}$ is positivity preserving. Consequently, it determines a Feller Markov process in Ω , which is our process $\omega_z(t)$.

The key moment of this argument is the existence of the limit operator A with desired properties. Surprisingly enough, we can prove this by purely algebraic tools, without any analytical machinery. Specifically, we are dealing with a nice algebra of functions on the set $\mathbb{Y} := \cup \mathbb{Y}_n$ of all Young diagrams. This algebra, introduced in [KO], is isomorphic to the algebra Λ of symmetric functions, and it turns out that all necessary computations can be carried out by manipulations with symmetric functions. Here we substantially use the so-called Frobenius–Schur symmetric functions introduced in [OIRV] (they are essentially the same as the shifted Schur functions [OO]). The domain of A , $\text{Dom}(A) \subset C(\Omega)$, can be identified with a quotient Λ° of the algebra Λ by an ideal; this quotient can be embedded into $C(\Omega)$ as a dense subspace. It turns out that both A and all T_n 's can be viewed as operators in $\Lambda^\circ \subset C(\Omega)$ preserving a natural ascending filtration of Λ° by finite-dimensional subspaces. Due to this fact, the limit transition (0.1) can be understood at the level of linear algebra: as convergence of operators in these finite-dimensional subspaces.

Results about the limit processes. We prove that $\omega_z(t)$ is a diffusion process, that is, a strong Markov process with continuous sample paths.

We show that $\omega_z(t)$ has the z -measure P_z as a unique stationary distribution. Moreover P_z is also a symmetrizing measure, so that the process is reversible.

We describe explicitly the spectrum of the generator, which turns out to be discrete. Due to existence of a spectral gap, the process is ergodic.

We describe the pre-generator $A = A_z$ in two different ways: (1) as an operator in the algebra Λ° acting on the (images of) the Schur functions and (2) as an infinite-dimensional differential operator in appropriate coordinates, which we call the moment coordinates. It is worth noting that these are *not* the natural coordinates α_i, β_i in Ω .

We represent the quadratic form associated to A_z as an integral against P_z of a “carré du champs”, which does not depend on the parameter and admits a simple expression in the moment coordinates.

Lifting. Let $\tilde{\Omega}$ denote the cone over Ω . Each of the processes $\omega_z(t)$ admits a “lifting” to the cone $\tilde{\Omega}$. The result is a Markov process $\tilde{\omega}_z(t)$ in $\tilde{\Omega}$ which is related to $\omega_z(t)$ by a skew product construction, like the Brownian motions in the Euclidean space and in its unit sphere. It turns out that, as an equilibrium process, $\tilde{\omega}_z(t)$ is a

time dependent *determinantal process*: its dynamical (i.e., space–time) correlation functions are principal minors of a kernel. That kernel appeared in our paper [BO6] as the result of a scaling limit transition from some continuous time jump Markov processes on Young diagrams. The paper [BO6] left open the question whether the Markov property persists in that limit transition; now we can settle this question affirmatively. We aim to discuss this issue in a subsequent paper.

Diffusions in Kingman’s simplex. Diffusion processes in simplices (both in finite and infinite dimensions) were extensively discussed in the literature in connection to mathematical models of population genetics. The paper by Ethier and Kurtz [EK1] is of special interest to us. These authors studied diffusions on the so-called Kingman simplex, which can be identified with the subspace of points in Ω with all beta-coordinates equal to 0. On Kingman’s simplex, there exists a remarkable family of probability measures, the Poisson–Dirichlet distributions; the Ethier–Kurtz diffusions preserve these distributions. As shown in Petrov [P], the Ethier–Kurtz diffusions can be constructed by the limit transition from appropriate up–down Markov chains; moreover, the same approach can be extended to a larger family of diffusions on Kingman’s simplex, which are associated to Pitman’s generalization of the Poisson–Dirichlet distributions. Note that the original construction of [EK1] was quite different. As will be shown in a subsequent paper, both the Ethier–Kurtz diffusions and our processes $\omega_z(t)$ are particular cases of a more general construction. This explains the origin of a striking similarity between the both families. We would like to gratefully acknowledge the influence of the paper [EK1] (and of the subsequent paper by Schmuland [S]) on our work.

Organization of the paper. In §1 we present the construction of Markov processes from up–down Markov chains in an abstract form. Our aim here is to single out the formal part of the argument, which can be applied in other situations, outside the context of the present paper. In §2 we illustrate the abstract formalism on a toy example: here the limit Markov process is a diffusion on $[0, 1]$. In §3 we introduce the necessary material related to the z -measures. In §4 we compute the action of the transition operators $p_{n+1,n}^\downarrow$ and $p_{n,n+1}^\uparrow$ on symmetric functions realized as functions on Young diagrams. Using this computation, we show in §5 that the assumptions of §1 are verified for the Markov chain related to the z -measures; this proves the existence of the limit processes $\omega_z(t)$. In §6 we derive an alternative expression for the pre-generator, which is used in the next section. In §7, we prove that $\omega_z(t)$ has continuous sample paths and establish other properties of this process.

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1. THE ABSTRACT FORMALISM

Let L be a graded set, that is, L is the disjoint union of subsets L_n , where $n = 0, 1, 2, \dots$. Elements of L will be denoted by the letters λ, μ, ν . If $\lambda \in L_n$ then we set $|\lambda| = n$. We assume that L_0 is a singleton and all L_n are finite.⁴ We also

⁴The latter assumption could be relaxed but it is sufficient for the purpose of the present work.

assume that we are given a function $p^\downarrow(\lambda, \mu)$ on $L \times L$ such that:

- $p^\downarrow(\lambda, \mu) \geq 0$ for all λ, μ ;
- $p^\downarrow(\lambda, \mu)$ vanishes unless $|\lambda| = |\mu| + 1$;
- for every fixed λ with $|\lambda| \geq 1$, we have

$$\sum_{\mu \in L_{|\lambda|-1}} p^\downarrow(\lambda, \mu) = 1.$$

Thus, the restriction of $p^\downarrow(\lambda, \mu)$ to $L_n \times L_{n-1}$ is a stochastic matrix for every $n \geq 1$. We may view this matrix as a transition function from L_n to L_{n-1} . For this reason we call p^\downarrow the “down” transition function.

A sequence $M = \{M_0, M_1, \dots\}$, where M_n is a probability measure on L_n , will be called a *coherent system* (of measures) if for any $n \geq 1$, the measures M_n and M_{n-1} are consistent with the transition function from L_n to L_{n-1} :

$$\sum_{\lambda \in L_n} M_n(\lambda) p^\downarrow(\lambda, \mu) = M_{n-1}(\mu) \quad \text{for any } \mu \in L_{n-1}, \quad (1.1)$$

where $M_n(\lambda)$ means the measure of the singleton $\{\lambda\}$.

Let $M = \{M_n\}$ be a coherent system such that $\text{supp } M_n$, the support of M_n , is the whole L_n for each n , that is $M_n(\lambda) > 0$ for all n and all $\lambda \in L_n$. Then we define the “up” transition function p^\uparrow as follows:

$$p^\uparrow(\lambda, \nu) = \frac{M_{n+1}(\nu)}{M_n(\lambda)} p^\downarrow(\nu, \lambda), \quad n = |\lambda|, \quad \nu \in L_{n+1}. \quad (1.2)$$

Note that $M = \{M_n\}$ determines a family of random variables indexed by $n = 0, 1, \dots$ such that the n th variable $\lambda(n)$ takes the values in L_n and has the law M_n , and, moreover,

$$\text{Prob}\{\lambda(n-1) = \mu \mid \lambda(n) = \lambda\} = p^\downarrow(\lambda, \mu)$$

for each $n \geq 1$. In these terms, the “up” transition function is interpreted as the conditional probability

$$p^\uparrow(\lambda, \nu) = \text{Prob}\{\lambda(n+1) = \nu \mid \lambda(n) = \lambda\}.$$

This implies that p^\uparrow is a stochastic matrix of format $L_n \times L_{n+1}$ for any $n = 0, 1, \dots$.

Note also that each couple (M_n, M_{n+1}) is consistent with the “up” transition function:

$$\sum_{\lambda \in L_n} M_n(\lambda) p^\uparrow(\lambda, \nu) = M_{n+1}(\nu) \quad \text{for any } \nu \in L_{n+1}. \quad (1.3)$$

Definition 1.1. Let $M = \{M_n\}$ be a coherent system with $\text{supp } M_n = L_n$. For any n we define a Markov operator T_n on the set L_n as the composition of the “up” and “down” transition functions, from L_n to L_{n+1} and then back to L_n . The matrix of T_n is given by:

$$T_n(\lambda, \tilde{\lambda}) = \sum_{\nu \in L_{n+1}} p^\uparrow(\lambda, \nu) p^\downarrow(\nu, \tilde{\lambda}), \quad \lambda, \tilde{\lambda} \in L_n.$$

The operator T_n defines a Markov chain on the set L_n . We call this Markov chain the (n th level) *up-down chain*.

Proposition 1.2. *The measure M_n is an invariant distribution for the n th level up–down Markov chain. Moreover, the chain is reversible with respect to M_n .*

Proof. The first claim is evident from (1.1) and (1.3). Indeed, $p_{n,n+1}^\uparrow$ sends M_n to M_{n+1} and then $p_{n+1,n}^\downarrow$ returns M_{n+1} back to M_n . To prove the second claim we have to check that the $L_n \times L_n$ matrix $M_n(\lambda)T_n(\lambda, \tilde{\lambda})$ is symmetric. From the definition of T_n and using (1.2) we get

$$M_n(\lambda)T_n(\lambda, \tilde{\lambda}) = \sum_{\nu \in L_{n+1}} M_{n+1}(\nu)p_{n+1,n}^\downarrow(\nu, \lambda)p_{n+1,n}^\uparrow(\nu, \tilde{\lambda}),$$

which is symmetric. \square

We aim to study the behavior of the up–down chains as $n \rightarrow \infty$. To do this we choose a topological space \bar{L} and embeddings $\iota_n : L_n \hookrightarrow \bar{L}$, and we make an appropriate limit transition inside \bar{L} . As a result we obtain a continuous time Markov process in \bar{L} .⁵

An excellent reference about limit transitions from Markov chains to continuous time Markov processes is Ethier–Kurtz’s book [EK2]. We use some general facts from [EK2] but we also explore some specific properties of our model.⁶ Namely, in our concrete situation the following assumptions hold true:

(A1) The ambient space \bar{L} is a compact, metrizable, separable topological space. Below we denote by $C(\bar{L})$ the Banach space of continuous real functions with the canonical norm

$$\|f\| = \sup_{\omega \in \bar{L}} |f(\omega)|.$$

(A2) The sets $\iota_n(L_n)$ approximate \bar{L} in the following sense: any open subset of \bar{L} has a nonempty intersection with $\iota_n(L_n)$ for all n large enough.

(A3) There is a distinguished dense subspace \mathcal{F} of the Banach space $C(\bar{L})$ and an ascending exhaustive filtration (\mathcal{F}^m) of \mathcal{F} by finite–dimensional subspaces such that each \mathcal{F}^m is invariant under the Markov operators T_n in the following sense:

Denote by $C(L_n)$ the (finite–dimensional) Banach space of functions on L_n with the norm

$$\|g\|_n = \sup_{\lambda \in L_n} |g(\lambda)|$$

and let $\pi_n : C(\bar{L}) \rightarrow C(L_n)$ be defined by

$$(\pi_n(f))(\lambda) = f(\iota_n(\lambda)), \quad \lambda \in L_n, \quad f \in C(\bar{L}).$$

Observe that π_n is injective on \mathcal{F}^m provided that n is large enough, where the necessary lower bound on n depends on m .⁷ Then the invariance property means that, for any m , $\pi_n(\mathcal{F}^m)$ is invariant under T_n , at least for large enough n .

