

# CLT for the visited proportion of vertices in random walk system on a complete graph

F. Machado, H. Mashurian, H. Matzinger

## Abstract

We prove a Central Limit Theorem for the visited proportion of individuals in a discrete time system of simple random walks on a complete graph with  $n$  vertices. A random walk duplicates each time it hits a unvisited vertex, dying if they hit an already visited vertex. The process stops as soon as there is no more active random walks.

## 1 Introduction

We prove a Central Limit Theorem for the visited proportion of vertices a discrete time system of simple random walks on  $K_n$ , a complete graph with  $n$  vertices. A comprehensive introduction on random walks on graphs can be found in [1]. This model, known as frog model, has been considered on infinite graphs, in particular hypercubic lattices and homogeneous trees, for which results as shape theorem and phase transition, have been proved. See for instance [2], [3], [4], [6], [7], [8] and the references therein. Our interest in this paper is to study the behavior of this model on complete graphs. For  $n \geq 3$ , let  $K_n$  be the  $n$ -complete graph - the graph with vertex set  $\mathcal{V} = \{1, 2, \dots, n\}$  and each pair of vertices linked by an edge.

It is a discrete time process on  $K_n$  evolving as follows. At time zero there is one inactive particle at each vertex of  $K_n$ . A particle is chosen to become active and by its turn that active particle chooses a vertex to jump at. From that time on, over their independent lifetime, each active particle perform a random walk on the vertices of  $K_n$ , activating all inactive particles it meets along its way. Each active particle lives while it chooses vertices with an inactive particle on it, dying at the first time it chooses to jump on a vertex which has been visited before by some active particle. At each time just one active particle makes a displacement. The process continues until there are no more active particles. We denote by  $V_\infty$  the number of vertices which have been visited by active particles when the process comes to an

end. We investigate the asymptotic distribution of the random variable  $V_\infty$ . The main result of this paper (Theorem 1.1) shows that properly re-scaled,  $V_\infty$  converges in distribution to a normal random variable.

Let us formally define the model. First let us define  $A_t$ ,  $D_t$  and  $I_t$  as the number of active particles at time  $t$ , the number of vertices whose original particles have already died up to time  $t$  and the number of particles still inactive at time  $t$ , respectively. In this sense,  $V_t = A_t + D_t$  and  $V_\infty = \lim_{t \rightarrow \infty} V_t$ . Note that  $\{(A_t, D_t, I_t)\}_{t \geq 0}$  is a Markov chain (in fact,  $\{(A_t, I_t)\}_{t \geq 0}$  is) such that  $A_0 = 1$ ,  $D_0 = 0$ , and  $I_0 = n - 1$ . Moreover,  $A_t + D_t + I_t = n$  for all  $t \geq 0$ . Besides, let  $\{S_t\}_{t \geq 0}$  denote a set of independent uniformly distributed random variables on  $\mathcal{V}$ , the set of vertices of  $K_n$ . At each time  $t$  one active particle (also uniformly chosen among the  $A_{t-1}$  active particles), choose the vertex  $S_t$  to jump to. It meets and activates a still inactive particle if and only if  $S_t \notin \{S_1, \dots, S_{t-1}\}$ . In this case  $A_t = A_{t-1} + 1$ . Otherwise that active particle dies, then  $A_t = A_{t-1} - 1$ . Observe that  $A_\infty := \lim_{t \rightarrow \infty} A_t = 0$ . For simulations and mean field analysis see [5]

Let  $q$  be the only non-zero solution to the equation

$$2p = -\ln(1 - p)$$

in  $[0, 1[$ . (See also lemma 2.2.) Let  $\mu_r$  be equal to

$$\mu_r := 2 - \frac{1}{1 - q}.$$

Finally let  $\sigma$  be equal to

$$\sigma := \frac{\sqrt{\int_0^q \frac{x}{(1-x)^2} dx}}{\mu_r}.$$

We are now ready to formulate the main theorem of this paper

**Theorem 1.1** *We have that*

$$\frac{V_\infty - qn}{\sigma\sqrt{n}} \rightarrow \mathcal{N}(0, 1)$$

*as  $n$  goes to infinity, where  $\rightarrow$  means convergence in law.*

This model can be viewed as an oriented dependent long range percolation model once one consider the analogous setup on an infinite connected graph. The main difficulty in answering the classical questions related to phase transition and shape theorem in this setup is that the classical coupling techniques cannot be applied, besides both FKG and BK inequalities fail.

Another possible approach and source of interest is to see this model as an option for modelling the spread of a disease in a population or spread of viruses in a computer network. Following the setup we use in this paper the virus duplicates any time it infects a susceptible individual. Once that happens the individual becomes immune. The virus dies the first time it tries to infect a immune individual. The population here is considered finite and have full contact as every individual can be contacted directly by any other individual. The main question we investigate in this paper corresponds to determine the distribution of the percentage of the population which escaped from the disease remaining not visited after all the random walks are dead.

## 2 Main Ideas

Let

$$T(s) = \min\{t : V_t = s\}$$

and

$$\rho = \min\{t : A_t = 0\}.$$

Observe that  $A_\infty := \lim_{t \rightarrow \infty} A_t = A_\rho$ . For all  $s$  such that the process has reached the level of  $s$  visited vertices, we define  $\bar{X}_s$  as the time the process spent at that level. Besides  $\bar{X}_s$  can also be seen as the (random) number of active particles which have to jump in so that the number of visited vertices either goes from  $s$  to  $s + 1$  or the process finishes.

