A liquid vapor phase transition in quantum statistical mechanics

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Abstract. We prove a liquid vapor phase transition for a quantum system of particles in the continuum. The system is the quantum version with Boltzmann statistics of the point particles model introduced in [**8**].

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1. Introduction

In a recent paper, [**8**], Lebowitz, Mazel and Presutti, have introduced a model of identical point particles in the continuum, proving a first order phase transition with the particles density as order parameter. The interaction is given by two and four body, finite range, translational invariant, Kac potentials and the phase transition and its proof are based on a Pirogov-Sinai scheme. Ground states are replaced by the minimizers of the non local free energy functional, which describes the system in the Lebowitz-Penrose limit; the small parameter, which in traditional Pirogov-Sinai models is the temperature, in LMP is replaced by an effective temperature $\beta^{-1}\gamma^d$, $\gamma > 0$ the Kac scaling parameter, $d \geq 2$ the space dimensions and β^{-1} the true temperature of the system.

As originally argued by Kac, Uhlenbeck and Hemmer, [**6**], Kac potentials are supposed to model the van der Waals theory of liquid vapor phase transitions, and indeed the limit $\gamma \to 0$, after the thermodynamic limit, gives rise, in a large variety of models, to a phase diagram which agrees with the one proposed by van der Waals (with the Maxwell, equal area law included). For a true proof of the van der Waals theory in a statistical mechanics

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setting, however, we would need γ fixed and indeed this is what accomplished in LMP, where the phase transition is proved at a fixed (but small) value of $\gamma > 0$.

In this paper we study the quantum extension of the result. Intuitively one may think the transition to quantum as a delocalization of particles, whose positions in the classical case are just points. This notion can be made quantitative in the Feynman-Kac representation, in which points are replaced by loops, namely brownian motions which travel for a time β and are conditioned to come back to their initial position at the final time. On the other hand, a characteristic feature of the LMP analysis is coarse-graining, which exploits the fact that the Kac potential energy is quite insensitive to the exact positions of the particles. This matches beautifully with the loops description, and it makes conceivable that quantum delocalization effects could be handled in this way. Indeed this is what we prove in the present paper, where we conclude that a phase transition is present also in the quantum model.

We study here the quantum Boltzmann statistics, the extension to the more realistic Bose and Fermi statistics should however be possible as the liquid vapor phase transitions that we consider are away from the range of parameters where Bose condensation effects and the Fermi surface structure are relevant. Technically however, the extension requires a considerable amount of work and this paper is already rather long and complex, so we leave the issue to future works.

The Pirogov-Sinai scheme consists mainly of two stages. The starting point is the notion of contours and here is where quantum effects are more relevant. Our contours in fact take into account both local deviations from the liquid and vapor densities (as in LMP), but also the occurrence of exceptionally long loops. The joint combination of the two mechanisms could in principle produce dangerous domino effects and the heart of our proof is to exclude such catastrophic events. Long loops, in the end, will give the largest contribution to the weight of the contours and hence cannot be handled simultaneously with the density deviations. With the notion of contours established, the first stage of the P-S scheme is based on the reduction to restricted ensembles. It is at this point that the correct value of the chemical potential for phase coexistence is found by equalizing the pressures in the plus and minus restricted ensembles. This is all done in the present paper where we also prove some energy estimates, which constitute the main term in the Peierls bounds. They are derived by solving variational problems for the non local, free energy functional of the Lebowitz-Penrose limit.

The second stage (in the P-S scheme) where the proof of the Peierls bounds is completed, involves an analysis of the finite volume, surface corrections to the pressure. Traditionally in Pirogov-Sinai models this follows from cluster expansion, but, as in LMP, we do not know whether a cluster expansion is valid in the range of parameters we are using. In LMP, cluster expansion has been replaced by Dobrushin uniqueness and here we follow the same approach, but with a quite different proof. The reduction to restricted ensembles, places the problem in a phase-transition free setting and in such a space the high temperature, Dobrushin uniqueness condition is satisfied, but not for all boundary conditions. The bad ones, however, have small probability, hence the necessity of a relativized Dobrushin uniqueness theorem. In a companion paper by the same authors, [**1**], the issue is discussed in a more general setting and a uniqueness criterion is first proved and then checked to hold in a variety of models including the present one. With the help of this, the surface corrections to the pressure are estimated and the Peierls bounds proved. Phase transition then follow in the usual way.

The paper is organized as follows. In Section 2 we define the model and state the main theorem, Theorem 2.1. In Section 3 we introduce the fundamental notion of contours and in Section 4 we write the $\pm$, diluted (in the sense of Pirogov-Sinai) partition functions in terms of contours, introducing then the $\pm$ restricted ensembles with cutoff contour weights. In Section 5 we choose the value of the chemical potential, by equating the pressures of the plus and of the minus ensembles. In Section 6 we divide the proof of the Peierls bounds in a sequence of intermediate estimates which are then proved in the remaining sections and in the appendices.

2. Model and main results

2.1. Classical model. In classical statistical mechanics the phase space of a system of identical point particles is the space of all locally finite subsets of $\mathbb{R}^d$. It is convenient here to consider a larger phase space Q where the particle configurations are sequences $q = (q_i)$, $q_i \in \mathbb{R}^d$, which put finitely many elements in any bounded set of $\mathbb{R}^d$. The order of the elements in the sequence is unimportant and physical observables in this space are represented by symmetric functions. Accordingly a sequence with n elements has a statistical weight with a factor $1/n!$, see (2.5) below.

We will denote by $q \sqcap \Lambda$ the subsequence of q obtained by discarding the elements of q which are not in Λ; we call $|q|$ the cardinality of q, thus $|q \sqcap \Lambda|$ is the number of particles of the configuration q which are in Λ. We denote by $Q_{\text{fin}} = \{q \in Q : |q| < \infty\}$, $Q^\Lambda = \{q \in Q : q = q \sqcap \Lambda\}$, $\Lambda \subset \mathbb{R}^d$, $Q^\Lambda_n = \{q \in Q^\Lambda : |q| = n\}$, n a non negative integer, obviously $Q^\Lambda = \bigsqcup_{n\geq 0} Q^\Lambda_n$.

In the LMP model, the energy of a configuration $q \in Q_{\text{fin}}$ is

$$h_{\gamma,\lambda}(q) := \int_{\mathbb{R}} e_\lambda\left(j_\gamma * q(r)\right) dr \tag{2.1}$$

where $\gamma > 0$ is the Kac scaling parameter,

$$j_\gamma * q(r) = \sum_{q_i \in q} j_\gamma(r, q_i), \qquad j_\gamma(r, r') := \gamma^d j(\gamma r, \gamma r') \tag{2.2}$$

with j a bounded, symmetric, translation-invariant probability kernel on $\mathbb{R}^d$, $j(0,\cdot)$ supported by the unit ball, and with some regularity properties stated in Subsection A.1 of Appendix A; $\lambda \in \mathbb{R}$ is the chemical potential and for $x \geq 0$ the function e_λ is given by

$$e_\lambda(x) := -\lambda x - \frac{x^2}{2!} + \frac{x^4}{4!} \tag{2.3}$$

The conditional energy of the configuration q given $\bar{q}$, ($q, \bar{q} \in Q_{\text{fin}}$, $(q, \bar{q})$ the configuration which collects all the particles of q and $\bar{q}$) is

$$h_{\gamma,\lambda}(q|\bar{q}) := h_{\gamma,\lambda}(q, \bar{q}) - h_{\gamma,\lambda}(\bar{q}) \tag{2.4}$$

The free measure ν_Λ, Λ a bounded, measurable set, is the Lebesgue-Poisson measure on Q^Λ, defined for bounded measurable functions $G : Q^\Lambda \to \mathbb{R}$ by

$$\int_{Q^\Lambda} G(q)\nu_\Lambda(dq) := \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} G(q_1, \ldots, q_n) dq_1 \ldots dq_n \tag{2.5}$$

2.2. Euclidean model. The quantum version of the LMP model with Boltzmann statistics becomes, in the Feynman-Kac representation, "an Euclidean model" which is like the classical one, but with configurations of loops rather than configurations of particles. The latter are defined below. This description was introduced by J. Ginibre, see e.g. [**5**], the concept of Euclidean Gibbs measures in [**7**].

The configurations of loops are sequences $\underset{\sim}{q} = (\underset{\sim}{q}_i)$, where a single loop $\underset{\sim}{q}_i$ is a continuous function $\underset{\sim}{q}_i : [0, \beta] \to \mathbb{R}^d$ with $\underset{\sim}{q}_i(0) = \underset{\sim}{q}_i(\beta)$ and $\{\underset{\sim}{q}_i\}(0) \in Q$. We will denote by $\underset{\sim}{Q}$ the space of loops configurations, by $\underset{\sim}{Q}_{\text{fin}}$ the subset of $\underset{\sim}{Q}$ with finitely many loops, by $\underset{\sim}{Q}_1$ the space of single loops. $\underset{\sim}{Q}^\Lambda$, $\Lambda \sqsubset \mathrm{R}^d$, is the space of all loops whose starting point is in Λ, $\underset{\sim}{Q}_n^\Lambda$ the subset of Q^Λ with n loops.

In the Euclidean model, the energy $H_{\beta,\gamma,\lambda}(\underset{\sim}{q})$, $\underset{\sim}{q} \in \underset{\sim}{Q}_{\text{fin}}$, is

$$H_{\beta,\gamma,\lambda}(\underset{\sim}{q}) := \int_0^\beta h_{\gamma,\lambda}(\underset{\sim}{q}(t))dt \tag{2.6}$$

and the conditional energy

$$H_{\beta,\gamma,\lambda}(\underset{\sim}{q}|\bar{\underset{\sim}{q}}) = H_{\beta,\gamma,\lambda}(\underset{\sim}{q}, \bar{\underset{\sim}{q}}) - H_{\beta,\gamma,\lambda}(\bar{\underset{\sim}{q}}), \qquad \underset{\sim}{q}, \bar{\underset{\sim}{q}} \in \underset{\sim}{Q}_{\text{fin}} \tag{2.7}$$

The conditional Wiener measure (Brownian bridge) $W^\beta_{x|x}$, $x \in \mathbb{R}^d$, is the probability on $\underset{\sim}{Q}_1$ supported by continuous loops $\omega(t), 0 \le t \le \beta$, which start and end in x; moreover on a cylinder set

$$A := \left\{\omega \in \underset{\sim}{Q}_1 \,|\omega(t_1) \in B_1, \ldots, \omega(t_n) \in B_n\right\}$$

(with $0 < t_1 < \ldots < t_n < \beta$ and $B_1, \ldots, B_n$ measurable subsets of $\mathbb{R}^d$), is

$$\begin{aligned} W^\beta_{x|x}(A) \;&=\; (2\pi\beta)^{d/2} \int_{B_1} \cdots \int_{B_n} p_{t_1}(r_1 - x) \cdot p_{t_2 - t_1}(r_2 - r_1) \\ &\quad \cdot \ldots \cdot p_{t_n - t_{n-1}}(r_n - r_{n-1}) \cdot p_{\beta - t_n}(x - r_n) dr_1 \ldots dr_n \end{aligned} \tag{2.8}$$

where

$$p_t(r) := \left(\frac{1}{2\pi t}\right)^{d/2} \exp\left(-\frac{r^2}{2t}\right)$$

The Lebesgue-Poisson measure $\nu_{\beta,\Lambda}(d\underline{q})$ on $\underline{Q}^{\Lambda}$, Λ a bounded measurable set, is such that, for any bounded measurable function $G : \underline{Q}^{\Lambda} \to \mathbb{R}$,

$$\int_{\underline{Q}^{\Lambda}} G(\underline{q})\nu_{\beta,\Lambda}(d\underline{q}) := \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \int_{(\underline{Q}_1)^n} G(\underline{q}_1, \dots, \underline{q}_n) \times W_{x_1|x_1}(d\underline{q}_1) \dots W_{x_n|x_n}(d\underline{q}_n) dx_1 \dots dx_n \tag{2.9}$$

2.3. Gibbs and DLR measures. In Appendix A, Subsection A.2, it is proved that for any $q \in Q_{\text{fin}}$ and $\bar{q} \in Q$,

$$h_{\gamma,\lambda}(q|\bar{q}) \geq b\ |q|, \qquad b = \inf_{x \geq 0} e'_{\lambda}(x) \tag{2.10}$$

(2.10) is a stability property of the hamiltonian, much stronger than usual stability because of the uniformity on the conditioning; it is a very peculiar feature of the LMP model, in many respects unessential in [**8**], but technically quite important in our proofs.

Given q we will denote by q_Δ the particles of q which are in Δ. Then, the Gibbs measure in the bounded, measurable region Λ with boundary conditions $\bar{q}$ is the probability on Q

$$\mu_{\beta,\gamma,\lambda,\Lambda}(dq|\bar{q}) = \frac{e^{-\beta h_{\gamma,\lambda}(q_\Lambda|\bar{q})}}{Z_{\beta,\gamma,\lambda}(\Lambda|\bar{q})}\, \nu_\Lambda(dq_\Lambda)\delta_{\bar{q}_{\Lambda^c}}(dq_{\Lambda^c}) \tag{2.11}$$

(δ_x the delta measure at x) which, by (2.10), is well defined for any $\bar{q} \in Q$.

We will extend the previous notation to loops, thus calling $\underline{q}_\Delta$ the set of loops of $\underline{q}$ which start in Δ. Then, in the Euclidean model, the Gibbs measure in Λ with boundary conditions $\bar{\underline{q}}$ is

$$\mu_{\beta,\gamma,\lambda}(d\underline{q}|\bar{\underline{q}}) = \frac{e^{-H_{\beta,\gamma,\lambda}(\underline{q}_\Lambda|\bar{\underline{q}})}}{Z_{\beta,\gamma,\lambda,\Lambda}(\Lambda|\bar{\underline{q}})}\, \nu_{\beta,\Lambda}(d\underline{q}_\Lambda)\ \delta_{\bar{\underline{q}}_{\Lambda^c}}(d\underline{q}_{\Lambda^c}) \tag{2.12}$$

whenever $\bar{\underline{q}}$ verifies the property that in each ball of $\mathbb{R}^d$ the number of particles of $\bar{\underline{q}}(t)$ is bounded uniformly in t. By (2.10), (2.12) is then well defined.

A DLR measure μ for given values of β, γ, λ is a probability on Q, resp. $\underline{Q}$, such that, for any bounded, measurable region Λ, the conditional probability of μ with respect to the σ-algebra $\mathcal{F}_{\Lambda^c}$, is μ-almost surely (2.11), resp. (2.12); $\mathcal{F}_{\Lambda^c}$ being the σ-algebra generated by functions which depend on q_{Λ^c}, in the classical case, and on $\underline{q}_{\Lambda^c}$ in the Euclidean model, i.e. on the loops which have origin in Λ^c. In the latter case, consistency of the above definition requires that, for any n,

$$\mu\Big(\underline{q} : \sup_{0\leq t\leq \beta} |\underline{q}(t) \sqcap B_n| < \infty\Big) = 1, \qquad B_n = \{r \in \mathbb{R}^d : |r| \leq n\} \tag{2.13}$$

In the context of the DLR measures, phase transitions means that there are distinct DLR measures for a same value of β, γ, λ. Existence of phase transitions is trivial at the level of a mean field approximation of the model.

2.4. Mean-field model. The mean field approximation of the LMP model is described by the mean field free energy density

$$f_{\beta,\lambda}(x) = e_\lambda(x) - \frac{s(x)}{\beta}, \qquad s(x) = -x(\ln x - 1) \tag{2.14}$$

where $e_\lambda(x)$ is the energy density defined in (2.3), $s(\cdot)$ the entropy density and $x \ge 0$ the particles density.

For $\beta \le \beta_c := (3/2)^{3/2}$, $f''_{\beta,\lambda}(x) \ge 0$ for all λ and x and $f_{\beta,\lambda}$ is a convex function. For any $\beta > \beta_c$ there is an interval of λ's in which the function $f_{\beta,\lambda}(\cdot)$ has two local minima and a unique value, $\lambda = \lambda_\beta$, where they are equal (thus being global minima). The two corresponding minimizers are denoted by $\rho^-_\beta < \rho^+_\beta$, elsewhere the minimizer is unique. Thus phase transitions at the mean field level (i.e. non uniqueness of the minimizers) occur at $\beta > \beta_c$ and $\lambda = \lambda_\beta$; the order parameter of the transition is the density and $\rho^\pm_\beta$ are the equilibrium densities in the $\pm$ phases.

2.5. Phase transitions. As in [**8**], we restrict hereafter to $\beta \in (\beta_c, \beta_0)$, where $\beta_0 > \beta_c$ is defined in (C.4). In [**8**] it is proved that for any $\gamma > 0$ small enough, there is $\lambda_{\rm class}(\beta,\gamma)$, and two distinct DLR measures at $\beta, \gamma, \lambda_{\rm class}(\beta,\gamma)$. Here we will prove the quantum extension of the result.

Theorem 2.1. *For any $d \ge 2$ and any $\beta \in (\beta_c, \beta_0)$ and any $\gamma > 0$ small enough, there is $\lambda(\beta,\gamma)$ and two distinct DLR measures at $\beta, \gamma, \lambda(\beta,\gamma)$.*

As in the classical case, we will prove Theorem 2.1 by constructing the two distinct DLR measures via a thermodynamic limit procedure, after “imposing $+$ and $-$ boundary conditions”. The persistent memory of the boundary conditions follows from Peierls estimates obtained by extending the Pirogov-Sinai methods. The proof yields a detailed control on the structure of the typical configurations, showing that in the two distinct DLR measures of Theorem 2.1, the coarse grained image of the particle configurations $\underline{q}(0)$ have approximately a homogeneous density; moreover the values of the density are respectively close to the mean field values $\rho^\pm_\beta$. We will also show that the typical loops are “short” so that the loop remains confined around their starting points and the energy of the loops is for γ small, typically very close to β times the classical energy of the starting points of the loops.

3. Coarse graining, contours

In this section we introduce the important notion of contours. Contours are designed to indicate location and nature of the "large deviations" from equilibrium which may occur in a configuration. The definition involves a coarse graining procedure from where we start, after recalling that throughout the sequel the inverse temperature β is kept fixed inside the interval (β_c, β_0) and often dropped from the notation; λ instead varies in the interval $\lambda_\beta \pm 1$, but it will be eventually fixed equal to $\lambda(\beta,\gamma)$, the value at which a phase transition occurs.

- For any $\ell \in \{2^n, n \in \mathbb{Z}\}$, we set

$$C_0^{(\ell)} = \Big\{ r \in \mathbb{R}^d : 0 \le r_i < \ell,\ 1 \le i \le d \Big\} \tag{3.1}$$

and call $\mathcal{D}^{(\ell)}$ the partition of $\mathbb{R}^d$ made of cubes which are translates of $C_0^{(\ell)}$ by vectors whose coordinates are integer multiples of ℓ. In this way $\mathcal{D}^{(\ell)}$ is coarser than $\mathcal{D}^{(\ell')}$ if $\ell \ge \ell'$. We will denote by $C_r^{(\ell)}$, $r \in \mathbb{R}^d$, the cube of $\mathcal{D}^{(\ell)}$ which contains r.

- *Abstract formulation.* Given $\ell_- < \ell_+$ both in $\{2^n, n \in \mathbb{Z}\}$, we are going to define three families of functions on $\mathbb{R}^d$. As we will see later, $1 \ll \ell_- \ll \ell_+$ and each configuration $\underline{q}$ will generate three functions, one for each one of the above families. The functions in the first family are denoted by η, they belong to $L^\infty(\mathbb{R}^d, \{0, \pm 1\})$ and are $\mathcal{D}^{(\ell_-)}$ measurable. Namely $\eta(r)$ is constant on each cube of $\mathcal{D}^{(\ell_-)}$ where it may have value $0, \pm 1$. The functions in the second class are denoted by $\sigma \in L^\infty(\mathbb{R}^d, \{0,1\})$ and they are $\mathcal{D}^{(\ell_+)}$ measurable. Given η and σ as above we then define a third function Θ by setting

$$\Theta(\eta,\sigma;r) = \begin{cases} \pm 1 & \text{if } \eta(r') \equiv \pm 1 \text{ and } \sigma(r') = 1 \text{ for all } r' \in C_r^{(\ell_+)} \sqcup \delta_{\text{out}}^{\ell_+}[C_r^{(\ell_+)}] \\ \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

We will frequently use in the paper the following notation: $\delta_{\text{out}}^{\ell}[A]$, A a $\mathcal{D}^{(\ell)}$-measurable set, is the union of all cubes in $\mathcal{D}^{(\ell)}$ which are in A^c and are connected to A, (two sets being connected if their closures have non empty intersection); $\delta_{\text{in}}^{\ell}[A] = \delta_{\text{out}}^{\ell}[A^c]$.

Notice that Θ as defined in (3.2) is a $\mathcal{D}^{(\ell_+)}$ measurable function.

- *Phase indicator.* We will associate here to each loop configuration $\underline{q}$ three functions $\eta(\underline{q};r)$, $\sigma(\underline{q};r)$ and $\Theta(\underline{q};r)$ which belong to the classes specified above, $\Theta(\underline{q};r)$ being "the indicator of phases" of the configuration $\underline{q}$. The choice of the functions depends on β, γ, ℓ_-, ℓ_+ and a positive parameter ζ: some of them may be added as superscripts, to remind dependence on such parameters.

The function $\sigma(\underline{q};r)$ is an indicator of the existence or absence of long loops, a loop $\omega(t), 0 \le t \le \beta$, being "long" if

$$\sup_{0 \le t \le \beta} |\omega(t) - \omega(0)| > \gamma^{-1/2} \tag{3.3}$$

and "short" in the opposite case; $\underline{q}^<$ and $\underline{q}^>$ denote the loop configurations obtained from $\underline{q}$ by selecting only its short and, respectively, its long loops.

With such a notion, we set $\sigma(\underline{q};r) = 0$, if there are a long loop $\underline{q}_i \in \underline{q}$ and a time $t \in [0,\beta]$ so that $\underline{q}_i(t) \in C_r^{(\ell_+)}$; otherwise, $\sigma(\underline{q};r) = 1$. In other words, $\{\sigma(\underline{q};\cdot) = 0\}$ is the minimal $\mathcal{D}^{(\ell_+)}$ region which covers all the long loops $\underline{q}^>$, while on its complement all loops are short.

The function $\eta(\underline{q};r)$ indicates the regions where the empirical density is close to the liquid (plus) and vapor (minus) densities, or where it deviates from both. Given ℓ_- and a positive "accuracy" parameter ζ, we set

$$\eta(\underline{q};r) = \begin{cases} \pm 1 & \text{if } \Big| \ell^{-d} |\underline{q}^<(0) \sqcap C_r^{(\ell)}| - \rho_\beta^\pm \Big| \le \zeta \\ \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

(notice that the definition of $\eta(\underline{q};r)$ depends also on β).

Finally, the phase indicator $\Theta(\underline{q};r)$ is defined by (3.2), using for η and σ the functions $\eta(\underline{q};r)$ and $\sigma(\underline{q};r)$, namely

$$\Theta(\underline{q};r) = \Theta\Big(\eta(\underline{q};\cdot), \sigma(\underline{q};\cdot); r\Big) \tag{3.5}$$

The function $\Theta(\underline{q};r)$ is "a phase indicator" in the sense that the two regions $\{r : \Theta(\underline{q};\cdot) = \pm 1\}$ are interpreted as those where the configuration is in the liquid, $+$, and in the vapor, $-$, phases. The complement is where "large deviations from equilibrium are concentrated".

- *Contours.* Let η and σ be two functions as above (all the parameters $\zeta, \ell_-, \ell_+, \beta, \gamma$ having been fixed), then the contours $\{\Gamma_i\}$ of the pair (η,σ) are the triples $(\mathrm{sp}(\Gamma_i), \eta_{\Gamma_i}, \sigma_{\Gamma_i})$, where $\mathrm{sp}(\Gamma_i)$ are the maximal connected components of $\{\Theta(\eta,\sigma;\cdot) = 0\}$, and η_{Γ_i} and σ_{Γ_i} are the restrictions of η and σ to $\mathrm{sp}(\Gamma_i)$.

It follows directly from the definition that:

Lemma 3.1. *If Γ is one of the contours of (η,σ) then, on each one of the maximal connected components of $\delta_{\mathrm{out}}^{\ell_+}[sp(\Gamma)] \sqcup \delta_{\mathrm{in}}^{\ell_+}[sp(\Gamma)]$, $\sigma = 1$ and η is constant and non zero.*

- *The contours of $\underline{q}$* are defined as the contours $\{\Gamma_i\}$ of the pair $(\eta(\underline{q};\cdot), \sigma(\underline{q};\cdot))$. We denote by $\underline{\Gamma}$ the map which associates $\{\Gamma_i\}$ to $\underline{q}$, writing $\{\Gamma_i\} = \underline{\Gamma}(\underline{q})$. Finally $\Gamma = (\mathrm{sp}(\Gamma), \eta_\Gamma, \sigma_\Gamma)$ is a contour, if it is an element of $\underline{\Gamma}(\underline{q})$ for some $\underline{q}$.

4. Contour models

As we will see, there are choices of the parameters ζ and $\ell_\pm$ for which the contours become rare and small, provided γ is small enough and the chemical potential is suitably chosen. By classical arguments this will imply the existence of two distinct Gibbs measures, each one selected by its appropriate boundary conditions.

4.1. Choice of parameters. We set

$$\zeta = \gamma^a, \quad \ell_{\pm,\gamma} = \gamma^{-(1\pm\alpha)} \tag{4.1}$$

where $1 \gg \alpha \gg a > 0$, the precise requirements will come out from the proofs.

By default and unless otherwise specified, contours are relative to the parameters (4.1).

4.2. Partition functions with constraint. A special role is played in the sequel by the plus and minus, "dilute" partition functions, whose definition imposes restrictions not only on the choice of boundary conditions but also on the possible contours which may appear.