Thus, identifying \mathcal{F}^m and $\pi_n(\mathcal{F}^m)$, which makes sense for large n , one may say that the operators T_n leave the spaces \mathcal{F}^m invariant.

⁵Likewise, interchanging the transition functions, one could introduce the down–up chains. They are slightly different from the up–down ones but have the same limit.

⁶The motivation for the concrete choice of the data $(L = (L_n), p^\downarrow, M = \{M_n\}, \bar{L}, \iota_n)$ comes from Vershik–Kerov’s theory [VK], [K].

⁷This is true because of (A2) and the fact that $\dim \mathcal{F}^m < \infty$.

(A4) There is a sequence $\{\varepsilon_n\}$ of positive numbers converging to 0 such that, under the identification $\mathcal{F}^m = \pi_n(\mathcal{F}^m)$, the limit

$$\lim_{n \rightarrow \infty} \varepsilon_n^{-1}(T_n - \mathbf{1})f = Af.$$

exists in any finite-dimensional space \mathcal{F}^m . Here $\mathbf{1}$ stands for the identity operator. Clearly, we get in this way a limit operator $A : \mathcal{F} \rightarrow \mathcal{F}$ which preserves each subspace \mathcal{F}^m .

(A5) \mathcal{F} contains the constant function 1.

Recall that a *conservative Markov semigroup* in $C(\bar{L})$ is a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ of contractive operators in $C(\bar{L})$ preserving the cone of non-negative functions and the constant function 1.

Definition 1.3. Following [EK2, Ch. 1, Section 6], let us say that a sequence of functions $\{f_n \in C(L_n)\}$ converges to a function $f \in C(\bar{L})$ if $\|f_n - \pi_n(f)\|_n \rightarrow 0$. Then we write $f_n \rightarrow f$.

Observe that $\|\pi_n(f)\|_n \leq \|f\|$ and, by virtue of (A2), $\|\pi_n(f)\|_n \rightarrow \|f\|$. Again by (A2), a sequence $\{f_n \in C(L_n)\}$ may have at most one limit in $C(\bar{L})$.

Proposition 1.4. *Assume that the assumptions (A1)–(A5) stated above are satisfied.*

(1) *The operator $A : \mathcal{F} \rightarrow \mathcal{F}$ defined in (A4) is closable in the Banach space $C(\bar{L})$.*

(2) *The closure \bar{A} of A generates a conservative Markov semigroup $\{T(t)\}$ in $C(\bar{L})$.*

(3) *The discrete semigroup $\{1, T_n, T_n^2, T_n^3, \dots\}$ converges, as $n \rightarrow \infty$, to $\{T(t)\}$ in the following sense:*

$$T_n^{[\varepsilon_n^{-1}t]} \pi_n(f) \rightarrow T(t)f, \quad \forall f \in C(\bar{L}), \quad (1.4)$$

for all $t \geq 0$, uniformly on bounded intervals, where the limit is understood according to Definition 1.3.

We will call A the *pre-generator* of the semigroup $\{T(t)\}$.

Proof. Step 1. The operator $A : \mathcal{F} \rightarrow \mathcal{F}$ is dissipative, that is, $\|(s\mathbf{1} - A)f\| \geq s\|f\|$ for any $s \geq 0$ and $f \in \mathcal{F}$.

Indeed, fix m so large that $f \in \mathcal{F}^m$. Assuming n large enough, we may identify \mathcal{F}^m with its image under π_n and view T_n as an operator in \mathcal{F}_m . Set $A_n = \varepsilon_n^{-1}(T_n - \mathbf{1})$. Since T_n is a contraction with respect to the norm $\|\cdot\|_n$, the operator $T_n - \mathbf{1}$ is dissipative with respect to this norm, whence the same holds for A_n . Since $A_n \rightarrow A$ in the finite-dimensional space \mathcal{F}^m and since $\|g\|_n \rightarrow \|g\|$ for any $g \in \mathcal{F}^m$ we conclude that A is dissipative.

Step 2. By virtue of step 1, for any $s > 0$ and any m , the operator $s\mathbf{1} - A$ maps the finite-dimensional subspace \mathcal{F}^m onto itself. Thus, $(s\mathbf{1} - A)\mathcal{F} = \mathcal{F}$.

Step 3. Since A is dissipative (step 1) and its domain \mathcal{F} is dense, A is closable in $C(\bar{L})$; moreover, for any $s > 0$, the closure of the range of $s\mathbf{1} - A$ coincides with the range of $s\mathbf{1} - \bar{A}$, where \bar{A} denotes the closure of A ([EK2, Ch. 1, Lemma 2.11]). Applying step 2 we see that the range of $s\mathbf{1} - \bar{A}$ coincides with the whole space $C(\bar{L})$. Thus, the operator \bar{A} satisfies the assumptions of the Hille–Yosida theorem

and therefore it generates a strongly continuous contractive semigroup $\{T(t)\}$ in $C(\bar{L})$, see [EK2, Ch. 1, Thm. 2.6].

Step 4. It follows from (A5) that the constant function 1 is in the domain of \bar{A} and $\bar{A}1 = 0$, whence $T(t)1 = 1$.

Step 5. All the assumptions of [EK2, Ch. 1, Thm. 6.5] are satisfied. Namely:

- $\{T(t)\}$ is a strongly continuous semigroup of contractions with generator \bar{A} ;
- the subspace \mathcal{F} is an essential domain for \bar{A} , that is, the operator \bar{A} coincides with the closure of its restriction to \mathcal{F} ;
- each T_n is a contraction, and we have $\varepsilon_n^{-1}(T_n - \mathbf{1})\pi_n(f) \rightarrow \bar{A}f$ for any $f \in \mathcal{F}$ in the sense of Definition 1.2.

Applying this theorem from [EK2], we obtain (1.4).

Step 6. Finally, (1.4) implies that the operators $T(t)$ preserve nonnegative functions, because the T_n 's possess this property (here we again use (A2)). Thus, $\{T(t)\}$ is a Markov semigroup. \square

Proposition 1.5. *The semigroup $\{T(t)\}$ constructed in Proposition 1.3 gives rise to a strong Markov process $X(t)$ in \bar{L} . The process has càdlàg sample paths and can start from any point or any probability distribution.*

Proof. This is a well-known general fact, see e.g. [EK2, Ch. 4, Thm. 2.7]. \square

Proposition 1.6. *Assume additionally that the measures $\nu_n(M_n)$ weakly converge to a measure P on \bar{L} . Then P is an invariant distribution for the process $X(t)$.*

Proof. This directly follows from (1.4). Indeed, let $\langle \cdot \rangle_P$ or $\langle \cdot \rangle_{M_n}$ means expectation with respect to P or M_n . The invariance property of P means that

$$\langle T(t)f \rangle_P = \langle f \rangle_P, \quad \forall f \in C(\bar{L}), \quad \forall t \geq 0.$$

Since $\nu_n(M_n)$ weakly converges to P , this is equivalent to

$$\lim_{n \rightarrow \infty} \langle T(t)f \rangle_{\nu_n(M_n)} = \lim_{n \rightarrow \infty} \langle f \rangle_{\nu_n(M_n)},$$

which can be rewritten as

$$\lim_{n \rightarrow \infty} \langle \pi_n(T(t)f) \rangle_{M_n} = \lim_{n \rightarrow \infty} \langle \pi_n(f) \rangle_{M_n}.$$

By virtue of (1.4), as n gets large, $\pi_n(T(t)f)$ is close in norm to $T_n^{[\varepsilon_n^{-1}t]}\pi_n(f)$, whence the last limit relation is equivalent to

$$\lim_{n \rightarrow \infty} \langle T_n^{[\varepsilon_n^{-1}t]}\pi_n(f) \rangle_{M_n} = \lim_{n \rightarrow \infty} \langle \pi_n(f) \rangle_{M_n},$$

which holds for trivial reasons, because

$$\langle T_n^{[\varepsilon_n^{-1}t]}\pi_n(f) \rangle_{M_n} = \langle \pi_n(f) \rangle_{M_n},$$

due to invariance of M_n with respect to T_n . \square

Proposition 1.7. *Under the hypothesis of Proposition 1.6, the pre-generator $A : \mathcal{F} \rightarrow \mathcal{F}$ is symmetric with respect to the inner product*

$$(f, g) := \langle f \cdot g \rangle_P.$$

Proof. The argument is similar to that used in Proposition 1.6. Let us show that

$$\langle Af \cdot g \rangle_P = \lim_{n \rightarrow \infty} \langle \varepsilon_n^{-1}(T_n - \mathbf{1})\pi_n(f) \cdot \pi_n(g) \rangle_{M_n}, \quad f, g \in \mathcal{F}.$$

By virtue of Proposition 1.2, the right-hand side is symmetric with respect to $f \leftrightarrow g$, hence the above equality implies the desired symmetry of the left-hand side. We have

$$\begin{aligned} \langle Af \cdot g \rangle_P &= \lim_{n \rightarrow \infty} \langle Af \cdot g \rangle_{\iota_n(M_n)} = \lim_{n \rightarrow \infty} \langle \pi_n(Af \cdot g) \rangle_{\iota_n(M_n)} \\ &= \lim_{n \rightarrow \infty} \langle \pi_n(Af) \cdot \pi_n(g) \rangle_{\iota_n(M_n)} = \lim_{n \rightarrow \infty} \langle \varepsilon_n^{-1}(T_n - \mathbf{1})\pi_n(f) \cdot \pi_n(g) \rangle_{M_n}, \end{aligned}$$

where the last step is justified using (1.4). \square

Proposition 1.8. *Under the hypothesis of Proposition 1.6, consider $X(t)$ as an equilibrium process with respect to its invariant distribution P . Likewise, consider the up-down Markov chains in equilibrium with respect to the invariant distributions M_n .*

Then the finite-dimensional distributions for the n th chain converge, as $n \rightarrow \infty$, to the corresponding finite-dimensional distributions of the process $X(t)$. Here we assume a natural scaling of time: one step of the n th Markov chain corresponds to a small time interval of order $\Delta t = \varepsilon_n$.

Proof. The argument is similar to that used in Proposition 1.6. \square

2. A TOY EXAMPLE: THE PASCAL TRIANGLE

Here we illustrate the above formalism on a simple example: the Pascal triangle.

In this example, the set L consists of arbitrary couples $\lambda = (a, b)$ of nonnegative integers. The grading is defined as $|(a, b)| = a + b$. Thus, L_0 consists of the single point $(0, 0)$, L_1 consists of two points $(0, 1)$ and $(1, 0)$, \dots , L_n consists of $n + 1$ points $(0, n), \dots, (n, 0)$.

The “down” transition function is defined as follows

$$p^\downarrow((a, b), (a - 1, b)) = \frac{a}{a + b}, \quad p^\downarrow((a, b), (a, b - 1)) = \frac{b}{a + b},$$

with all other transitions being of probability 0.

As M_n we take the uniform measure on the set L_n . The coherency condition (1.1) is immediately checked. Thus, $M = \{M_n\}$ is a coherent system. The nonzero values of the “up” transition function are

$$p^\uparrow((a, b), (a + 1, b)) = \frac{a + 1}{a + b + 2}, \quad p^\uparrow((a, b), (a, b + 1)) = \frac{b + 1}{a + b + 2}.$$

The Markov operator T_n of the up-down Markov chain is given by the following matrix (we list the nonzero entries only and assume $a + b = n$)

$$\begin{aligned} T_n((a, b), (a + 1, b - 1)) &= \frac{(a + 1)b}{(a + b + 2)(a + b + 1)} \\ T_n((a, b), (a - 1, b + 1)) &= \frac{a(b + 1)}{(a + b + 2)(a + b + 1)} \\ T_n((a, b), (a, b)) &= \frac{(a + 1)^2 + (b + 1)^2}{(a + b + 2)(a + b + 1)}. \end{aligned}$$

The ambient compact space is the closed unit interval $[0, 1]$. The embeddings $L_n \hookrightarrow \bar{L}$ are defined as $(a, b) \mapsto x$ with $x = \frac{a}{a+b} \in [0, 1]$.