So,

$$\bar{X}_s = (T(s+1) - T(s))(1_{\{T(s)=\infty\}} + 1_{\{T(s+1) < \infty\}}) + (\rho - T(V_\infty))1_{\{T(s+1) - T(s) = \infty\}}$$

( $\infty - \infty = 0$ ). Note that when there are  $s$  visited vertices, each active particle which jumps has a probability of  $s/n$  to die and a probability of  $(n - s)/n$  to hit an inactive particle. For the number of visited vertices to go from  $s$  to  $s + 1$ , we need one additional unvisited vertex to be chosen. Hence,

$$\bar{X}_s \sim \min\left\{\mathcal{G}\left(\frac{n - s}{n}\right), A_{T(s)}\right\}$$

$\mathcal{G}$  stands for the Geometric probability distribution. To be more precise, for all  $k = 1, 2, \dots, V_\infty - 1$

$$\bar{X}_{V_\infty - k} \sim \mathcal{G}\left(\frac{n - V_\infty + k}{n}\right)$$

while

$$\bar{X}_{V_\infty} = A_{T(V_\infty)}$$

Going from  $s$  to  $s + 1$  visited vertices, the change in the amount of active particles is designated by  $\bar{Y}_s$  where

$$\bar{Y}_s := A_{T(s+1)} - A_{T(s)} = 2 - \bar{X}_s.$$

So we have that

$$A_{T(s)} = \sum_{i=1}^{s-1} \bar{Y}_i$$

Besides  $\bar{Y}_{V_\infty} = A_{T(V_\infty)}$ . Note that the variables  $\bar{Y}_1, \bar{Y}_2, \dots, \bar{Y}_{V_\infty-1}$  are independent but not identically distributed. We now make up an approximation for the model by considering  $X_s \sim \mathcal{G}(\frac{n-s}{n})$  and  $Y_s = 2 - X_s$  for  $s = 1, 2, \dots$ . Moreover we consider

$$W_s = \sum_{i=1}^s Y_i$$

$$\tau := \min\{s : W_s \leq 1\}.$$

Observe that it is possible to make a coupling such that  $X_i = \bar{X}_i$  with probability 1 and from this we have that  $\rho = \tau - 1$ . So, for what comes next we are interested in the random variable  $\tau$ . We show that  $\tau$  has expectation of order  $n$ , standard deviation of order  $n$  and when re-scaled properly converges to a normal variable.

Let  $\mu_s := E[Y_s]$ , so that

$$\mu_s = 2 - \frac{1}{1 - \frac{s}{n}}. \tag{2.1}$$

Note that up to  $s < n/2$  we have  $\mu_s > 0$ . On the other hand for  $s > n/2$  we find  $\mu_s < 0$ . This means that about up to  $s = n/2$  the random map  $s \mapsto W_s$  increases and after  $s = n/2$  it decreases.

Let  $w_s := E[W_s]$  and let  $W_s^* := W_s - w_s$ , whilst  $Y_i^* = Y_i - E[Y_i]$ . By these definitions, we get

$$W_s^* = Y_1^* + Y_2^* + \dots + Y_s^*.$$

The variables  $Y_1^*, Y_2^*, \dots$  are independent. Let  $c < 1$  be any constant not depending on  $n$ . Then for  $s \leq cn$  the variables  $Y_i$  with  $i \leq s$  are stochastically

uniformly bounded by a geometric variable. Hence,  $W_s^*$  is typically of order  $\sqrt{s}$  when  $s \leq cn$ . On the other hand,  $s \rightarrow w_s$  takes on values which are of order  $n$ . Hence, “the main shape” of  $s \rightarrow W_s$  is “determined” by  $s \rightarrow w_s$  whilst  $W_s^*$  only represents a smaller fluctuation.

We have for  $s < n$ ,

$$w_s = \sum_{i=1}^s \left( 2 - \frac{1}{1 - \frac{i}{n}} \right), \quad (2.2)$$

which implies that,

$$w_s \approx n \int_0^{s/n} 2 - \frac{1}{1-x} dx.$$

The integral in the expression on the right side of the above approximation, is equal

$$2s/n + \ln(1 - s/n).$$

The next lemma gives the precision of our approximation for  $w_s$ .

**Lemma 2.1** *For all  $n$  and all  $s < n$ , we have:*

$$|w_s - n [2(s/n) + \ln(1 - (s/n))] | \leq 3 + \frac{1}{1 - (s/n)}. \quad (2.3)$$

**Proof.** Let  $f$  denote a decreasing function on the interval  $[a, b]$ . Note that we have

$$\frac{1}{n} \sum_{i=1}^n f \left( a + (b-a) \frac{i}{n} \right) \leq \int_a^b f(y) dy \leq \frac{1}{n} \sum_{i=0}^{n-1} f \left( a + (b-a) \frac{i}{n} \right)$$

and hence

$$\left| \frac{1}{n} \sum_{i=1}^n f \left( a + (b-a) \frac{i}{n} \right) - \int_a^b f(y) dy \right| \leq \frac{1}{n} (|f(a)| + |f(b)|). \quad (2.4)$$