Throughout the paper we use the notation

$$\underline{q}_A = \{\underline{q}_i \in \underline{q} : \underline{q}_i(0) \in A\} \tag{4.2}$$

• Recalling the definition of $\underline{\Gamma}(\underline{q})$ from the previous section, we define for any $\mathcal{D}^{(\ell_-,\gamma)}$-measurable region Λ, the maps $\underline{\Gamma}^\pm_\Lambda$ by setting $\underline{\Gamma}^\pm_\Lambda(\underline{q}) = \{\Gamma_i\}$, where $\{\Gamma_i\}$ is the set of all contours of the pair $(\eta_{\Lambda,\pm}, \sigma_{\Lambda,+})$, where

$$\sigma_{\Lambda,+}(r) = \sigma(\underline{q}_\Lambda; r), \qquad \eta_{\Lambda,\pm}(r) = \begin{cases} \eta(\underline{q}_\Lambda; r) & \text{if } r \in \Lambda \\ \pm 1 & \text{otherwise} \end{cases} \tag{4.3}$$

For any set $\mathcal{A}$ contained in the range of $\underline{\Gamma}^\pm_\Lambda$, we define the partition function with constraint $\mathcal{A}$ as

$$Z_{\gamma,\lambda}\Big(\Lambda; \mathcal{A}|\bar{\underline{q}}\Big) = \int_{\underline{\Gamma}^\pm_\Lambda(\underline{q}) \in \mathcal{A}} \nu_\Lambda(d\underline{q}) e^{-H_{\gamma,\lambda}(\underline{q}|\bar{\underline{q}}_{\Lambda^c})} \tag{4.4}$$

• A configuration $\bar{\underline{q}}$ is a $\pm$ boundary condition for Λ, if $\{\underline{\Gamma}^\pm_{\Lambda^c}(\bar{\underline{q}}) \sqsubset \Lambda^c\}$.

• $Z^\pm_{\gamma,\lambda}(\Lambda|\bar{\underline{q}})$ is the $\pm$ dilute partition function in Λ with b.c. $\bar{\underline{q}}$, if $\bar{\underline{q}}$ is a $\pm$ b.c. for Λ and

$$Z^\pm_{\gamma,\lambda}(\Lambda|\bar{\underline{q}}) := Z_{\gamma,\lambda}\Big(\Lambda; \{\underline{\Gamma}^\pm_\Lambda \sqsubset \Lambda\}\big|\bar{\underline{q}}\Big) \tag{4.5}$$

where, for any subset $B \sqsubset \Lambda$ (including Λ itself)

$$\{\underline{\Gamma}^\pm_\Lambda \sqsubset B\} = \Big\{\{\Gamma_i\} \in \text{range of } \underline{\Gamma}^\pm_\Lambda : \text{sp}(\Gamma_i) \sqsubset B, \text{ for all } \Gamma_i \text{ in } \{\Gamma_i\}\Big\} \tag{4.6}$$

We may also write (see for instance (4.14) below) $Z^{\pm}_{\gamma,\lambda}(\Lambda;\mathcal{A}|\bar{\underline{q}})$ for $Z_{\gamma,\lambda}\Big(\Lambda;\mathcal{A}|\bar{\underline{q}}\Big)$ if $\bar{\underline{q}}$ is a $\pm$ b.c. and $\mathcal{A}$ is a subset of $\{\underline{\Gamma}^{\pm}_{\Lambda} \sqsubset \Lambda\}$.

Remarks. We first need some notation. Call

$$A = \delta^{\ell_+,\gamma}_{\text{in}}[\Lambda], \qquad \Lambda_0 = \Lambda \setminus A \tag{4.7}$$

$$q \sqsubset B = q_i \in B, \;\; \text{for all } q_i \in q \tag{4.8}$$

$$\underline{q} \sqsubset B \equiv \underline{q}(t) \sqsubset B, 0 \le t \le \beta \tag{4.9}$$

With these notation

$$\{\underline{\Gamma}^{\pm}_{\Lambda}(\underline{q}) \sqsubset \Lambda\} = \{\eta(\underline{q};r) = 1 \;\text{ for all } r \in A\} \sqcap \{\sigma(\underline{q};r) = 1 \;\text{ for all } r \in A\} \tag{4.10}$$

This shows that the constraint in the partition function (4.5) is actually made of two conditions. The first one (which refers to η) is local and involves only loops starting from A; the second one is instead global, as it it requires that any long loop $\underline{q}_i$ must stay always inside Λ_0.

A second remark about (4.5) is that the loops of $\bar{\underline{q}}$ which originate outside $\delta^{\ell_+,\gamma}_{\text{out}}[\Lambda]$ do not interact (for γ small enough) with the loops of $\underline{q}$: in fact to get within interaction range, a loop should be long and enter into $\delta^{\ell_+,\gamma}_{\text{out}}[\Lambda]$, against the condition that the contours of $\underline{\Gamma}^{\pm}_{\Lambda^c}(\underline{q})$ are inside Λ^c. Thus $Z^{+}_{\gamma,\lambda}(\Lambda|\bar{\underline{q}})$ is insensitive to changes or removals of loops of $\bar{\underline{q}}_{\Lambda^c}$ which originate in the complement of $\delta^{\ell_+,\gamma}_{\text{out}}[\Lambda]$, provided $\bar{\underline{q}}$ remains a $\pm$ boundary condition.

4.3. Weight of contours. The goal is to write first the partition function as the partition function of "a gas of contours" and then to prove that the gas has a low density. Each contour will contribute to the partition function with its statistical weight. The weight of a contour is the ratio of its probability over the probability of it being absent; the probabilities being the conditional probabilities specified below. We first need a few extra notation about contours, others will be added when needed.

Given a contour Γ, we call

$$K = \delta^{\ell_+,\gamma/2}_{\text{in}}[\text{sp}(\Gamma)], \qquad K^{\pm} = \Big\{r \in K : \eta_{\Gamma}(r) = \pm 1\Big\} \tag{4.11}$$

noticing that, by Lemma 3.1, $K = K^+ \sqcup K^-$, with K^+ and K^- mutually disconnected.

Γ is a plus [minus] contour if K^+ [K^-] is connected to the unbounded component of $\text{sp}(\Gamma)^c$. Let us proceed by supposing that Γ is a plus contour, the definitions for minus contours are completely analogous and omitted. We call $\text{int}^-_i(\Gamma)$ the maximal connected components of $\text{sp}(\Gamma)^c$ which are connected to K^- and set

$$\text{int}^-(\Gamma) = \bigsqcup_i \text{int}^-_i(\Gamma), \qquad A^- = \delta^{\ell_+,\gamma}_{\text{in}}[\text{int}^-(\Gamma)] \tag{4.12}$$

noticing that, by Lemma 3.1, $\eta = -1$ on A^-. By $\text{int}^+_i(\Gamma)$ we then denote the remaining maximal connected components of $\text{sp}(\Gamma)^c$, except the one which is unbounded, $\text{int}^+(\Gamma)$ is the union of all $\text{int}^+_i(\Gamma)$.

The weight of a plus contour Γ with b.c. $\underset{\sim}{q}$, is

$$W^+_{\gamma,\lambda}(\Gamma;\underset{\sim}{q}) = \frac{N_{\gamma,\lambda}(\underset{\sim}{q})}{D_{\gamma,\lambda}(\underset{\sim}{q})} \tag{4.13}$$

where, shorthanding below $\Delta^- = \text{int}^-(\Gamma)$,

$$N_{\gamma,\lambda}(\underset{\sim}{q}) = \int_{\eta=\eta_\Gamma,\sigma=\sigma_\Gamma} \nu_{\text{sp}(\Gamma)\setminus K^+}(d\underset{\sim}{q}')\, e^{-H_{\gamma,\lambda}(\underset{\sim}{q}'|\underset{\sim}{q}_{K^+})} Z^-_{\gamma,\lambda}\Big(\Delta^-; \{\underline{\Gamma}^-_{\Delta^-} \sqsubset \Delta^- \setminus A^-\}\big|\underset{\sim}{q}'\Big) \tag{4.14}$$

$$D_{\gamma,\lambda}(\underset{\sim}{q}) = \int_{\eta=\sigma=1} \nu_{\text{sp}(\Gamma)\setminus K^+}(d\underset{\sim}{q}')\, e^{-H_{\gamma,\lambda}(\underset{\sim}{q}'|\underset{\sim}{q}_{K^+})} Z^+_{\gamma,\lambda}\Big(\Delta^-; \{\underline{\Gamma}^+_{\Delta^-} \sqsubset \Delta^- \setminus A^-\}\big|\underset{\sim}{q}'\Big) \tag{4.15}$$

4.4. Partition function as a gas of contours. Let Λ be a bounded region,

$$\mathcal{C}^+_\Lambda = \Big\{\{\Gamma_i\}: \text{ all } \Gamma_i \text{ are } + \text{ contours, sp}(\Gamma_i) \sqsubset \Lambda, \\ \text{dist}(\text{ sp}(\Gamma_i), \text{sp}(\Gamma_j)) \ge \ell_{+,\gamma}, \text{ for all } j \neq i \Big\} \tag{4.16}$$

with $\mathcal{C}^-_\Lambda$ defined analogously. By a simple inductive argument, by now classical in the Pirogov-Sinai theory,

$$Z^\pm_{\gamma,\lambda}(\Lambda|\bar{\underset{\sim}{q}}) = \sum_{\{\Gamma_i\}\in\mathcal{C}^\pm_\Lambda} \int_{\underset{\sim}{Q}^\Lambda_\pm} \nu_\Lambda(d\underset{\sim}{q}) e^{-H_{\gamma,\lambda}(\underset{\sim}{q}|\bar{\underset{\sim}{q}}_{\Lambda^c})} \prod_i W^\pm_{\gamma,\lambda}(\Gamma_i;\underset{\sim}{q}) \tag{4.17}$$

where $\underset{\sim}{Q}^\Lambda_\pm$ are the plus, minus, restricted ensembles in Λ, namely

$$\underset{\sim}{Q}^\Lambda_\pm = \Big\{\underset{\sim}{q} \in \underset{\sim}{Q}^\Lambda : \underline{\Gamma}^\pm_\Lambda(\underset{\sim}{q}) = \emptyset\Big\} \tag{4.18}$$

and $W^+_{\gamma,\lambda}(\Gamma;\underset{\sim}{q})$ is the weight defined in (4.13).

In the classical n.n. Ising model, (4.17) becomes much simpler: $\underset{\sim}{Q}^\Lambda_\pm$ is a single configuration, the ground state configuration with all spins equal to +1 (resp. −1). The contours are just surfaces (in the dual lattice) and their weights are small (equal to the exponential of minus β times the perimeter of the contour, if the spin-spin couplings are set equal to 1). Thus (4.17) becomes the partition function of a non interacting gas of contours (except for the compatibility condition). For β large, the statistical weight of each contour is so depressed that cluster expansion techniques become available. In our case (4.17) is not as simple, the gas of contours is non-trivially coupled to a "restricted ensemble of loops configurations", $\underset{\sim}{Q}^\Lambda_\pm$. A possible approach is to fix the configuration $\{\Gamma_i\}$ in (4.17) and integrate over ν_Λ: as a result, we have a true gas of contours, but the contours do now interact with each other.

We will not follow this approach and study instead the combined system of loops configurations in the restricted ensembles $\underset{\sim}{Q}^{\Lambda}_{\pm}$ coupled with a gas of contours.

The strategy for bounding $W^{+}_{\gamma,\lambda}(\Gamma; \underset{\sim}{q})$ is based on two points: first, to prove that the two partition functions $Z^{\pm}_{\gamma,\lambda}$ in (4.14) and (4.15) have approximately the same value; secondly, that the integral on the r.h.s. of (4.14) is much smaller than its analogue on the r.h.s. of (4.15). While the second point is robust, as it follows from an analysis of the cost of deviations from equilibrium (as imposed by the definition of contours), the first one is much more delicate, and it rests on a special choice of the chemical potential λ, as a function of the inverse temperature β and of the Kac parameter γ. Technically, the most difficult point is to show that the difference between the two partition functions (which is a surface term) is smaller than the cost of the deviations from equilibrium, cf. Section 11.

4.5. Abstract contour models. Following the Zahradnik's version of the Pirogov-Sinai theory, we introduce an abstract contour model where the weights of the contours are given a priori (our choice is (4.25) below), thus they are not necessarily equal to the true ones.

We then define an abstract partition function, using (4.17) with the weights (4.25) replacing the true ones: let Λ be a bounded $\mathcal{D}^{(\ell_-,\gamma)}$-measurable region, $\mathcal{A} \sqsubset \mathcal{C}^{\pm}_{\Lambda}$, then

$$\hat{Z}^{\pm}_{\gamma,\lambda}(\Lambda; \mathcal{A}|\bar{\underset{\sim}{q}}) = \sum_{\{\Gamma_i\}\in\mathcal{A}} \int_{\underset{\sim}{Q}^{\Lambda}_{\pm}} \nu_{\Lambda}(d\underset{\sim}{q}) e^{-H_{\gamma,\lambda}(\underset{\sim}{q}|\bar{\underset{\sim}{q}}_{\Lambda^c})} \prod_i \hat{W}^{\pm}_{\gamma,\lambda}(\Gamma_i; \underset{\sim}{q}) \tag{4.19}$$

(the hat superscript referring to the new partition functions and the new weights) and simply write $\hat{Z}^{\pm}_{\gamma,\lambda}(\Lambda|\bar{\underset{\sim}{q}})$ for $\hat{Z}^{\pm}_{\gamma,\lambda}(\Lambda; \mathcal{C}^{\pm}_{\Lambda}|\bar{\underset{\sim}{q}})$.

As mentioned, we regard (4.19) as defining $\hat{Z}^{\pm}_{\gamma,\lambda}$ for given $\hat{W}^{\pm}_{\gamma,\lambda}$, so that if the weights are not the real ones, the partition functions are fictitious. It is however possible (without a direct comparison with the true quantities) to understand if the weights from where we started are the real ones. To this end we go back to (4.14)–(4.15) that we regard now as a definition of the l.h.s. thus getting, for given Γ,

$$\hat{N}^{+}_{\gamma,\lambda}(\underset{\sim}{q}) := \int_{\eta=\eta_\Gamma,\sigma=\sigma_\Gamma} \nu_{\mathrm{sp}(\Gamma)\setminus K^+}(d\underset{\sim}{q}')\, e^{-H_{\gamma,\lambda}(\underset{\sim}{q}'|\underset{\sim}{q}_{K^+})} \hat{Z}^{-}_{\gamma,\lambda}\Big(\mathrm{int}^-(\Gamma); \mathcal{C}^{-}_{\mathrm{int}^-(\Gamma)\setminus A^-}\big|\underset{\sim}{q}'\Big) \tag{4.20}$$

$$\hat{D}^{+}_{\gamma,\lambda}(\underset{\sim}{q}) := \int_{\eta=\sigma=1} \nu_{\mathrm{sp}(\Gamma)\setminus K^+}(d\underset{\sim}{q}')\, e^{-H_{\gamma,\lambda}(\underset{\sim}{q}'|\underset{\sim}{q}_{K^+})} \hat{Z}^{+}_{\gamma,\lambda}\Big(\mathrm{int}^-(\Gamma); \mathcal{C}^{+}_{\mathrm{int}^-(\Gamma)\setminus A^-}\big|\underset{\sim}{q}'\Big) \tag{4.21}$$

and analogous expressions for $\hat{N}^{-}_{\gamma,\lambda}$ and $\hat{D}^{-}_{\gamma,\lambda}$. Notice that $\hat{N}^{+}_{\gamma,\lambda}$ and $\hat{D}^{+}_{\gamma,\lambda}$ are functions of the weights $\hat{W}^{\pm}_{\gamma,\lambda}$ via (4.19).

By an inductive procedure on the volume of the regions, which, being by now classical in Pirogov-Sinai models, is omitted, we then have:

Theorem 4.1. *Suppose that for all* Γ *and* $\underline{q}$ *as above,*

$$\hat{W}^{\pm}_{\gamma,\lambda}(\Gamma;\underline{q}) = \frac{\hat{N}^{+}_{\gamma,\lambda}(\underline{q})}{\hat{D}^{+}_{\gamma,\lambda}(\underline{q})} \tag{4.22}$$

then

$$\hat{W}^{\pm}_{\gamma,\lambda}(\Gamma;\underline{q}) = W^{\pm}_{\gamma,\lambda}(\Gamma;\underline{q}); \qquad \hat{Z}^{\pm}_{\gamma,\lambda}(\Lambda|\bar{\underline{q}}) = Z^{\pm}_{\gamma,\lambda}(\Lambda|\bar{\underline{q}}) \tag{4.23}$$

for all Λ *and* $\bar{\underline{q}}$.

4.6. Cutoff weights. At the very end of our analysis we will prove that there are $\lambda(\beta,\gamma)$ and a constant $c_f > 0$ so that, for all γ small enough, the true weights satisfy

$$W^{\pm}_{\gamma,\lambda(\beta,\gamma)}(\Gamma;\underline{q}) \le e^{-2c_f\gamma^{-1}N_\Gamma}, \qquad \ell^d_{+,\gamma}N_\Gamma = \big|\mathrm{sp}(\Gamma)\big| \tag{4.24}$$

uniformly in Γ and $\underline{q}$, (the bound improves if the deviations defining the contour Γ are due to the η variable). With this in mind, we then specify the weights $\hat{W}^{\pm}_{\gamma,\lambda}$ to be the cutoff weights

$$\hat{W}^{\pm}_{\gamma,\lambda}(\Gamma;\underline{q}) = \min\Big\{e^{-c_f\gamma^{-1}N_\Gamma} \;\; ; \;\; \frac{\hat{N}^{\pm}_{\gamma,\lambda}(\underline{q})}{\hat{D}^{\pm}_{\gamma,\lambda}(\underline{q})}\Big\} \tag{4.25}$$

Notice that the r.h.s. depends on the weights $\hat{W}^{\pm}_{\gamma,\lambda}(\Gamma';\cdot)$, with Γ' such that $\mathrm{sp}(\Gamma')$ is contained in $\mathrm{int}^-(\Gamma)$ (supposing for instance Γ a plus contour). Thus (4.25) becomes a true (non circular) definition through an inductive procedure on the volume of the regions.

Notice also that the factor 2 present in the exponential in (4.24) is instead missing in (4.25).

Using again an inductive procedure on the volume of the regions, which is also omitted, we then have:

Theorem 4.2. *Suppose that for some* $\lambda(\beta,\gamma)$*, the cutoff weights* $\hat{W}^{\pm}_{\gamma,\lambda}(\Gamma;\underline{q})$ *defined by* (4.25) *satisfy the bound*

$$\hat{W}^{\pm}_{\gamma,\lambda}(\Gamma;\underline{q}) < e^{-c_f\gamma^{-1}N_\Gamma} \tag{4.26}$$

for all Γ *and* $\underline{q}$. *Then* (4.23) *is verified.*

We have thus overcome the hardest point of the analysis, namely we can work with weights which are small and, by Theorem 4.2, we only need to check at the end the consistency property (4.26).

5. Choice of chemical potential

The aim is to prove that the cutoff weights defined in (4.25) are (at least for γ small enough) strictly smaller than $e^{-c_f \gamma^{-1} N_\Gamma}$. By Theorem 4.2 this will imply that they are equal to the true weights and hence that the Peierls estimates hold; the step from this to the proof of Theorem 2.1 is then reported in Section 11.

In this and in the next section we will prove the Peierls estimates, under the validity of some intermediate theorems which constitute the main body of the paper and which will be proved in the successive sections and in the appendices. The main point of the whole argument is a correct choice of the chemical potential $\lambda(\beta, \gamma)$, which, according to Pirogov-Sinai, must equalize the pressures in the two restricted ensembles. We will in fact prove that with this choice there is a phase transition. In this section we will prove that such a chemical potential indeed exists, we will see later where this choice is specifically needed.

We will consider here the $\pm$ restricted ensembles $\underset{\sim}{Q}^\Lambda_\pm$ defined in (4.18), with Λ a bounded $\mathcal{D}^{(\ell_-,\gamma)}$-measurable region and eventually a $\mathcal{D}^{(\ell_+,\gamma)}$ cube. By default, η, σ and Θ without superscripts, are defined with the parameters (4.1).

By taking advantage of the fact that the thermodynamic pressure is independent of the boundary conditions, we choose a special class of boundary conditions:

$$\hat{Z}^\pm_{\gamma,\lambda}(\Lambda|\rho^\pm_\beta \mathbf{1}_{\Lambda^c}) := \sum_{\{\Gamma_i\}\in \mathcal{C}^\pm_\Lambda} \int_{\underset{\sim}{Q}^\Lambda_\pm} \nu_\Lambda(d\underset{\sim}{q}) e^{-H_{\gamma,\lambda}(\underset{\sim}{q}|\rho^\pm_\beta \mathbf{1}_{\Lambda^c})} \prod_i \hat{W}^+_{\gamma,\lambda}(\Gamma_i;\underset{\sim}{q}) \tag{5.1}$$

where,

$$H_{\gamma,\lambda}\big(\underset{\sim}{q}|\rho^\pm_\beta \mathbf{1}_{\Lambda^c}\big) = \int_0^\beta h_{\gamma,\lambda}\big(\underset{\sim}{q}(t)|\rho^\pm_\beta \mathbf{1}_{\Lambda^c}\big)dt,$$

$$h_{\gamma,\lambda}\big(\underset{\sim}{q}(t)|\rho^\pm_\beta \mathbf{1}_{\Lambda^c}\big) = \int e_\lambda\big(j_\gamma * [\underset{\sim}{q}(t) + \rho^\pm_\beta \mathbf{1}_{\Lambda^c}]\big) - e_\lambda\big(\rho^\pm_\beta j_\gamma * \mathbf{1}_{\Lambda^c}\big)\, dr$$

and, if q is a particle configuration and $\rho(r)$, $r \in \mathbb{R}^d$, a measurable (non negative) function,

$$j_\gamma * (q+\rho)(r) = j_\gamma * q(r) + j_\gamma * \rho(r) = \sum_{q_i \in q} j_\gamma(r, q_i) + \int j_\gamma(r, r')\rho(r')dr' \tag{5.2}$$

It is sufficient for our purposes to prove the existence of the thermodynamic limit for the pressure over cubes Λ_n of side $2^n \ell_{+,\gamma}$ and with the above special boundary conditions (but the existence of the thermodynamic limit is valid with much greater generality). In Section 7 we will prove the following theorems.

Theorem 5.1 (Existence of pressure)**.** *The limits*

$$\lim_{n\to\infty} \frac{\log \hat{Z}^\pm_{\gamma,\lambda}(\Lambda_n|\rho^\pm_\beta \mathbf{1}_{\Lambda^c})}{\beta|\Lambda_n|} =: \hat{P}^\pm_{\gamma,\lambda} \tag{5.3}$$

exist and are both continuous function of λ *in the interval* $|\lambda - \lambda_\beta| \le 1$.

Theorem 5.2. *There is a constant* $c > 0$ *and for all* γ *small enough there is* $\lambda(\beta,\gamma) \in [\lambda_\beta - c\gamma^{1/2}, \lambda_\beta + c\gamma^{1/2}]$ *so that*

$$\hat{P}^{+}_{\gamma,\lambda(\beta,\gamma)} = \hat{P}^{-}_{\gamma,\lambda(\beta,\gamma)} \tag{5.4}$$

6. Peierls estimates: scheme of proof

Having found the "right" chemical potential, hereafter denoted by $\lambda_\gamma = \lambda(\beta,\gamma)$, we begin the proof that the cutoff weights in (4.25) with $\lambda = \lambda_\gamma$ are (for γ small enough) strictly smaller than $e^{-c_f\gamma^{-1}N_\Gamma}$. *Notational remark:* by default, when temperature and/or chemical potential are missing as subscripts, they are meant to have the above values. Thus, for instance, $H_\gamma \equiv H_{\beta,\gamma,\lambda(\beta,\gamma)}$.

For the sake of definiteness, we suppose that Γ is a plus contour and, recalling (4.25) for notation, we want to show that

$$\hat{N}^{+}_{\gamma}(\underline{q}_{K^+}) \le e^{-2c_f\gamma^{-1}N_\Gamma}\hat{D}^{+}_{\gamma}(\underline{q}_{K^+}) \tag{6.1}$$

where l.h.s. and r.h.s. are defined respectively in (4.20) and (4.21).

Once (6.1) has been proved, then, by (4.25),

$$\hat{W}^{\pm}_{\gamma,\lambda}(\Gamma;\underline{q}) = \frac{\hat{N}^{\pm}_{\gamma,\lambda}(\underline{q})}{\hat{D}^{\pm}_{\gamma,\lambda}(\underline{q})}$$

and, by Theorem 4.1,

$$\hat{W}^{\pm}_{\gamma,\lambda}(\Gamma;\underline{q}) = W^{\pm}_{\gamma,\lambda}(\Gamma;\underline{q}) \tag{6.2}$$

We will prove (6.1) after a chain of inequalities for $\hat{N}^{+}_{\gamma}$, recovering (6.1) at the last step.

6.1. Contribution of the long loops. The long loops produce the first gain term, denoted by $g_{1,\gamma}$, g standing for "gain". Let

$$g_{1,\gamma} = e^{-c_1\gamma^{-1}N^{(\sigma)}_{\Gamma}}, \quad N^{(\sigma)}_{\Gamma} = \frac{\big|\{r \in \mathrm{sp}(\Gamma) : \sigma_\Gamma(r) = 0\}\big|}{\ell^d_{+,\gamma}} \tag{6.3}$$

and

$$\mathrm{sp}_{\pm}(\Gamma) = \Big\{r \in \mathrm{sp}(\Gamma) : \Theta(\eta_\Gamma, 1; r) = \pm 1\Big\} \tag{6.4}$$

Thus $\mathrm{sp}_{\pm}(\Gamma)$ are those regions in $\mathrm{sp}(\Gamma)$ which belong to the contour only because of the long loops. In Appendix B we will prove that for a suitable value of $c_1 > 0$ and shorthanding below $\Delta_- = \text{int}^-(\Gamma) \sqcup \text{sp}_-(\Gamma)$,

$$\hat{N}^+_\gamma(\underline{q}_{K^+}) \;\le\; g_{1,\gamma} \int_{\eta=\eta_\Gamma,\sigma=1} \nu_{\text{sp}(\Gamma)\setminus[K^+\sqcup\text{sp}_-(\Gamma)]}(d\underline{q})\, e^{-H_\gamma(\underline{q}|\underline{q}_{K^+})} \times \hat{Z}^-_\gamma\big(\Delta_-\,;\,\mathcal{C}^-_{\text{int}^-(\Gamma)\setminus A^-}\big|\,\underline{q}\big) \tag{6.5}$$

where $\hat{Z}^-_\gamma\big(\Delta_-\,;\,\mathcal{C}^-_{\text{int}^-(\Gamma)\setminus A^-}\big|\,\underline{q}\big)$ is the partition function with the constraint described by the second of its arguments, namely the sum in (4.19) is restricted to all $\{\Gamma_j\} \in \mathcal{C}^-_{\Delta^-}$ so that $\text{sp}(\{\Gamma_j\}) \sqcap [A^- \sqcup \text{sp}_-(\Gamma)] = \emptyset$, see (4.16).