The dense subspace $\mathcal{F} \subset C(\bar{L}) = C([0, 1])$ is the space of polynomials with the canonical filtration by degree.

The fulfilment of (A1), (A2), and (A5) is evident, let us verify (A3) and (A4).

The nontrivial one-step transitions of our Markov chain $(a, b) \rightarrow (a + 1, b)$ and $(a, b) \rightarrow (a, b + 1)$ turn into $x \rightarrow x \pm \Delta x$ with $\Delta x = n^{-1}$, where we assume $n = a + b$. According to the above formulas for T_n ,

$$\begin{aligned} \text{Prob}\{x \rightarrow x + \Delta x\} &= \frac{(a + 1)b}{(a + b + 2)(a + b + 1)} \\ \text{Prob}\{x \rightarrow x - \Delta x\} &= \frac{a(b + 1)}{(a + b + 2)(a + b + 1)}, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \text{Prob}\{x \rightarrow x + \Delta x\} &= \frac{n^2}{(n + 1)(n + 2)} x(1 - x) + \frac{n}{(n + 1)(n + 2)} (1 - x) \\ \text{Prob}\{x \rightarrow x - \Delta x\} &= \frac{n^2}{(n + 1)(n + 2)} x(1 - x) + \frac{n}{(n + 1)(n + 2)} x. \end{aligned}$$

It follows that for a polynomial $f(x)$,

$$\begin{aligned} ((T_n - \mathbf{1})f)(x) &= \frac{n^2}{(n + 1)(n + 2)} x(1 - x)(f(x + \Delta x) + f(x - \Delta x) - 2f(x)) \\ &+ \frac{n}{(n + 1)(n + 2)} (1 - x)(f(x + \Delta x) - f(x)) + \frac{n}{(n + 1)(n + 2)} x(f(x - \Delta x) - f(x)) \end{aligned}$$

(recall that $\mathbf{1}$ stands for the identity operator). Therefore,

$$\begin{aligned} ((T_n - \mathbf{1})f)(x) &= \frac{n^2 \cdot (\Delta x)^2}{(n + 1)(n + 2)} x(1 - x)(f''(x) + \dots) \\ &+ \frac{n \cdot \Delta x}{(n + 1)(n + 2)} (1 - x)(f'(x) + \dots) + \frac{n \cdot \Delta x}{(n + 1)(n + 2)} x(-f'(x) + \dots), \end{aligned}$$

where dots mean higher derivatives multiplied by suitable nonzero powers of Δx .

From this expression we see that the operator T_n is well defined on polynomials and it does not raise the degree, so that assumption (A3) holds true.

Next, we see that if $\varepsilon_n \sim n^{-2}$ then, as $n \rightarrow \infty$,

$$\varepsilon_n^{-1}((T_n - \mathbf{1})f)(x) \rightarrow x(1-x)f''(x) + (1-2x)f'(x).$$

Therefore, assumption (A4) holds with

$$A = x(1-x)\frac{d^2}{dx^2} + (1-2x)\frac{d}{dx}.$$

Thus, all necessary assumptions are satisfied and one can apply Proposition 1.4 to conclude that our Markov chains converge to a continuous time Markov process $X(t)$ on $[0, 1]$. The generator \bar{A} of $X(t)$ is the closure of the differential operator A initially defined on polynomials.

3. THE Z-MEASURES

Here we specify the abstract data described in §1 and introduce related extra notation. For more detail, see [Ol1], [KOV2], [KOO], [BO3].

3.1. Young diagrams and modified Frobenius coordinates. As L we take the set \mathbb{Y} of all Young diagrams including the empty diagram \emptyset . The subset $L_n \subset L$ becomes the subset $\mathbb{Y}_n \subset \mathbb{Y}$ of diagrams with n boxes.

Given $\lambda \in \mathbb{Y}_n$, denote by $a_1, \dots, a_d, b_1, \dots, b_d$ its *modified Frobenius coordinates*: here d is the number of diagonal boxes in λ , a_i equals $\frac{1}{2}$ plus the number of boxes in the i th row to the right of the diagonal, and b_i equals $\frac{1}{2}$ plus the number of boxes in the i th column below the diagonal. Note that $\sum(a_i + b_i) = n$. We write $\lambda = (a_1, \dots, a_d \mid b_1, \dots, b_d)$.

If λ and μ are two Young diagrams then we write $\mu \nearrow \lambda$ or, equivalently, $\lambda \searrow \mu$ if $\mu \subset \lambda$ and $|\lambda| = |\mu| + 1$. That is, λ is obtained from μ by adding a box. This box is then denoted as λ/μ . In terms of the modified Frobenius coordinates, $\mu \nearrow \lambda$ means that λ is obtained from μ either by adding 1 to one of the coordinates or by creating a new pair of coordinates $(\frac{1}{2}; \frac{1}{2})$ (the latter happens if the new box λ/μ lies on the diagonal).

More generally, for any $\mu \subset \lambda$ we denote by λ/μ the corresponding skew Young diagram.

3.2. The “down” transition functions. The choice of the “down” transition function p^\downarrow is motivated by the representation theory of the symmetric groups S_n . Recall that \mathbb{Y}_n is the set of labels of irreducible representations of S_n . Given $\lambda \in \mathbb{Y}_n$, we denote by π_λ the corresponding irreducible representation of S_n and we write $\dim \lambda = \dim \pi_\lambda$. Here is an explicit expression for this quantity in terms of the modified Frobenius coordinates:

$$\frac{\dim \lambda}{n!} = \frac{\prod_{1 \leq i < j \leq d} (a_i - a_j)(b_i - b_j)}{\prod_{1 \leq i, j \leq d} (a_i + b_j) \prod_{1 \leq i \leq d} (a_i - \frac{1}{2})!(b_i - \frac{1}{2})!}.$$

We realize S_n as the group of permutations of the set $\{1, \dots, n\}$, and we embed S_{n-1} into S_n as the subgroup fixing the point n . The Young rule says that the

restriction of π_λ (where $\lambda \in \mathbb{Y}_n$) to $S_{n-1} \subset S_n$ decomposes into the multiplicity free direct sum of the representations π_μ such that $\mu \nearrow \lambda$. Consequently,

$$\dim \lambda = \sum_{\mu: \mu \nearrow \lambda} \dim \mu, \quad |\lambda| \geq 2.$$

We use this identity to define p^\downarrow :

$$p^\downarrow(\lambda, \mu) = \begin{cases} \frac{\dim \mu}{\dim \lambda}, & \mu \nearrow \lambda \\ 0, & \text{otherwise} \end{cases}$$

and we also set $p^\downarrow(\lambda, \mu) = 1$ when $|\lambda| = 1$ and $|\mu| = 0$, that is, when λ consists of a single box and μ is empty.

3.3. The Thoma simplex. As \bar{L} we take the *Thoma simplex*. Recall that this is the subspace $\Omega \subset [0, 1]^\infty \times [0, 1]^\infty$ formed by couples $\omega = (\alpha, \beta)$ such that

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq 0) \in [0, 1]^\infty, \quad \beta = (\beta_1 \geq \beta_2 \geq \dots \geq 0) \in [0, 1]^\infty, \\ \sum_{i=1}^{\infty} (\alpha_i + \beta_i) \leq 1.$$

The embedding $\iota_n : \mathbb{Y}_n \hookrightarrow \Omega$ is defined as follows. For $\lambda = (a_1, \dots, a_d \mid b_1, \dots, b_d) \in \mathbb{Y}_n$ (here we wrote λ in terms of the modified Frobenius coordinates), its image $\iota_n(\lambda) = (\alpha, \beta)$ is given by

$$\alpha_i = \begin{cases} a_i/n, & 1 \leq i \leq d, \\ 0, & i > d; \end{cases} \quad \beta_i = \begin{cases} b_i/n, & 1 \leq i \leq d, \\ 0, & i > d. \end{cases}$$

The embeddings ι_n satisfy the assumption (A2) of §1.

3.4. Thoma's measures and moment coordinates. To any point $\omega = (\alpha, \beta) \in \Omega$ one can assign a probability measure ν_ω on the closed interval $[-1, 1]$:

$$\nu_\omega = \sum_{i=1}^{\infty} \alpha_i \delta_{\alpha_i} + \sum_{i=1}^{\infty} \beta_i \delta_{-\beta_i} + \gamma \delta_0, \quad \gamma := 1 - \sum \alpha_i - \sum \beta_i,$$

where δ_x denotes the Dirac measure at x . The measure ν_ω is called the *Thoma measure* corresponding to ω .

Denote by $q_k = q_k(\omega)$ the moments of ν_ω :

$$q_k := \int x^k \nu_\omega(dx) = \sum_{i=1}^{\infty} \alpha_i^{k+1} + (-1)^k \sum_{i=1}^{\infty} \beta_i^{k+1}, \quad k = 1, 2, \dots$$

and note that the 0th moment is always equal to 1. We call q_1, q_2, \dots the *moment coordinates* of ω . Observe that they are continuous functions in ω . Indeed, since α_i 's decrease, the condition $\sum \alpha_i \leq 1$ implies $\alpha_i \leq i^{-1}$ for any $i = 1, 2, \dots$,

whence $\alpha_i^{k+1} \leq i^{-k-1}$. Similarly, $\beta_i^{k+1} \leq i^{-k-1}$. It follows that the both series are uniformly convergent in $\omega \in \Omega$, which implies their continuity as functions on Ω .⁸

Let $\mathcal{M}_1[-1, 1]$ denote the space of probability Borel measures on $[-1, 1]$ equipped with the weak topology. Since this topology is determined by convergence of moments, the assignment $\omega \mapsto \nu_\omega$ determines a homeomorphism of the Thoma simplex on a compact subset of $\mathcal{M}_1[-1, 1]$.

On the other hand, the assignment $\omega \mapsto (q_1, q_2, \dots)$ determines a homeomorphism of Ω on a compact subset of $[-1, 1]^\infty$.

Thinking of Ω as of a subspace of $\mathcal{M}_1[-1, 1]$ or $[-1, 1]^\infty$ turns out to be useful even though we cannot describe the image of Ω in $[-1, 1]^\infty$ explicitly.

Note also that the moment coordinates are algebraically independent as functions on Ω . Indeed, this holds even we restrict them on the subset with all β_i 's equal to 0. It follows that the algebra of polynomials $\mathbb{R}[q_1, q_2, \dots]$ can be viewed as a subalgebra of $C(\Omega)$, the (real) Banach algebra of continuous functions on Ω with pointwise operations and the supremum norm. Since this subalgebra separates points, it is dense in $C(\Omega)$.

3.5. Symmetric functions. Let Λ be the algebra of symmetric functions [Ma]. Recall that Λ is freely generated (as a commutative unital algebra) by the Newton power sums p_1, p_2, \dots . As the base field, it is convenient for us to take \mathbb{R} . Let $\Lambda^\circ = \Lambda/(p_1 - 1)$ be the quotient of the algebra Λ by the ideal generated by $p_1 - 1$. The algebra Λ° has a natural structure of a filtered algebra inherited from Λ , and the graded algebra associated to Λ° is isomorphic to $\Lambda/(p_1)$.