The last inequality also holds for increasing functions. Note that the map  $x \mapsto 2 - \frac{1}{1-x}$  is everywhere monotone on  $[0, 1]$ . Hence we can apply to it inequality (2.4) and find

$$\left| \sum_{i=1}^s \left( 2 - \frac{1}{1 - \frac{i}{n}} \right) - n \int_0^{s/n} 2 - \frac{1}{1-y} dy \right| \leq 3 + \frac{1}{1 - \frac{s}{n}}. \quad (2.5)$$

The integral in the expression above can be calculated explicitly:

$$\int_0^{s/n} 2 - \frac{1}{1-y} dy = 2(s/n) + \ln(1 - (s/n)).$$

Plugging the expression into inequality (2.5) yields the desired result. ■

We will see that we only need to consider values of  $s$  for which  $s \leq cn$  where  $c < 1$  is a constant not depending on  $n$ . Hence the bound on the right side of (2.3) can be treated as a constant bound.

The main result in this paper is concerned with finding the (random) zero  $\tau$  of the map  $W_s$ . In the next lemma, we start by investigating the zeros of the map  $p \rightarrow 2p + \ln(1 - p)$ , which is our first approximation of  $W_s$ .

**Lemma 2.2** *The map*

$$p \mapsto 2p + \ln(1 - p) \quad ; \quad ]0, 1[ \rightarrow \mathbb{R}$$

*has only one zero  $q \in ]0, 1[$ . Furthermore*

$$0.796 < q < 0.798 \tag{2.6}$$

**Proof.** The derivative of our map is  $2 - 1/(1 - p)$ . It is strictly positive for  $p \in [0; 1/2[$ . So our map  $h(p) := 2p + \ln(1 - p)$  first increases from the value  $h(0) = 0$  to the positive value  $h(1/2) = 1 - \ln 2 > 0$ . After then the derivative of  $h(p)$  is strictly negative. Since  $h(1/2) > 0$  and  $h(1) = -\infty$ , we infer that there is only one zero of the map  $h(p)$  in  $]0; 1[$ . The bounds (2.6) were obtained by numeric approximation from above and below. ■

Let  $r$  be equal to

$$r := nq.$$

Due to lemma 2.1, we have that  $w_r$  is close to zero up to a constant. In other words,  $r$  is approximatively equal to the zero of the map  $s \mapsto w_s$ . By definition,  $W_s$  is equal to  $w_s + W_s^*$ , where typically  $w_s$  takes on values of order  $n$  and  $W_s^*$  takes on values of order  $\sqrt{n}$ . This implies that the zero of  $s \mapsto W_s$  is equal to the zero of  $w_s$  plus/minus a term of order  $\sqrt{n}$ . Hence, the stopping time  $\tau$  is typically equal to  $r = nq$  plus a random term with standard deviation of order  $\sqrt{n}$ .

How big is the standard deviation of  $\tau$ ? For this, let us quickly look at another, related problem: assume that the variables  $\tilde{Y}_1, \tilde{Y}_2, \dots$  are i.i.d. variables with finite second moment and  $E[\tilde{Y}_1] < 0$ . Let  $K > 0$  be a large number, and let  $\tilde{\tau}$  be

$$\tilde{\tau} := \min\{s | K + \tilde{Y}_1 + \tilde{Y}_2 + \dots + \tilde{Y}_s < 0\}.$$

We find that  $\tilde{\tau}$  takes typically values which are about equal to  $K/|E[\tilde{Y}_1]|$  with a fluctuation of order  $\sqrt{K}$ . (The proof is identical to the proof of the

Law of Large Numbers and Central Limit Theorem for Renewal Processes). In our case, the variables  $Y_i$  are not i.i.d but only independent. However, the variables with  $s$  close to  $r = nq$  have all about the same distribution that is geometric with expectation  $\mu_r$ . Note that

$$\mu_r = \mu_{qn} = 2 - \frac{1}{1 - q},$$

is a number not depending on  $n$ .

We saw that up to a constant factor,  $w_r$  is approximately equal to zero and hence we have  $W_r^* \approx W_r$ . Assume that  $W_r^* > 0$ . Then, for  $W_s$  to become zero after  $s = r$ , we need about

$$-\frac{W_r}{\mu_r} \approx -\frac{W_r^*}{\mu_r}$$

additional ‘‘steps’’, i.e. additional variables  $Y_s$ . (The argument goes like the argument presented above for the variables  $\tilde{Y}_i$ ). This yields the approximation

$$\tau \approx r - \frac{W_r^*}{\mu_r} = qn - \frac{Y_1^* + \dots + Y_{qn}^*}{\mu_r}. \quad (2.7)$$