6.2. Contours with only short loops. We rewrite (6.5) in terms of the contours generated only by the density fluctuations. Let

$$\{\,\text{sp}(\Gamma_i)\}\ \text{the maximal connected components of}\ \{r \in \text{sp}(\Gamma) : \Theta(\eta_\Gamma, 1; r) = 0\} \tag{6.6}$$

and, correspondingly,

$$K^+_i = \delta^{\ell_+,\gamma/2}_{\text{in}}[\text{sp}(\Gamma_i)] \sqcap \{\eta_\Gamma = 1\}, \qquad A^-_i = \delta^{\ell_+,\gamma}_{\text{in}}[\text{int}^-(\Gamma_i)] \tag{6.7}$$

Since for any i, $A^-_i \sqsubset \{A^- \sqcup \text{sp}_-(\Gamma)\}$, shorthanding $\Delta_- = \text{int}^-(\Gamma) \sqcup \text{sp}_-(\Gamma)$ as in (6.5),

$$\mathcal{C}^-_{\Delta_-\setminus\{A^-\sqcup\text{sp}_-(\Gamma)\}} \;\sqsubset\; \mathcal{C}^-_{\Delta_-\setminus\sqcup_i A^-_i}$$

hence

$$\hat{N}^+_\gamma(\underline{q}_{K^+}) \;\le\; g_{1,\gamma} \int_{\eta=\sigma=1} \nu_{[\sqcup_i K^+_i \sqcup \text{sp}_+(\Gamma)]\setminus K^+}(d\underline{q})\, e^{-H_\gamma(\underline{q}|\underline{q}_{K^+})} \prod_i M^{(i)}_\gamma(\underline{q}, \underline{q}_{K^+}) \tag{6.8}$$

where

$$M^{(i)}_\gamma(\underline{q}, \underline{q}_{K^+}) \;=\; \int_{\sigma=1;\,\eta=\eta_\Gamma\ \text{on}\ \text{sp}(\Gamma_i),\ \eta=-1\ \text{on}\ A^-_i} \nu_{A^-_i\sqcup\text{sp}(\Gamma_i)\setminus K^+_i}(d\underline{q}')\, e^{-H_\gamma(\underline{q}'|\underline{q}_{K^+},\underline{q})} \times \hat{Z}^-_\gamma\Big(\text{int}^-_i(\Gamma) \setminus A^-_i\ \Big|\ \underline{q}'\Big) \tag{6.9}$$

is the expression for $\hat{N}^+_\gamma$ relative to a contour Γ which has only short loops. Let $c > 0$ and

$$g^{(i)}_{2,\gamma} = e^{-c\gamma^{-(1-\alpha)d+2a} N_{\Gamma_i}} \tag{6.10}$$

6.3. Main estimate. There exists a constant $c > 0$ so that

$$M^{(i)}_\gamma(\underline{q}, \underline{q}_{K^+}) \le\ g^{(i)}_{2,\gamma}\, \hat{Z}^+_\gamma\Big(\text{int}^-_i(\Gamma) \sqcup \text{sp}(\Gamma_i) \setminus K^+_i\,;\,\mathcal{C}^+_{\text{int}^-_i(\Gamma)\setminus A^-_i}\ \Big|\underline{q}_{K^+_i}\Big) \tag{6.11}$$

Postponing the proof of (6.11), we have:

6.4. Conclusion of the proof. Calling $g_{2,\gamma} = \prod_i g^{(i)}_{2,\gamma}$ we get, shorthanding $\Delta = \mathrm{sp}(\Gamma) \sqcup \mathrm{int}^-(\Gamma) \setminus K^+$,

$$\hat N^+_\gamma(\underline{q}_{K^+}) \le g_{1,\gamma} g_{2,\gamma}\, \hat Z^+_\gamma\Big(\mathrm{sp}(\Gamma)\sqcup \mathrm{int}^-(\Gamma)\setminus K^+\,;\, \mathcal{C}^+_{\Delta\setminus\mathcal{B}}\,\Big|\underline{q}_{K^+}\Big)$$
$$\mathcal{B} = \mathrm{sp}_+(\Gamma) \;\sqcup\; \{\sqcup_i[\mathrm{sp}(\Gamma_i)\sqcup A^-_i]\} \tag{6.12}$$

In order to reconstruct $\hat D^+_\gamma(\underline{q}_{K^+})$ (see (4.21)), we need to replace $\mathcal{B}$ by $\mathrm{sp}(\Gamma)\sqcup A^-$. We have

$$\mathrm{sp}(\Gamma) = \mathrm{sp}_-(\Gamma)\sqcup \mathrm{sp}_+(\Gamma)\sqcup_i \mathrm{sp}(\Gamma_i) \tag{6.13}$$

$$A^-(\Gamma) \sqsubset \bigsqcup_i \Big(A^-_i \sqcup \delta^{\ell_{+,\gamma}}_{\mathrm{out}}[\mathrm{sp}_-(\Gamma)]\Big) \tag{6.14}$$

Thus we need to add the constraint of no contour to a region Δ contained in $\mathrm{sp}_-(\Gamma)\sqcup \delta^{\ell_{+,\gamma}}_{\mathrm{out}}[\mathrm{sp}_-(\Gamma)]$. The number of $\mathcal{D}^{(\ell_{+,\gamma})}$ cubes in such a region is bounded by $3^d|\mathrm{sp}_-(\Gamma)|\ell^{-d}_{+,\gamma}$ (recalling that the number of cubes contiguous to a given one is 3^d-1). Then, by Lemma G.2,

$$\hat N^+_\gamma(\underline{q}_{K^+}) \le g_{1,\gamma} g_{2,\gamma}\; e^{3^d|\mathrm{sp}_-(\Gamma)|\ell^{-d}_{+,\gamma} e^{-c_f\gamma^{-1/2}}}\;\; \hat D^+_\gamma(\underline{q}_{K^+}) \tag{6.15}$$

which, for γ small enough, yields (6.1) with $c_f < c_1$, c_1 as in (6.3)-(6.5).

Proof of main estimate

It remains to show (6.11). In Section 9 we will prove that with a small cost we can replace in the energy $\lambda(\beta,\gamma)$ by the mean field value λ_β. The idea of the proof is that using stability we can reduce to configurations whose density in the $\mathcal{D}^{(\ell_{-,\gamma})}$-cubes is bounded by $e^{2-\beta b}$. Recalling from Theorem 5.2 that $|\lambda_\beta - \lambda(\beta,\gamma)| \le c\gamma^{1/2}$, we then deduce that there is a constant c_2 so that

$$\begin{aligned} M^{(i)}_\gamma(\underline{q},\underline{q}_{K^+}) \;\;\le\;\; & b^{(i)}_{2,\gamma}\int_{\sigma=1;\,\eta=\eta_\Gamma \text{ on } \mathrm{sp}(\Gamma_i),\ \eta=-1 \text{ on } A^-_i} \nu_{A^-_i\sqcup \mathrm{sp}(\Gamma_i)\setminus K^+_i}(d\underline{q}')\; e^{-H_{\beta,\gamma,\lambda_\beta}(\underline{q}'|\underline{q}_{K^+},\underline{q})} \\ & \times \hat Z^-_\gamma\Big(\mathrm{int}^-_i(\Gamma)\setminus A^-_i\;\Big|\;\underline{q}'\Big), \qquad b^{(i)}_{2,\gamma} = e^{c_2\gamma^{1/2}|\mathrm{sp}(\Gamma_i)|} \end{aligned} \tag{6.16}$$

(we denote by $b^{(i)}_{n,\gamma}$ the loss terms, b standing for "bad").

6.5. Separating corridors. In Section 8 we will prove a "small deviations result", namely that "well inside" regions where $\eta\equiv 1$ (or $\eta\equiv -1$), the density is in fact" much closer to" $\rho^\pm_\beta$ than what implied by η being 1 (or -1). For this we introduce two corridors

C_i^+ and C_i^- of width $\ell_{+,\gamma}/6$,

$$C_i^- = \delta_{\rm in}^{\ell_{+,\gamma}/6}[{\rm sp}(\Gamma_i)] \sqcap \{\eta_{\Gamma_i} = -1\}, \quad C_i^+ = \Big(\delta_{\rm in}^{5\ell_{+,\gamma}/6}[{\rm sp}(\Gamma_i)] \sqcap \{\eta_{\Gamma_i} = 1\}\Big) \setminus B_i^+ \tag{6.17}$$
$$B_i^+ = \Big(\delta_{\rm in}^{4\ell_{+,\gamma}/6}[{\rm sp}(\Gamma_i)] \sqcap \{\eta_{\Gamma_i} = 1\}\Big)$$

Thus C_i^- is at the minus boundary of ${\rm sp}(\Gamma_i)$, C_i^+ is away from K_i^+ by $\ell_{+,\gamma}/6$ (and hence also away by the same quantity from the "interior" ${\rm sp}(\Gamma) \setminus \delta_{\rm in}^{\ell_{+,\gamma}}[{\rm sp}(\Gamma)]$ of the spatial support of Γ_i). The important point for what follows is that these corridors are well inside the regions where $\eta = \pm 1$, respectively.

We then decompose

$${\rm sp}(\Gamma_i) = B_i^+ \sqcup C_i^+ \sqcup T_i \sqcup C_i^- \tag{6.18}$$

the identity defining T_i (under the requirement that (6.18) is a decomposition of ${\rm sp}(\Gamma_i)$). Calling

$$b_{3,\gamma}^{(i)} = \exp\Big\{c_3 \gamma^{1/2}(|C_i^-| + |C_i^+|)\Big\} \tag{6.19}$$

there are c_3 and ω positive so that, denoting by $F_{\beta,\lambda}(\cdot)$ the functional defined in (C.10),

$$\begin{aligned} M_\gamma^{(i)}(\underline{q}, \underline{q}_{K^+}) \;\le\;& b_{2,\gamma}^{(i)}\, b_{3,\gamma}^{(i)} \, \exp\big\{ -\beta\gamma^{-d}[F_{\beta,\lambda_\beta}(\rho_\beta^- \mathbf{1}_{\gamma C_i^-}) + F_{\beta,\lambda_\beta}(\rho_\beta^+ \mathbf{1}_{\gamma C_i^+})]\big\} \\ & \times \int_{\eta=\sigma=1} \nu_{B_i^+ \setminus K_i^+}(d\underline{q}')\, e^{-H_{\beta,\gamma,\lambda_\beta}(\underline{q}' | \underline{q}_{K_i^+}, \rho_\beta^+ \mathbf{1}_{C_i^+})} \\ & \times \int_{\eta=\eta_\Gamma,\sigma=1} \nu_{T_i}(d\underline{q}_{T_i})\, e^{-H_{\beta,\gamma,\lambda_\beta}(\underline{q}_{T_i} | \rho_\beta^- \mathbf{1}_{C_i^-}, \rho_\beta^+ \mathbf{1}_{C_i^+})} \\ & \times \hat{Z}_\gamma^-\Big({\rm int}_i^-(\Gamma)\,;\, \mathcal{C}^-_{{\rm int}_i^-(\Gamma)\setminus A_i^-} \;\Big|\; \rho_\beta^- \mathbf{1}_{C_i^-}\Big) \end{aligned} \tag{6.20}$$

We have in this step used the Lebowitz-Penrose coarse graining technique and properties of the non local functional, we are going to use the argument again and thus postpone comments on this point.

6.6. A variational problem. The gain term $g_{2,\gamma}^{(i)}$ comes from the integral over $\underline{q}_{T_i}$ in (6.20). Following the classical Lebowitz-Penrose approach, we will first bound the integral in terms of a minimization problem for the non local functional $F_{\beta,\lambda_\beta}(\rho)$ (see Appendix F) and then prove lower bounds for such a problem which exploit the imposed deviations from equilibrium present in the integral. All that is done in Section 9 and in some appendices, where we prove that

$$\begin{aligned} &\int_{\eta=\eta_\Gamma,\sigma=1} \nu_{T_i}(d\underline{q}_{T_i})\, e^{-H_{\beta,\gamma,\lambda_\beta}(\underline{q}_{T_i} | \rho_\beta^- \mathbf{1}_{C_i^-}, \rho_\beta^+ \mathbf{1}_{C_i^+})} \\ &\qquad \le\; (g_{2,\gamma}^{(i)})^2 \, \exp\Big\{ -\beta\gamma^{-d}\Big(f_{\beta,\lambda_\beta}(\rho_\beta^+)|\gamma T_i| + I^-_{\gamma T_i, \gamma C_i^-} + I^+_{\gamma T_i, \gamma C_i^+}\Big)\Big\} \end{aligned} \tag{6.21}$$

with $I^{\pm}_{A,B}$ defined in (C.18). The terms $\beta I^{\pm}_{\mathrm{int}_i^-(\Gamma),C_i^-}$ are the surface corrections to the pressure in the limit $\gamma = 0$.

6.7. Ratio of partition functions. Using the two identities

$$
\begin{aligned}
&I^-_{\gamma T_i,\gamma C_i^-} + F_{\beta,\lambda_\beta}(\rho^-_\beta \mathbf{1}_{\gamma C_i^-}) = f_{\beta,\lambda_\beta}(\rho^+_\beta)|\gamma C_i^-| - I^-_{\gamma A_i^-,\gamma C_i^-} \\
&f_{\beta,\lambda_\beta}(\rho^+_\beta)|\gamma C_i^-| = I^+_{\gamma T_i,\gamma C_i^-} + F_{\beta,\lambda_\beta}(\rho^+_\beta \mathbf{1}_{\gamma C_i^-}) + I^+_{\gamma A_i^-,\gamma C_i^-}
\end{aligned}
\tag{6.22}
$$

proved in Proposition C.3, we then get from (6.20) and (6.21)

$$
\begin{aligned}
M^{(i)}_\gamma(\underline{q},\underline{q}_{K^+}) \le\ & (g^{(i)}_{2,\gamma})^2 b^{(i)}_{2,\gamma}\, b^{(i)}_{3,\gamma} \int_{\eta=\sigma=1} \nu_{B_i^+\setminus K_i^+}(d\underline{q}')\, e^{-H_{\beta,\gamma,\lambda_\beta}(\underline{q}'|\underline{q}_{K_i^+},\rho^+_\beta \mathbf{1}_{C_i^+})} \\
& \times \exp\Big\{ -\beta\gamma^{-d}\Big(F_{\beta,\lambda_\beta}(\rho^+_\beta \mathbf{1}_{\gamma[C_i^+\sqcup T_i \sqcup C_i^-]}) - I^-_{\gamma A_i^-,\gamma C_i^-} + I^+_{\gamma A_i^-,\gamma C_i^-}\Big)\Big\} \\
& \times\ \hat{Z}^-_\gamma\Big(\mathrm{int}_i^-(\Gamma)\,;\, \mathcal{C}^-_{\mathrm{int}_i^-(\Gamma)\setminus A_i^-}\ \Big|\ \rho^-_\beta \mathbf{1}_{C_i^-}\Big)
\end{aligned}
\tag{6.23}
$$

Maybe the most delicate part of the proof is the following bound, proved in Section 10

$$
\frac{\exp\{\beta\gamma^{-d} I^-_{\gamma A_i^-,\gamma C_i^-}\}\ \hat{Z}^-_\gamma\Big(\mathrm{int}_i^-(\Gamma)\,;\,\mathcal{C}^-_{\mathrm{int}_i^-(\Gamma)\setminus A_i^-}\ \Big|\ \rho^-_\beta \mathbf{1}_{C_i^-}\Big)}{\exp\{\beta\gamma^{-d} I^+_{\gamma A_i^-,\gamma C_i^-}\}\ \hat{Z}^+_\gamma\Big(\mathrm{int}_i^-(\Gamma)\,;\,\mathcal{C}^+_{\mathrm{int}_i^-(\Gamma)\setminus A_i^-}\ \Big|\ \rho^+_\beta \mathbf{1}_{C_i^-}\Big)} \le\ b^{(i)}_{4,\gamma}
\tag{6.24}
$$

$$
b^{(i)}_{4,\gamma} = \exp\{c_4 \gamma^{a'} |A_i^-|\}
\tag{6.25}
$$

with c_4 a positive constant and $1 \gg a' \gg \alpha \gg a > 0$.

This is the point where the choice of the chemical potential is important. In fact, (6.24) is an estimate on the finite volume corrections to the pressure: the log of the partition functions are to first order given by $\beta P^{\pm}_{\beta,\lambda(\beta,\gamma)}|\mathrm{int}_i^-(\Gamma)|$, hence by (5.4) these contributions cancel between numerator and denominator. As said before, the terms $\beta I^{\pm}_{\mathrm{int}_i^-(\Gamma),C_i^-}$ are the surface corrections in the limit $\gamma = 0$, and $b^{(i)}_{4,\gamma}$ bounds its finite γ correction. Hidden in this description is an estimate of exponential decay of correlations which allows to localize around and close to the surface, the corrections to the pressure (which are then computed by taking the limit $\gamma = 0$ value and then estimating its $\gamma > 0$ correction).

We have so far shown that

$$
\begin{aligned}
M^{(i)}_\gamma(\underline{q},\underline{q}_{K^+}) \le\ & (g^{(i)}_{2,\gamma})^2 b^{(i)}_{2,\gamma} \cdots b^{(i)}_{4,\gamma} \int_{\eta=\sigma=1} \nu_{B_i^+\setminus K_i^+}(d\underline{q}')\, e^{-H_{\beta,\gamma,\lambda_\beta}(\underline{q}'|\underline{q}_{K_i^+},\rho^+_\beta \mathbf{1}_{C_i^+})} \\
& \times e^{-\beta\gamma^{-d} F_{\beta,\lambda_\beta}(\rho^+_\beta \mathbf{1}_{\gamma[C_i^+\sqcup T_i\sqcup C_i^-]})}\ \hat{Z}^+_\gamma\Big(\mathrm{int}_i^-(\Gamma)\,;\,\mathcal{C}^+_{\mathrm{int}_i^-(\Gamma)\setminus A_i^-}\ \Big|\ \rho^+_\beta \mathbf{1}_{C_i^-}\Big)
\end{aligned}
\tag{6.26}
$$

6.8. Reconstruction of a plus partition function. We will reconstruct $\hat{D}^+_\gamma$ by doing in reverse the previous steps. Call $F_\gamma = F_{\beta,\lambda(\beta,\gamma)}$ and

$$b^{(i)}_{5,\gamma} = e^{c_5(|C^+_i|+|T_i|+|C^-_i|)\gamma^{1/2}} \tag{6.27}$$

Then, as in (6.16), there is $c_5 > 0$ so that

$$\begin{aligned} M^{(i)}_\gamma(\underline{q}, \underline{q}_{K^+}) \le \; & (g^{(i)}_{2,\gamma})^2 b^{(i)}_{2,\gamma} \cdots b^{(i)}_{5,\gamma} \int_{\eta=\sigma=1} \nu_{B^+_i\setminus K^+_i}(d\underline{q}')\, e^{-H_{\beta,\gamma,\lambda(\beta,\gamma)}(\underline{q}'|\underline{q}_{K^+_i},\rho^+_\beta \mathbf{1}_{C^+_i})} \\ & \times e^{-\beta\gamma^{-d} F_\gamma(\rho^+_\beta \mathbf{1}_{\gamma[C^+_i\sqcup T_i\sqcup C^-_i]})}\, \hat{Z}^+_\gamma\Big(\mathrm{int}^-_i(\Gamma)\,;\, \mathcal{C}^+_{\mathrm{int}^-_i(\Gamma)\setminus A^-_i} \;\Big|\; \rho^+_\beta \mathbf{1}_{C^-_i}\Big) \end{aligned} \tag{6.28}$$

Using Appendix F we want to replace the free energy functional by a partition function. The r.h.s. of (6.28) can be rewritten as

$$\begin{aligned} & (g^{(i)}_{2,\gamma})^2 b^{(i)}_{2,\gamma} \cdots b^{(i)}_{5,\gamma} \int_{\eta=\sigma=1} \nu_{B^+_i\setminus K^+_i}(d\underline{q}') \int_{\eta=\sigma=1} \nu_{A^-_i}(d\underline{q}'')\, e^{-H_{\beta,\gamma,\lambda(\beta,\gamma)}(\underline{q}'|\underline{q}_{K^+_i},\rho^+_\beta \mathbf{1}_{C^+_i})} \\ & \quad \times e^{-\beta\gamma^{-d} F_\gamma(\rho^+_\beta \mathbf{1}_{\gamma[C^+_i\sqcup T_i\sqcup C^-_i]})} e^{-H_{\beta,\gamma,\lambda(\beta,\gamma)}(\underline{q}''|\rho^+_\beta \mathbf{1}_{C^-_i})}\, \hat{Z}^+_\gamma\Big(\mathrm{int}^-_i(\Gamma) \setminus A^-_i \;\Big|\; \underline{q}''_{A^-_i}\Big) \end{aligned} \tag{6.29}$$

Moreover,

$$\begin{aligned} & H_{\beta,\gamma,\lambda(\beta,\gamma)}(\underline{q}'_{B^+_i\setminus K^+_i}|\underline{q}_{K^+_i}, \rho^+_\beta \mathbf{1}_{C^+_i}) + \beta\gamma^{-d} F_\gamma(\rho^+_\beta \mathbf{1}_{\gamma[C^+_i\sqcup T_i\sqcup C^-_i]}) + H_{\beta,\gamma,\lambda(\beta,\gamma)}(\underline{q}''_{A^-_i}|\rho^+_\beta \mathbf{1}_{C^-_i}) \\ = \; & H_{\beta,\gamma,\lambda(\beta,\gamma)}(\underline{q}'_{B^+_i\setminus K^+_i}|\underline{q}_{K^+_i}) + \beta\gamma^{-d} F_\gamma(\rho^+_\beta \mathbf{1}_{\gamma[C^+_i\sqcup T_i\sqcup C^-_i]}|\underline{q}'_{B^+_i\setminus K^+_i}, \underline{q}''_{A^-_i}) + H_{\beta,\gamma,\lambda(\beta,\gamma)}(\underline{q}''_{A^-_i}) \end{aligned}$$

Adding subscripts to remind the regions from where the loops start, by (F.9),

$$\begin{aligned} -\beta\gamma^{-d} F_\gamma(\rho^+_\beta \mathbf{1}_{\gamma[C^+_i\sqcup T_i\sqcup C^-_i]}|\underline{q}'_{B^+_i\setminus K^+_i}, \underline{q}''_{A^-_i}) & \le \log Z_\gamma(T_i \sqcup C^-_i \sqcup C^+_i|\underline{q}'_{B^+_i\setminus K^+_i}, \underline{q}''_{A^-_i}) \\ & \quad + c_6\gamma^{1/2}(|T_i| + |C^-_i| + |C^+_i|) \end{aligned} \tag{6.30}$$

hence we proved that there is a constant $b^{(i)}_{6,\gamma} := e^{c_6\gamma^{1/2}\mathrm{sp}(\Gamma_i)}$ so that

$$M^{(i)}_\gamma(\underline{q}, \underline{q}_{K^+}) \le \; (g^{(i)}_{2,\gamma})^2 b^{(i)}_{2,\gamma} \cdots b^{(i)}_{6,\gamma}\, \hat{Z}^+_\gamma\Big(\mathrm{int}^-_i(\Gamma) \sqcup \mathrm{sp}(\Gamma_i) \setminus K^+_i\,;\, \mathcal{C}^+_{\mathrm{int}^-_i(\Gamma)\setminus A^-_i} \;\Big|\underline{q}_{K^+_i}\Big) \tag{6.31}$$

Summarizing, for γ small enough there exists a constant $c_7 = c_2 + \cdots + c_6$ such that

$$(g^{(i)}_{2,\gamma})^2 b^{(i)}_{2,\gamma} \cdots b^{(i)}_{6,\gamma} \le e^{-\left(2c\gamma^{-(1-\alpha)d+2a} - c_7 \ell^d_{+,\gamma}\gamma^{a'}\right) N_{\Gamma_i}} \tag{6.32}$$

As $d\alpha + 2a < -d\alpha + a'$ we obtain (6.11).

7. Equality of pressures

In this section we will prove Theorems 5.1 and 5.2.

7.1. Proof of Theorem 5.1. For notational simplicity we only refer to the + case. Let

$$D_{\gamma,\lambda}(n) := \frac{\log \hat Z^+_{\gamma,\lambda}(\Lambda_n|\rho^+_\beta \mathbf{1}_{\Lambda_n^c})}{\beta|\Lambda_n|} - \frac{\log \hat Z^+_{\gamma,\lambda}(\Lambda_{n-1}|\rho^+_\beta \mathbf{1}_{\Lambda_{n-1}^c})}{\beta|\Lambda_{n-1}|} \tag{7.1}$$

We claim that there is a constant c such that

$$\sup_{|\lambda-\lambda_\beta|\le 1} \Big| D_{\gamma,\lambda}(n) \Big| \le c 2^{-n} \tag{7.2}$$

Since for any n

$$\frac{\log \hat Z^+_{\gamma,\lambda}(\Lambda_n|\rho^+_\beta \mathbf{1}_{\Lambda_n^c})}{\beta|\Lambda_n|} \tag{7.3}$$

is a continuous function of λ, existence of the limit (5.3) and its continuity then follow from (7.2)–(7.3), thus we need to prove (7.2).

Lower bound. Decomposing Λ_n into cubes $\Lambda_{n-1}(i)$ of side 2^{n-1}, see right above (A.19), we call Λ^0 the union over all i of $\Lambda_{n-1}(i) \setminus \delta_{\text{in}}^{\ell_{+,\gamma}}[\Lambda_{n-1}(i)]$. We have

$$\hat Z^+_{\gamma,\lambda}(\Lambda_n|\rho^+_\beta \mathbf{1}_{\Lambda_n^c}) \ge \sum_{\{\Gamma_i\}\in \mathcal{C}^+_{\Lambda^0}} \int_{\underset{\sim}{Q}_+^{\Lambda_n}} \nu_\Lambda(d\underset{\sim}{q}) e^{-H_{\gamma,\lambda}(\underset{\sim}{q}|\rho^+_\beta \mathbf{1}_{\Lambda_n^c})} \prod_i \hat W^+_{\gamma,\lambda}(\Gamma_i;\underset{\sim}{q}) \tag{7.4}$$

Exploiting that the configurations are in the restricted ensemble, we gain a control of the interaction energy. Referring to Lemma A.6 for details,

$$\log \hat Z^+_{\gamma,\lambda}(\Lambda_n|\rho^+_\beta \mathbf{1}_{\Lambda_n^c}) \ge 2^d \log \hat Z^+_{\gamma,\lambda}\Big(\Lambda_{n-1}; \mathcal{C}^+_{\Lambda^0_{n-1}} \,|\rho^\pm_\beta \mathbf{1}_{\Lambda_{n-1}^c}\Big) - c(2^n \ell_{+,\gamma})^{d-1}\gamma^{-1} \tag{7.5}$$

where $\Lambda^0_{n-1} = \Lambda_{n-1} \setminus \delta_{\text{in}}^{\ell_{+,\gamma}}[\Lambda_{n-1}]$.