Given $f \in \Lambda$ we denote its image in Λ° by f° . In particular, we will be dealing with the elements $s_\mu^\circ \in \Lambda^\circ$ coming from the Schur functions and the elements $p_k^\circ \in \Lambda^\circ$ coming from the Newton power sums. Clearly, $p_1^\circ = 1$ and Λ° is freely generated (as a unital commutative algebra) by $p_2^\circ, p_3^\circ, \dots$.⁹

Setting $p_2^\circ \rightarrow q_1, p_3^\circ \rightarrow q_2, \dots$, where $q_k = q_k(\omega)$ are the moment coordinates defined above, we define an algebra isomorphism between Λ° and the subalgebra $\mathbb{R}[q_1, q_2, \dots] \subset C(\Omega)$. Thus, each element of $f^\circ \in \Lambda^\circ$ becomes a continuous function $f^\circ(\omega)$ on Ω , in particular,

$$p_k^\circ(\omega) = \sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} \sum_{i=1}^{\infty} \beta_i^k, \quad \omega = (\alpha, \beta) \in \Omega, \quad k = 2, 3, \dots$$

Note that $p_1^\circ \equiv 1$.

We will take Λ° as the dense subspace in $C(\Omega)$ required in the assumption (A3) of §1.

3.6. Boundary measures. Let $\{M_n\}$ be an arbitrary coherent system of probability measures on the sets \mathbb{Y}_n with respect to the “down” transition functions introduced in §3.2; see §1 for the general definition of coherent systems. Denote by $\iota_n\{M_n\}$ the measure on Ω obtained as the push-forward of M_n with respect to the embedding $\iota_n : \mathbb{Y}_n \rightarrow \Omega$ defined in §3.3.

⁸This argument substantially relies on the fact that $k+1 \geq 2$. Note that the function $\omega \mapsto \sum \alpha_i + \sum \beta_i$ is not continuous on Ω .

⁹One could identify Λ° with the subalgebra in Λ generated by p_2, p_3, \dots but we do not want to do this.

Theorem. *There exists a weak limit $P = \lim_{n \rightarrow \infty} \iota_n \{M_n\}$ on Ω . Conversely, $\{M_n\}$ can be reconstructed from P by means of the equation*

$$M_n(\nu) = \dim \nu \int_{\Omega} s_{\nu}^{\circ}(\omega) P(d\omega), \quad \nu \in \mathbb{Y}_n,$$

where The correspondence $\{M_n\} \rightarrow P$ is a bijection between coherent systems on $\mathbb{Y} = \cup \mathbb{Y}_n$ and probability measures on Ω .

We call P the *boundary measure* of the coherent system $\{M_n\}$.

This fundamental result is a refinement of Thoma's theorem [T]. It is essentially due to Vershik and Kerov [VK], [K]. See [KOO] for a detailed proof.

3.7. The z -measures. Now we proceed to the definition of a distinguished family of coherent systems. Introduce the notation

$$(z)_{\lambda} = \prod_{(i,j) \in \lambda} z + j - i, \quad \lambda \in \mathbb{Y}, \quad z \in \mathbb{C},$$

where “ $(i, j) \in \lambda$ ” means that the product is taken over the boxes of λ ; here and below we denote by (i, j) the box with row number i and column number j . The difference $j - i$ is called the *content* of a box (i, j) .

We define likewise $(z)_{\lambda/\mu}$ for skew diagrams λ/μ (then the product is taken over the boxes in λ/μ). This is a generalization of the Pochhammer symbol

$$(z)_n = z(z + 1) \dots (z + n - 1)$$

which is obtained in the particular case when λ is (n) , the one-row diagram with n boxes.

Let z and z' be complex numbers such that $zz' \notin \{0, -1, -2, \dots\}$. The z -measure on the finite set \mathbb{Y}_n is the complex measure $M_{z,z'}^{(n)}$, with the weights

$$M_{z,z'}^{(n)}(\lambda) = \frac{(z)_{\lambda} (z')_{\lambda}}{(zz')_n} \frac{(\dim \lambda)^2}{n!}, \quad \lambda \in \mathbb{Y}_n.$$

The z -measure does not change under transposition $z \leftrightarrow z'$. Thus, instead of (z, z') , one can also take as parameters zz' and $z + z'$.

It is known that the weights sum to 1,

$$\sum_{\lambda \in \mathbb{Y}_n} M_{z,z'}^{(n)}(\lambda) = 1,$$

and that the z -measures satisfy (1.1):

$$\sum_{\lambda \searrow \mu} M_{z,z'}^{(n)}(\lambda) p^{\downarrow}(\lambda, \mu) = M_{z,z'}^{(n-1)}(\mu).$$

This follows from the representation-theoretical construction of [KOV2]. For a direct proof, see, e.g., [O11].

The weights $M_{z,z'}^{(n)}(\lambda)$ are strictly positive for all n if and only if (z, z') belongs to one of the following two sets in \mathbb{C}^2 :

- *Principal series*: Both z and z' are not real and are conjugate to each other.
- *Complementary series*: Both z and z' are real and are contained in the same open interval of the form $(N, N + 1)$, where $N \in \mathbb{Z}$.

The union of these two sets admits a nice description in terms of the coordinates

$$x = \frac{z + z'}{2}, \quad y = zz' - x^2 = -\left(\frac{z - z'}{2}\right)^2.$$

Namely, this is the domain D in the real (x, y) -plane bounded from below by the piecewise smooth curve C built from countably many smooth arcs C_N : here N ranges over \mathbb{Z} and each C_N is an arc of a parabola: $y = -(x - N)^2$, $|x - N| \leq \frac{1}{2}$. The principal series and the complementary are described by points $(x, y) \in D$ with $y > 0$ and $y \leq 0$, respectively.

Thus, to each value (z, z') of the principal/complementary series (equivalently, to each $(x, y) \in D$), a coherent family $\{M_n = M_{z, z'}^{(n)}\}$ of probability measures is attached. Note that the support of M_n is the whole set \mathbb{Y}_n .

The “up” transition function for the z -measures looks as follows: for $\lambda \in \mathbb{Y}_n$

$$p_{z, z'}^{\uparrow}(\lambda, \lambda^\bullet) = \begin{cases} \frac{(z)_{\lambda^\bullet/\lambda} (z')_{\lambda/\lambda^\bullet}}{zz' + n} \frac{\dim \lambda^\bullet}{(n + 1) \dim \lambda}, & \lambda^\bullet \searrow \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

We will often use the symbol λ^\bullet to denote a diagram ν such that $\nu \searrow \lambda$. Likewise λ_\bullet will denote a diagram μ such that $\mu \nearrow \lambda$. Note that λ^\bullet/λ is a single box.

3.8. The boundary z -measures. Given a coherent system $\{M_{z, z'}^{(n)}\}$ of z -measures, we denote by $P_{z, z'}$ the corresponding boundary measure on Ω and call it the *boundary z -measure*.

Theorem. *Except the equality $P_{z, z'} = P_{z', z}$, the boundary z -measures with different parameters are mutually singular with respect to each other.*

See [KOV2] for a proof.

4. THE “UP” AND “DOWN” OPERATORS FOR THE Z -MEASURES

Let $\text{Fun}(\mathbb{Y})$ denote the algebra of all functions on \mathbb{Y} with pointwise operations. We define an algebra morphism $\Lambda \rightarrow \text{Fun}(\mathbb{Y})$ by specifying it on the generators p_k , as follows

$$p_k(\lambda) = \sum_{i=1}^d a_i^k + (-1)^{k-1} \sum_{i=1}^d b_i^k, \quad \lambda \in \mathbb{Y},$$

where a_1, \dots, a_d and b_1, \dots, b_d are the modified Frobenius coordinates of λ . It is readily verified that the images of the p_k 's in $\text{Fun}(\mathbb{Y})$ are algebraically independent, so that our morphism is injective. Thus, we may view Λ as a subalgebra of $\text{Fun}(\mathbb{Y})$. We will denote the value at λ of the function on \mathbb{Y} corresponding to an element $f \in \Lambda$ as $f(\lambda)$.

A distinguished basis in Λ is formed by the *Schur functions* s_μ (here and below μ ranges over \mathbb{Y}). The Schur functions are homogeneous, the degree of s_μ is equal to $|\mu|$, see [Ma].

There is another important basis, formed by the Frobenius–Schur functions FS_μ . These are inhomogeneous elements such that FS_μ differs from s_μ by lower degree terms. The crucial property of the FS_μ 's is expressed by the formula

$$FS_\mu(\lambda) = n^{\downarrow m} \frac{\dim(\mu, \lambda)}{\dim \lambda}, \quad n = |\lambda|, \quad m = |\mu|, \quad (4.1)$$

where we use the notation

$$n^{\downarrow m} = n(n-1)\dots(n-m+1).$$

In particular, $FS_\mu(\lambda)$ vanishes unless λ contains μ .

For more detail about the realization of symmetric functions as functions on \mathbb{Y} and about the Frobenius–Schur functions, see [KO], [OO], [OIRV], [IO].

Given $f \in \Lambda$, we denote by f_n the restriction of the function $f(\cdot)$ to $\mathbb{Y}_n \subset \mathbb{Y}$. It is readily checked that the subalgebra $\Lambda \subset \text{Fun}(\mathbb{Y})$ separates points, which implies that for each n , the functions of the form f_n , with $f \in \Lambda$, exhaust the space $C(\mathbb{Y}_n)$.

Let $D_{n+1,n} : C(\mathbb{Y}_n) \rightarrow C(\mathbb{Y}_{n+1})$ and $U_{n,n+1} : C(\mathbb{Y}_{n+1}) \rightarrow C(\mathbb{Y}_n)$ be the “down” and “up” operators acting on functions:

$$\begin{aligned} (D_{n+1,n}f)(\nu) &= \sum_{\lambda \in \mathbb{Y}_n} p^\downarrow(\nu, \lambda) f(\lambda), \quad f \in C(\mathbb{Y}_n), \quad \nu \in \mathbb{Y}_{n+1}, \\ (U_{n,n+1}g)(\lambda) &= \sum_{\nu \in \mathbb{Y}_{n+1}} p_{z,z'}^\uparrow(\lambda, \nu) g(\nu), \quad g \in C(\mathbb{Y}_{n+1}), \quad \lambda \in \mathbb{Y}_n. \end{aligned}$$

Note that $U_{n,n+1}$ depends on z and z' while $D_{n+1,n}$ does not.

In this section, we prove the following claim.