The above approximation is typically precise up to a term of order  $n^{1/4}$ . This will be proven by introducing some events  $B_0^n$ ,  $B_1^n$  and  $B_2^n$  and showing that they when they hold (lemma 3.1) then the error in the last approximation above is of order  $n^{1/4}$ . In the last section, we prove that the events  $B_0^n$ ,  $B_1^n$  and  $B_2^n$  have their probabilities going to one when  $n$  goes to infinity. The expression on the right side of approximation (2.7) gives the asymptotic behavior of the standard deviation of  $\tau$ . The reason is that the term

$$\frac{Y_1^* + \dots + Y_{qn}^*}{\mu_r} \quad (2.8)$$

has a standard deviation of order  $\sqrt{n}$ , whilst the error term of the approximation (2.7) is of order  $n^{1/4}$ . Hence, the standard deviation of (2.8) is asymptotically equal to the standard deviation of  $\tau$  up to a much smaller error term. Let us calculate the variance of the expression (2.8). We have that the variables  $Y_i^*$  are re-centered geometric variables with parameter  $(n-i)/n$ . The variance of  $Y_i^*$  is thus

$$\frac{i/n}{(1 - i/n)^2}$$

Hence we find that the variance of the sum (2.8) is equal to

$$\frac{1}{\mu_r^2} \sum_{i=1}^{qn} VAR[Y_i^*] = \frac{1}{\mu_r^2} \sum_{i=1}^{qn} \frac{i/n}{(1 - i/n)^2} \quad (2.9)$$

The sum in the above expression can be approximated by an integral. This is the content of the next lemma:

**Lemma 2.3** *We have for all  $n$  and all  $q < 1$  that*

$$\left| \sum_{i=1}^{qn} \frac{i/n}{(1-i/n)^2} - n \int_0^q \frac{x}{(1-x)^2} dx \right| \leq \frac{q}{(1-q)^2}. \quad (2.10)$$

**Proof.** Let  $h(x) := x/(1-x)^2$ . We find that the derivative is equal to

$$h'(x) = \frac{1+x}{(1-x)^3}$$

which is positive for all  $x \in [0, 1[$ . Hence, inequality (2.4) can be applied and we find that inequality (2.10) holds. ■

The last lemma above implies that the standard deviation of (2.8) is approximately equal to  $\sigma\sqrt{n}$ , where

$$\sigma := \frac{\sqrt{\int_0^q \frac{x}{(1-x)^2} dx}}{\mu_r}.$$

Note that this is exactly the re-scaling factor used in our main theorem 1.1!

### 3 Combinatorics

The first event  $B_0^n$  is the event that the first  $n^{1/4}$  active random walks which jump in, do not get killed:

$$B_0^n := \{\forall i, j \leq n^{1/4}, \text{ with } i \neq j \text{ we have, } S_i \neq S_j\}.$$

The next event  $B_1^n$  is the event that the approximation of  $W_s$  by  $w_s$  does not exceed the size  $\ln s\sqrt{s}$ . More precisely,  $B_1^n$  is the event that for all  $s$  with  $n^{1/4} \leq s \leq qn$ , we have

$$|W_s^*| \leq \ln s\sqrt{s}.$$

The next event  $B_2^n$  says that for all  $i$  such that  $0 \leq i \leq (\ln n)^2\sqrt{n}$  we have that

$$|Y_{qn+1} + Y_{qn+2} + \dots + Y_{qn+i} - i\mu_r| \leq (\ln n)^3 \cdot n^{1/4}$$

and

$$|Y_{qn-1} + Y_{qn-2} + \dots + Y_{qn-i} - i\mu_r| \leq (\ln n)^3 \cdot n^{1/4}$$

Next comes our main combinatorial lemma

**Lemma 3.1** *Assume that  $B_0^n$ ,  $B_1^n$  and  $B_2^n$  all hold, then we have*

$$\left| \tau - \left( qn - \frac{Y_1^* + Y_2^* + \dots + Y_{qn}^*}{\mu_r} \right) \right| \leq 2(\ln n)^3 \cdot n^{1/4}$$

**Proof.** First, note that when  $B_0^n$  holds, then  $\tau$  is not in the interval  $[0, n^{1/4}]$ . Second, according to lemma 3.2, we have for all  $s$  contained in the interval

$$[n^{1/4}, nq - (\ln n)^2 \sqrt{n}] \quad (3.1)$$

that  $w_s > \ln s \sqrt{s}$ . Hence, when  $B_1^n$  holds, and since by definition  $W_s = W_s^* + w_s$ , we get that  $\tau$  is not in the interval (3.1).

Let  $s \mapsto f(s)$  be the (random) linear map

$$f(s) = W_{qn} + (s - qn)\mu_r.$$

Note that the map  $f$  has a zero at

$$qn - \frac{W_{qn}}{\mu_r}.$$

(Note that  $\mu_r$  is negative.) Let  $f^+$ , resp.  $f^-$  be the linear map  $f + (\ln n)^3 \cdot n^{1/4}$ , resp.  $f - (\ln n)^3 \cdot n^{1/4}$ . The zero of  $f^+$ , resp.  $f^-$  is at

$$qn - \frac{W_{qn} + (\ln n)^3 \cdot n^{1/4}}{\mu_r},$$

resp.

$$qn - \frac{W_{qn} - (\ln n)^3 \cdot n^{1/4}}{\mu_r}$$

Let  $I$  be the interval

$$I := [qn - (\ln n)^2 \sqrt{n}, qn + (\ln n)^2 \sqrt{n}]$$

and let  $J$  be the interval

$$J := \left[ qn - \frac{W_{qn} - (\ln n)^3 \cdot n^{1/4}}{\mu_r}, qn - \frac{W_{qn} + (\ln n)^3 \cdot n^{1/4}}{\mu_r} \right].$$