As the weights are small (by definition) we can bound the cost of extending their presence to the whole Λ_{n-1}. Referring to Lemma G.1 for details, we have

$$\log \hat Z^+_{\gamma,\lambda}\Big(\Lambda_{n-1}; \mathcal{C}^+_{\Lambda^0_{n-1}} \,|\rho^+_\beta \mathbf{1}_{\Lambda_{n-1}^c}\Big) \ge \log \hat Z^+_{\gamma,\lambda}\big(\Lambda_{n-1}|\rho^+_\beta \mathbf{1}_{\Lambda_{n-1}^c}\big) - 2d(2^{n-1})^{d-1} e^{-c_f \gamma^{-1}/2} \tag{7.6}$$

By (7.5) and (7.6), there is a constant $c > 0$ so that

$$D_{\gamma,\lambda}(n) \ge -c2^{-n} \tag{7.7}$$

Upper bound. Using Lemma G.1, instead of (7.4) we get

$$\begin{aligned}\hat Z^{+}_{\gamma,\lambda}(\Lambda_n|\rho^+_\beta \mathbf{1}_{\Lambda_n^c}) \le\ & e^{2^d 2d(2^{n-1})^{d-1} e^{-c_f\gamma^{-1}/2}} \\ & \times \sum_{\{\Gamma_i\}\in\mathcal{C}^+_{\Lambda 0}} \int_{\underline{Q}_+^{\Lambda_n}} \nu_\Lambda(d\underline{q}) e^{-H_\gamma(\underline{q}|\rho^+_\beta \mathbf{1}_{\Lambda_n^c})} \prod_i \hat W^+_{\gamma,\lambda}(\Gamma_i;\underline{q})\end{aligned}$$

and, instead of (7.5),

$$\begin{aligned}\log \hat Z^{+}_{\gamma,\lambda}(\Lambda_n|\rho^+_\beta \mathbf{1}_{\Lambda_n^c}) \ \le\ & 2^d\Big[\log \hat Z^+_{\gamma,\lambda}\Big(\Lambda_{n-1};\mathcal{C}^+_{\Lambda^0_{n-1}}\,|\rho^+_\beta \mathbf{1}_{\Lambda^c_{n-1}}\Big) \\ & + 2d(2^{n-1})^{d-1}e^{-c_f\gamma^{-1}/2}\Big] + c(2^n\ell_{+,\gamma})^{d-1}\gamma^{-1}\end{aligned}$$

Hence, $D_{\gamma,\lambda}(n) \le c2^{-n}$, which together with (7.7) proves (7.2) and the theorem. □

7.2. Proof of Theorem 5.2. In traditional Pirogov-Sinai models, the statement is proved using the implicit function theorem after establishing bounds on the derivatives w.r.t. λ; such an approach yields local uniqueness. To avoid bounds on derivatives, which are here not so straightforward, we will use a different, weaker approach, showing that the difference $\hat P^+_{\gamma,\lambda} - \hat P^-_{\gamma,\lambda}$ is a continuous function of λ which undergoes a change of sign. Continuity in λ has been proved in Theorem 5.1, the change of sign property follows from its validity in the mean field limit together with closeness to mean field for γ small enough. We restrict λ to the interval $|\lambda-\lambda_\beta| \le \gamma^{2a}$, recalling that $\zeta=\gamma^a$ and $\gamma^{2a} \gg \gamma^{1/2}$, and look for a change of sign of $\hat P^+_{\gamma,\lambda} - \hat P^-_{\gamma,\lambda}$ for λ in such an interval.

Using Lemma G.1, we can drop, with a "small error", the sum over $\{\Gamma_i\}$ in (5.1), retaining only the term without contours. We then use Proposition F.1 to get

$$\hat P^{\pm}_{\gamma,\lambda} \le -\lim_{n\to\infty}\inf_{\rho\in\mathcal{A}^\pm_n} \frac{F_{\beta,\lambda,\gamma\Lambda_n}(\rho|\rho^\pm_\beta \mathbf{1}_{\Lambda_n^c})}{\gamma^d|\Lambda_n|} + \frac{c}{\beta}\gamma^{1/2} \tag{7.8}$$

where $\mathcal{A}^\pm_n$ is the set of all $\rho \in L^\infty(\gamma\Lambda_n,\mathbb{R}_+)$ such that the averages in the cubes of $\mathcal{D}^{(\gamma\ell_{-,\gamma})}$ are close to $\rho^\pm_\beta$ within $\zeta=\gamma^a$. By (C.16)

$$\hat P^{\pm}_{\gamma,\lambda} \le \frac{c}{\beta}\gamma^{1/2} - \lim_{n\to\infty}\inf_{\rho\in\mathcal{A}^\pm_n}\frac{1}{\gamma^d|\Lambda_n|}\int_{\gamma\Lambda_n} f_{\beta,\lambda}(j*[\rho_{\Lambda_n}+\rho^\pm_\beta \mathbf{1}_{\Lambda_n^c}])dr \tag{7.9}$$

By (A.6), for $\rho\in\mathcal{A}^\pm_n$, there is $c'>0$ so that

$$j*[\rho_{\Lambda_n}+\rho^\pm_\beta \mathbf{1}_{\Lambda_n^c}] \in [\rho^\pm_\beta - \zeta - c'\gamma\ell_{-,\gamma}, \rho^\pm_\beta+\zeta+c'\gamma\ell_{-,\gamma}] \tag{7.10}$$

Then, by (7.9),

$$\hat P^{\pm}_{\gamma,\lambda} \le \frac{c}{\beta}\gamma^{1/2} - \inf_{|s-\rho^\pm_\beta|\le\zeta+c'\gamma\ell_{-,\gamma}} f_{\beta,\lambda}(s)$$

Recalling that $f_{\beta,\lambda_\beta}(\cdot)$ is a double well function with two minimizers, $\rho^\pm_\beta$, we deduce that for $\lambda-\lambda_\beta$ small enough, $f_{\beta,\lambda}(\cdot)$ is still double well with local minima at $\rho^\pm_{\beta,\lambda}$. Moreover,

$|\rho^\pm_{\beta,\lambda} - \rho^\pm_\beta| \le c|\lambda-\lambda_\beta| \le c\gamma^{2a}$, for γ small then, $|\rho^\pm_{\beta,\lambda} - \rho^\pm_\beta| \le \zeta + c'\gamma\ell_{-,\gamma}$ and

$$\hat P^\pm_{\gamma,\lambda} \le \frac{c}{\beta}\gamma^{1/2} - f_{\beta,\lambda}(\rho^\pm_{\beta,\lambda}) \tag{7.11}$$

For the lower bound we use (F.7) getting

$$\hat P^\pm_{\gamma,\lambda} \ge -\lim_{n\to\infty} \frac{F_{\beta,\lambda,\gamma\Lambda_n}(\rho^{(n)}|\rho^\pm_\beta \mathbf{1}_{\Lambda^c_n})}{\gamma^d|\Lambda_n|} - \frac{c}{\beta}\gamma^{1/2}, \qquad \rho^{(n)} \in \mathcal{A}^\pm_n$$

By the previous argument, for γ small enough $\rho^{(n)}(r) = \rho^\pm_{\beta,\lambda}\mathbf{1}_{\Lambda_n} \in A^\pm_n$, so that

$$\hat P^\pm_{\gamma,\lambda} \ge -f_{\beta,\lambda}(\rho^\pm_{\beta,\lambda}) - \frac{c}{\beta}\gamma^{1/2} \tag{7.12}$$

(7.11)–(7.12) show that

$$\Big|[\hat P^+_{\gamma,\lambda} - \hat P^-_{\gamma,\lambda}] - [f_{\beta,\lambda}(\rho^-_{\beta,\lambda}) - f_{\beta,\lambda}(\rho^+_{\beta,\lambda})]\Big| \le \frac{2c}{\beta}\gamma^{1/2}$$

which completes the proof of Theorem 5.2, because

$$f_{\beta,\lambda_\beta}(\rho^-_\beta) = f_{\beta,\lambda_\beta}(\rho^+_\beta), \qquad \frac{d}{d\lambda}[f_{\beta,\lambda}(\rho^-_{\beta,\lambda}) - f_{\beta,\lambda}(\rho^+_{\beta,\lambda})]\Big|_{\lambda=\lambda_\beta} = \rho^+_\beta - \rho^-_\beta > 0$$

□

8. Small deviations

In this section we will prove (6.20). Let Λ be a bounded $\mathcal{D}^{(\ell_{-,\gamma})}$-measurable region, which satisfies the "fatness property":

$$\Big|\Big\{r : \mathrm{dist}(r,\Lambda) \le 4\gamma^{-1}\Big\}\Big| \le 3^d|\Lambda| \tag{8.1}$$

(3^d-1 the connectivity of $\mathbb{Z}^d$). Let then

$$Z_{\gamma,\lambda_\beta}(\Lambda; \underset{\sim}{Q}^\Lambda_\pm|\bar{\underset{\sim}{q}}) = \int_{\underset{\sim}{Q}^\Lambda_\pm} \nu_\Lambda(d\underset{\sim}{q})\, e^{-H_{\beta,\gamma,\lambda_\beta}(\underset{\sim}{q}|\bar{\underset{\sim}{q}})} \tag{8.2}$$

and suppose $\sigma(\bar{\underset{\sim}{q}}) \equiv 1$, $\bar{\underset{\sim}{q}}(0) \sqsubset \Lambda^c$ and that $\mathrm{Av}^{(\ell_{-,\gamma})}(\bar{\underset{\sim}{q}}(0);\cdot) \le 2\rho_{\beta^+}$, conditions which are fulfilled in the case of (6.20).

Let finally Δ be a $\mathcal{D}^{(\ell_{-,\gamma})}$-measurable subset of Λ and suppose

$$\mathrm{dist}(\Delta,\Lambda^c) \ge 10\gamma^{-1} + \gamma^{-1}\ell, \quad \ell > 0 \tag{8.3}$$

In the proof of (6.20), we take for Λ a maximal connected component either of

$$A^-_i \sqcup \Big(\delta^{\ell_+,\gamma}_{\mathrm{in}}[\mathrm{sp}(\Gamma_i)] \sqcap \{r \in \mathrm{sp}(\Gamma_i) : \eta_{\Gamma_i} = -1\}\Big)$$

or else of

$$\left(\delta_{\rm in}^{\ell_{+,\gamma}}[\mathrm{sp}(\Gamma_i)] \sqcap \{r \in \mathrm{sp}(\Gamma_i) : \eta_{\Gamma_i} = 1\}\right) \setminus K_i^+$$

In either case (8.1) is verified.

Δ in the proof of (6.20) is either C_i^- or C_i^+. Since $\gamma^{-1}\ell > \ell_{+,\gamma}/100$, then (6.20) follows from (8.4) below.

Proposition 8.1. *There are $\omega > 0$ and $c > 0$ so that for any Λ and Δ as above*

$$\begin{aligned} Z_{\gamma,\lambda_\beta}(\Lambda; \underset{\sim}{Q}_{\pm}^{\Lambda} \,|\, \bar{\underset{\sim}{q}}) \;&\le\; e^{c(\gamma^{1/2}|\Lambda| + e^{-\omega\ell}|\Delta|)} e^{-\beta\gamma^{-d} F_{\beta,\lambda_\beta}(\rho_\beta^{\pm}\mathbf{1}_{\gamma\Delta})} \\ &\times Z_{\gamma,\lambda_\beta}\Big(\Lambda \setminus \Delta; \underset{\sim}{Q}_{\pm}^{\Lambda\setminus\Delta} \,|\, \bar{\underset{\sim}{q}}_{\Lambda^c}, \rho_\beta^{\pm}\mathbf{1}_\Delta\Big) \end{aligned} \tag{8.4}$$

Proof. For the sake of definiteness we restrict to the plus case. By (F.6) and using the assumption (8.1), there is $c > 0$ so that

$$\log Z_{\gamma,\lambda_\beta}(\Lambda; \underset{\sim}{Q}_{\pm}^{\Lambda} \,|\, \bar{\underset{\sim}{q}}) \le -\inf_{\rho \in \mathcal{A}^*} \beta\gamma^{-d} F_{\beta,\lambda_\beta}(\rho|\bar\rho) + c\gamma^{1/2}|\Lambda| \tag{8.5}$$

where

$$\mathcal{A}^* := \Big\{\rho \in L^\infty(\gamma\Lambda, \mathbb{R}_+) : \eta^{(\zeta,\gamma\ell_{-,\gamma})}(\rho; r) = 1\Big\} \tag{8.6}$$

and $\eta^{(\zeta,\ell)}(\rho; r)$ is defined as in (3.4) with $|q \sqcap C_r^{(\ell)}|$ replaced by $\int_{C_r^{(\ell)}} \rho(r')\,dr$. For $\epsilon > 0$, let $\rho_\epsilon \in \mathcal{A}^*$ be such that

$$F_{\beta,\lambda_\beta}(\rho_\epsilon|\bar\rho) \le\; \epsilon + \inf_{\rho\in\mathcal{A}^*} F_{\beta,\lambda_\beta}(\rho|\bar\rho) \tag{8.7}$$

Calling

$$D = \delta_{\rm in}^{10\gamma^{-1}}[\Lambda], \quad \mathcal{B}^* = \Big\{\rho \in L^\infty(\gamma(\Lambda\setminus D), \mathbb{R}_+) : \eta^{(\zeta,\gamma\ell_{-,\gamma})}(\rho; r) = 1\Big\} \tag{8.8}$$

$$\inf_{\rho\in\mathcal{A}^*} F_{\beta,\lambda_\beta}(\rho|\bar\rho) \ge -\epsilon + F_{\beta,\lambda_\beta}(\rho_\epsilon\mathbf{1}_D|\bar\rho) + \inf_{\rho\in\mathcal{B}^*} F_{\beta,\lambda_\beta}(\rho|\rho_\epsilon\mathbf{1}_D) \tag{8.9}$$

In Theorem D.3 it is proved that the inf on the r.h.s. is actually a minimum, the minimizer is unique, say $\hat\rho_\epsilon$, and there are $\omega > 0$ and c_ω such that for all $r \in \Lambda \setminus D$:

$$|\hat\rho_\epsilon(r) - \rho_\beta^+| \le c_\omega e^{-\omega\,\mathrm{dist}(r,D)} \tag{8.10}$$

Thus, calling $\tilde\rho_\epsilon = \rho_\epsilon\mathbf{1}_D + \hat\rho_\epsilon$,

$$\begin{aligned} \inf_{\rho\in\mathcal{A}^*} F_{\beta,\lambda_\beta}(\rho|\bar\rho) + \epsilon &\ge F_{\beta,\lambda_\beta}(\tilde\rho_\epsilon|\bar\rho) \ge F_{\beta,\lambda_\beta}(\hat\rho_\epsilon\mathbf{1}_{\gamma\Delta}) + F_{\beta,\lambda_\beta}(\tilde\rho_\epsilon\mathbf{1}_{\gamma(\Lambda\setminus\Delta)}|\bar\rho + \hat\rho_\epsilon\mathbf{1}_{\gamma\Delta}) \\ &\ge -ce^{-\omega\ell}|\gamma\Delta| + F_{\beta,\lambda_\beta}(\rho_\beta^+\mathbf{1}_{\gamma\Delta}) + F_{\beta,\lambda_\beta}\Big(\tilde\rho_\epsilon\mathbf{1}_{\gamma(\Lambda\setminus\Delta)}\Big|\bar\rho + \rho_\beta^+\mathbf{1}_{\gamma\Delta}\Big) \end{aligned}$$

and choosing $\epsilon < ce^{-\omega\ell}|\gamma\Delta|$,

$$\inf_{\rho\in\mathcal{A}^*} F_{\beta,\lambda_\beta}(\rho|\bar\rho) \ge -2ce^{-\omega\ell}|\gamma\Delta| + F_{\beta,\lambda_\beta}(\rho_\beta^+\mathbf{1}_{\gamma\Delta}) + F_{\beta,\lambda_\beta}\Big(\tilde\rho_\epsilon\mathbf{1}_{\gamma(\Lambda\setminus\Delta)}\Big|\bar\rho + \rho_\beta^+\mathbf{1}_{\gamma\Delta}\Big) \tag{8.11}$$

By (F.7)

$$\exp\Big\{ - \beta\gamma^{-d} F_{\beta,\lambda_\beta}\Big(\tilde\rho_\epsilon \mathbf{1}_{\gamma(\Lambda\setminus\Delta)}\Big|\bar\rho + \rho^+_\beta \mathbf{1}_{\gamma\Delta}\Big)\Big\}$$
$$\le e^{c\gamma^{1/2}|\Lambda|}\, Z_{\gamma,\lambda_\beta}\Big(\Lambda\setminus\Delta;\, Q^\Lambda_\pm\,\Big|\bar{q}, \rho^+_\beta \mathbf{1}_\Delta\Big) \tag{8.12}$$

By (8.5), (8.11) and (8.12) we then obtain (8.4). □

9. Large deviations

In this section we will prove the statements in Section 6 about the "large deviations" from equilibrium. We will repeatedly use Lemma 9.1 below, where

$$N := e^{-\beta b}\ell^d_{-,\gamma} e^2 \tag{9.1}$$

and, given a configuration q,

$$n(r) = |q(0) \sqcap C^{(\ell_-,\gamma)}_r| \tag{9.2}$$

Lemma 9.1. *Let Δ be a bounded $\mathcal{D}^{(\ell_-,\gamma)}$-measurable region, η^* a $\{0,\pm1\}$-valued function on Δ, $\bar q$ any finite loop configuration. Then*

$$\frac{Z_{\gamma,\beta,\lambda(\beta,\gamma)}\Big(\Delta; \{\eta=\eta^*, \sigma=1\}\big|\bar q\Big)}{Z_{\gamma,\beta,\lambda(\beta,\gamma)}\Big(\Delta; \{\eta=\eta^*, \sigma=1, n(r)\le N, r\in\Delta\}\big|\bar q\Big)} \le\ 3^{|\Delta|/\ell^d_{-,\gamma}} \tag{9.3}$$

Proof. Let $\{X\}$ be the set of the centers of the cubes of $\mathcal{D}^{(\ell_-,\gamma)}$ which are in Δ. Then

$$Z_{\gamma,\beta,\lambda(\beta,\gamma)}\Big(\Delta; \{\eta=\eta^*,\sigma=1\}\big|\bar q\Big) = \sum_{Y\sqsubset X} Z_{\gamma,\beta,\lambda(\beta,\gamma)}\Big(\Delta; \{\eta=\eta^*,\sigma=1,$$
$$n(x)\le N, x\in X\setminus Y, n(y)>N, y\in Y\}\big|\bar q\Big) \tag{9.4}$$

and

$$\frac{Z_{\gamma,\beta,\lambda(\beta,\gamma)}\Big(\Delta; \{\eta=\eta^*,\sigma=1,n(x)\le N, x\in X\setminus Y, n(y)>N, y\in Y\}\big|\bar q\Big)}{Z_{\gamma,\beta,\lambda(\beta,\gamma)}\Big(\Delta; \{\eta=\eta^*,\sigma=1,n(x)\le N, x\in X\setminus Y, n(y)=N, y\in Y\}\big|\bar q\Big)}$$
$$\le \Big(\sum_{p>0}\frac{N!}{(N+p)!}[e^{-\beta b}\ell^d_{-,\gamma}]^p\Big)^{|Y|} =: C(Y) \tag{9.5}$$

To derive (9.5), we first write

$$\nu_{\beta,\Delta}(d\underline{q}) = \prod_{x\in X} \nu_{\beta, C_x^{(\ell_-,\gamma)}}(d\underline{q}_{C_x^{(\ell_-,\gamma)}})$$

Then in each cube $C_y^{(\ell_-,\gamma)}$, $y \in Y$, which has $N+p$ particles, $p>0$, we pick up p particles and call $\underline{q}'$ the collection of all such particles over all the cubes with $y \in Y$. Calling $\underline{q}$ the remaining ones, by (2.10)

$$H_\gamma(\underline{q}+\underline{q}'|\bar{\underline{q}}) - H_\gamma(\underline{q}|\bar{\underline{q}}) = H_\gamma(\underline{q}'|\underline{q},\bar{\underline{q}}) \geq b|\underline{q}'| \tag{9.6}$$

which yields (9.5) because $\eta(\underline{q}(0);x) = \eta(n(x))$ is a function of the number $n(x)$, and, by (C.3), $\eta(n(x))$ is constant for $n(x) \geq N$.

Recalling the Stirling formula

$$n! = n^{n+1/2} e^{-n}\sqrt{2\pi}\Big(1 + 0\left(\frac{1}{\sqrt{n}}\right)\Big) \tag{9.7}$$

we write

$$(N+p)! \geq N^{(N+p)+1/2} e^{-(N+p)}\sqrt{2\pi}\Big(1 + 0\left(\frac{1}{\sqrt{N+p}}\right)\Big) \geq \frac{1}{2}\ N! N^p e^{-p}$$

for N large enough (hence, recalling (9.1), for γ small enough). Then

$$\frac{[e^{-\beta b}\ell^d_{-,\gamma}]^p}{(N+p)!} \leq \frac{2}{N!}\left(\frac{[e^{-\beta b}\ell^d_{-,\gamma}]e}{N}\right)^p \leq \frac{2}{N!}e^{-p}$$

so that, recalling that $C(Y)$ has been defined in (9.5),

$$C(Y) \leq 2^{|Y|} \tag{9.8}$$

and, by (9.4)-(9.5),

$$Z_{\gamma,\beta,\lambda(\beta,\gamma)}\Big(\Delta; \{\eta=\eta^*, \sigma = 1\}\big|\bar{\underline{q}}\Big) \leq 3^{|X|} Z_{\gamma,\beta,\lambda(\beta,\gamma)}\Big(\Delta; \{\eta=\eta^*, \sigma=1, \\ n(x) \leq N, x \in X\}\big|\bar{\underline{q}}\Big) \tag{9.9}$$

Lemma 9.1 is proved. □

Proof of (6.16). By Lemma 9.1 with $\Delta = A_i^- \sqcup \mathrm{sp}(\Gamma_i) \setminus K_i^+$,

$$M_\gamma^{(i)}(\underline{q}, \underline{q}_{K^+}) \leq 3^{|\Delta|/\ell^d_{-,\gamma}} \int_{\sigma=1;\eta=\eta_\Gamma \text{ on } \mathrm{sp}(\Gamma_i),\ \eta=-1 \text{ on } A_i^-;\ n(r)\leq N, r\in\Delta} \nu_\Delta(d\underline{q}') \\ \times e^{-H_{\beta,\gamma,\lambda(\beta,\gamma)}(\underline{q}'|\underline{q}_{K^+},\underline{q})} \hat{Z}^-_\gamma\Big(\mathrm{int}_i^-(\Gamma)\setminus A_i^- \ \Big|\ \underline{q}'\Big)$$

(6.16) then follows from

$$H_{\beta,\gamma,\lambda_\beta}(\underline{q}'|\underline{q}_{K^+},\underline{q}) - H_{\beta,\gamma,\lambda(\beta,\gamma)}(\underline{q}'|\underline{q}_{K^+},\underline{q}) = |\underline{q}'|\big(\lambda(\beta,\gamma)-\lambda_\beta\big)$$

$|\lambda_\beta - \lambda(\beta,\gamma)| \le c\gamma^{1/2}$ and $|\underline{q}'| \le [e^{-\beta b} e^2]|\Delta|$ and then dropping the condition $n(\cdot) \le N$. $\square$

Proof of (6.21). To have lighter notation we drop the suffix i and write T, $C^\pm$ for T_i, $C_i^\pm$. By (9.3),

$$\text{l.h.s. of (6.21)} \le 3^{|T|/\ell^d_{-,\gamma}} \int_{\eta=\eta_\Gamma,\sigma=1,n(r)\le N, r\in T} \nu_T(d\underline{q}_T)\, e^{-H_{\beta,\gamma,\lambda_\beta}(\underline{q}_T|\rho^-_\beta \mathbf{1}_{C^-},\rho^+_\beta \mathbf{1}_{C^+})}$$

By (F.10), there is a constant $c > 0$ so that,

$$\text{l.h.s. of(6.21)} \le -\inf_{\rho\in\mathcal{A}^*} \beta\gamma^{-d} F_{\beta,\lambda_\beta}(\rho|\bar\rho) + c\gamma^{1/2}|T| \tag{9.10}$$

where

$$\mathcal{A}^* = \Big\{\rho \in L^\infty(\gamma T, \mathbb{R}_+) : \rho(r) \le X = e^{-\beta b+2},\ \eta^{(\zeta,\gamma\ell_{-,\gamma})}(\rho; r) = \eta_\Gamma(\gamma r), r \in \gamma T\Big\} \tag{9.11}$$

$$\bar\rho = (\rho^-_\beta \mathbf{1}_{\gamma C^-}, \rho^+_\beta \mathbf{1}_{\gamma C^+}) \tag{9.12}$$

A bound for the minimization problem of (9.10) is proved in Theorem E.2, which we apply taking $\Lambda = \gamma T$ and $\ell = \gamma\ell_{-,\gamma}$ and supposing γ small enough. We have

$$N_0 + N_\pm \ge 3^{-d} N_T \tag{9.13}$$

where N_0 and $N_\pm$ are defined in (E.2)-(E.3), while N_T is the number of cubes of $\mathcal{D}^{(\ell_+,\gamma)}$ which are contained in T. In fact if C is a $\mathcal{D}^{(\gamma\ell_+,\gamma)}$ cube in T and C^* is the union of C with its neighbor ones, it cannot happen, by the definition of contour, that $\eta^{(\zeta,\gamma\ell_{-,\gamma})}(\rho; r)$ is constantly equal to 1 or to -1 in C^*, hence C^* must contain at least an element of $\{C_i^0\}$ or of $\{C_j^\pm\}$. Since the number of the cubes contiguous to a given one is $3^d - 1$, (9.13) follows.