Theorem 4.1. (1) *There exists a unique operator $\tilde{D} : \Lambda \rightarrow \Lambda$ such that*

$$D_{n+1,n}f_n = \frac{1}{n+1}(\tilde{D}f)_{n+1}, \quad \text{for all } n = 0, 1, \dots \text{ and all } f \in \Lambda.$$

In the basis $\{FS_\mu\}$ it is given by

$$\tilde{D}FS_\mu = (p_1 - |\mu|)FS_\mu, \quad \mu \in \mathbb{Y}.$$

(2) *There exists a unique operator $\tilde{U} : \Lambda \rightarrow \Lambda$ depending on z, z' , such that*

$$U_{n,n+1}f_{n+1} = \frac{1}{zz' + n}(\tilde{U}f)_n, \quad \text{for all } n = 0, 1, \dots \text{ and all } f \in \Lambda.$$

In the basis $\{FS_\mu\}$ it is given by

$$\tilde{U}FS_\mu = \sum_{\mu_\bullet \nearrow \mu} (z)_{\mu/\mu_\bullet} (z')_{\mu/\mu_\bullet} FS_{\mu_\bullet} + (p_1 + zz' + |\mu|)FS_\mu, \quad \mu \in \mathbb{Y}.$$

Proof of (1). Uniqueness follows from the fact that $\Lambda \rightarrow \text{Fun}(\mathbb{Y})$ is an embedding. Let us check the required relation:

Let $m = |\mu|$ and $\nu \in \mathbb{Y}_{n+1}$. We assume $n \geq m$, otherwise $(FS_\mu)_n = 0$. We have

$$\begin{aligned}
D_{n+1,n}(FS_\mu)_n(\nu) &= \sum_{\nu_\bullet \nearrow \nu} \frac{\dim \nu_\bullet}{\dim \nu} FS_\mu(\nu_\bullet) \\
&= \sum_{\nu_\bullet \nearrow \nu} \frac{\dim \nu_\bullet}{\dim \nu} n^{\downarrow m} \frac{\dim(\mu, \nu_\bullet)}{\dim \nu_\bullet} \quad \text{by (4.1)} \\
&= n^{\downarrow m} \sum_{\nu_\bullet \nearrow \nu} \frac{\dim(\mu, \nu_\bullet)}{\dim \nu} \\
&= n^{\downarrow m} \frac{\dim(\mu, \nu)}{\dim \nu} \\
&= \frac{n^{\downarrow m}}{(n+1)^{\downarrow m}} (FS_\mu)_{n+1}(\nu) \quad \text{by (4.1)} \\
&= \frac{n+1-m}{n+1} (FS_\mu)_{n+1}(\nu).
\end{aligned}$$

This coincides with

$$\frac{1}{n+1} \tilde{D} FS_\mu(\nu) = \frac{1}{n+1} ((p_1 - m) FS_\mu)(\nu),$$

because $p_1(\nu) = n+1$. \square

The proof of (2) is less trivial and depends on the lemma below which is essentially due to Sergei Kerov (see Okounkov [Ok]).

Let $\text{Fun}_0(\mathbb{Y}) \subset \text{Fun}(\mathbb{Y})$ be the space of functions with finite support, and let $\{\delta_\lambda\}$ be its natural basis: $\delta_\lambda(\lambda) = 1$ and $\delta_\lambda(\nu) = 0$ for $\nu \neq \lambda$. Consider the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ with its basis

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Lemma 4.2. *For any complex z and z' , the following action of E , F , and H on the basis $\{\delta_\lambda\}$ defines a representation of $\mathfrak{sl}(2, \mathbb{C})$ in $\text{Fun}_0(\mathbb{Y})$*

$$E\delta_\lambda = \sum_{\lambda_\bullet \searrow \lambda} (z)_{\lambda_\bullet/\lambda} (z')_{\lambda_\bullet/\lambda} \delta_{\lambda_\bullet}, \quad F\delta_\lambda = - \sum_{\lambda_\bullet \nearrow \lambda} \delta_{\lambda_\bullet}, \quad H\delta_\lambda = (zz' + 2|\lambda|)\delta_\lambda.$$

Proof. The only nontrivial commutation relation to be checked is $[E, F] = H$. We have

$$[E, F]\delta_\lambda = \sum_{\varkappa \nearrow \lambda_\bullet} \sum_{\lambda_\bullet \searrow \lambda} (z)_{\lambda_\bullet/\lambda} (z')_{\lambda_\bullet/\lambda} \delta_\varkappa - \sum_{\varkappa \searrow \lambda_\bullet} \sum_{\lambda_\bullet \nearrow \lambda} (z)_{\varkappa/\lambda_\bullet} (z')_{\varkappa/\lambda_\bullet} \delta_\varkappa$$

The right-hand side is a linear combination of the vectors δ_\varkappa such that either $\varkappa = \lambda$ or \varkappa is obtained from λ by adding a box \square_1 and removing another square $\square_2 \neq \square_1$. In the latter case, the coefficient of δ_\varkappa in each double sum equals $(z)_{\square_1} (z')_{\square_1}$, so that the total coefficient is 0.

Examine now the coefficient of δ_λ , which is equal to

$$\sum_{\lambda^\bullet \searrow \lambda} (z_{\lambda^\bullet/\lambda})(z'_{\lambda^\bullet/\lambda}) - \sum_{\lambda^\bullet \nearrow \lambda} (z_{\lambda/\lambda^\bullet})(z'_{\lambda/\lambda^\bullet})$$

Denoting by $\{x_i\}$ and $\{y_j\}$ the contents of the boxes¹⁰ that can be added to the diagram λ or removed from it, respectively, we write the above expression as

$$\begin{aligned} & \sum_i (z + x_i)(z' + x_i) - \sum_j (z + y_j)(z' + y_j) \\ &= \left(\sum_i 1 - \sum_j 1 \right) z z' + \left(\sum_i x_i - \sum_j y_j \right) (z + z') + \left(\sum_i x_i^2 - \sum_j y_j^2 \right). \end{aligned}$$

As was first observed by Kerov (see his book [K, Ch. IV, §1]), the x_i 's and the y_j 's form two interlacing sequences

$$x_1 < y_1 < x_2 < \cdots < x_k < y_k < x_{k+1}$$

such that

$$\sum_i x_i - \sum_j y_j = 0, \quad \sum_i x_i^2 - \sum_j y_j^2 = 2|\lambda|.$$

The easiest way to prove this is to proceed by induction on $|\lambda|$, by consecutively adding a box to the diagram.

It follows that in our expression, the coefficient of $z z'$ equals 1 (because the number of x 's is greater than the number of y 's by 1), that of $z + z'$ equals 0, and the last term equals $2|\lambda|$. This completes the proof. \square

The next lemma is a simple observation:

Lemma 4.3. *Let $\square_{k,l}$ be the rectangular diagram with k rows and l columns, and $V_{k,l} \subset \text{Fun}_0(\mathbb{Y})$ stand for the finite dimensional subspace spanned by the basis vectors δ_λ such that $\lambda \subseteq \square_{k,l}$.*

If $z = k$ and $z' = -l$ then $V_{k,l}$ is $\mathfrak{sl}(2, \mathbb{C})$ -invariant and the action of $\mathfrak{sl}(2, \mathbb{C})$ in $V_{k,l}$ lifts to a representation of the group $SL(2, \mathbb{C})$.

Proof. If a diagram λ is contained in $\square_{k,l}$ while a diagram λ^\bullet , such that $\lambda^\bullet \searrow \lambda$, is not, then the square λ^\bullet/λ may be only one of the boxes $(1, l+1)$ or $(k+1, 1)$. In the former case, $(z')_{\lambda^\bullet/\lambda}$ vanishes, and in the latter case $(z)_{\lambda^\bullet/\lambda}$ vanishes. Therefore, the coefficient of δ_{λ^\bullet} in the expansion of $E\delta_\lambda$ equals 0. It follows that the subspace $V_{k,l}$ is E -invariant, and its invariance with respect to F and H is obvious. Thus, $V_{k,l}$ is an $\mathfrak{sl}(2, \mathbb{C})$ -module. Since it has finite dimension, it generates a representation of the group $SL(2, \mathbb{C})$. Note that this representation is irreducible but we do not need this fact. \square

We proceed to the proof of the second claim of Theorem 4.1.

¹⁰The *content* of a box (i, j) is defined as $j - i$.

Proof of (2). As in (1), the uniqueness part of the claim is evident. The remaining (nontrivial) part of the claim means that for any $n \geq |\mu|$ and any $\lambda \in \mathbb{Y}_n$,

$$(zz' + n)(U_{n,n+1}(FS_\mu)_{n+1})(\lambda) = \sum_{\mu_\bullet \nearrow \mu} (z)_{\mu/\mu_\bullet} (z')_{\mu/\mu_\bullet} FS_{\mu_\bullet}(\lambda) + (n + zz' + |\mu|)FS_\mu(\lambda). \quad (4.2)$$

For $\mu = \emptyset$, FS_μ reduces to the constant function 1, the sum in the right-hand side disappears, and (4.2) reduces to the tautology $zz' + n = zz' + n$.

Assume now $|\mu| = m \geq 1$. Using the definition of $U_{n,n+1}$ and the basic formula (4.1), one can reduce (4.2) to the following combinatorial identity

$$\sum_{\lambda^\bullet \searrow \lambda} (z)_{\lambda^\bullet/\lambda} (z')_{\lambda^\bullet/\lambda} \dim(\mu, \lambda^\bullet) = \sum_{\mu_\bullet \nearrow \mu} (z)_{\mu/\mu_\bullet} (z')_{\mu/\mu_\bullet} \dim(\mu_\bullet, \lambda) + (n + zz' + m)(n - m + 1) \dim(\mu, \lambda), \quad (4.3)$$

where $\lambda \in \mathbb{Y}_n$, $n \geq m$.

Observe that the identity is satisfied if λ does not contain μ . Indeed, in such a case $\dim(\mu, \lambda) = 0$, and the last summand disappears. If the set difference $\mu \setminus \lambda$ contains 2 or more boxes then no λ^\bullet contains μ and no μ_\bullet is contained in λ , so that both sides vanish. Examine now the case when $\mu \setminus \lambda$ consists of a single square \square . Then the only nonzero contribution to the left-hand side comes from the summand with $\lambda^\bullet = \lambda \cup \square$, and the only nonzero contribution to the right-hand side comes from $\mu_\bullet = \mu \setminus \square$. Since $\lambda^\bullet/\lambda = \mu/\mu_\bullet = \square$, the identity is reduced to $\dim(\mu, \lambda^\bullet) = \dim(\mu_\bullet, \lambda)$, which is obvious, because the skew diagrams λ^\bullet/μ and λ/μ_\bullet coincide.

Thus, we may assume $\mu \subseteq \lambda$. We will check the identity using Lemma 4.2. Since both sides are polynomials in z and z' , we may assume that $z = k$ and $z' = -l$, where k and l are so large that all diagrams λ^\bullet are contained in $\square_{k,l}$.

Let us multiply the left-hand side of (4.3) by $(z)_{\lambda/\mu} (z')_{\lambda/\mu}$. Due to our assumptions this quantity is well defined and is nonzero. We obtain

$$\begin{aligned} (z)_{\lambda/\mu} (z')_{\lambda/\mu} \sum_{\lambda^\bullet \searrow \lambda} (z)_{\lambda^\bullet/\lambda} (z')_{\lambda^\bullet/\lambda} \dim(\mu, \lambda^\bullet) &= \sum_{\lambda^\bullet \searrow \lambda} (z)_{\lambda^\bullet/\mu} (z')_{\lambda^\bullet/\mu} \dim(\mu, \lambda^\bullet) \\ &= \sum_{\lambda^\bullet \searrow \lambda} (E^{n-m+1} \delta_\mu, \delta_{\lambda^\bullet}) = (n+1-m)! \sum_{\lambda^\bullet \searrow \lambda} (e^E \delta_\mu, \delta_{\lambda^\bullet}) \\ &= (n+1-m)! (e^E \delta_\mu, \sum_{\lambda^\bullet \searrow \lambda} \delta_{\lambda^\bullet}), \end{aligned}$$

where all operators act in the finite-dimensional subspace $V_{k,l}$ described in Lemma 4.3, and (\cdot, \cdot) is the natural inner product inherited from $\ell^2(\mathbb{Y})$.