Note that when the event  $B_1^n$  holds, then

$$|W_{qn}^*| \leq \ln n \sqrt{n}. \quad (3.2)$$

By definition

$$W_{qn} = W_{qn}^* + w_{qn}. \quad (3.3)$$

But by equality (2.2) and by lemma 2.1, we have for the constant  $k := 3 + 1/(1 - q)$

$$\left| w_{qn} - n \int_0^q 2 - \frac{1}{1-x} dx \right| \leq k. \quad (3.4)$$

By definition of  $q$ , we have

$$\int_0^q 2 - \frac{1}{1-x} dx = 0,$$

so that with inequality (3.4), we obtain

$$|w_{qn}| \leq k. \quad (3.5)$$

The last inequality together with (3.2) and (3.3) implies

$$|W_{qn}| \leq k + \ln n \sqrt{n}. \quad (3.6)$$

Using inequality (3.6), we obtain that for  $n$  large enough

$$J \subset I.$$

Now, when the event  $B_2^n$  holds, then in the interval  $I$  we have that  $W_s$  is between  $f^-$  and  $f^+$ , that is  $f^-(s) \leq W_s \leq f^+(s)$  for  $s \in I$ . Hence, in the interval  $I$ , the map  $s \mapsto W_s$  has its zero between the zeros of  $f^-$  and  $f^+$ . More precisely, this means that  $W_s$  has a zero somewhere in the interval  $I$  and furthermore we have that all zeros of  $W_s$  in the interval  $I$  are located in  $J$ .

We can now summarize what we found so far: when  $B_0^n$ ,  $B_1^n$  and  $B_2^n$  all hold, then the map  $s \mapsto W_s$  has no zero before the interval  $I$ , but within  $I$  all the zeros are located in the subinterval  $J$ . Hence,  $\tau \in J$  which implies

$$\left| \tau - \left( qn - \frac{W_{qn}}{\mu_r} \right) \right| \leq -(\ln n)^3 \cdot n^{1/4} / \mu_r$$

Using the last equation together with (3.5) and (3.3), we find

$$\left| \tau - \left( qn - \frac{W_{qn}^*}{\mu_r} \right) \right| \leq -(\ln n)^3 \cdot n^{1/4} / \mu_r + k$$

Note that for  $n$  large enough, the right side of the last inequality is smaller than  $2(\ln n)^3 \cdot n^{1/4}$ . This finishes proving our lemma ■

**Lemma 3.2** *For all  $n$  large enough: every  $s$  contained in the interval*

$$[ n^{1/4}, nq - (\ln n)^2 \sqrt{n} ] \quad (3.7)$$

*satisfies*

$$w_s > \ln s \sqrt{s}. \quad (3.8)$$

**Proof.** We consider the three intervals  $I_1 = [n^{1/4}, n/3]$ ,  $I_2 := [n/3, n/2]$  and  $I_3 := [n/2, nq - (\ln n)^2 \sqrt{n}]$ . We are going to prove that inequality (3.8) holds for each one of them. Let  $h$  designate the map  $h(x) := 2x + \ln(1 - x)$ . Note that the second derivative of  $h$  is negative everywhere on  $I_1$  for  $x = s/n$ . Hence,  $h'(x) \geq h'(1/3) > 0$  for all  $x \in [0, 1/3]$ . Since  $h(0) = 0$ , the mean value theorem implies that for all  $x \in [0, 1/3]$ , we have  $h(x) \geq x \cdot h'(1/3)$ . When  $s \in I_1$  then  $(s/n) \in [0, 1/3]$  so that

$$h(s/n) \geq (s/n) \cdot h'(1/3). \quad (3.9)$$

According to inequality (2.3), we have

$$w_s \geq nh(s/n) - (3 + 1/(1 - (s/n)))$$

and for  $s \in I_1$  since  $(s/n) \leq 1/3$ , we obtain

$$w_s \geq nh(s/n) - 4.5.$$

The last inequality above together with inequality (3.9) then implies

$$w_s \geq s \cdot h'(1/3) - 4.5. \quad (3.10)$$

The expression on the right side of the last inequality above is larger than  $\ln s\sqrt{s}$  for  $s$  large enough. However for  $s \in I_1$ , we have  $s \geq n^{1/4}$ , so that for  $n$  large enough,  $s$  will be large enough and

$$s \cdot h'(1/3) - 4.5 \geq \ln s\sqrt{s}.$$

From the last inequality above and (3.10), we have that inequality (3.8) follows.

Next we need to prove (3.8) for  $s$  in  $I_2$ . Using inequality (2.3) together with the fact that  $s/n \leq 1/2$  for  $s \in I_2$ , we find

$$w_s \geq nh(s/n) - 5. \quad (3.11)$$

When  $s \in I_2$  we have that  $s/n \in [1/3, 1/2]$ . But on the interval  $[1/3, 1/2]$  the map  $h$  is everywhere increasing. Hence for  $s \in I_2$ , we have that  $h(s/n) \geq h(1/3)$ . Plugging the last inequality into (3.11) gives

$$w_s \geq nh(1/3) - 5. \quad (3.12)$$

For  $s \in I_2$  we have  $s \leq n$ . Hence

$$\ln s\sqrt{s} \leq \ln n\sqrt{n}. \quad (3.13)$$

For  $n$  large enough,  $\ln n\sqrt{n}$  is less than  $nh(1/3)-5$ . From this and inequalities (3.12) and (3.13) inequality (3.8) follows.