Thus, by (E.4), calling c' the constant c in that equation,

$$\begin{aligned}\text{l.h.s. of (6.21)} \le -\beta\gamma^{-d}\Big\{&[c'\zeta^2(\gamma\ell_{-,\gamma})^d]3^{-d}N_T \\ &+ f_{\beta,\lambda_\beta}(\rho^+_\beta)|\gamma T| + I^-_{\gamma T,\gamma C^-} + I^+_{\gamma T,\gamma C^+}\Big\} + c\gamma^{1/2}|T|\end{aligned} \tag{9.14}$$

10. Finite volume corrections to the pressure

We need to prove that the absolute value of the difference between

$$\log\Big\{e^{\beta\gamma^{-d} I^+_{\gamma\Lambda,\gamma\Lambda^c}} \hat Z^+_\gamma\Big(\Lambda\,;\,\mathcal{C}^+_{\Lambda\setminus A}\ \Big|\ \rho^+_\beta \mathbf{1}_{\Lambda^c}\Big)\Big\} \tag{10.1}$$

(shorthanding $\Lambda = \mathrm{int}_i^-(\Gamma)$ and $A = A_i^-$) and the corresponding term relative to the minus ensemble, is bounded by $c|A|\gamma^{a'}$, with c a constant and a' as in (6.24).

10.1. Main result. This will be achieved by proving that

$$(10.1) = \beta \hat{P}^{+}_{\gamma,\lambda(\beta,\gamma)}|\Lambda^0| - \beta f_{\beta,\lambda_\beta}(\rho^+_\beta)|\Lambda \setminus \Lambda^0| + \int_{\Lambda \sqcup \delta^{\ell_+,\gamma}_{\text{out}}[\Lambda]} L_{\gamma,\Lambda}(r)\, dr \tag{10.2}$$

where

$$\Lambda^0 := \Delta_{10} \tag{10.3}$$

having called $\Delta_k = \Delta_{k-1} \setminus \delta^{\ell_+,\gamma}_{\text{in}}[\Delta_k]$, $\Delta_0 = \Lambda$, 10 is not optimal. The remainder term $L_{\gamma,\Lambda}(r)$ is "small" in the following sense. There are ω_0, c and c' positive so that

$$\big|L_{\gamma,\Lambda}(r)\big| \le c e^{-\omega_0 \gamma \operatorname{dist}(r,\Lambda^c)}, \quad r \in \Lambda^0; \tag{10.4}$$

$$\Big|\int_{\Lambda \sqcup \delta^{\ell_+,\gamma}_{\text{out}}[\Lambda] \setminus \Lambda^0} L_{\gamma,\Lambda}(r)\, dr\Big| \le c' \gamma^{a'} |A| \tag{10.5}$$

After proving the analogous bounds in the minus case, we exploit the identities $\hat{P}^{+}_{\gamma,\lambda(\beta,\gamma)} = \hat{P}^{-}_{\gamma,\lambda(\beta,\gamma)}$ and $f_{\beta,\lambda_\beta}(\rho^+_\beta) = f_{\beta,\lambda_\beta}(\rho^-_\beta)$, to conclude that

$$\text{l.h.s. of (6.24)} \le \int_{\Lambda^0} 2c e^{-\omega_0 \gamma \operatorname{dist}(r,\Lambda^c)}\, dr + 2c' \gamma^{a'} |A| \tag{10.6}$$

which proves (6.24).

10.2. Strategy of proof. (10.2)-(10.5) is an estimate of the "finite volume corrections to the pressure", a result usually proved using cluster expansion techniques to exponentiate the partition function. One then obtains a sum (or an integral) of quasi local terms, which differ from the pressure by a term exponentially small with the distance from the boundary of the domain. The terms which are close to the boundaries however, giving a contribution which is proportional to the surface times the range of the interaction, are a priori larger than the gain term from the energy of the contour. The special choice of the boundary conditions, given by the constant value ρ^+_β, allows to compute such dangerous terms to leading order as $\gamma \to 0$, giving $e^{-\beta\gamma^{-d} I^{+}_{\gamma\Lambda,\gamma\Lambda^c}}$, which exactly cancels with the prefactor in (10.1). The next order correction in γ gives rise to the term $c'\gamma^{a'}|A|$ in (10.5), which covers also the exponentially small corrections to the bulk terms and it is infinitesimal w.r.t. the gain term from the contour as $\gamma \to 0$.

In order to prove (10.2)-(10.5), we will choose among the several Dobrushin-Shlosman versions of cluster expansion, [**4**], the one based on an interpolation of the hamiltonian. $L_{\gamma,\Lambda}(r)$ will then be the expectation of a local function translated by r; the expectation is with respect to a Gibbs measure for a convex combination of hamiltonians. Exponential decay of correlations for such measures is then responsible for (10.4), while (10.5) will be proved using a Lebowitz-Penrose argument.

10.3. An hamiltonian representing contours. By using cluster expansion we can regard the contribution of the contours to the partition function as an additional term in the hamiltonian:

$$\sum_{\{\Gamma_i\}\sqsubset\mathcal{C}^+_{\Lambda\setminus A}} \prod_i \hat{W}^+_{\gamma,\lambda}(\Gamma_i; \underline{q}) = e^{-K^+_{\Lambda\setminus A}(\underline{q}_{\Lambda\setminus A})} \tag{10.7}$$

Referring to Appendix G for more details, we mention here that the hamiltonian $K^+_{\Lambda\setminus A}(\underline{q}_{\Lambda\setminus A})$ is expressed in terms of potentials U^+_Δ:

$$K^+_{\Lambda\setminus A}(\underline{q}_{\Lambda\setminus A}) = \sum_{\Delta\in\Lambda\setminus A} U^+_\Delta(\underline{q}_\Delta) \tag{10.8}$$

with U^+_Δ as in (G.6) and (G.7). We then call

$$H^+_\gamma(\underline{q}|\rho^+_\beta \mathbf{1}_{\Lambda^c}) = H_{\beta,\gamma,\lambda(\beta,\gamma)}(\underline{q}|\rho^+_\beta \mathbf{1}_{\Lambda^c}) + K^+_{\Lambda\setminus A}(\underline{q}_{\Lambda\setminus A}) \tag{10.9}$$

and have

$$\hat{Z}^+_\gamma\Big(\Lambda\,;\,\mathcal{C}^+_{\Lambda\setminus A}\;\Big|\;\rho^+_\beta\mathbf{1}_{\Lambda^c}\Big) = \int_{\underline{Q}^\Lambda_+} \nu_\Lambda(d\underline{q}) e^{-H^+_\gamma(\underline{q}|\rho^+_\beta\mathbf{1}_{\Lambda^c})} \tag{10.10}$$

10.4. Interpolating hamiltonians. Given a a reference hamiltonian $H_0(\underline{q})$, we set

$$H^+_{\gamma,u} = uH^+_\gamma + (1-u)H_0, \qquad u \in [0,1] \tag{10.11}$$

with H^+_γ defined in (10.9). $H^+_{\gamma,u}$ is the interpolating hamiltonian and u the interpolating parameter. Then

$$\begin{aligned}\log \hat{Z}^+_\gamma\Big(\Lambda\,;\,\mathcal{C}^+_{\Lambda\setminus A}\;\Big|\;\rho^+_\beta\mathbf{1}_{\Lambda^c}\Big) &= \log \hat{Z}^{+,0}_\gamma\Big(\Lambda\;\Big|\;\rho^+_\beta\mathbf{1}_{\Lambda^c}\Big)\\ &\quad - \int_0^1 du\; \mu^+_{\gamma,u,\Lambda}\left[H^+_\gamma(\underline{q}|\rho^+_\beta\mathbf{1}_{\Lambda^c}) - H_0(\underline{q})\right]\end{aligned} \tag{10.12}$$

where $\hat{Z}^{+,0}_\gamma$ is given by the r.h.s. of (10.10) with H^+_γ replaced by H_0 and $\mu^+_{\gamma,u,\Lambda}$ is defined as it is described right before (10.15).

H_0 will be the hamiltonian of a free gas, for which the representation (10.2) is easily derived. The difference $H^+_\gamma(\underline{q}|\rho^+_\beta\mathbf{1}_{\Lambda^c}) - H_0(\underline{q})$ in (10.12) will be a sum of local potentials, and the crucial point will be a proof that the measure $\mu^+_{\gamma,u,\Lambda}$ is well approximated by its thermodynamic limit and that this is translational invariant.

The typical example where the interpolation method works is at high temperatures. In that case, the reference hamiltonian H_0 is taken equal to zero. The interpolating hamiltonian is then the same original hamiltonian, with $\beta \to u\beta$, i.e. at higher temperatures, and the whole scheme works. We are in a sense not too far from this case, as we are working in the restricted ensemble with a single well situation, but the extension requires some care.

To choose H_0 (that will be called $H_{0,\Lambda}$) we first approximate

$$H^+_\gamma(\underline{q}|\rho^+_\beta\mathbf{1}_{\Lambda^c}) \longrightarrow \beta h_{\lambda_\beta}(\underline{q}(0)|\rho^+_\beta\mathbf{1}_{\Lambda^c}), \qquad \underline{q} = \underline{q}_\Lambda$$

and then expand the latter to first order in $j_\gamma * (\underline{q}(0) - \rho_\beta^+ \mathbf{1}_\Lambda)$,

$$
\begin{aligned}
h_{\lambda_\beta}(\underline{q}(0)|\rho_\beta^+\mathbf{1}_{\Lambda^c}) &\approx h_{\lambda_\beta}(\rho_\beta^+\mathbf{1}_\Lambda|\rho_\beta^+\mathbf{1}_{\Lambda^c}) + e'_{\lambda_\beta}(\rho_\beta^+)\int j_\gamma * (\underline{q}(0) - \rho_\beta^+\mathbf{1}_\Lambda)\, dr \\
&=: \beta^{-1}H_{0,\Lambda}(\underline{q})
\end{aligned}
\tag{10.13}
$$

Using the identity

$$h_{\lambda_\beta}(\rho_\beta^+\mathbf{1}_\Lambda|\rho_\beta^+\mathbf{1}_{\Lambda^c}) = e_{\lambda_\beta}(\rho_\beta^+)|\Lambda| + \gamma^{-d} I^+_{\gamma\Lambda,\gamma\Lambda^c}$$

proved in (C.19) and

$$\int j_\gamma * (\underline{q}(0) - \rho_\beta^+\mathbf{1}_\Lambda)\, dr = |\underline{q}(0)| - \rho_\beta^+|\Lambda|$$

we can rewrite

$$
\begin{aligned}
&H_{0,\Lambda}(\underline{q}) = -\beta\lambda_{\text{eff}}|\underline{q}| + \beta E_\Lambda \\
&\lambda_{\text{eff}} := -e'_{\lambda_\beta}(\rho_\beta^+), \qquad E_\Lambda = [e_{\lambda_\beta}(\rho_\beta^+) + \lambda_{\text{eff}}\rho_\beta^+]|\Lambda| + \gamma^{-d} I^+_{\gamma\Lambda,\gamma\Lambda^c}
\end{aligned}
\tag{10.14}
$$

The hamiltonian $H_{0,\Lambda}$ has only a one body potential and it describes an ideal gas, which is thus easy to study. The measure $\mu^+_{\gamma,u,\Lambda}$ is Gibbs on $\underline{Q}^\Lambda_+$ with hamiltonian

$$H^+_{\gamma,u}(\underline{q}|\rho_\beta^+\mathbf{1}_{\Lambda^c}) = uH^+_\gamma(\underline{q}|\rho_\beta^+\mathbf{1}_{\Lambda^c}) + (1-u)(-\beta\lambda_{\text{eff}}|\underline{q}|) \tag{10.15}$$

having dropped from $H_{0,\Lambda}$ in (10.15) the constant term βE_Λ present in (10.14).

10.5. Pressure of reference hamiltonian. We are going to prove that

$$\log\Big\{e^{\beta\gamma^{-d}I^+_{\gamma\Lambda,\gamma\Lambda^c}}\hat{Z}^{+,0}_\gamma\Big\} = p^{+,0}_\gamma|\Lambda| \tag{10.16}$$

$$|p^{+,0}_\gamma + \beta f_{\beta,\lambda_\beta}(\rho_\beta^+)| \le e^{-c\gamma^{-1}} \tag{10.17}$$

Calling C a cube of side $\ell_{-,\gamma}$

$$\hat{Z}^{+,0}_\gamma = \int_{\underline{Q}^\Lambda_+} \nu_\Lambda(d\underline{q}) e^{-H_{0,\Lambda}(\underline{q})} = \Big(\int_{\eta=1} \nu^<_C(d\underline{q}) e^{\beta\lambda_{\text{eff}}|\underline{q}|}\Big)^{|\Lambda|/\ell^d_{-,\gamma}} e^{-\beta E_\Lambda}$$

Then, recalling the definition of E_Λ in (10.14), we get (10.16) with

$$e^{p^{+,0}_\gamma \ell^d_{-,\gamma}} = \Big(\int_{\eta=1} \nu^<_C(d\underline{q}) e^{\beta\lambda_{\text{eff}}|\underline{q}|}\Big) e^{-\beta(e_{\lambda_\beta}(\rho_\beta^+) + \lambda_{\text{eff}}\ \rho_\beta^+)\ell^d_{-,\gamma}} \tag{10.18}$$

It remains to prove (10.17). We have

$$\int \nu_C(d\underline{q}) e^{\beta\lambda_{\text{eff}}|\underline{q}|} = \exp\Big\{e^{\beta\lambda_{\text{eff}}}\ell^d_{-,\gamma}\Big\} = e^{\rho_\beta^+\ell^d_{-,\gamma}} \tag{10.19}$$

because $\lambda_{\rm eff} = (\log \rho_\beta^+)/\beta$, by the mean field equation. Recalling the definition of $p_\gamma^{+,0}$ in (10.18), we then get

$$e^{p_\gamma^{+,0}\ell_{-,\gamma}^d} = e^{(\rho_\beta^+ - \beta e_{\lambda_\beta}(\rho_\beta^+) + \beta\lambda_{\rm eff}\rho_\beta^+))\ell_{-,\gamma}^d} R$$

$$R = \frac{\displaystyle\int_{\eta=1} \nu_C^<(d\underline{q}) e^{\beta\lambda_{\rm eff}|\underline{q}|}}{\displaystyle\int \nu_C(d\underline{q}) e^{\beta\lambda_{\rm eff}|\underline{q}|}}$$

Since $\rho_\beta^+ - \beta\lambda_{\rm eff}\rho_\beta^+ = s(\rho_\beta^+)$, we have

$$e^{p_\gamma^{+,0}\ell_{-,\gamma}^d} = e^{-\beta f_{\beta,\lambda_\beta}(\rho_\beta^+)\ell_{-,\gamma}^d} e^{\log(1-(1-R))} \tag{10.20}$$

and (10.17) follows once we show that $1-R$ is bounded by $e^{-c\gamma^{-1}}$, $c > 0$.

There are two contributions to $1-R$: one comes from density deviations, $\eta \neq 1$. By (10.19) in fact the average number N of particles is $\rho_\beta^+\ell_{-,\gamma}^d$, thus the probability that $|N - \rho_\beta^+\ell_{-,\gamma}^d| \ge \gamma^a\ell_{-,\gamma}^d$ is a large deviation estimate (for an ideal gas) and it is bounded by $\exp\{-c'\gamma^{2a}\ell_{-,\gamma}^d\}$. The second contribution to $1-R$ is due to the occurrence of a long loop, this is bounded by $\epsilon\rho_\beta^+\ell_{-,\gamma}^d$, where ϵ is the probability that a brownian bridge is longer than $\gamma^{-1/2}$, and $\rho_\beta^+\ell_{-,\gamma}^d$ is the mean number of particles. Hence $1-R \le e^{-c\gamma^{-1}}$ and (10.16)-(10.17) are proved.

10.6. Energy difference. We have proved so far that

$$(10.1) = p_\gamma^{+,0}|\Lambda| - \int_0^1 \mu_{\gamma,u,\Lambda}^+ \left[H_\gamma^+(\underline{q}|\rho_\beta^+\mathbf{1}_{\Lambda^c}) - H_{0,\Lambda}(\underline{q})\right] du \tag{10.21}$$

with $p_\gamma^{+,0}$ verifying (10.17). By the help of (10.14), we can write more explicitly the energy difference:

$$\begin{aligned} H_{\beta,\gamma,\lambda(\beta,\gamma)}(\underline{q}|\rho_\beta^+\mathbf{1}_{\Lambda^c}) - H_{0,\Lambda}(\underline{q}) &= \\ &= \int \phi(\underline{q},r)\mathbf{1}_{\Lambda^0} + \psi_\Lambda(\underline{q},r)\mathbf{1}_{(\Lambda^0)^c} + v_\Lambda(\underline{q},r)\mathbf{1}_{\Lambda\setminus A}\,dr \end{aligned} \tag{10.22}$$

where

$$\phi(\underline{q},\cdot) = \int_0^\beta e_{\lambda(\beta,\gamma)}(j_\gamma * \underline{q}(t)) - e_{\lambda_\beta}(\rho_\beta^+) + \lambda_{\rm eff} j_\gamma * [\underline{q}(0) - \rho_\beta^+]\,dt$$

$$\begin{aligned}\psi_\Lambda(\underline{q};\cdot) = \int_0^\beta &\{e_{\lambda(\beta,\gamma)}(j_\gamma * [\underline{q}(t) + \rho_\beta^+\mathbf{1}_{\Lambda^c}]) - e_{\lambda(\beta,\gamma)}(j_\gamma * [\rho_\beta^+\mathbf{1}_{\Lambda^c}]) - \mathbf{1}_{\Lambda\setminus\Lambda^0} e_{\lambda_\beta}(\rho_\beta^+)\}dt \\ &-\beta\gamma^{-d} I_{\gamma\Lambda,\gamma\Lambda^c}^+ + \int_0^\beta \lambda_{\rm eff} j_\gamma * [\underline{q}(0) - \rho_\beta^+\mathbf{1}_\Lambda]\,dt\end{aligned}$$

$$v_\Lambda(\underline{q},r) = \sum_{\Delta\ni r,\Delta\sqsubset\Lambda\setminus A} \frac{1}{|\Delta|} U_\Delta^+(\underline{q}_\Delta) \tag{10.23}$$

Notice that $\psi_\Lambda(\underline{q};r) = \phi(\underline{q},r)$ if $r \in \Lambda^0$ and $\psi_\Lambda(\underline{q};r) = 0$ if $r \in \Big(\Lambda \sqcup \delta_{\rm out}^{\ell_{+,\gamma}}[\Lambda]\Big)^c$. According to Lemma A.5, ϕ and ψ_Λ are uniformly bounded.

10.7. Decay of correlations. In [**1**] , Theorem 5.2, it is proved that for γ small enough the system of conditional probabilities $\{\mu^+_{\gamma,u,C_i}(\cdot|\underline{q}_{C_i^c}), C_i \in \mathcal{D}^{(\ell_{-,\gamma})}\}$ satisfies the Dobrushin uniqueness condition, uniformly in $u \in [0,1]$. In particular, it is shown that for each $u \in [0,1]$ there exists a unique measure $\mu^+_{\gamma,u}$ which is DLR w.r.t. the above system of conditional probabilities; moreover $\mu^+_{\gamma,u}$ is invariant w.r.t. translations by integer multiples of $\ell_{+,\gamma}$ and there are c and c_0 positive so that for any bounded measurable function f which depends only on $\underline{q}_\Delta$ and any $u \in [0,1]$,

$$\Big| \mu^+_{\gamma,u,\Lambda}[f] - \mu^+_{\gamma,u}[f]\Big| \le c|\Delta| e^{-c_0\gamma\,\mathrm{dist}(\Delta,\Lambda^c)}\|f\|_\infty \tag{10.24}$$

10.8. The thermodynamic pressure. Take $\Lambda = \Lambda_n$ in (10.21), Λ_n a cube of side $2^n\ell_{+,\gamma}$, i.e. one of those used in Theorem 5.1 to define $\hat P^+_{\gamma,\lambda(\beta,\gamma)}$. By letting $n \to \infty$, and using the results of Subsection 10.7, we can then identify the thermodynamic pressure as

$$\beta \hat P^+_{\gamma,\lambda(\beta,\gamma)} = p^{+,0}_\gamma - \int_0^1 \frac{1}{|C|}\int_C \mu^+_{\gamma,u}\Bigg[\phi(\underline{q},r) + \sum_{\substack{\Delta\in\mathcal{D}^{(\ell_{+,\gamma})}\\ r\in\Delta}} \frac{1}{|\Delta|}U^+_\Delta(\underline{q}_\Delta)\Bigg]\,dr\;du \tag{10.25}$$

having used that ψ_Λ and U_Δ are uniformly bounded and that U_Δ decays exponentially, see (G.7). C denotes the cube from $\mathcal{D}^{(\ell_{+,\gamma})}$ which contains 0.

10.9. Bounds on remainders. We use (10.25) to express $p^{+,0}_\gamma$ in terms of the other quantities appearing in (10.25) itself. We insert such an expression in (10.21) and get

$$(10.1) = \beta \hat P^+_{\gamma,\lambda(\beta,\gamma)}|\Lambda^0| - \beta f_{\beta,\lambda_\beta}(\rho^+_\beta)|\Lambda\setminus\Lambda^0| + \int L_1(r)+L_2(r)+L_3(r)+L_4(r)\,dr$$

where the remainders $L_1,\dots,L_4$ are

$$L_1(r) = \big(p^{+,0}_\gamma + \beta f_{\beta,\lambda_\beta}(\rho^+_\beta)\big)\,\mathbf{1}_{\Lambda\setminus\Lambda^0}(r) \tag{10.26}$$

$$L_2(r) = -\int_0^1 \mu^+_{\gamma,u,\Lambda}\left[\phi(\cdot;r)+v_\Lambda(\cdot;r)\right] - \mu^+_{\gamma,u}\left[\phi(\cdot;r)+v_\Lambda(\cdot;r)\right]\,du\,\mathbf{1}_{\Lambda^0}(r)$$

$$L_3(r) = \int_0^1 \mu^+_{\gamma,u}\Big[\sum_{\Delta\ni r,\Delta\not\subset\Lambda\setminus A}\frac{1}{|\Delta|}U_\Delta\Big]\,du\,\mathbf{1}_{\Lambda^0}(r) - \int_0^1 \mu^+_{\gamma,u}\big[v_\Lambda(\cdot;r)\big]\,du\,\mathbf{1}_{\Lambda\setminus[A\sqcup\Lambda^0]}(r)$$

$$L_4(r) = \quad -\int_0^1 \mu^+_{\gamma,u,\Lambda}\left[\psi_\Lambda(\cdot,r)\right]\,du\,\mathbf{1}_{(\Lambda^0)^c}(r)$$

By (10.17), L_1 satisfies the bound (10.5). By Subsection 10.7 and (G.7), L_2 satisfies (10.4). Using again (G.7), we prove that L_3 satisfies (10.4) and (10.5), and it only remains to bound L_4.