Since the operator $\delta_\lambda \mapsto \sum_{\lambda^\bullet \searrow \lambda} \delta_{\lambda^\bullet}$ is adjoint to $-F$, our expression can be rewritten simply as

$$-(n+1-m)! (Fe^E \delta_\mu, \delta_\lambda).$$

A simple computation in $SL(2, \mathbb{C})$ shows that

$$-Fe^F = -e^E (e^{-E} Fe^E) = e^E (-F + E + H),$$

and due to the last claim of Lemma 4.3 we may interpret the above identity as a relation between operators in $V_{k,l}$. Then we obtain

$$\begin{aligned} -(Fe^E\delta_\mu, \delta_\lambda) &= -(e^E F\delta_\mu, \delta_\lambda) + (e^E H\delta_\mu, \delta_\lambda) + (e^E E\delta_\mu, \delta_\lambda) \\ &= \sum_{\mu_\bullet \nearrow \mu} (e^E \delta_{\mu_\bullet}, \delta_\lambda) + (zz' + 2m)(e^E \delta_\mu, \delta_\lambda) + (Ee^E \delta_\mu, \delta_\lambda) \end{aligned} \quad (4.4)$$

It remains to check that multiplying (4.4) by $(n - m + 1)!$ gives the right-hand side of (4.3) multiplied by $(z)_{\lambda/\mu}(z')_{\lambda/\mu}$. We expression (4.4) comprises three terms.

The first term gives

$$\begin{aligned} (n - m + 1)! \sum_{\mu_\bullet \nearrow \mu} (e^E \delta_{\mu_\bullet}, \delta_\lambda) &= \sum_{\mu_\bullet \nearrow \mu} (z)_{\lambda/\mu_\bullet} (z')_{\lambda/\mu_\bullet} \dim(\mu_\bullet, \lambda) \\ &= (z)_{\lambda/\mu} (z')_{\lambda/\mu} \sum_{\mu_\bullet \nearrow \mu} (z)_{\mu/\mu_\bullet} (z')_{\mu/\mu_\bullet} \dim(\mu_\bullet, \lambda). \end{aligned} \quad (4.5)$$

Next, the second term gives

$$(n - m + 1)!(zz' + 2m)(e^E \delta_\mu, \delta_\lambda) = (z)_{\lambda/\mu} (z')_{\lambda/\mu} (zz' + 2m)(n - m + 1) \dim(\mu, \lambda),$$

the third term gives

$$(n - m + 1)!(Ee^E \delta_\mu, \delta_\lambda) = (z)_{\lambda/\mu} (z')_{\lambda/\mu} (n - m + 1)(n - m) \dim(\mu, \lambda),$$

and their sum equals

$$(z)_{\lambda/\mu} (z')_{\lambda/\mu} (zz' + n + m)(n - m + 1) \dim(\mu, \lambda). \quad (4.6)$$

We see that the sum of (4.5) and (4.6) is indeed equal to the right-hand side of (4.3) multiplied by $(z)_{\lambda/\mu}(z')_{\lambda/\mu}$. \square

5. CONVERGENCE OF MARKOV SEMIGROUPS

Fix arbitrary fixed parameters (z, z') of principal or complementary series. Let $\{M_n\} = \{M_{z, z'}^{(n)}\}$ be the corresponding coherent family of probability measures on \mathbb{Y} . According to the general formalism of §1, we form, for each $n = 1, 2, \dots$, the n th level up-down Markov chain on the set \mathbb{Y}_n of Young diagrams with n boxes. Since the move “up” consists in appending a box to a Young diagram, while the move “down” consists in removing a box, any nontrivial change of our up-down chain under one step reduces to moving one of the boxes of a Young diagram to a new position.

The measure M_n is an invariant measure of the chain. It is readily seen that all the states are communicating, so that M_n is a unique invariant measure.

As explained in §3, we consider the embeddings $\iota_n : Y_n \rightarrow \Omega$ determined by the normalized modified Frobenius coordinates. Let π_n be the corresponding linear map $C(\Omega) \rightarrow C(\mathbb{Y}_n)$.

As the space $\mathcal{F} \subset C(\Omega)$ we take the algebra Λ° . It is dense in $C(\Omega)$ by virtue of [KOO, Lemma 5.3]. The filtration in Λ° is inherited from Λ .

The two limit relations in the claim below are understood in the sense of §1.

Theorem 5.1. *With these data, all the assumptions of Proposition 1.4 are satisfied provided that the scaling of time is determined by the factors $\varepsilon_n \sim n^{-2}$. Thus, denoting by T_n the Markov operator of the n th level Markov chain, we have convergence to a conservative Markov semigroup $\{T(t)\}$ in the Banach space $C(\Omega)$, as in (1.4):*

$$\lim_{n \rightarrow \infty} T^{[n^2 t]} \pi_n(f) = T(t)f$$

for any fixed $f \in C(\Omega)$ and all $t \geq 0$, uniformly on bounded intervals. Furthermore, the generator of the limit semigroup $\{T(t)\}$ is the closure of the operator A with domain $\Lambda^\circ \subset C(\Omega)$, defined by

$$Af = \lim_{n \rightarrow \infty} n^2(T_n - \mathbf{1})f, \quad f \in \Lambda^\circ.$$

Proof. By virtue of Proposition 1.4, it suffices to check assumptions (A1)–(A5) stated before the formulation of the proposition.

The fulfilment of assumptions (A1), (A2), and (A5) is obvious.

We proceed to verifying assumption (A3).

Recall that the Markov operator $T_n : C(\mathbb{Y}_n) \rightarrow C(\mathbb{Y}_n)$ is defined as the composition $U_{n,n+1} \circ D_{n+1,n}$ of the up and down operators. As shown in §4, the latter operators are implemented by certain operators in Λ . Here we interpret Λ as a subalgebra in $\text{Fun}(\mathbb{Y})$ and consider the restriction map $\text{Fun}(\mathbb{Y}) \rightarrow C(\mathbb{Y}_n)$ turning elements $f \in \Lambda$ to functions f_n on \mathbb{Y}_n .

Lemma 5.2. *Let μ be a Young diagram and $m = |\mu|$. The operator $T_n - \mathbf{1}$ acts on $(FS_\mu)_n$ as follows*

$$\begin{aligned} (T_n - \mathbf{1})(FS_\mu)_n &= -\frac{m(m-1+zz')}{(n+1)(zz'+n)}(FS_\mu)_n \\ &\quad + \frac{n+1-m}{(n+1)(zz'+n)} \sum_{\mu_\bullet \nearrow \mu} (z)_{\mu/\mu_\bullet} (z')_{\mu/\mu_\bullet} (FS_{\mu_\bullet})_n. \end{aligned}$$

Proof. This follows directly from the computation of §4. Indeed, we have

$$\begin{aligned} T_n(FS_\mu)_n &= U_{n,n+1} D_{n+1,n} (FS_\mu)_n, \\ D_{n+1,n} (FS_\mu)_n &= \frac{n+1-m}{n+1} (FS_\mu)_{n+1}, \\ U_{n,n+1} (FS_\mu)_{n+1} &= \frac{1}{zz'+n} \left(\sum_{\mu_\bullet \nearrow \mu} (z)_{\mu/\mu_\bullet} (z')_{\mu/\mu_\bullet} (FS_\mu)_n + (n+zz'+m)(FS_\mu)_n \right), \end{aligned}$$

which implies the desired expression. Note that in the last equality we used the fact that $(p_1 FS_\mu)_n = n(FS_\mu)_n$. \square

Corollary 5.3. *T_n preserves the filtration in \mathcal{F} .*

Let us verify assumption (A4) with $\varepsilon_n = n^{-2}$:

Consider the map $\Lambda \rightarrow C(\mathbb{Y}_n)$, defined as restriction to $\iota_n(\mathbb{Y}_n) \subset \Omega$. We denote it as $f \mapsto f_{[n]}$. By definition, if $\lambda = (a_1, \dots, a_d \mid b_1, \dots, b_d) \in \mathbb{Y}_n$ (the modified Frobenius coordinates) then

$$f_{[n]}(\lambda) = f\left(\frac{a_1}{n}, \dots, \frac{a_d}{n}, 0, 0, \dots; \frac{b_1}{n}, \dots, \frac{b_d}{n}, 0, 0, \dots\right), \quad f \in \Lambda.$$

In the notation of §1,

$$f_{[n]} = \pi_n(f^\circ), \quad f \in \Lambda.$$

Let $G : \Lambda \rightarrow \Lambda$ denote the operator acting on the m th homogeneous component of Λ as multiplication by m ($m = 0, 1, 2, \dots$). According to this definition, we denote by s^G , where $s \neq 0$, the automorphism of the algebra Λ that reduces to multiplication by s^m on the homogeneous component of degree m . Then we have

$$f_n(\lambda) = (n^G f)_{[n]}(\lambda).$$

Lemma 5.4. *There exists a linear operator $A_{z, z'}$ in $\mathcal{F} = \Lambda^\circ$ which is the limit of the operators $n^2(T_n - \mathbf{1})$ as $n \rightarrow 1$. Specifically,*

$$A_{z, z'} s_\mu^\circ = -(mzz' + m(m-1))s_\mu^\circ + \sum_{\mu_\bullet \nearrow \mu} (z)_{\mu/\mu_\bullet} (z')_{\mu/\mu_\bullet} s_{\mu_\bullet}^\circ, \quad \mu \in \mathbb{Y}, \quad m := |\mu|. \quad (5.1)$$

Proof. By Lemma 5.2, we have

$$\begin{aligned} (T_n - \mathbf{1})(n^G(FS_\mu)_{[n]}) &= -\frac{m(m-1+zz')}{(n+1)(zz'+n)} (n^G(FS_\mu)_{[n]}) \\ &\quad + \frac{n+1-m}{(n+1)(zz'+n)} \sum_{\mu_\bullet \nearrow \mu} (z)_{\mu/\mu_\bullet} (z')_{\mu/\mu_\bullet} (n^G(FS_{\mu_\bullet})_{[n]}). \end{aligned}$$

Multiply both sides by n^{-m} , where $m = |\mu|$, and observe that

$$\lim_{n \rightarrow \infty} n^{-|\mu|} n^G FS_\mu = s_\mu$$

because FS_μ and s_μ differ in lower order terms only. This implies the claim of the lemma. \square

Note that for $\mu = \emptyset$, the sum in (5.1) disappears and m vanishes, so that A sends $s_\emptyset^\circ = 1$ to 0, as it should be.

This concludes the proof of Theorem 5.1. \square

Looking at formula (5.1), it is not obvious that it defines an operator in Λ° , because the elements s_μ° are not linearly independent. Of course, correctness of (5.1) follows from the computation in the proof of Lemma 5.4. On the other hand, this also can be proved directly:

Proposition 5.5. *The operator $B_{z, z'} : \Lambda \rightarrow \Lambda$ determined in the basis of Schur functions by the expression*

$$B_{z, z'} s_\mu = -|\mu|(|\mu| - 1 + zz')s_\mu + p_1 \sum_{\mu_\bullet \nearrow \mu} (z)_{\mu/\mu_\bullet} (z')_{\mu/\mu_\bullet} s_{\mu_\bullet} \quad (5.2)$$

preserves the principal ideal generated by $p_1 - 1$, and the reduction of B modulo this ideal coincides with the expression (5.1).

Proof. The second claim is obvious from the comparison of (5.2) with (5.1); note that the prefactor p_1 in front of the sum in (5.2) will disappear after the reduction. Note also that the operator B preserves the grading in Λ .

To prove the first claim it suffices to check that B commutes with the operator of multiplication by p_1 .

We use the representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ in the space $\text{Fun}_0(\mathbb{Y})$ defined in Lemma 4.2. Let H^* , E^* , and F^* be the adjoint operators to H , E , and F , respectively. We interpret them again as operators in $\text{Fun}_0(\mathbb{Y})$, given by adjoint matrices in the basis $\{\delta_\mu\}$ (equivalently, by transposed matrices, because all the matrices in questions have real entries). Note that $H^* = H$.