Now, it only remains to prove inequality (3.8) for  $s \in I_3$ . When  $s \in I_3$  we have that  $s/n \leq q < 1$ . This together with inequality (2.3) yields

$$w_s \geq nh(s/n) - (3 + 1/(1 - q)). \quad (3.14)$$

When  $s \in I_3$ , we have that

$$s/n \in \left[ 0.5, q - \frac{(\ln n)^2}{\sqrt{n}} \right].$$

On the interval on the right side of the last inclusion above the map  $h$  is everywhere decreasing. Hence, for  $s/n \in I_3$  we have

$$h(s/n) \geq h\left(q - \frac{(\ln n)^2}{\sqrt{n}}\right). \quad (3.15)$$

Note that by definition  $h(q) = 0$ . Furthermore,  $h'(q) < 0$ . Hence, using the mean value theorem applied to (3.15), we obtain that for all  $n$  large enough

$$h(s/n) \geq -h'(q)\frac{(\ln n)^2}{2\sqrt{n}}. \quad (3.16)$$

The last inequality together with (3.14), gives

$$w_s \geq -h'(q)\sqrt{n}(\ln n)^2 + (3 - 1/(1 - q)). \quad (3.17)$$

For  $n$  large enough, the right side of the last inequality above is larger than  $\ln n\sqrt{n}$  which is larger than  $\ln s\sqrt{s}$  when  $s \in I_3$ . Hence inequality (3.8) holds. ■

## 4 Probabilities

**Lemma 4.1** *We have that  $P(B_0^n) \rightarrow 1$  as  $n \rightarrow \infty$ .*

**Proof.** Let  $B_{0i}^n$  be the event that the  $i$ -th frog jumping in does not die. Hence,  $B_0^n$  is the event that  $NS_i \neq S_t$  for all  $t < i$ . For an event  $A^n$  we designate by  $A^{nc}$  its complement. We have that

$$B_0^n = \bigcap_{i=1}^{n^{1/4}} B_{0i}^n$$

and hence

$$P(B_0^{nc}) \leq \sum_{i=1}^{n^{1/4}} P(B_{0i}^{nc}). \quad (4.1)$$

Now, for  $i \leq n^{1/4}$  there are no more than  $n^{1/4}$  vertices and hence the probability for the  $i$ -th jumping frog to die is not more than  $n^{1/4}/n = n^{-3/4}$ . This immediately implies that

$$P(B_{0i}^{nc}) \leq \frac{1}{n^{3/4}}.$$

Using the last inequality with inequality (4.1), we find

$$P(B_0^{nc}) \leq \frac{n^{1/4}}{n^{3/4}} = \frac{1}{\sqrt{n}}.$$

This finishes to prove our lemma. ■

**Lemma 4.2** *There exist two constants  $\kappa > 0$  and  $c > 0$  such that for every geometric variable  $X$  with parameter  $p$  satisfying*

$$p \in [1 - q, 1] \quad (4.2)$$

and every  $\Delta \in [0, c]$ , we have

$$E[e^{(X - (1/p) - \Delta) \cdot \kappa \Delta}] \leq e^{-0.5\Delta^2 \kappa} \quad (4.3)$$

**Proof.** Let  $\kappa$  be equal to

$$\kappa := \min \frac{p^3}{1.1(1-p)(0.1+p)}, \quad (4.4)$$

where the minimum is taken over all  $p \in [1 - q, 1]$ . Note that  $p$  is bounded away from zero and  $\kappa > 0$ .

Let  $c_1 > 0$  be a number such that for all  $\Delta \in [0, c_1]$  we have

$$\ln \left( 1 - \frac{\Delta^2 \kappa^2 (1-p)(0.1+p)}{2p^3} \right) \geq -1.1 \frac{\Delta^2 \kappa^2 (1-p)(0.1+p)}{2p^3}. \quad (4.5)$$

Such a number  $c_1 > 0$  exists since for all  $s > 0$  small enough we have  $\ln(1 - s) \geq -1.1s$  and since  $\kappa^2(1-p)(0.1+p)/(2p^3)$  admits a uniform finite upper bound for  $p \in [1 - q, 1]$ .

Let  $c_2 > 0$  be a number such that for all  $\Delta \in [0, c_2]$  we have

$$e^{\Delta \kappa / p} \leq 1 + \Delta \kappa / p + 1.1 \Delta^2 \kappa^2 / (2p^2). \quad (4.6)$$

Such a number  $c_2 > 0$  exists since for all  $s > 0$  small enough we have  $e^s \leq 1 + s + 1.1s^2/2$ . Let  $c = \min\{c_1, c_2\}$ . Hence when  $\Delta \in [0, c]$  we have that both conditions (4.5) and (4.6) are satisfied.