By Corollary A.4,

$$\Big|\int_{(\Lambda^0)^c} \psi_\Lambda(\underline{q},r) - \psi_\Lambda^{(\gamma^{-1/2})}(\underline{q}(0),r)dr\Big| \le c\gamma^{1/2}|A| \tag{10.27}$$

where $\psi_\Lambda^{(\gamma^{-1/2})}(\underline{q}(0),r)$ is defined as in the second one of the equations (10.23) (i.e. the one relative to $\psi_\Lambda(\underline{q},r)$), with $\underline{q}(t)$ on the r.h.s. replaced by $\underline{q}(0)$ and j_γ by $j_\gamma^{(\gamma^{-1/2})}$, see (A.3). Letting

$$L_5(r) := -\int_0^1 \mu^+_{\gamma,u,\Lambda}\Big[\psi_\Lambda^{(\gamma^{-1/2})}(\underline{q}(0),r)\Big]\,du\,\mathbf{1}_{(\Lambda^0)^c}(r) \tag{10.28}$$

we have

$$\int |L_4(r)|\,dr \le \int |L_5(r)|\,dr + c\gamma^{1/2}|A| \tag{10.29}$$

and it only remains to prove the bound

$$\int |L_5(r)|\,dr \le c\gamma^{a'}|A| \tag{10.30}$$

For any $\underline{q}(0) = \underline{q}_\Lambda(0)$, let

$$\mathcal{B} := \Big\{ r \notin \Lambda^0 : |j_\gamma^{(\gamma^{-1/2})} * [\underline{q}(0) + \rho_\beta^+ \mathbf{1}_{\Lambda^c}](r) - \rho_\beta^+| \ge \gamma^{a'} \Big\} \tag{10.31}$$

Recalling that $|\lambda_{\beta,\gamma} - \lambda_\beta| \le c\gamma^{1/2}$, there is a constant k_1 such that

$$\Big|\int_{(\Lambda^0)^c} \psi_\Lambda^{(\gamma^{-1/2})}(\underline{q}(0),r)\,dr\Big| \le k_1\gamma^{2a'}\big|(\Lambda \sqcup \delta_{\text{out}}^{(\ell_+,\gamma)}[\Lambda]) \setminus \Lambda^0\big|, \quad r \notin \mathcal{B}$$

We distinguish in the expectation in (10.28) whether $|\mathcal{B}|$ is either smaller or larger than $|A|\gamma^{a'}$, getting

$$\int |L_5(r)|\,dr \le k_1\gamma^{2a'}\big|(\Lambda \sqcup \delta_{\text{out}}^{(\ell_+,\gamma)}[\Lambda]) \setminus \Lambda^0\big| + \|\psi_\Lambda^{(\gamma^{-1/2})}\|_\infty |A|\gamma^{a'}$$
$$+\mu^+_{\gamma,u,\Lambda}\Big[\mathbf{1}_{\underline{q}(0):|\mathcal{B}|\ge|A|\gamma^{a'}} \int_{(\Lambda^0)^c} \psi_\Lambda^{(\gamma^{-1/2})}(\underline{q}(0),r)\,dr\Big]\,du$$

There are therefore constants k_2, k_3 so that

$$\int |L_5(r)|\,dr \le k_2\gamma^{a'}|A| + k_3|A| \sup_{u\in[0,1]} \mu^+_{\gamma,u,\Lambda}\Big[\{\underline{q}(0) : |\mathcal{B}| \ge |A|\gamma^{a'}\}\Big] \tag{10.32}$$

Call $\Delta = \Lambda^0 \setminus \delta_{\text{in}}^{\ell_+,\gamma}[\Lambda^0]$, then, by the DLR property,

$$\mu^+_{\gamma,u,\Lambda}\Big[\{|\mathcal{B}| \ge |A|\gamma^{a'}\}\Big] = \mu^+_{\gamma,u,\Lambda}\Big[\mu^+_{\gamma,u,\Lambda\setminus\Delta}\Big[\{|\mathcal{B}| \ge |A|\gamma^{a'}\}\big|\underline{q}_\Delta, \rho_\beta^+\mathbf{1}_{\Lambda^c}\Big]\Big] \tag{10.33}$$

where $\mu^{+}_{\gamma,u,\Lambda\setminus\Delta}[\,\cdot\,|\underline{q}_\Delta,\rho^+_\beta \mathbf{1}_{\Lambda^c}]$ is the finite volume Gibbs measure in $\Lambda\setminus\Delta$ in the restricted ensemble of configurations $\underline{Q}_+^{\Lambda\setminus\Delta}$ with interaction determined by the Hamiltonian $H^+_{\gamma,u}$. The contribution to the corresponding Gibbs factor coming from $K^+_{\Lambda\setminus A}$ can be bounded using (G.7). We then get

$$\mu^{+}_{\gamma,u,\Lambda\setminus\Delta}\Big[\{|\mathcal{B}|\ge |A|\gamma^{a'}\}\big|\underline{q}_\Delta,\rho^+_\beta\mathbf{1}_{\Lambda^c}\Big]\le \exp\big\{c|A|e^{-c_f\gamma^{-1/2}}\big\}$$
$$\times p^{+}_{\gamma;u;\Lambda\setminus\Delta}\Big[\{|\mathcal{B}|\ge |A|\gamma^{a'}\}\big|\underline{q}_\Delta,\rho^+_\beta\mathbf{1}_{\Lambda^c}\Big] \tag{10.34}$$

where p on the r.h.s. denotes the Gibbs measure on $\underline{Q}_+^{\Lambda\setminus\Delta}$ with hamiltonian

$$uH_{\beta,\gamma,\lambda(\beta,\gamma)}(\underline{q}_{\Lambda\setminus\Delta}|\rho^+_\beta\mathbf{1}_{\Lambda^c},\underline{q}_\Delta)+(1-u)(-\beta\lambda_{\rm eff})|\underline{q}_{\Lambda\setminus\Delta}(0)| \tag{10.35}$$

By an error bounded by $e^{c_2\gamma^{1/2}|\Lambda\setminus\Delta|}$, see (6.16), we can replace $\lambda(\beta,\gamma)$ by λ_β, we call $p^+_{\gamma,\lambda_\beta;u;\Lambda\setminus\Delta}$ the corresponding Gibbs measure. By the argument used in Subsection 6.5

$$p^{+}_{\gamma,\lambda_\beta;u;\Lambda\setminus\Delta}\Big[\{|\mathcal{B}|\ge |A|\gamma^{a'}\}\big|\underline{q}_\Delta,\rho^+_\beta\mathbf{1}_{\Lambda^c}\Big]$$
$$\le e^{c_3\gamma^{1/2}|A|}p^{+}_{\gamma,\lambda_\beta;u;\Lambda\setminus\Lambda^0}\Big[\{|\mathcal{B}|\ge |A|\gamma^{a'}\}\big|\rho^+_\beta\mathbf{1}_{(\Lambda\setminus\Lambda^0)^c}\Big] \tag{10.36}$$

To bound the r.h.s. we perform the continuum approximation using the analogue of Proposition F.1. Due to the presence of the reference hamiltonian, the free energy functional is modified into a u-dependent functional. Call $D=\gamma(\Lambda\setminus\Lambda^0)$ and define

$$F^0_\beta(\rho_D)=\int_D f^0_\beta(\rho_D)\,dr,\qquad f^0_\beta(x)=-\lambda_{\rm eff}x-\frac{s(x)}{\beta} \tag{10.37}$$

$$B=\Big\{\rho\in L^\infty(D,\mathbb{R}_+):\eta^{(\gamma^a,\ell_-)}(\rho;r)=1,r\in D;$$
$$|\{r:|j^{(\gamma^{1/2})}*(\rho+\rho^+_\beta\mathbf{1}_{D^c})(r)-\rho^+_\beta|>\gamma^{a'}\}|>\gamma^{a'}(\gamma^d|A|)\Big\} \tag{10.38}$$

Then, by the analogue of Proposition F.1,

$$p^{+}_{\gamma,\lambda_\beta;u;\Lambda\setminus\Lambda^0}\Big[\{|\mathcal{B}|\ge |A|\gamma^{a'}\}\big|\rho^+_\beta\mathbf{1}_{(\Lambda\setminus\Lambda^0)^c}\Big]\le\exp\Big\{c\gamma^{1/2}|A|$$
$$-\gamma^{-d}\inf_{\rho_D\in B}\Big(u\mathcal{F}_{\beta,\lambda_\beta}(\rho_D|\rho^+_\beta\mathbf{1}_{D^c})+(1-u)\mathcal{F}^0_\beta(\rho_D)\Big)\Big\} \tag{10.39}$$

where the functionals on the r.h.s. are the excess free energies, namely $\mathcal{F}_{\beta,\lambda_\beta}(\rho_D|\rho^+_\beta\mathbf{1}_{D^c})=F_{\beta,\lambda_\beta}(\rho_D|\rho^+_\beta\mathbf{1}_{D^c})-F_{\beta,\lambda_\beta}(\rho^+_\beta\mathbf{1}_D|\rho^+_\beta\mathbf{1}_{D^c})$ and similarly, $\mathcal{F}^0_\beta(\rho_D)=F^0_\beta(\rho_D)-F^0_\beta(\rho^+_\beta\mathbf{1}_D)$.

Using as in (E.7) that ρ^+_β is a minimum of f_{β,λ_β} we obtain for $\rho_D\in\mathcal{B}$

$$F_{\beta,\lambda_\beta}(\rho_D|\rho^+_\beta\mathbf{1}_{D^c})\ge F_{\beta,\lambda_\beta}(\rho^+_\beta\mathbf{1}_D|\rho^+_\beta\mathbf{1}_{D^c})+c\gamma^{3a'}\gamma^d|A| \tag{10.40}$$

Since

$$F^0_\beta(\rho_D)\ge F^0_\beta(\rho^+_\beta\mathbf{1}_D)+c'\gamma^{3a'}\gamma^d|A| \tag{10.41}$$

from (10.39) we get

$$p^{+}_{\gamma,\lambda_\beta;u;\Lambda\setminus\Lambda^0}\Big[\{|\mathcal{B}|\ge |A|\gamma^{a'}\}\big|\rho^{+}_{\beta}\mathbf{1}_{(\Lambda\setminus\Lambda^0)^c}\Big]\le \exp\Big\{c\gamma^{1/2}|A|-c''\gamma^{3a'}|A|\Big\} \quad (10.42)$$

and (10.30) is proved.

11. Conclusions

So far we have proved that given $\beta\in(\beta_c,\beta_0)$ there exists $\gamma_\beta>0$ and for all $\gamma\le\gamma_\beta$ there is $\lambda(\gamma,\beta)$ so that the weights $W^{\pm}_{\gamma,\lambda(\beta,\gamma)}(\Gamma;\underline{q})$ defined in (4.13) satisfy the bound

$$W^{\pm}_{\gamma,\lambda(\beta,\gamma)}(\Gamma;\underline{q})\le e^{-2c_f\gamma^{-1}N_\Gamma} \quad (11.1)$$

(The bound (11.1) follows from (6.1)-(6.2)).

Given a contour Γ call

$$\phi(\Gamma)=(\mathrm{sp}(\Gamma)\sqcup \mathrm{int}(\Gamma))\setminus K^{+} \quad (11.2)$$

Let $\mu^{\pm,\underline{q}}_{\gamma,\Lambda}$ shorthand the "dilute" Gibbs measure in Λ with $\pm$ b.c. $\underline{q}$ and parameters β, γ, $\lambda(\beta,\gamma)$, namely the Gibbs measure in Λ with with $\pm$ b.c. $\underline{q}$, conditioned on $\{\Theta(\cdot;r)=1$ for all $r\in\delta^{\ell_+,\gamma}_{\mathrm{out}}[\Lambda]\}$. Then, for γ sufficiently small,

$$\mu^{\pm,\underline{q}}_{\gamma,\Lambda}\Big(\ \{\text{there is }\Gamma\text{ so that }\phi(\Gamma)\ni 0\ \}\Big)\le e^{-c_f\gamma^{-1}} \quad (11.3)$$

(11.3) follows from bounding the probability of occurrence of a contour Γ by $e^{-2c_f\gamma^{-1}N_\Gamma}$ and then summing over all Γ with $\phi(\Gamma)\ni 0$. For $\gamma>0$ small enough, by a classical counting argument which is omitted, we then get (11.3). Thus for $d>1$

$$\mu^{+,\underline{q}}_{\gamma,\Lambda}\Big(\ \{\eta(\cdot;0)=1\ \}\Big)\ge 1-e^{-c_f\gamma^{-1}};\qquad \mu^{-,\underline{q}}_{\gamma,\Lambda}\Big(\ \{\eta(\cdot;0)=1\ \}\Big)\le e^{-c_f\gamma^{-1}} \quad (11.4)$$

which shows that the effect of the boundary conditions persists in the thermodynamic limit.

The implication that there are two distinct DLR measures requires a proof that the above dilute, finite volume, Gibbs measures have limit points in the thermodynamic limit and that they are DLR measures. The information gathered in the previous sections yield such conclusions and much more, as we will see in the present section. We start from a realization of the dilute Gibbs measure which is nice because it makes quantitative the idea that the configurations are "an ocean of plus" (in the plus case) perturbed by a collection of "islands" which are typically rare and small.

11.1. A realization of dilute Gibbs measures. For simplicity let us just refer to the measure $\mu^{+,\bar{q}}_{\gamma,\Lambda}$, with plus boundary conditions. Let $p^{+,\bar{q}}_{\gamma,\Lambda}(dq)$ be the Gibbs measure on Q_+ (supported by $\bar{q}_{\Lambda^c}$ outside Λ) with hamiltonian $H_\gamma(q_\Lambda|\bar{q}_{\Lambda^c})+K^+_\Lambda(q_\Lambda)$, the latter defined in (10.7); recall that the cutoff weights coincide with the true ones. By the last remark and using the representation (4.17), the partition function of the above Gibbs measure is the same as the true dilute partition function, i.e. the partition function with only $H_\gamma(q_\Lambda|\bar{q}_{\Lambda^c})$, but without the restriction of the configurations being in the restricted ensemble (the only restriction coming from the dilute condition, namely that $\Theta = 1$ on $\delta^{\ell_{+,\gamma}}_{\text{out}}[\Lambda]$).

$p^{+,\bar{q}}_{\gamma,\Lambda}(dq)$ does not contain all the information necessary to recover $\mu^{+,\bar{q}}_{\gamma,\Lambda}$, but the latter can be easily reconstructed from the former, as we are going to show. To this end we denote by $\pi^+_{\gamma,\Lambda}(\cdot|q)$, $q \in Q_+$, , the law on $\mathcal{C}^+_\Lambda$, defined as

$$\pi^+_{\gamma,\Lambda}\Big(\{\Gamma_i\}\Big|q_\Lambda\Big) = \frac{\prod_i W^+_\gamma(\Gamma_i, q_\Lambda)}{\sum_{\{\Gamma'_j\}\in\mathcal{C}^+_\Lambda}\prod_j W^+_\gamma(\Gamma'_j, q_\Lambda)}, \quad \{\Gamma_i\} \in \mathcal{C}^+_\Lambda \tag{11.5}$$

The skew product of $p^{+,\bar{q}}_\gamma(dq_\Lambda)$ and $\pi^+_{\gamma,\Lambda}(\cdot|q_\Lambda)$ defines a measure on $Q_+ \times \mathcal{C}^+_\Lambda$, denoted by

$$m^{+,\bar{q}}_{\gamma,\Lambda}(dq, \{\Gamma_i\}) = \pi^+_{\gamma,\Lambda}\Big(\{\Gamma_i\}\Big|q_\Lambda\Big)p^{+,\bar{q}}_{\gamma,\Lambda}(dq_\Lambda) \tag{11.6}$$

In words, this measure is obtained by sampling q according to $p^{+,\bar{q}}_{\gamma,\Lambda}(dq)$, this is the "ocean of plus" in the previous heuristic description. Then, given q, we "throw independently except for exclusion", contours Γ taken from $\mathcal{C}^+_\Lambda$, with rate $W^+_\gamma(\Gamma, q_\Lambda)$.

A last step is still needed to reconstruct $\mu^{+,\bar{q}}_{\gamma,\Lambda}$. We will introduce a transformation $T^+_{q,\{\Gamma_i\}}$ which, for each $(q, \{\Gamma_i\}) \in Q_+ \times \mathcal{C}^+$, maps bounded measurable functions on Q into bounded measurable functions on $Q_+ \times \mathcal{C}^+$ such that the following formula holds

$$\mu^{+,\bar{q}}_{\gamma,\Lambda}(f) = \sum_{\{\Gamma_i\}\in\mathcal{C}^+_\Lambda}\int_{Q^\Lambda_+} T^+_{q,\{\Gamma_i\}}(f)\, m^{+,\bar{q}}_{\gamma,\Lambda}(dq, \{\Gamma_i\}) \tag{11.7}$$

where $\mathcal{C}^+ = \Big\{\{\Gamma_i\} \,:\, |\mathrm{sp}(\Gamma_i)| < \infty \quad \mathrm{dist}(\mathrm{sp}(\Gamma_i), \mathrm{sp}(\Gamma_j)) \geq \ell_{+,\gamma}\Big\}$ denotes the projective limit of the sets of contours $\mathcal{C}^+_\Lambda$. Shorthanding $\phi(\{\Gamma_i\}) = \bigsqcup_i \phi(\Gamma_i)$ and

$$\{\Gamma_i\}_{\text{ext}} = \{\Gamma_i \in \{\Gamma_i\} : \phi(\Gamma_i) \quad \text{is not contained in any } \phi(\Gamma_j), j \neq i\}$$

we set

$$T^+_{q,\{\Gamma_i\}}(f) = \int f(q_{\phi(\{\Gamma_i\})^c}, q'_{\phi(\{\Gamma_i\})})\, \mu_{\gamma,\phi(\{\Gamma_i\})}\Big(dq'\Big|q_{\phi(\{\Gamma_i\})^c}; \{\underline{\Gamma}(q_{\phi(\{\Gamma_i\})^c}, q'_{\phi(\{\Gamma_i\})}) \sqsupset \{\Gamma_i\}_{\text{ext}}\}\Big) \tag{11.8}$$

where the measure on the r.h.s. is the Gibbs measure in $\phi(\{\Gamma_i\})$ with b.c. $\underline{q}$ and conditioned on the set of loops configurations which have among their contours all the contours $\{\Gamma_i\}_{\text{ext}}$.

The proof that (11.7) holds with the choice (11.8) is based on the fact that conditioned on the external contours $\{\Gamma_i\}_{\text{ext}}$ the measures $m^{+,\bar{\underline{q}}}_{\gamma,\Lambda}$ and $p^{+,\bar{\underline{q}}}_{\gamma,\Lambda}$ are identical outside of $\Phi(\{\Gamma_i\}_{\text{ext}})$. (11.8) just states that inside of $\Phi(\{\Gamma_i\}_{\text{ext}})$ we use the conditional expectation of the Gibbs measure instead of the conditional expectation of $m^{+,\bar{\underline{q}}}_{\gamma,\Lambda}$. One reconstructs the normalization of $m^{+,\bar{\underline{q}}}_{\gamma,\Lambda}$ using that the partition functions of $\mu^{+,\bar{\underline{q}}}_{\gamma,\Lambda}$ coincides actually with the partition functions of the contour model, cf. (4.17). If f is cylindrical w.r.t. B, i.e. depends only on $\underline{q}_B$, then $T^{+}_{\underline{q},\{\Gamma_i\}}(f)$ depends only on

$$\{\Gamma_i\}_B = \{\Gamma_i |\, \text{sp}(\Gamma_i) \sqcap B \neq \emptyset\}$$

and on $\underline{q}_{B\sqcup\phi(\{\Gamma_i\}_B)}$.

11.2. Couplings. Let Λ' and Λ be bounded, $\mathcal{D}^{(\ell_+,\gamma)}$-measurable sets, $\Lambda' \sqsupset \Lambda$, and let $\underline{q}$ and $\underline{q}'$ be plus b.c. relative to Λ and Λ' and call m and m' the corresponding measures defined by (11.6). In Appendix C of [1] it is proved that for any bounded set B of Λ there is a coupling $dP(\xi,\xi')$ of m and m', $\xi = (\underline{q},\{\Gamma_i\})$, $\xi' = (\underline{q}',\{\Gamma'_j\})$ with the following property.

- Call Δ a set of agreement for (ξ,ξ') if $\underline{q}_\Delta = \underline{q}'_\Delta$, all $\phi(\Gamma_i)$ and $\phi(\Gamma'_j)$ are either in Δ or in Δ^c and,

$$\Big\{\Gamma_i \in \{\Gamma_i\} : \text{sp}(\Gamma_i) \sqsubset \Delta\Big\} = \Big\{\Gamma'_j \in \{\Gamma'_j\} : \text{sp}(\Gamma'_j) \sqsubset \Delta\Big\} \tag{11.9}$$

- There are then positive constants c_0, c_2 so that

$$P\Big(\{B \sqsubset \Delta,\, \Delta \text{ is a set of agreement}\}\Big) \geq 1 - c_2|B|e^{-c_0\gamma \text{dist}(B,\Lambda^c)} \tag{11.10}$$

- As a consequence if f is a bounded function, cylindrical in B (i.e. which depends on $\underline{q}_B$), then

$$\big|m(f) - m'(f)\big| \leq 2\|f\|_\infty P\Big(\{B \sqsubset \Delta,\, \Delta \text{ is a set of agreement}\}^c\Big) \leq 2c_2|B|\|f\|_\infty e^{-c_0\gamma \text{dist}(B,\Lambda^c)} \tag{11.11}$$

and the same holds for the dilute Gibbs measures, because if there is an agreement set Δ which contains B then all $\phi(\Gamma_i)$ which intersect B are in Δ, by definition of agreement set.

11.3. Thermodynamic limits. By letting $\Lambda \nearrow \mathbb{R}^d$ (for $\mathcal{D}^{(\ell_+,\gamma)}$ measurable Λ), the measures $m^{\pm,\bar{\underline{q}}}_{\gamma,\Lambda}(d\underline{q},\{\Gamma_i\})$ converge weakly, independently of the b.c. to limit measures $m^{\pm}$

on the space $\underline{Q}_+ \times \mathcal{C}^+$. The result is a direct consequence of (11.10) and, by (11.7), also the Gibbs measures $\mu^{\pm,\underline{q}}_{\gamma,\Lambda}$ converge weakly to limit measures $\mu^\pm_\gamma$, which by the argument presented in the beginning of the section, are distinct from each other. In particular, (11.7) holds also in the thermodynamic limit. To conclude the proof of Theorem 2.1 we thus need to prove that $\mu^\pm_\gamma$ are DLR.

11.4. The DLR property. Referring for the sake of definiteness to μ^+_γ, we have to prove that for every bounded Borel sets $A \sqsubset \mathbb{R}^d$, $A' \sqsubset \mathbb{R}^d$ and every bounded, cylindrical function $f(\underline{q}) \equiv f(\underline{q}_{A'})$

$$\int\int f(\underline{q}')\mu_{\gamma,A}(d\underline{q}'|\underline{q}_{A^c})\mu^+_\gamma(d\underline{q}) = \int f(\underline{q})\mu^+_\gamma(d\underline{q}) \tag{11.12}$$

where $\mu_{\gamma,A}(d\underline{q}'|\underline{q}_{A^c})$ is the Gibbs measure in A with b.c. $\underline{q}_{A^c}$ (when the parameters are γ, β, $\lambda(\gamma,\beta)$).

The usual proof is based on the fact that μ^+_γ is limit of finite volume Gibbs measures for which (11.12) holds, so that the equality, which holds at finite volumes, is preserved under weak convergence and holds as well in the limit. The difficulty here is that dilute Gibbs measures do not satisfy (11.12). We instead have

$$\int\int f(\underline{q}')\mu^+_{\gamma,\Lambda;A}(d\underline{q}'|\underline{q}_{A^c})\mu^{+,\bar{\underline{q}}}_{\gamma,\Lambda}(d\underline{q}) = \int f(\underline{q})\mu^{+,\bar{\underline{q}}}_{\gamma,\Lambda}(d\underline{q}) \tag{11.13}$$

where $\mu^+_{\gamma,\Lambda;A}(d\underline{q}'|\underline{q}_{A^c})$ is the conditional probability of $\mu^{+,\bar{\underline{q}}}_{\gamma,\Lambda}(d\underline{q})$ given $\underline{q}_{A^c}$ outside A. Namely

$$\mu^+_{\gamma,\Lambda;A}(d\underline{q}'|\underline{q}_{A^c}) = \mu_{\gamma,A}\big(d\underline{q}'|\underline{q}_{A^c};\{\Theta(\underline{q}';r)=1, r\in\delta^{\ell_{+,\gamma}}_{\text{out}}[\Lambda]\}\big) \tag{11.14}$$

For Λ large enough, $A \sqsubset (\Lambda \setminus \delta_{\text{in}}{}^{\ell_{+,\gamma}}[\Lambda])$ and the condition $\Theta(\underline{q}';r) = 1, r \in \delta^{\ell_{+,\gamma}}_{\text{out}}[\Lambda]$ is a condition both on $\underline{q}'_{A^c} = \underline{q}_{A^c}$ (satisfied almost surely w.r.t. $\mu^{+,\bar{\underline{q}}}_{\gamma,\Lambda}$) and a condition on $\underline{q}'_A$. The latter is the requirement that no long loop in $\underline{q}'_A$ reaches $\delta^{\ell_{+,\gamma}}_{\text{in}}[\Lambda]$. Let $\{C_n\}$ be an increasing sequence of $\mathcal{D}^{(\ell_{+,\gamma})}$ cubes which invades $\mathbb{R}^d$. We postpone the proof that

$$\mu_{\gamma,A}\Big(\{\underline{q}'(t)\sqcap C^c_n \neq \emptyset, \text{ for some } t\in[0,\beta]\}\Big|\underline{q}_{A^c}\Big) \le e^{-c'\,\text{dist}(A,C^c_n)} \tag{11.15}$$

Then the cylinder functions

$$g_n := \int_{\underline{q}'(t)\sqsubset C_n, \text{ for all } t\in[0,\beta]} f(\underline{q}')\mu_{\gamma,A}(d\underline{q}'|\underline{q}_{A^c}) - f(\underline{q}) \tag{11.16}$$

approximate g (defined as g_n, but without the restriction on the integral) in the following sense

$$\int |g_n(\underline{q}) - g(\underline{q})|\mu^+_\gamma(d\underline{q}) \le \|g\|_\infty e^{-c'\,\text{dist}(A,C^c_n)}. \tag{11.17}$$

(11.13) implies $\int g_n(\underline{q})\mu_\gamma^+(d\underline{q}) = 0$ and (11.12) holds due to the following calculation

$$\left|\int g(\underline{q})\mu_\gamma^+(d\underline{q})\right| \le \lim_{n\to\infty} \int |g_n(\underline{q}) - g(\underline{q})|\mu_\gamma^+(d\underline{q}) = 0 \tag{11.18}$$

Proof of (11.15).

$$\mu_{\gamma,A}\Big(\{\underline{q}'(t) \sqcap C_n^c \ne \emptyset, t \in [0,\beta]\}\Big|\underline{q}_{A^c}\Big) \le e^{\beta b} W_{0|0}^\beta\big(\sup_{0\le t\le\beta} |\omega(t) - \omega(0)| > \mathrm{dist}(A, C_n^c)\big)$$

By (B.12) with $h = 1$,

$$W_{0|0}^\beta\big(\sup_{0\le t\le\beta} |\omega(t) - \omega(0)| > R\big) \le c e^{-R^2/(2\beta)}$$

from which (11.15) follows.

11.5. Structure of DLR measures. We have proved that there are two distinct DLR measures, namely the measures $\mu_\gamma^\pm$ which are limits of finite volume, dilute, $\pm$ Gibbs measures. We have also uniqueness, in the sense that any sequence of dilute plus Gibbs measures converges to μ_γ^+ and any sequence of minus measures converges to μ_γ^-. Hence the measures $\mu_\gamma^\pm$ are invariant under translations by integer multiples of $\ell_{+,\gamma}$ and, by the same argument used in [**3**], any translational invariant DLR measure is a convex combination of $\mu_\gamma^\pm$.

A representation like (11.7) holds also in the infinite volume. In fact the measures $p_{\gamma,\Lambda}^{+,\bar{\underline{q}}}(d\underline{q})$ converge in the thermodynamic limit, as the coupling property is a fortiori verified. Call the limit measure $p_\gamma^+(d\underline{q})$. For each $\underline{q} \in \mathcal{Q}^+$ a process is well defined where contours $\Gamma \in \{\Gamma_i\}^+$ are placed with rate $W_\gamma(\Gamma; \underline{q})$ independently except for exclusion (using Kolmogorov's theorem for projective limits of probability measures).

The skew product of the process $p_\gamma^+(d\underline{q})$ with the latter we call the law of m_γ^+ and, analogously to (11.7),

$$\mu_\gamma^+(f) = \sum_{\{\Gamma_i\}\in\mathcal{C}^+} \int T^+_{\underline{q},\{\Gamma_i\}}(f)\; m_\gamma^+(d\underline{q}, \{\Gamma_i\}) \tag{11.19}$$

Appendix A. Properties of the hamiltonian

A.1. Assumptions on the interaction. We suppose that $j(r,r')$ is a bounded, symmetric probability kernel on $\mathbb{R}^d$, translation invariant, i.e. $j(r,r') = j(0,r'-r)$ for all r,r', and supported by the unit ball, i.e. $j(r,r') = 0$ if $|r-r'| > 1$. Letting

$$A^{(\ell)}(r;r') = \sup_{r_1,r_2 \in C^{(2\ell)}_{r'} \sqcup \delta^{2\ell}_{\text{out}}[C^{(2\ell)}_{r'}]} \big| j(r,r_1) - j(r,r_2) \big|, \qquad \ell = 2^n, n \in \mathbb{Z} \tag{A.1}$$

we also suppose that there is a constant c so that for any r, r' and ℓ as above,

$$\int A^{(\ell)}(r;r')dr' \le c\ell, \qquad \int A^{(\ell)}(r;r')dr \le c\ell \tag{A.2}$$

(A.2) is obviously satisfied if $j(r,r')$ is differentiable with bounded derivative or if $j(r,r') = c\mathbf{1}_{|r-r'|\le 1}$ (c the normalization constant), the case considered in [**8**].