Now identify $\text{Fun}_0(\mathbb{Y})$ and Λ (as vector spaces) via the correspondence $\delta_\mu \leftrightarrow s_\mu$. Then we may interpret $H^* = H$, E^* , and F^* as operators in Λ . From the definition of F and the well-known identity

$$p_1 s_\lambda = \sum_{\lambda \bullet \searrow \lambda} s_{\lambda \bullet}$$

it follows that the operator of multiplication by p_1 equals $-F^*$. Next, the operator

$$s_\mu \mapsto \sum_{\mu \bullet \nearrow \mu} (z)_{\mu/\mu \bullet} (z')_{\mu/\mu \bullet} s_{\mu \bullet}$$

equals E^* . Finally, recall the operator G introduced just before Lemma 5.4 and note that

$$G : s_\mu \mapsto |\mu| s_\mu.$$

Therefore,

$$B_{z, z'} = -G(G - 1 + \tau) - F^* E^*, \quad \tau := z z'.$$

We have to check that this operator commutes with $-F^*$, which is equivalent to

$$[G(G - 1 + \tau) + EF, F] = 0.$$

Observe that $H = 2G + \tau$. Therefore

$$G(G - 1 + \tau) + EF = \frac{1}{4}H^2 - \frac{1}{2}H + EF - \frac{1}{4}\tau^2 + \frac{1}{2}\tau.$$

It is readily checked that this expression gives a central element in the universal enveloping algebra of $\mathfrak{sl}(2, \mathbb{C})$. \square

6. THE PRE-GENERATOR $A_{z, z'}$ AS A DIFFERENTIAL OPERATOR

Our aim is to represent the operator $A : \Lambda^\circ \rightarrow \Lambda^\circ$ defined by (5.1) as a differential operator with polynomial coefficients.¹¹

¹¹Note that *any* linear operator in the vector space of polynomials with countably many indeterminates can be written as a differential operator, that is, as an infinite sum of differential monomials with polynomial coefficients.

Theorem 6.1. *In the moment coordinates $q_1 = p_2^\circ, q_2 = p_3^\circ, \dots$, the operator $A_{z,z'}$ defined by (5.1) can be written as the differential operator*

$$\begin{aligned} A_{z,z'} = & \sum_{i,j=1}^{\infty} (i+1)(j+1)(q_{i+j} - q_i q_j) \frac{\partial^2}{\partial q_i \partial q_j} \\ & - z z' \sum_{i=1}^{\infty} (i+1) q_i \frac{\partial}{\partial q_i} + (z+z') \sum_{i=1}^{\infty} (i+1) q_{i-1} \frac{\partial}{\partial q_i} \\ & + \sum_{i,j=0}^{\infty} (i+j+3) q_i q_j \frac{\partial}{\partial q_{i+j+2}} - \sum_{i=1}^{\infty} (i+1) i q_i \frac{\partial}{\partial q_i}, \end{aligned} \quad (6.1)$$

where, by agreement, $q_0 = 1$.

Proof. We will show that the operator $B_{z,z'} : \Lambda \rightarrow \Lambda$ (see (5.2)) can be written as the following differential operator in the variables p_1, p_2, \dots :

$$\begin{aligned} B_{z,z'} = & \sum_{i,j=2}^{\infty} i j (p_1 p_{i+j-1} - p_i p_j) \frac{\partial^2}{\partial p_i \partial p_j} \\ & - z z' \sum_{i=2}^{\infty} i p_i \frac{\partial}{\partial p_i} + (z+z') \sum_{i=2}^{\infty} i p_1 p_{i-1} \frac{\partial}{\partial p_i} \\ & + \sum_{i,j=1}^{\infty} (i+j+1) p_1 p_i p_j \frac{\partial}{\partial p_{i+j+1}} - \sum_{i=2}^{\infty} i(i-1) p_i \frac{\partial}{\partial p_i} \end{aligned} \quad (6.2)$$

Note that (6.2) does not involve $\partial/\partial p_1$, so that we can reduce (6.2) modulo the relation $p_1 = 1$, by sending p_i to $p_i^\circ = q_{i-1}$, with the understanding that $p_1^\circ = q_0 = 1$. By virtue of Proposition 5.5 this will give us (6.1).

The main difficulty is to handle the sum in the right-hand side of (5.2), that is, the operator in Λ defined by

$$s_\mu \mapsto \sum_{\mu_\bullet \nearrow \mu} (z)_{\mu/\mu_\bullet} (z')_{\mu/\mu_\bullet} s_{\mu_\bullet}.$$

Given a box \square with the row coordinate i and the column coordinate j , we denote by $c(\square)$ its content $j-i$. The latter operator can be written as a linear combination of three operators,

$$C'_2 + (z+z')C'_1 + z z' C'_0,$$

where by $C'_k : \Lambda \rightarrow \Lambda$ we denote the operator

$$C'_k : s_\mu \mapsto \sum_{\mu_\bullet \nearrow \mu} (c(\mu/\mu_\bullet))^k s_{\mu_\bullet}, \quad k = 0, 1, 2, \dots$$

In this notation the operator $B_{z,z'}$ is written as follows

$$B_{z,z'} = -G(G - \mathbf{1} + z z' \mathbf{1}) + p_1 (C'_2 + (z+z')C'_1 + z z' C'_0). \quad (6.3)$$

By the very definition of G we have

$$G = \sum_{i=1}^{\infty} i p_i \frac{\partial}{\partial p_i}. \quad (6.3)$$

The operator C'_0 is also easy to write:

$$C'_0 = \partial/\partial p_1. \quad (6.5)$$

Indeed, to see this, one can use the fact that $\partial/\partial p_1$ is adjoint to the operator of multiplication by p_1 , which has the form

$$s_\mu \mapsto \sum_{\mu_\bullet \searrow \mu} s_{\mu_\bullet}.$$

The operator C'_1 and especially the operator C'_2 are more involved.

To handle them it is convenient to introduce auxiliary operators $C_k : \Lambda \rightarrow \Lambda$ by

$$C_k : s_\mu \mapsto \left(\sum_{\square \in \mu} (c(\square))^k \right) s_\mu,$$

summed over all boxes contained in μ , and observe that $C'_0 = \partial/\partial p_1$ implies

$$C'_k = \left[\frac{\partial}{\partial p_1}, C_k \right].$$

We will employ the following results proved in [LT, Proposition 3.3]:

Lemma 6.2. *Let t and u be formal variables. The exponential generating series for the operators C_k has the form*

$$\sum_{k=1}^{\infty} C_k \frac{t^k}{k!} = \frac{V_0 - 1}{(e^t - 1)(1 - e^{-t})} - G,$$

where V_0 is the constant term of the “vertex operator”

$$\begin{aligned} V(u) &= \sum_{m=-\infty}^{+\infty} V_m u^{-m} \\ &:= \exp \left(\sum_{k=1}^{\infty} (e^{kt} - 1) \frac{u^k}{k} p_k \right) \exp \left(\sum_{k=1}^{\infty} (1 - e^{-kt}) u^{-k} \frac{\partial}{\partial p_k} \right). \end{aligned}$$

Moreover, the following commutation relations hold:

$$\left[V_m, k \frac{\partial}{\partial p_k} \right] = (1 - e^{kt}) V_{m+k}.$$

In particular,

$$\left[\frac{\partial}{\partial p_1}, V_0 \right] = (e^t - 1) V_1.$$

From Lemma 6.2 we deduce explicit expressions for the operators C'_1 and C'_2 :

Lemma 6.3. *We have*

$$C'_1 = \sum_{i=1}^{\infty} (i+1)p_i \frac{\partial}{\partial p_{i+1}} \quad (6.6)$$

$$C'_2 = \sum_{i,j=1}^{\infty} ij p_{i+j-1} \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} + \sum_{i,j=1}^{\infty} (i+j+1)p_i p_j \frac{\partial}{\partial p_{i+j+1}}. \quad (6.7)$$

Note that this particular result can also be obtained by the more elementary approach used in [FW].

Proof of Lemma 6.3. From Lemma 6.2 we get

$$\left[\frac{\partial}{\partial p_1}, C_k \right] = \text{coefficient of } t^k u^{-1} \text{ in } k! \frac{V(u)}{1 - e^{-t}}, \quad k = 1, 2, \dots$$

Next, write

$$V(u) = \exp \left(\sum_{r=1}^{\infty} a_r t^r \right) \exp \left(\sum_{r=1}^{\infty} b_r t^r \right),$$

where

$$a_r = \sum_{k=1}^{\infty} \frac{k^r u^k}{r! k} p_k, \quad b_r = (-1)^{r-1} \sum_{k=1}^{\infty} \frac{k^r u^{-k}}{r!} \frac{\partial}{\partial p_k}. \quad (6.8)$$

Then

$$C'_1 = \left[\frac{\partial}{\partial p_1}, C_1 \right] = \text{the coefficient of } t^2 u^{-1} \text{ in} \\ \frac{\exp(a_1 t + a_2 t^2 + \dots) \exp(b_1 t + b_2 t^2 + \dots)}{1 - \frac{1}{2}t + \frac{1}{6}t^2 + \dots} \quad (6.9)$$

and

$$C'_2 = \left[\frac{\partial}{\partial p_1}, C_2 \right] = \text{the coefficient of } t^3 u^{-1} \text{ in} \\ 2 \cdot \frac{\exp(a_1 t + a_2 t^2 + a_3 t^3 + \dots) \exp(b_1 t + b_2 t^2 + b_3 t^3 + \dots)}{1 - \frac{1}{2}t + \frac{1}{6}t^2 - \frac{1}{24}t^3 + \dots}, \quad (6.10)$$

where the dependence in u is hidden in (6.8). First, we compute the coefficients of t^2 and t^3 in (6.9) and (6.10), respectively.

The coefficient of t^2 in (6.9) equals

$$a_2 + \frac{1}{2}a_1^2 + a_1 b_1 + \frac{1}{2}a_1 + b_2 + \frac{1}{2}b_1^2 + \frac{1}{2}b_1 + \frac{1}{12}.$$

From (6.8) it immediately follows that a nonzero contribution to the term with u^{-1} can come from $a_1 b_1$, b_2 , and $\frac{1}{2}b_1$ only. It turns out that the total contribution of $b_2 + \frac{1}{2}b_1$ equals 0. This gives (6.6).

Likewise, coefficient of t^3 in (6.10) equals

$$\begin{aligned} & 2a_3 + a_2 + 2a_1a_2 + 2a_2b_1 + \frac{1}{2}a_1^2 + \frac{1}{3}a_1^3 + a_1^2b_1 + 2a_1b_2 + a_1b_1 \\ & + a_1b_1^2 + \frac{1}{6}a_1 + 2b_3 + b_2 + 2b_1b_2 + \frac{1}{2}b_1^2 + \frac{1}{3}b_1^3 + \frac{1}{6}b_1 \end{aligned}$$

and a nonzero contribution to the term with u^{-1} can come from

$$2a_2b_1 + a_1^2b_1 + 2a_1b_2 + a_1b_1 + a_1b_1^2 + 2b_3 + b_2 + \frac{1}{6}b_1$$

only. It turns out that the total contribution of $a_1b_1 + 2a_2b_1 + 2a_1b_2$ equals 0, and the same holds for $2b_3 + b_2 + \frac{1}{6}b_1$. Finally, we get (6.7). \square

Substituting the expressions (6.4), (6.5), (6.6), and (6.7) into (6.3) we get after cancellations the desired expression (6.2). \square

Let $\bar{A}_{z,z'}$ denote the closure of the operator $A_{z,z'}$ in $C(\Omega)$ (recall that the closure exists according to Proposition 1.4 (1)).