Now for the geometric variable  $X$  with parameter  $p$  we have that

$$E[e^{(X-(1/p)-\Delta)t}] = \sum_{m=1}^{\infty} e^{mt-t/p-\Delta t}(1-p)^{m-1}p.$$

Using the formula  $\sum_{m=1}^{\infty} a^m = a/(1-a)$ , we find that for  $t$  small enough

$$E[e^{(X-(1/p)-\Delta)t}] = pe^{-t/p-t\Delta} \frac{e^t}{1-(1-p)e^t} = \frac{pe^{t(1-(1/p)-\Delta)}}{1-e^t(1-p)}.$$

For  $t = \kappa\Delta$ , we obtain

$$E[e^{(X-(1/p)-\Delta)\Delta\kappa}] = \frac{pe^{\Delta\kappa(1-(1/p)-\Delta)}}{1-e^{\Delta\kappa}(1-p)} = \frac{pe^{-\Delta^2\kappa}}{e^{\Delta\kappa(1-p)/p} - (1-p)e^{\Delta\kappa/p}}. \quad (4.7)$$

Note that for any  $s > 0$  we have  $e^s \geq 1 + s + s^2/2$ . Hence for  $s = \Delta\kappa(1-p)/p$  we find

$$e^{\Delta\kappa(1-p)/p} \geq 1 + \Delta\kappa(1-p)/p + \Delta^2\kappa^2(1-p)^2/2p^2 \quad (4.8)$$

Applying inequalities (4.8) and (4.6) to the expression on the right side of inequality (4.9), we find

$$\begin{aligned} E[e^{(X-(1/p)-\Delta)\Delta\kappa}] &\leq \frac{pe^{-\Delta^2\kappa}}{1 + \frac{\Delta\kappa(1-p)}{p} + \frac{\Delta^2\kappa^2(1-p)^2}{2p^2} - (1-p) - \frac{(1-p)\Delta\kappa}{p} - \frac{1.1(1-p)\Delta^2\kappa^2}{2p^2}} \\ &= \frac{pe^{-\Delta^2\kappa}}{p + \frac{\Delta^2\kappa^2(1-p)^2}{2p^2} - \frac{1.1(1-p)\Delta^2\kappa^2}{2p^2}} \\ &= \frac{e^{-\Delta^2\kappa}}{1 + \frac{\Delta^2\kappa^2(1-p)^2}{2p^3} - \frac{1.1(1-p)\Delta^2\kappa^2}{2p^3}} \\ &= \exp(-\Delta^2\kappa) \cdot \exp(-\ln(1 - \Delta^2\kappa^2(1-p)(0.1+p)/2p^3)) \end{aligned}$$

Applying inequality (4.5) to the most right expression in the last chain of inequalities above we find

$$E[e^{(X-(1/p)-\Delta)\Delta\kappa}] \leq e^{-\Delta^2\kappa} \cdot e^{1.1\Delta^2\kappa^2(1-p)(0.1+p)/2p^3} \quad (4.9)$$

$$\leq e^{-\Delta^2\kappa(1-1.1\kappa(1-p)(0.1+p)/2p^3)}. \quad (4.10)$$

By the definition (4.4) of  $\kappa$ , we have

$$\kappa \leq \frac{p^3}{1.1(1-p)(0.1+p)}$$

and hence

$$1 - \frac{1.1\kappa(1-p)(0.1+p)}{2p^3} \geq 0.5.$$

The last inequality above applied to (4.10) yields

$$E[e^{(X-(1/p)-\Delta)\Delta\kappa}] \leq e^{-0.5\Delta^2\kappa}.$$

■

We can prove the same type of inequality as the one in the lemma above for the variable  $-X$ . Hence, we assume that there exist  $c > 0$  and  $\kappa > 0$  such that for all  $p \in [1-q, 1]$  we have that condition (4.3) is satisfied as well as

$$E[e^{(-X+(1/p)-\Delta)\kappa\Delta}] \leq e^{-0.5\Delta^2\kappa} \quad (4.11)$$

where again  $X$  is a geometric variable with parameter  $p$ .

**Lemma 4.3** *We have that  $P(B_1^n) \rightarrow 1$  as  $n \rightarrow \infty$ .*

**Proof.** Let  $B_{11s}$  be the event

$$B_{11s} := \{Y_1^* + Y_2^* + \dots + Y_s^* \leq \ln s\sqrt{s}\}$$

and let

$$B_{12s} := \{-Y_1^* - Y_2^* - \dots - Y_s^* \leq \ln s\sqrt{s}\}$$

We have that

$$B_1^n = \bigcap_{s=n^{1/4}}^{nq} (B_{11s} \cap B_{12s})$$

and hence

$$P(B_1^{nc}) \leq \sum_{s=n^{1/4}}^{nq} P(B_{11s}^c) + \sum_{s=n^{1/4}}^{nq} P(B_{12s}^c) \quad (4.12)$$

Recall that for every  $t > 0$  and any variable  $Z$  we have

$$P(Z \geq 0) \leq E[e^{tZ}]. \quad (4.13)$$

We have

$$P(B_{11s}^c) = P((Y_1^* - \Delta) + (Y_2^* - \Delta) + \dots + (Y_s^* - \Delta) > 0)$$

where  $\Delta := \ln s / \sqrt{s}$ . Using inequality (4.13) yields

$$P(B_{11s}^c) \leq E[e^{t((Y_1^* - \Delta) + (Y_2^* - \Delta) + \dots + (Y_s^* - \Delta))}] = \prod_{i=1}^s E[e^{t(Y_i^* - \Delta)}] \quad (4.14)$$