A.2. Stability of energy. We will next prove (2.10). Let $q \in Q_n$ and $\bar q \in Q$. Recalling that $b = \inf_{x\ge 0} e'_\lambda(x)$,

$$e_\lambda(x+y) - e_\lambda(x) \ge b\, y, \qquad x,y \ge 0$$

we obtain

$$e_\lambda\left(j_\gamma * [q+\bar q]\right) - e_\lambda\left(j_\gamma * \bar q(r)\right) \ge b\; j_\gamma * q(r) = b\sum_{i=1}^n j_\gamma(r,q_i)$$

so that

$$h_{\gamma,\lambda}(q|\bar q) \ge \int_{\mathbb{R}^d} b \sum_{i=1}^n j_\gamma(r,q_i)dr = bn$$

because $j_\gamma(r,r') = \gamma^d j(\gamma r,\gamma r')$ is a symmetric probability kernel. □

A.3. Energy coarse-graining. For any $\ell = 2^n, n \in \mathbb{Z}$, we set

$$\text{Av}^{(\ell)}(f;r) = \ell^{-d}\int_{C^{(\ell)}_r} f(r')dr', \quad j^{(\ell)}(r,r') = \text{Av}^{(\ell)}(j(r,\cdot);r') \tag{A.3}$$

Lemma A.1. *Let $\rho \in L^\infty(\mathbb{R}^d, \mathbb{R}_+)$, $\ell = 2^n, n \in \mathbb{Z}$, c as in* (A.2)*, then, calling $\rho^{(\ell)}(r) = \mathrm{Av}^{(\ell)}(\rho; r)$*

$$|j^{(\ell)}(r,r') - j(r,r')| \le A^{(\ell)}(r;r') \tag{A.4}$$

$$j * \rho^{(\ell)} = j^{(\ell)} * \rho \tag{A.5}$$

$$\Big| j * \rho(r) - j^{(\ell)} * \rho(r) \Big| \le c\ell \sup_{|r'-r| \le 1+10\ell} |\mathrm{Av}^{(\ell)}(\rho; r')| \tag{A.6}$$

Proof. (A.4) is an immediate consequence of (A.3) and (A.1). (A.5) follows from

$$j * \rho^{(\ell)} = j^{(\ell)} * \rho^{(\ell)} = j^{(\ell)} * \rho$$

and (A.6) from (A.4) and (A.2).

□

The Lebowitz-Penrose limit involves a coarse graining of the hamiltonian, the coarse-grained version of the hamiltonian with mesh ℓ being the hamiltonian with kernel

$$j_\gamma^{(\ell)}(r,r') := \gamma^d j^{(\gamma\ell)}(\gamma r, \gamma r') \tag{A.7}$$

namely

$$H_{\gamma,\lambda}^{(\ell)}(\underline{q}) = \int_0^\beta h_{\gamma,\lambda}^{(\ell)}(\underline{q}(t))dt, \qquad h_{\gamma,\lambda}^{(\ell)}(q) := \int e_{\lambda_\beta}(j_\gamma^{(\ell)} * q)dr \tag{A.8}$$

In Appendix F, we will actually use the mesh $\ell = \gamma^{-1/2}$, and replace $H_{\gamma,\lambda}(\underline{q}|\bar{\underline{q}})$ by $\beta h_{\gamma,\lambda}^{(\gamma^{-1/2})}(\underline{q}(0)|\bar{\underline{q}}(0))$ with $\underline{q} = \underline{q}_\Lambda$, $\bar{\underline{q}} = \bar{\underline{q}}_{\Lambda^c}$; Λ is a bounded $\mathcal{D}^{(\ell_{-,\gamma})}$-measurable region, $\sigma(\underline{q} + \bar{\underline{q}}; \cdot) \equiv 1$ (i.e. all loops are short) and there is a constant, namely

$$X_0 = \max\{\sqrt{2}, e^{-\beta b}\} \tag{A.9}$$

such that

$$|\underline{q}(0) \sqcap C^{(\ell_{-,\gamma})}| \le X_0 \ell_{-,\gamma}^d, \quad |\bar{\underline{q}}(0) \sqcap C^{(\ell_{-,\gamma})}| \le X_0 \ell_{-,\gamma}^d, \qquad \text{for all } C^{(\ell_{-,\gamma})} \in \mathcal{D}^{(\ell_{-,\gamma})} \tag{A.10}$$

Given Λ in $\mathbb{R}^d$ and $R > 0$, we write

$$\partial_R \Lambda = \{r \in \Lambda : \mathrm{dist}(r, \Lambda^c) \le R\} \sqcup \{r \in \Lambda^c : \mathrm{dist}(r, \Lambda) \le R\} \tag{A.11}$$

Proposition A.2. *There are c and γ^* positive so that, for any $\gamma \le \gamma^*$, Λ, $\underline{q}$ and $\bar{\underline{q}}$ as above,*

$$\Big| H_{\gamma,\lambda}(\underline{q}|\bar{\underline{q}}) - \beta h_{\gamma,\lambda}^{(\gamma^{-1/2})}(\underline{q}(0)|\bar{\underline{q}}(0)) \Big| \le c\gamma^{1/2}\big(|\Lambda| + |\partial_{4\gamma^{-1}}\Lambda|\big) \tag{A.12}$$

Proof. By assumption, the loops of $\underline{q}$ and $\bar{\underline{q}}$ are all short, then, if γ is small enough, for all r such that $\mathrm{dist}(r,\Lambda) > 2\gamma^{-1}$, and for all $t \in [0,\beta]$,

$$j_\gamma * [\underline{q}(t)+\bar{\underline{q}}(t)](r) = [j_\gamma * \bar{\underline{q}}(t)](r), \qquad j_\gamma^{(\gamma^{-1/2})} * [\underline{q}(t)+\bar{\underline{q}}(t)](r) = [j_\gamma^{(\gamma^{-1/2})} * \bar{\underline{q}}(t)](r) \quad \text{(A.13)}$$

where $j_\gamma * [\underline{q}+\underline{q}'] = j_\gamma * \underline{q} + j_\gamma * \underline{q}'$, which is the same as defining $\underline{q}+\underline{q}'$ as the configuration which collects all loops of $\underline{q}$ and $\underline{q}'$.

Recalling that j is bounded and supported by the unit ball, by (A.10) there is a constant c so that, for any $t \in [0,\beta]$,

$$j_\gamma * [\underline{q}(t)+\bar{\underline{q}}(t)] \le c, \qquad j_\gamma^{(\gamma^{-1/2})} * [\underline{q}(t)+\bar{\underline{q}}(t)] \le c \quad \text{(A.14)}$$

Since $|e'_\lambda(x)|$ is bounded on the compacts, there is a new constant c so that

$$\text{l.h.s. of (A.12)} \le c\int_0^\beta \int_{\Lambda\sqcup\partial_{2\gamma^{-1}}\Lambda} \Big(\Big|j_\gamma * [\underline{q}(t)+\bar{\underline{q}}(t)] - j_\gamma^{(\gamma^{-1/2})} * [\underline{q}(0)+\bar{\underline{q}}(0)]\Big| + \Big|j_\gamma * \bar{q}(t) - j_\gamma^{(\gamma^{-1/2})} * \bar{\underline{q}}(0)\Big|\Big) dr dt \quad \text{(A.15)}$$

Since all the loops of $\underline{q}$ and $\bar{\underline{q}}$ are short, recalling the definition (A.1) of $A^{(\ell)}$, we get

$$\begin{aligned}\int_{\Lambda\sqcup\partial_{2\gamma^{-1}}\Lambda} &\Big|j_\gamma * [\underline{q}(t)+\bar{\underline{q}}(t)] - j_\gamma^{(\gamma^{-1/2})} * [\underline{q}(0)+\bar{\underline{q}}(0)]\Big| dr \\ &\le \sum_{q_i \in \underline{q}(0)+\bar{\underline{q}}(0)} \gamma^d \int_{\Lambda\sqcup\partial_{2\gamma^{-1}}\Lambda} A^{(\gamma^{1/2})}(\gamma r, \gamma q_i)\, dr \\ &\le c\gamma^{1/2} |(\underline{q}(0)+\bar{\underline{q}}(0)) \sqcap (\Lambda \sqcup \partial_{4\gamma^{-1}}\Lambda)|\end{aligned}$$

because $A^{(\gamma^{1/2})}(\gamma r, \gamma q_i) = 0$, $r \in \Lambda \sqcup \partial_{2\gamma^{-1}}\Lambda$, if $q_i \notin \Lambda \sqcup \partial_{4\gamma^{-1}}\Lambda$.

The same procedure is used for the last term with $\bar{\underline{q}}$ alone in (A.15) and the proposition is proved. □

The same proof applies for the following variants:

Corollary A.3. *There are c and γ^* positive so that for any $\gamma \le \gamma^*$, Λ, $\underline{q}$ and $\bar{\underline{q}}$ as above*

$$\Big|H_{\gamma,\lambda}(\underline{q}|\bar{\underline{q}}) - H_{\gamma,\lambda}\big(\mathrm{Av}^{(\gamma^{-1/2})}(\underline{q}(0))\big|\ \bar{\underline{q}}\big)\Big| \le c\gamma^{1/2}\big(|\Lambda| + |\partial_{4\gamma^{-1}}\Lambda|\big) \quad \text{(A.16)}$$

Corollary A.4. *Let Λ be a cube of side $2^n\ell_{+,\gamma}$, $\underline{q} = \underline{q}_\Lambda$ and $\Theta\big(\underline{q} + \rho^+_\beta \mathbf{1}_{\Lambda^c}\big) = 1$ then,*

$$\Big|H_{\gamma,\lambda}(\underline{q}|\rho^+_{\beta,\lambda}\mathbf{1}_{\Lambda^c}) - \beta h_{\gamma,\lambda}^{(\gamma^{-1/2})}(\underline{q}(0)|\rho^+_{\beta,\lambda}\mathbf{1}_{\Lambda^c})\Big| \le c\gamma^{1/2}|\Lambda| \quad \text{(A.17)}$$

[the analogous result holds for the minus case].

Lemma A.5. *There is a constant c so that if $\eta(\underline{q}(0);\cdot)\neq 0$ and $\sigma(\underline{q};\cdot)\equiv 1$, then*

$$j_\gamma * \underline{q}(t) \le c\rho_\beta^+, \qquad 0\le t\le \beta \tag{A.18}$$

Proof. Since $[j_\gamma * \underline{q}(t)](r) \le \gamma^d \|j\|_\infty \big|\underline{q}(t)\sqcap B(r,\gamma^{-1})\big|$, $B(r,\gamma^{-1})$ the ball with center r and radius γ^{-1}, for γ small enough, $[j_\gamma * \underline{q}(t)](r)$ is bounded by the sum of $\big|\underline{q}(0)\sqcap C_{r'}^{(\ell_-,\gamma)}\big|$ over all cubes with $|r'-r|\le 3\gamma^{-1}$ (because, by assumption, all loops are short). Such a sum is bounded by $(3\gamma^{-1})^d(\rho_\beta^+ + \zeta)$, because $\eta^{(\zeta,\ell_-,\gamma)}(\underline{q}(0);\cdot)\neq 0$. The lemma is proved. □

The next lemma is used in Section 7 in the following context: Λ_n is the cube of side $2^n\ell_{+,\gamma}$, it is union of 2^d cubes $\Lambda_{n-1}(i)$, $i=1,..,2^d$, of side $2^{n-1}\ell_{+,\gamma}$. $\underline{q}\in \underline{Q}^{\Lambda_n}$ is such that $\sigma(\underline{q};\cdot)\equiv 1$ and $\eta(\underline{q}(0);r)=1$ for all $r\in\Lambda_n$. We then set $\underline{q}^{(i)} = \underline{q}_{\Lambda_{n-1}(i)}$ and

$$U_\gamma^\pm(\underline{q}) := H_{\gamma,\lambda}(\underline{q}|\rho_\beta^\pm \mathbf{1}_{\Lambda_n^c}) - \sum_{i=1}^{2^d} H_{\gamma,\lambda}(\underline{q}^{(i)}|\rho_\beta^\pm \mathbf{1}_{\Lambda_{n-1}(i)^c}) \tag{A.19}$$

Lemma A.6. *There is a constant $c>0$ so that with the above notation*

$$\big|U_\gamma^\pm(\underline{q})\big| \le c\big(2^n\ell_{+,\gamma}\big)^{d-1}\gamma^{-1} \tag{A.20}$$

Proof. Using repeatedly the relation

$$H_{\gamma,\lambda}(\underline{q}+\underline{q}'|\underline{q}'') = H_{\gamma,\lambda}(\underline{q}|\underline{q}'+\underline{q}'') + H_{\gamma,\lambda}(\underline{q}'|\underline{q}'')$$

we have

$$H_{\gamma,\lambda}(\underline{q}|\rho_\beta^\pm \mathbf{1}_{\Lambda_n^c}) = \sum_{i=1}^{2^d} H_{\gamma,\lambda}(\underline{q}^{(i)}|\underline{q}^{(i+1)}+\cdots+\underline{q}^{(2^d)}+\rho_\beta^\pm \mathbf{1}_{\Lambda_n^c}) \tag{A.21}$$

Writing $\underline{q}$ for $\underline{q}^{(i)}$ and $\underline{q}'$ for $\underline{q}^{(i+1)}+\cdots+\underline{q}^{(2^d)}+\rho_\beta^\pm\mathbf{1}_{\Lambda_n^c}$, we have, calling Σ the boundary of $\Lambda_{n-1}(i)$ and using the fact that all loops are short,

$$\begin{aligned} H_{\gamma,\lambda}(\underline{q}|\underline{q}') - H_{\gamma,\lambda}(\underline{q}) \;=\; & \int_0^\beta dt \int_{\mathrm{dist}(r,\Sigma)\le \gamma^{-1}+\gamma^{-1/2}} \Big\{ e_\lambda\big(j_\gamma * [\underline{q}(t)+\underline{q}'(t)](r)\big) \\ & -e_\lambda\big(j_\gamma * \underline{q}'(t)(r)\big) - e_\lambda\big(j_\gamma * \underline{q}(t)(r)\big)\Big\} dr \end{aligned} \tag{A.22}$$

Thus, by Lemma A.5,

$$\big|H_{\gamma,\lambda}(\underline{q}|\underline{q}') - H_{\gamma,\lambda}(\underline{q})\big| \le c'|\Sigma|\gamma^{-1}$$

Analogously

$$\big|H_{\gamma,\lambda}(\underline{q}^{(i)}|\rho_\beta^\pm \mathbf{1}_{\Lambda_{n-1}(i)^c}) - H_{\gamma,\lambda}(\underline{q}^{(i)})\big| \le c'|\Sigma|\gamma^{-1}$$

and the lemma is proved. □

Appendix B. Bounds on long loops

In this Appendix we will prove (6.5). We shorthand

$$\nu_\Lambda^>(d\underline{q}) = \nu_\Lambda(d\underline{q})\,\mathbf{1}_{\underline{q}=\underline{q}^>} \tag{B.1}$$

where, for any $\underline{q}$, $\underline{q}^>$ is the collection of long loops in $\underline{q}$, so that $\nu_\Lambda^>$ is supported by configurations with only long loops. Calling $\Lambda = \{r \in \mathrm{sp}(\Gamma) : \sigma_\Gamma(r) = 0\}$, (6.5) becomes a consequence of the following proposition:

Proposition B.1. *There is $c_1 > 0$ so that for any bounded, $\mathcal{D}^{(\ell_+,\gamma)}$-measurable region Λ and any $\bar{\underline{q}} \in Q_{\rm fin}$ (i.e. $\bar{\underline{q}}$ is a configuration in $\mathbb{R}^d$ with finitely many loops not necessarily starting from Λ^c)*

$$\int_{\sigma(\underline{q};\cdot)=0 \text{ on } \Lambda} \nu_\Lambda^>(d\underline{q}) e^{-H_\gamma(\underline{q}|\bar{\underline{q}})} \;\le\; e^{-c_1\gamma^{-1}N_\Lambda} \tag{B.2}$$

where $N_\Lambda = |\Lambda|/\ell_{+,\gamma}^d$

Proof. By (2.10),

$$\text{l.h.s. of (B.2)} \le \int_{\sigma(\underline{q};\cdot)=0 \text{ on } \Lambda} \nu_\Lambda^>(d\underline{q}) e^{-\beta b|\underline{q}|} \tag{B.3}$$

Denoting by Q_1 the space of single loops, namely the space of periodic, continuous, $\mathbb{R}^d$-valued functions $\omega(t)$, $t \in [0,\beta]$, we define iteratively an increasing sequence of stopping times T_k, $k \ge 0$, with values in $[0,\beta] \sqcup \{+\infty\}$, as follows. We set $T_0 = 0$ and, for $k \ge 1$, $T_k = \infty$ if $T_{k-1} \ge \beta$, while, if $T_{k-1} < \beta$,

$$T_k = \begin{cases} t \in (T_{k-1},\beta] & \text{if } |\omega(s)-\omega(T_{k-1})| < \gamma^{-1/2} \text{ for all } s \in [T_{k-1},t), \\ & \quad\text{and } |\omega(t)-\omega(T_{k-1})| = \gamma^{-1/2} \\ +\infty & \text{if } |\omega(s)-\omega(T_{k-1})| < \gamma^{-1/2} \text{ for all } s \in [T_{k-1},\beta] \end{cases} \tag{B.4}$$

Written differently: $T_k = \inf\{t > T_{k-1}|\ \sup_{s\in(T_{k-1},t]} |\omega(s)-\omega(T_{k-1})| \ge \gamma^{-1/2}\}$

By (B.1), $\nu_\Lambda^>(d\underline{q})$ is supported by configurations $\underline{q} = (\underline{q}_1, ..\underline{q}_n)$ with $T_1(\underline{q}_i) < \infty$, $i = 1,,.,n$. We decompose

$$\{T_1 < \infty\} = \bigsqcup_{k\ge2} \{T_k = \infty, T_{k-1} < \infty\} \tag{B.5}$$

hence

$$\nu_\Lambda^>(d\underline{q}) = \sum_{n,k_1,..,k_n} \frac{1}{n!} \prod_{i=1}^n W_{x_i,x_i}(d\omega_i)\, dx_i \mathbf{1}_{x_i\in\Lambda}\, \mathbf{1}_{\{T_{k_i}(\omega_i)=\infty, T_{k_i-1}(\omega_i)<\infty\}} \tag{B.6}$$

where $W_{x,x}(d\omega)$ is the law on Q_1 of a Brownian bridge starting at x.

We are going to show that the condition $\{\sigma(\underline{q};\cdot) = 0 \text{ on } \Lambda\}$ in (B.2) implies that

$$k_1 + .. + k_n \ge 2^{-d} N_\Lambda \tag{B.7}$$

In fact, if $|\omega(s)-\omega(a)| \le \gamma^{-1/2}$ for all $a \le s \le b$, then the set $\{\omega(s), a \le s \le b\}$ is contained in a ball of radius $\gamma^{-1/2}$, which is covered by at most 2^d cubes of $\mathcal{D}^{(\gamma^{-1/2})}$, hence by at most 2^d cubes of $\mathcal{D}^{(\ell_+,\gamma)}$. Thus if $\omega \in \{T_k = \infty, T_{k-1} < \infty\}$, then $\{\omega(t), 0 \le t \le \beta\}$ touches at most $2^d k$ cubes of $\mathcal{D}^{(\ell_+,\gamma)}$ and (B.7) follows.

By (B.3)

$$\text{l.h.s. of (B.2)} \le \sum_{n=1}^{\infty} \sum_{\substack{k_i \ge 2, i=1,..,n \\ k_1+,..+k_n \ge 2^{-d} N_\Lambda}} \frac{(|\Lambda| e^{-\beta b})^n}{n!} \prod_{i=1}^{n} W_{0,0}\big(\{T_{k_i} = \infty, T_{k_i-1} < \infty\}\big) \tag{B.8}$$

We bound $W_{0,0}\big(\{T_{k_i} = \infty, T_{k_i-1} < \infty\}\big) \le W_{0,0}\big(\{T_{k_i-1} < \infty\}\big)$. We claim that for $h \ge 2$

$$W_{0,0}\big(T_h < \infty\big) \le W_{0,0}\big(T_{[\frac{h}{2}]} < \frac{\beta}{2}\big) + W_{0,0}\big(\{T_{[\frac{h}{2}]} \ge \frac{\beta}{2}\} \sqcap \{T_h < \infty\}\big) \le 2W_{0,0}\big(S_{[\frac{h}{2}]} < \frac{\beta}{2}\big) \tag{B.9}$$

where S_k are the stopping times defined as in (B.4) but having replaced $\gamma^{-1/2}$ by $\gamma^{-1/2}/2$.

Let $\tilde\omega(s) = \omega(\beta - s)$, then $\beta - S_1(\tilde\omega) \ge T_{h-1}(\omega)$ and, by iteration, $\beta - S_k(\tilde\omega) \ge T_{h-k}(\omega)$. Thus, if $T_{[h/2]}(\omega) \ge \beta/2$, then $\beta - S_{h-[h/2]}(\tilde\omega) \ge \beta/2$, and $S_{h-[h/2]}(\tilde\omega) \le \beta/2$. Then the last inequality in (B.9) follows the symmetry under time reflection of the law of the Brownian bridge. We also have

$$W_{0,0}\big(T_1 < \infty\big) \le 2W_{0,0}\big(T_1 < \frac{\beta}{2}\big) \le 2W_{0,0}\big(S_1 < \frac{\beta}{2}\big) \tag{B.10}$$

We can now reduce to a probability of a Brownian motion (starting from 0), whose law will be denoted by W_0. In fact, if $f(\omega)$ is a bounded function measurable on $\{\omega(s), s \le \beta/2\}$, then

$$\int W_{0,0}(d\omega) f(\omega) = (2\pi\beta)^{d/2} \int P_0(d\omega) f(\omega) e^{-\omega(\beta/2)^2/(2\beta/2)} (2\pi\beta/2)^{-d/2}$$

so that, for $h \ge 1$,

$$W_{0,0}\big(T_h < \infty\big) \le 2^{d/2+1} W_0\big(S_{p(h)} < \frac{\beta}{2}\big), \quad p(h) = \max\{[h/2], 1\} \tag{B.11}$$

By the strong Markov property of Brownian motion, the difference of the stopping times $S_k - S_{k-1}$ is independent of $S_l - S_{l-1}$, then, by classical properties of Brownian motion, see [**10**],

$$W_{0,0}\left(T_h < \infty\right) \le 2^{d/2+1} W_0\big(S_1 \le \frac{\beta}{2}\big)^{p(h)} \le \Big(c e^{-\gamma^{-1}/(2\beta)}\Big)^{p(h)} \tag{B.12}$$

for a suitable constant c.

Going back to (B.8), we get

$$\begin{aligned}\text{l.h.s. of (B.2)} &\le \sum_{n=1}^{\infty} \sum_{\substack{k_i \ge 2, i=1,..,n \\ k_1+..+k_n \ge 2^{-d} N_\Lambda}} \frac{(|\Lambda| e^{-\beta b})^n}{n!} \Big(c e^{-\gamma^{-1}/(2\beta)} \Big)^{p(k_1-1)+\cdots p(k_n-1)} \\ &\le \sum_{n=1}^{\infty} \sum_{\substack{k_i \ge 1, i=1,..,n \\ k_1+..+k_n+n \ge 2^{-d} N_\Lambda}} \frac{(|\Lambda| e^{-\beta b})^n}{n!} \Big(c e^{-\gamma^{-1}/(2\beta)} \Big)^{p(k_1)+\cdots+p(k_n)}\end{aligned}$$

and since $k_1 + .. + k_n \ge n$,

$$\text{l.h.s. of (B.2)} \le \sum_{n=1}^{\infty} \sum_{\substack{k_i \ge 1, i=1,..,n \\ k_1+..+k_n \ge 2^{-d-1} N_\Lambda}} \frac{(|\Lambda| e^{-\beta b})^n}{n!} \Big(c e^{-\gamma^{-1}/(2\beta)} \Big)^{p(k_1)+\cdots+p(k_n)} \tag{B.13}$$

Moreover

$$p(k_1) + \cdots + p(k_n) \ge \frac{1}{4}(k_1 + \cdots + k_n) \tag{B.14}$$

hence, calling

$$N = 2^{-d-1} N_\Lambda / 8 \tag{B.15}$$

and supposing γ small enough, (in particular γ such that $c e^{-\gamma^{-1}/(16\beta)} < 1/2$)

$$\begin{aligned}\text{l.h.s. of (B.2)} &\le [c e^{-\gamma^{-1}/(2\beta)}]^N \sum_{n=1}^{\infty} \frac{(|\Lambda| e^{-\beta b})^n}{n!} \Big(\sum_{k \ge 1} \big(c e^{-\gamma^{-1}/(2\beta)} \big)^{k/8} \Big)^n \\ &\le \exp\Big\{ - N[\gamma^{-1}/(2\beta) - \log c] + |\Lambda| e^{-\beta b} 2 c e^{-\gamma^{-1}/(16\beta)} \Big\} \qquad \text{(B.16)}\end{aligned}$$

which proves (B.2). □

Appendix C. Basic properties of mean field

C.1. Mean field free energy density. The mean field free energy density $f_{\beta,\lambda}(x)$, $x \ge 0$, defined in (2.14), has a phase diagram which exhibits a phase transition. There is in fact a critical [inverse] temperature $\beta_c = (3/2)^{3/2}$ such that, for $\beta < \beta_c$, $f''_{\beta,\lambda}(x) > 0$, for all λ and x, while, for any $\beta > \beta_c$ there is a unique value of λ, $\lambda = \lambda_\beta$, where f_{β,λ_β} has two distinct minimizers, $\rho^{\pm}_\beta$, elsewhere the minimizer is unique. However there is an interval of values of λ containing λ_β where there are still two local minima at $\rho^{\pm}_{\beta,\lambda}$, $\rho^{+}_{\beta,\lambda} > \rho^{-}_{\beta,\lambda}$. $\rho^{+}_{\beta,\lambda}$

is the global minimizer if $\lambda > \lambda_\beta$ and it is interpreted as the density of the liquid phase, which, for $\lambda > \lambda_\beta$ is the only equilibrium phase; for $\lambda < \lambda_\beta$ instead, $\rho^-_{\beta,\lambda}$ is the global minimizer and interpreted as the density of the gas phase. The two values, $\rho^+_\beta > \rho^-_\beta$, are then the densities of liquid and vapor at phase coexistence, occurring at $\lambda = \lambda_\beta$.