Corollary 6.4. *Let $f(\omega)$ be a smooth cylinder function in the moment coordinates, that is, $f(\omega) = g(q_1(\omega), \dots, q_m(\omega))$ for a certain $m = 1, 2, \dots$ and a certain smooth function $g(q_1, \dots, q_m)$ in m real variables, in a neighborhood of $[-1, 1]^m$. Then f enters the domain of $\bar{A}_{z,z'}$. Moreover, $\bar{A}_{z,z'}f$ is also a cylinder function, which can be obtained via application of the suitably truncated differential expression (6.1) to the function g .*

Proof. First of all, note that here “truncation” means that we keep in (6.1) only terms not containing derivatives $\partial/\partial q_i$ with $i > m$. It is worth noting that the resulting cylinder function depends on the larger number of variables, $2m$, because of the presence of the variables q_{k+l} in (6.1). However, this does not cause problems.

To prove the claim of the proposition we observe that the function $g(q_1, \dots, q_m)$, together with its partial derivatives of up to second order, can be approximated by a sequence $\{g_n(q_1, \dots, q_m)\}$ of polynomials, uniformly on the cube $[-1, 1]^m \subset \mathbb{R}^m$. Let $[A]$ stand for the truncated differential operator as explained above. The application of \bar{A} to the function $\omega \mapsto g_n(q_1(\omega), \dots, q_m(\omega))$ is reduced to the application of $[A]$ to g_n . Since $g_n \rightarrow g$ and $[A]g_n \rightarrow [A]g$ uniformly on the cube $[-1, 1]^{2m}$, we see that f belongs to the domain of $\bar{A}_{z,z'}$ and $\bar{A}_{z,z'}f$ is given by the cylinder function $[A]g$. \square

7. THE LIMIT PROCESS

The theorems of this section are almost direct consequences of the results established in sections 1, 5, and 6. Note that application of Propositions 1.6 and 1.7 is justified, because their hypothesis is satisfied due to the theorem of §3.6.

Theorem 7.1. *The Markov semigroup $\{T(t)\}$ in $C(\Omega)$ constructed in §5 gives rise to a diffusion process $\omega_{z,z'}(t)$ in Ω .*

By a diffusion process we mean a strong Markov process (which can start from any point or any probability distribution) with continuous sample paths.

Proof. Once the existence of a Markov semigroup $\{T(t)\}$ is established (Theorem 5.1), the existence of the corresponding strong Markov process is guaranteed by

a general theorem stated above as Proposition 1.5. It remains to prove that the sample paths are continuous almost surely.

Since our semigroup acts in the space of continuous functions on a compact space, the application of Riesz's theorem implies the existence of a transition function (see, e.g., [L, §7.7, Thm. 1]). The continuity property holds if the transition function obeys the Dynkin–Kinney condition ([W, §10.3] or [EK2, Ch. 4, (2.35)]). This condition in turn holds if for any point $\omega \in \Omega$ and any its neighborhood $U \ni \omega$ one can find a function $f \in C(\Omega)$ with the following properties (see [EK2, Ch. 4, Remark 2.10]):

- (1) f is contained in $D(\bar{A})$, the domain of the generator \bar{A} of the semigroup;
- (2) $\bar{A}f(\omega) = 0$;
- (3) $\|f\| = f(\omega)$ and the supremum of f outside U is strictly less than $f(\omega)$.

Such functions can be built using Corollary 6.4. Indeed, take m so large that one can find inside U a neighborhood of the form

$$\{\omega' \in \Omega : |q_i(\omega') - q_i(\omega)| < \varepsilon, \quad 1 \leq i \leq m\}.$$

Then take as f a cylinder function as in Corollary 6.4, where g equals 1 in a very small neighborhood of the point $(q_1(\omega), \dots, q_m(\omega)) \in [0, 1]^m$ and then rapidly decays to 0. Since the differential operator (6.1) does not have a constant term, $\bar{A}f$ vanishes about ω . \square

Let $P_{z,z'}$ be the boundary measure on Ω corresponding to the coherent system with parameters (z, z') , and consider the inner product in $\Lambda^\circ \subset C(\Omega)$ given by

$$(f, g)_{z,z'} = \langle f \cdot g \rangle_{P_{z,z'}}.$$

Theorem 7.2. (1) *The space Λ° can be decomposed into a direct sum of eigenspaces of the pre-generator $A_{z,z'}$, and this decomposition is orthogonal with respect to the above inner product.*

- (2) *The spectrum of $A_{z,z'}$ is $\{0\} \cup \{-\sigma_m : m = 2, 3, \dots\}$ where*

$$\sigma_m = m(m - 1 + zz'), \quad m = 2, 3, \dots$$

- (3) *The eigenvalue 0 is simple, and the multiplicity of $-\sigma_m$ equals the number of partitions of m without parts equal to 1.*

Proof. (1) This is a fact of linear algebra because the pre-generator $A_{z,z'}$ is symmetric (Proposition 1.7) and preserves the filtration of Λ° .

- (2) Let I denote the principal ideal in Λ generated by $p_1 - 1$. We have

$$\Lambda = \mathbb{R}[p_1, p_2, p_3, \dots] = \mathbb{R}[p_2, p_3, \dots] \oplus I$$

so that we may identify Λ° with $\mathbb{R}[p_2, p_3, \dots]$. It follows from (5.2) that for any homogeneous element $f \in \Lambda$ of degree m ,

$$B_{z,z'}f = -m(m - 1 + zz')f + g + h, \quad \text{where } g \in I \text{ and } \deg h < m.$$

In particular, this is true for any monomial $p_2^{m_2} p_3^{m_3} \dots$ from $\mathbb{R}[p_2, p_3, \dots]$. Taking into account Proposition 5.5 we conclude that the spectrum of $A_{z,z'}$ is as indicated in claim (2) and the multiplicity of $-\sigma_m$ equals the number of solutions in nonnegative integers of the equation

$$2m_2 + 3m_3 + \dots = m,$$

which proves claim (3). \square

Theorem 7.3. (1) *The process $\omega_{z,z'}(t)$ constructed in Theorem 7.1 has the boundary measure $P_{z,z'}$ as a unique stationary distribution.*

(2) *It is also a symmetrizing measure.*

(3) *The process is ergodic in the sense that for any $f \in C(\Omega)$,*

$$\lim_{t \rightarrow +\infty} \|T(t)f - \langle f \rangle_{P_{z,z'}} \cdot 1\| = 0,$$

where the norm is the supremum one and 1 is the constant function equal to one.

Proof. Consider the orthogonal decomposition of Λ° onto eigenspaces afforded by Theorem 7.2:

$$\Lambda^\circ = \mathbb{R}1 \oplus \bigoplus_{m=2}^{\infty} \Lambda_m^\circ. \quad (7.1)$$

The operator $T(t)$ leaves invariant the constant 1 and acts in Λ_m° as multiplication by $\exp(-\sigma_m t)$. Note that the direct sum in (7.1) is understood in purely algebraic sense: for any vector $f \in \Lambda^\circ$, its decomposition $f = f_0 + f_2 + f_3 + \dots$ has finitely many nonzero components.

(1) The fact that $P_{z,z'}$ is an invariant distribution follows from Proposition 1.6. To prove uniqueness we observe that if P is an invariant distribution then $\langle f \rangle_P = 0$ for any $f \in \Lambda_m^\circ$, $m = 2, 3, \dots$. Therefore, for any $f = f_0 + f_2 + f_3 + \dots \in \Lambda^\circ$ with $f_0 = c1$ we have

$$\langle f \rangle_P = \langle f_0 \rangle_P = c. \quad (7.2)$$

Since Λ° is dense in $C(\Omega)$, P is determined uniquely.

(2) The claim means that

$$\int_{\Omega} (\bar{A}_{z,z'} f(\omega)) g(\omega) P_{z,z'}(d\omega)$$

is symmetric under transposition $f \leftrightarrow g$, for any f and g in the domain of $\bar{A}_{z,z'}$. It suffices to check this for f and g in $\Lambda^\circ \subset C(\Omega)$ and with $A_{z,z'}$ replacing $\bar{A}_{z,z'}$, which follows from Proposition 1.7.

(3) From the decomposition (7.1) it is evident that for any $f \in \Lambda^\circ$, we have $T(t)f \rightarrow f_0$ as $t \rightarrow +\infty$. Here the convergence holds in a finite-dimensional space invariant under the semigroup $T(t)$, hence $T(t)f$ converges to f_0 in norm, too. Together with (7.2) this proves claim (3) for $f \in \Lambda^\circ$. Then it is evident that it also holds for any $f \in C(\Omega)$. \square

Recall that any function $f(\omega)$ from $\Lambda^\circ \subset C(\Omega)$ can be also viewed as a polynomial in the moment coordinates $q_1 = q_1(\omega)$, $q_2 = q_2(\omega)$, \dots . With this understanding, we set, for any two functions $f, g \in \Lambda^\circ$

$$\Gamma(f, g)(\omega) = \sum_{i,j=1}^{\infty} \Gamma_{ij}(\omega) \frac{\partial f}{\partial q_i}(\omega) \frac{\partial g}{\partial q_j}(\omega), \quad (7.3)$$

where

$$\Gamma_{ij}(\omega) = (i+1)(j+1)(q_{i+j}(\omega) - q_i(\omega)q_j(\omega)), \quad i, j = 1, 2, \dots \quad (7.4)$$

Note that the sum is actually finite, because the partial derivatives with sufficiently large indices vanish.

Theorem 7.4. For any $f, g \in \Lambda^\circ \subset C(\Omega)$

$$-\int_{\Omega} A_{z,z'} f(\omega) g(\omega) P_{z,z'}(d\omega) = \int_{\Omega} \Gamma(f, g)(\omega) P_{z,z'}(d\omega)$$

The point here is that both the pre-generator $A_{z,z'}$ and the boundary measure $P_{z,z'}$ depend on the parameters (z, z') while $\Gamma(f, g)$ does not (in the right-hand side, the parameters enter $P_{z,z'}$ only).

Proof. We abbreviate $A = A_{z,z'}$ and $\langle \cdot \rangle = \langle \cdot \rangle_{P_{z,z'}}$. Let us show that

$$2\Gamma(f, g) = A(fg) - (Af)g - f(Ag), \quad f, g \in \Lambda^\circ. \quad (7.3)$$

Indeed, by Theorem 6.1, A is a second order differential operator in the moment coordinates. Therefore, its first order terms do not contribute to the right-hand side of (7.3). Writing

$$A = \sum_{i,j=1}^{\infty} \Gamma_{ij} \frac{\partial^2}{\partial q_i \partial q_j} + \text{first order terms}$$

we see that the right-hand side of (7.3) is equal to

$$2 \sum_{i,j=1}^{\infty} \Gamma_{ij} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q_j},$$

which is precisely the definition of $\Gamma(f, g)$.

Next, as we already pointed out above, the expectation $\langle \cdot \rangle$ vanishes on all eigenspaces of A , except that corresponding to the eigenvalue 0. Consequently $\langle \cdot \rangle$ vanishes on the range of the operator A . Applying the expectation to the both sides of (7.3) and using the fact that A is symmetric, we get the desired formula. \square

Remark 7.5. Note that for any $\omega \in \Omega$, the infinite matrix $[\Gamma_{ij}(\omega)]_{i,j=1}^{\infty}$ is non-negative definite. Indeed, recall that $q_i = q_i(\omega)$ is the i th moment of the Thoma measure ν_ω on $[-1, 1]$. It follows that for any sequence c_1, c_2, \dots of real numbers with finitely many nonzero entries, the quadratic form $\sum_{i,j} (q_{i+j} - q_i q_j) c_i c_j$ equals the variance of the function $\sum_i c_i x^i$ with respect to ν_ω . Cf. Schmuland [S, p. 255]. It is tempting to regard the quantity $\Gamma(f, f)$ defined by (7.3)–(7.4) as a square field (carré du champs). Similar expressions already appeared in works on measure-valued diffusions, see, e.g., Overbeck–Röckner–Schmuland [OvRS].

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