But  $Y_i^* = -X_i + 1/p_i$ , where  $X_i$  is a geometric variable with parameter  $p_i$ , since by definition  $Y_i^* = Y_i - E(Y_i)$  and  $Y_i = 2 - X_i$ . Therefore taking  $t = \kappa\Delta$  and applying inequality (4.11) to (4.14), we obtain

$$P(B_{11s}^c) \leq e^{-0.5\kappa\Delta^2 s} = s^{-0.5\kappa \ln s}. \quad (4.15)$$

Similarly one can prove

$$P(B_{12s}^c) \leq e^{-0.5\kappa\Delta^2 s} = s^{-0.5\kappa \ln s}. \quad (4.16)$$

Applying inequalities (4.15) and (4.16) to inequality (4.12) finally gives

$$P(B_1^{nc}) \leq \sum_{s=n^{1/4}}^{nq} 2s^{-0.5\kappa \ln s}$$

and hence  $P(B_1^{nc})$  goes to zero as  $n \rightarrow \infty$ . ■

To prove that the event  $B_2^n$  has high probability we first need the following lemma:

**Lemma 4.4** *for all  $i$  such that  $0 \leq i \leq (\ln n)^2 \sqrt{n}$  we have that*

$$|\mu_{qn+1} + \mu_{qn+2} + \dots + \mu_{qn+i} - i\mu_r| \leq (\ln n)^5 \quad (4.17)$$

and

$$|\mu_{qn-1} + \mu_{qn-2} + \dots + \mu_{qn-i} - i\mu_r| \leq (\ln n)^5 \quad (4.18)$$

**Proof.** Let  $f$  be the map defined by  $f(x) := 2 - (1/(1-x))$ . Note that  $f$  is continuously differentiable in a neighborhood of  $x = q$ . Hence there exists  $\delta > 0$  such that for all  $\Delta \in [-\delta, \delta]$ , we have

$$|f(q + \Delta) - f(q)| \leq c \cdot \Delta \quad (4.19)$$

where  $c > 0$  is a constant not depending on  $\Delta$ . Note that when  $i$  satisfies  $0 \leq i \leq (\ln n)^2 \sqrt{n}$  then

$$\left| \frac{i}{n} \right| \leq \frac{(\ln n)^2}{\sqrt{n}}.$$

The right side of the last inequality above goes to zero as  $n \rightarrow \infty$  and hence for  $n$  large enough it is less than  $\delta$ . We assume now that  $n$  is large enough so that

$$\left| \frac{i}{n} \right| \leq \delta,$$

from which by (4.19) we get

$$|f(q + \frac{i}{n}) - f(q)| \leq c \frac{i}{n} \leq c \frac{(\ln n)^2}{\sqrt{n}}$$

and equivalently

$$|\mu_{qn+i} - \mu_r| \leq c \frac{(\ln n)^2}{\sqrt{n}}.$$

Applying the last inequality above to the expression on the left side of inequality (4.17) gives

$$|\mu_{qn+1} + \mu_{qn+2} + \dots + \mu_{qn+i} - i\mu_r| \leq ic \frac{(\ln n)^2}{\sqrt{n}} \leq c(\ln n)^4.$$

The term on the right side of the last inequality above for  $n$  large enough is less than  $(\ln n)^5$  which finishes proving (4.17). In a similar way we prove (4.18). ■

**Lemma 4.5** *We have that  $P(B_2^n) \rightarrow 1$  as  $n \rightarrow \infty$*

**Proof.** Hint: Use lemma 4.4 and the Hoeffding inequality. ■

**Proof of the main theorem 1.1** Lemma 3.1 states that when  $B_0^n$ ,  $B_1^n$  and  $B_2^n$  all hold then

$$\left| \frac{\tau - qn}{\sigma\sqrt{n}} - \frac{-Y_1^* - Y_2^* - \dots - Y_{qn}^*}{\sqrt{\int_0^q \frac{x}{(1-x)^2} dx} \sqrt{n}} \right| \leq \frac{2(\ln n)^3}{\sigma n^{1/4}}. \quad (4.20)$$

From lemma 2.3 and equality (2.9) it follows that the standard deviation of the sum

$$Y_1^* + Y_2^* + \dots + Y_{qn}^*$$

is equal up to a constant term to

$$\sqrt{\int_0^q \frac{x}{(1-x)^2} dx} \sqrt{n}.$$

Hence by the Central Limit Theorem for independent but non-identical variables we have that the re-scaled sum

$$\frac{-Y_1^* - Y_2^* - \dots - Y_{qn}^*}{\sqrt{\int_0^q \frac{x}{(1-x)^2} dx \sqrt{n}}}$$

converges weakly to a Standard Normal variable. From this and from the fact that inequality (4.20) holds with probability converging to one when  $n$  goes to infinity we get that  $(\tau - qn)/(\sigma\sqrt{n})$  converges weakly to a standard normal. We also used the fact that the right side of (4.20) goes to zero as  $n$  goes to infinity. Inequality (4.20) holds with probability going to one when  $n$  goes to infinity, because the events  $B_0^n$ ,  $B_1^n$  and  $B_2^n$ , which together imply (4.20), all have their probabilities going to one as  $n$  goes to infinity.

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