Minimizers are solutions of the mean field equation $f'_{\beta,\lambda}(x) = 0$, which, at $\lambda = \lambda_\beta$, reads as

$$x = K_\beta(x) := e^{-\beta e'_{\lambda_\beta}(x)} \tag{C.1}$$

When $\beta > \beta_c$, it has three solutions: $0 < \rho^-_\beta < \rho^0_\beta < \rho^+_\beta$, and

$$f''_{\beta,\lambda_\beta}(\rho^\pm_\beta) > 0 \tag{C.2}$$

Notice that

$$\rho^+_\beta = e^{-\beta e'_{\lambda_\beta}(\rho^+_\beta)} \le e^{-\beta b} \tag{C.3}$$

with b defined in (2.10).

There is $\beta_0 > \beta_c$ so that

$$K'_\beta(\rho^+_\beta) \in (-1, 1) \quad \text{for all } \beta \in (\beta_c, \beta_0) \tag{C.4}$$

while $K'_\beta(\rho^-_\beta) \in (0, 1)$ for all $\beta > \beta_c$. Therefore the linearization of the map $x \to K_\beta(x)$, at $x = \rho^\pm_\beta$ is a contraction when $\beta \in (\beta_c, \beta_0)$, this is the reason for restricting in the paper to $\beta \in (\beta_c, \beta_0)$.

Since

$$\inf_{x \ge 0} e'_\lambda(x) = e'_\lambda(\sqrt{2}) =: b \tag{C.5}$$

the maximum of $K_\beta(x)$ is reached at $x = \sqrt{2}$ and for any

$$X \ge X_0 := \max\{\sqrt{2}, K_\beta(\sqrt{2})\}, \qquad K_\beta(\sqrt{2}) = e^{-\beta b} \tag{C.6}$$

$$K_\beta(x) : [0, X] \to [a, X] \qquad a = \min\{K_\beta(0), K_\beta(X)\} \tag{C.7}$$

In Section 10 we have considered the plus and minus free energy densities, defined for $\beta > 1$ by

$$f^{0,\pm}_\beta(x) = -\lambda^\pm_{\text{eff}} x - \frac{s(x)}{\beta}, \quad \lambda^\pm_{\text{eff}} = -e'_{\lambda_\beta}(\rho^\pm_\beta) \tag{C.8}$$

$f^{0,+}_\beta(x)$, [resp. $f^{0,-}_\beta(x)$] is a strictly convex function with minimum at $x = \rho^+_\beta$, [$x = \rho^-_\beta$]. The interpolating mean field free energy is then

$$f^\pm_{\beta,u}(x) = u f_{\beta,\lambda_\beta}(x) + (1-u) f^{0,\pm}_\beta(x), \quad u \in [0, 1] \tag{C.9}$$

which, for $u < 1$ has ρ^+_β [resp. ρ^-_β] as its unique minimizer.

C.2. Mean field free energy functional. In terms of statistical mechanics, $f_{\beta,\lambda}$ is the limit canonical free energy density with a mean field interaction. For Kac potentials, in the "the Lebowitz-Penrose limit" (i.e. first the thermodynamic limit, then the limit $\gamma \to 0$), the relevant quantity is a free energy functional which, in the model considered in this paper, is given by

$$F_{\beta,\lambda}(\rho) = \int e_\lambda(j * \rho) - \frac{s(\rho)}{\beta}\, dr \tag{C.10}$$

which we regard as a [non local free energy] functional on $L_0^\infty(\mathbb{R}^d; \mathbb{R}_+)$, the space of non negative valued, bounded measurable functions with compact support. The integral on the r.h.s. is then well defined as $e_\lambda(0) = s(0) = 0$.

The relation with statistical mechanics will be recalled in Appendix F, notice also that on a torus Λ, $F_{\beta,\lambda}(x\mathbf{1}_\Lambda) = f_{\beta,\lambda}(x)|\Lambda|$.

It is often convenient to rewrite $F_{\beta,\lambda}(\rho)$ as

$$F_{\beta,\lambda}(\rho) = \int f_{\beta,\lambda}(j * \rho) + \frac{1}{\beta}\Big(s(j*\rho) - j * s(\rho)\Big)\, dr \tag{C.11}$$

Notice in fact that the second term is non negative for any ρ, by the concavity of $s(\cdot)$, so that the minimizer of the functional on a torus is a function constantly equal to the minimizer of $f_{\beta,\lambda}(\cdot)$.

The conditional free energy

$$F_{\beta,\lambda}(\rho|\bar\rho) = F_{\beta,\lambda}(\rho + \bar\rho) - F_{\beta,\lambda}(\bar\rho) \tag{C.12}$$

is the free energy of $\rho \in L_0^\infty(\mathbb{R}^d; \mathbb{R}_+)$ "conditioned to" $\bar\rho \in L_0^\infty(\mathbb{R}^d; \mathbb{R}_+)$.

Associated to the free energy densities $f_\beta^{0,\pm}(\cdot)$ and $f_{\beta,u}^{\pm}(x)$ defined in (C.8) and (C.9), we consider their corresponding free energy functionals, defined for $\beta > 1$ on $L_0^\infty(\mathbb{R}^d; \mathbb{R}_+)$ by

$$F_\beta^{0,\pm}(\rho) := \int f_\beta^{0,\pm}(\rho(r))\, dr \tag{C.13}$$

and

$$F_{\beta,u}^{\pm}(\rho) = uF_{\beta,\lambda}(\rho) + (1-u)F_\beta^{0,\pm}(\rho) = \int f_{\beta,u}^{\pm}(\rho(r))\, dr \tag{C.14}$$

C.3. Equilibrium. We will prove here that the conditional free energy "has range 2", Lemma C.1 below, and that equilibrium outside a region propagates inside, Lemma C.2 below. We will use throughout the sequel the following notation: given a function ρ and a set Λ,

$$\rho_\Lambda(r) = \rho(r)\mathbf{1}_{r\in\Lambda} \tag{C.15}$$

Lemma C.1. *Let* Λ *be a bounded measurable region and* $\bar\rho_{\Lambda^c} \in L^\infty_0(\Lambda^c;\mathbb{R}_+)$. *Then* $F_{\beta,\lambda}(\rho_\Lambda|\bar\rho_{\Lambda^c})$ *is continuous on* $L^\infty(\Lambda;\mathbb{R}_+)$, *it does not depend on* $\bar\rho(r)$, $r \in \Lambda^c \setminus \partial_2\Lambda$ *and there is a constant* c *independent of* Λ *and* $\bar\rho_{\Lambda^c}$ *so that*

$$F_{\beta,\lambda}(\rho_\Lambda|\bar\rho_{\Lambda^c}) \ge \int_\Lambda f_{\beta,\lambda}(j*[\rho_\Lambda+\bar\rho_{\Lambda^c}])dr - c(1+\|\rho_\Lambda+\bar\rho_{\Lambda^c}\|^4)|\partial_1\Lambda| \tag{C.16}$$

Proof. Continuity follows from the continuity of $t \to f_{\beta,\lambda}(t)$ and $t \to s(t)$. To prove (C.16), we use (C.11) to rewrite $F_{\beta,\lambda}$ in the following way, ($\partial_1\Lambda$ has been defined in (A.11)),

$$\begin{aligned} F_{\beta,\lambda}(\rho_\Lambda|\bar\rho_{\Lambda^c}) = &\int_{\Lambda\sqcup\partial_1\Lambda} f_{\beta,\lambda}(j*[\rho_\Lambda+\bar\rho_{\Lambda^c}])dr \\ &+\frac{1}{\beta}\int_{\Lambda\sqcup\partial_1\Lambda} s(j*[\rho_\Lambda+\bar\rho_{\Lambda^c}]) - j*s([\rho_\Lambda+\bar\rho_{\Lambda^c}])\,dr \\ &-\int_{\partial_1\Lambda} f_{\beta,\lambda}(j*\bar\rho_{\Lambda^c})dr - \frac{1}{\beta}\int_{\partial_1\Lambda} s(j*\bar\rho_{\Lambda^c}) - j*s(\bar\rho_{\Lambda^c})\,dr \end{aligned} \tag{C.17}$$

where we have used that $f_{\beta,\lambda}(0) = s(0) = 0$.

The r.h.s. of (C.17) does not change if we vary $\bar\rho_{\Lambda^c}$ outside $\Lambda\sqcup\partial_2\Lambda$. By concavity of $s(\cdot)$, the second term on the r.h.s. of (C.17) is non negative. Since there is c so that for any t, $|f_{\beta,\lambda}(t)| \le c(1+t^4)$, hence (C.16). The lemma is proved. □

Lemma C.2. *Let* Λ *be a bounded measurable region and* $\Lambda' = \Lambda\sqcup\partial_2\Lambda$. *Then, for* $\lambda \ge \lambda_\beta$, *[*$\lambda \le \lambda_\beta$*],* $\rho^+_{\beta,\lambda}\mathbf{1}_\Lambda$ *[*$\rho^-_{\beta,\lambda}\mathbf{1}_\Lambda$*] is the unique minimizer of* $F_{\beta,\lambda}(\rho_\Lambda|\rho^+_{\beta,\lambda}\mathbf{1}_{\Lambda'\setminus\Lambda})$ *[*$F_{\beta,\lambda}(\rho_\Lambda|\rho^-_{\beta,\lambda}\mathbf{1}_{\Lambda'\setminus\Lambda})$*].*

Proof. Suppose $\lambda \ge \lambda_\beta$ (the other case is analogous and omitted). Calling $\bar\rho_{\Lambda^c} = \rho^+_{\beta,\lambda}\mathbf{1}_{\Lambda^c}$, the first term on the r.h.s. of (C.17) is bounded from below by $|\Lambda\sqcup\partial_1\Lambda| f_{\beta,\lambda}(\rho^+_{\beta,\lambda})$, the inequality being strict unless $\rho_\Lambda = \rho^+_{\beta,\lambda}\mathbf{1}_\Lambda$. The second term is non negative, the other ones do not depend on ρ_Λ, thus the minimizer is unique and given by $\rho_\Lambda = \rho^+_{\beta,\lambda}\mathbf{1}_\Lambda$. □

C.4. Surface corrections. Here we suppose $\lambda = \lambda_\beta$. We have proved in Lemma C.2, that, if outside a region Λ there is equilibrium, i.e. $\bar\rho(r) = \rho^+_\beta$, then the conditional minimal free energy is obtained when $\rho(r) = \rho^+_\beta$ in the whole Λ (the analogous property holding in the minus case). This does not mean that the corresponding free energy is equal to the equilibrium free energy density $f_{\beta,\lambda_\beta}(\rho^+_\beta)$ times the volume, because

$$I^\pm_{\beta,\Lambda,\Lambda^c} = F_{\beta,\lambda_\beta}\Big(\rho^\pm_\beta\mathbf{1}_\Lambda\Big|\rho^\pm_\beta\mathbf{1}_{\Lambda^c}\Big) - f_{\beta,\lambda_\beta}(\rho^\pm_\beta)|\Lambda| \tag{C.18}$$

is not 0. $I^{\pm}_{\beta,\Lambda,\Lambda^c}$ is a "surface term". In fact using (C.10) instead of (C.17) we obtain instead of (C.11)

$$F_{\beta,\lambda}(\rho_\Lambda|\bar\rho_{\Lambda^c}) = \int_{\Lambda\sqcup\partial_1\Lambda} e_{\beta,\lambda}(j*[\rho_\Lambda+\bar\rho_{\Lambda^c}])dr - \frac{1}{\beta}\int_\Lambda s(\rho_\Lambda)\,dr$$
$$-\int_{\partial_1\Lambda} e_{\beta,\lambda}(j*\bar\rho_{\Lambda^c})dr$$

Using such an expression in (C.18), we get

$$\begin{aligned} I^{\pm}_{\beta,\Lambda,\Lambda^c} &= \int_{\partial_1\Lambda} \mathbf{1}_{\Lambda^c}\; e_{\lambda_\beta}(\rho^{\pm}_\beta) - e_{\lambda_\beta}(j*\rho^{\pm}_\beta\mathbf{1}_{\Lambda^c})\,dr \\ &= h(\rho^{\pm}_\beta\mathbf{1}_\Lambda|\rho^{\pm}_\beta\mathbf{1}_{\Lambda^c}) - |\Lambda|e_{\lambda_\beta}(\rho^{\pm}_\beta) \end{aligned} \tag{C.19}$$

An analogous formula, with Δ replacing Λ^c, defines $I^{\pm}_{\beta,\Lambda,\Delta}$. Notice that $I^{\pm}_{\beta,\Lambda,\Delta} \neq I^{\pm}_{\beta,\Delta,\Lambda}$.

Proposition C.3. *Let Λ be a bounded measurable region, then*

$$F_{\beta,\lambda_\beta}\Big(\rho^-_\beta\mathbf{1}_\Lambda\Big) + I^-_{\beta,\Lambda^c,\Lambda} = F_{\beta,\lambda_\beta}\Big(\rho^+_\beta\mathbf{1}_\Lambda\Big) + I^+_{\beta,\Lambda^c,\Lambda} = f_{\beta,\lambda_\beta}(\rho^{\pm}_\beta)|\Lambda| \tag{C.20}$$

Proof. Using (C.10) we can write $F_{\beta,\lambda_\beta}\Big(\rho^{\pm}_\beta\mathbf{1}_\Lambda\Big)$ as

$$\begin{aligned} &= \int e_{\lambda_\beta}(j*(\rho^{\pm}_\beta\mathbf{1}_\Lambda))dr - \frac{s(\rho^{\pm}_\beta)}{\beta}\;|\Lambda| \\ &= \int_\Lambda e_{\lambda_\beta}(\rho^{\pm}_\beta)dr - \frac{s(\rho^{\pm}_\beta)}{\beta}|\Lambda| + \int_{\partial_1\Lambda}\Big(e_{\lambda_\beta}(j*(\rho^{\pm}_\beta\mathbf{1}_\Lambda)) - \mathbf{1}_\Lambda\; e_{\lambda_\beta}(\rho^{\pm}_\beta)\Big)dr \\ &= f_{\beta,\lambda_\beta}(\rho^{\pm}_\beta) + I^{\pm}_{\beta,\Lambda^c,\Lambda} \end{aligned}$$

□

Remark. If in Proposition C.3 we take Λ equal to C^-_i or C^+_i, we then obtain (6.22).

C.5. Dynamics. For any bounded measurable region Λ, we consider the evolution equation

$$\begin{cases} \dfrac{d\rho}{dt} = -\rho + \Phi_\beta(\rho;\cdot) & \text{on } \Lambda \\ \rho(r,t) = \rho_0(r) & \text{on } \Lambda^c\times\mathbb{R}_+ \\ \rho(r,0) = \rho_0(r) & \text{on } \mathbb{R}^d \end{cases} \tag{C.21}$$

where

$$\Phi_\beta(\rho;r) = e^{-\beta j * e'_{\lambda_\beta}(j*\rho(r))} \tag{C.22}$$

and $\rho_0 \in L^\infty(\mathbb{R}^d, \mathbb{R}_+)$.

(C.21) has then a unique global solution $\rho(r,t)$, namely, for each r, $\rho(r,t)$ is differentiable for $t > 0$ and converges to $\rho_0(r)$ as $t \to 0$. Moreover, for each t, $\rho(\cdot,t) \in L^\infty(\mathbb{R}^d, \mathbb{R}_+)$. By setting $T_t^\Lambda(\rho_0) = \rho(\cdot,t)$ we thus define a semigroup T_t^Λ on $L^\infty(\mathbb{R}^d, \mathbb{R}_+)$.

The solution $\rho(r,t)$ of (C.21) solves also the integral version of (C.21), namely

$$\rho(r,t) = e^{-t}\rho_0(r) + \int_0^t e^{-(t-s)}\Phi_\beta(\rho(\cdot,s);r)ds \tag{C.23}$$

from where we deduce that

$$e^{-t}\rho_0(r) \le \rho(r,t) \le e^{-t}\rho_0(r) + (1-e^{-t})e^{-\beta b} \tag{C.24}$$

with b as in Subsection A.2.

The upper bound in (C.24) allows to improve the lower bound. In fact, since $|e'_{\lambda_\beta}(x)| \le |\lambda_\beta| + |x| + \frac{|x|^3}{3!}$, we can bound Φ_β from below by

$$\Phi_\beta(\rho(t,\cdot);r) \ge e^{-\beta\left(|\lambda_\beta| + \|\rho_0\|_{L^\infty} + D_\beta + \frac{1}{3!}(\|\rho_0\|_{L^\infty} + e^{-\beta b})^3\right)} =: D(\beta, \|\rho_0\|_{L^\infty}) > 0$$

which implies

$$\rho(r,t) \ge e^{-t}\rho_0(r) + (1-e^{-t})D(\beta, \|\rho_0\|_{L^\infty}) \tag{C.25}$$

Our main interest for dynamics is that it makes the free energy decrease. A direct calculation shows that

Proposition C.4. *Let $\rho \in L^\infty(\mathbb{R}^d)$ and Λ as above. Then for any $t \ge 0$*

$$F_{\beta,\lambda_\beta}\Big(T_t^\Lambda(\rho)\,\mathbf{1}_\Lambda\Big|\rho\,\mathbf{1}_{\Lambda^c}\Big) - F_{\beta,\lambda_\beta}\Big(\rho\,\mathbf{1}_\Lambda\Big|\rho\,\mathbf{1}_{\Lambda^c}\Big) = -\int_0^t \mathcal{I}_\beta(T_s^\Lambda(\rho);\Lambda)ds \tag{C.26}$$

$$\mathcal{I}_\beta(\rho;\Lambda) = \int_\Lambda \Big(-\rho + \Phi_\beta(\rho;\cdot)\Big)\frac{1}{\beta}\Big(-\log\rho + \log\Phi_\beta(\rho;\cdot)\Big)dr \tag{C.27}$$

$\mathcal{I}_\beta(\rho;\Lambda) \ge 0$ with equality iff

$$\rho(r) = \Phi_\beta(\rho;r), \qquad \textit{for all } r \in \Lambda \tag{C.28}$$

Proposition C.5. *Let $\rho \in L^\infty(\mathbb{R}^d)$ non-negative and Λ as above, then, as $t \to \infty$, $T_t^\Lambda(\rho)$ converges by subsequences in $L^\infty(\mathbb{R}^d)$. Moreover, any limit point u of $T_t^\Lambda(\rho)$ satisfies (C.28) and $F_\beta(u|\rho_{\Lambda^c}) \le F(\rho_\Lambda|\rho_{\Lambda^c})$.*

Proof. For each non-negative $\rho \in L^\infty(\mathbb{R}^d)$ the set $\{T_t^\Lambda(\rho) - e^{-t}\rho|\ t \in [0,\infty)\}$ is relative compact in the uniform norm. This is true, because the following estimate and Assumption A.2 shows that it is equi-continuous.

$$\left|\left(T_t^\Lambda(\rho)(r) - e^{-t}\rho(r)\right) - \left(T_t^\Lambda(\rho)(r') - e^{-t}\rho(r')\right)\right| \tag{C.29}$$

$$\leq \int_0^t e^{-(t-s)}\left|\Phi_\beta(T_s^\Lambda(\rho);r) - \Phi_\beta(T_s^\Lambda(\rho);r')\right| \tag{C.30}$$

$$\leq \beta b e^{-\beta b}\int_0^t e^{-(t-s)}\|T_s^\Lambda(\rho)\|_{L^\infty}\, ds \int |j(r,x) - j(r',x)| dx \tag{C.31}$$

Let $\rho \in L^\infty(\mathbb{R}^d, \mathbb{R}_+)$ and let $\left(T_{t_n}^\Lambda(\rho)\right)_{n\in\mathbb{N}}$ be a converging sequence. Recalling (C.24) and (C.25), the integrand in $F_{\beta,\lambda_\beta}\left(T_t^\Lambda(\rho)\,\mathbf{1}_\Lambda\middle|\rho\,\mathbf{1}_{\Lambda^c}\right)$ is uniformly integrable, (C.26) holds also for the limit point, $\mathcal{I}_\beta(T_t^\Lambda(\rho);\Lambda)$ is continuous in t, and hence for the limit point $\mathcal{I}_\beta(\cdot;\Lambda)$ is 0. □

For the interpolating free energy functional, dynamics is defined by replacing $\Phi_\beta(\rho;r)$ with

$$\Phi_{\beta,u}^\pm(\rho;r) = e^{-\beta\left(uj*e'_{\lambda_\beta}(j*\rho(r)) + (1-u)e'_{\lambda_\beta}(\rho_\beta^\pm)\right)} \tag{C.32}$$

recall in fact that $\lambda_{\text{eff}}^\pm = -e'_{\lambda_\beta}(\rho_\beta^\pm)$. The solutions to the evolution equation define a semigroup $T_t^{\Lambda,\pm,u}(\rho)$ for which the analogues of Proposition C.4 and Proposition C.5 hold.

Appendix D. Non local functionals, small deviations

The main result in this appendix is that if a density profile is in a neighborhood of the equilibrium value in a region Λ, then by decreasing its free energy it can be made closer to equilibrium at an exponential rate from the boundary of Λ, Theorem D.3 below.

By default, throughout the sequel ℓ denotes an element of $\{2^n, n \in \mathbb{Z}\}$. The "coarse grained image" $\mathrm{Av}^{(\ell)}(\rho;r)$, $\rho \in L^\infty(\mathbb{R}^d)$, see (A.3), is a bounded function constant on the cubes of $\mathcal{D}^{(\ell)}$. For any $\zeta > 0$ we then set

$$\eta^{(\zeta,\ell)}(\rho;r) = \begin{cases} \pm 1 & \text{if } |\mathrm{Av}^{(\ell)}(\rho;r) - \rho_{\beta,\pm}| \leq \zeta \\ 0 & \text{otherwise} \end{cases} \tag{D.1}$$

Lemma D.1. *There are ζ_0', d_0 and ϵ_0 all positive so that for any $\zeta \le \zeta_0'$, any $\ell \le d_0\zeta$ and in $\{2^n, n \in \mathbb{Z}\}$), any bounded $\mathcal{D}^{(\ell)}$-measurable region Λ and any $\rho \in L^\infty(\mathbb{R}^d, \mathbb{R}_+)$ such that $\eta^{(\zeta,\ell)}(\rho; r) = \pm 1$, dist$(r, \Lambda) \le 10$, ($\Phi_\beta$ below as in* (C.22)*)*

$$\left|\Phi_\beta(\rho; r) - \rho_\beta^\pm\right| \le (1-\epsilon_0)\zeta, \qquad r \in \Lambda \tag{D.2}$$

$$\left|j * \rho(r) - \rho_\beta^\pm\right| \le 2\zeta, \qquad \mathrm{dist}(r, \Lambda) \le 2 \tag{D.3}$$

Proof. As the proofs for the $+$ and the $-$ cases are the same, we will only consider the former. By (C.4), for $\beta \in (\beta_0, \beta_c)$

$$\left| \frac{d}{ds} e^{-\beta e'_{\lambda_\beta}(s)} \Big|_{s=\rho_\beta^+} \right| < 1$$

Then, for each $\epsilon > 0$ small enough, there is $\delta > 0$ such that, for all $s > 0$ with $|\rho_\beta^+ - s| \le \delta$,

$$\beta |e''_{\lambda_\beta}(s)| \le (1-\epsilon) e^{\beta e'_{\lambda_\beta}(\rho_\beta^+)} = \frac{1-\epsilon}{\rho_\beta^+} \tag{D.4}$$

Take $\zeta_0 \le \delta/2$. By (A.6) and for $\ell < 1$,

$$\left| j * \rho(r) - j^{(\ell)} * \rho(r) \right| \le (\rho_\beta^+ + \zeta) c\ell, \qquad \mathrm{dist}(r, \Lambda) \le 2$$

By (A.5) $j^{(\ell)} * \rho \in [\rho_\beta^+ - \zeta, \rho_\beta^+ + \zeta]$ so that

$$\left| j * \rho(r) - \rho_\beta^+ \right| \le c\ell(\rho_\beta^+ + \zeta) + \zeta \le (1+\epsilon)\zeta \tag{D.5}$$

for the choice $0 < d_0 < \frac{\epsilon}{c(\rho_\beta^+ + \zeta_0)}$. We have thus proved (D.3). (D.4) implies that

$$\beta |j * e'_{\lambda_\beta}(j * \rho) - e'_{\lambda_\beta}(\rho_\beta^+)| \le (1-\epsilon) e^{\beta e'_{\lambda_\beta}(\rho_\beta^+)} j * |j * \rho - \rho_\beta^+| \tag{D.6}$$

Using (D.5) we can bound this by

$$(1-\epsilon) e^{\beta e'_{\lambda_\beta}(\rho_\beta^+)} (1+\epsilon)\zeta \tag{D.7}$$

which is in particular bounded by

$$e^{\beta e'_{\lambda_\beta}(\rho_\beta^+)} \zeta \tag{D.8}$$

By the inequality $|e^x - 1| \le e^{|x|}|x|$, where x is the l.h.s. of (D.6), using (D.7) and (D.8) we get

$$|\Phi_\beta(\rho; r) - \rho_\beta^+| \;\le\; e^{\zeta e^{\beta e'_{\lambda_\beta}(\rho_\beta^+)}} (1-\epsilon^2)\zeta \tag{D.9}$$

Choose $\zeta_0 < e^{-\beta e'_{\lambda_\beta}(\rho_\beta^+)} \ln(1+\epsilon^2)$ then (D.9) is bounded by $(1-\epsilon^4)\zeta$. Choose ϵ small enough then the lemma is proved with $\zeta_0' = \min\{\delta/2, e^{-\beta e'_{\lambda_\beta}(\rho_\beta^+)} \ln(1+\epsilon^2)\}$. □

Given ζ, ℓ and a $\mathcal{D}^{(\ell)}$-measurable region Λ, we set

$$N^{\pm}_{\zeta,\ell,\Lambda} := \Big\{\rho \in L^\infty(\mathbb{R}^d,\mathbb{R}^+) : \eta^{(\zeta,\ell)}(\rho;r) = \pm 1, \text{whenever } \mathrm{dist}(r,\Lambda) \le 10\Big\} \tag{D.10}$$

Lemma D.2. *Let ζ and ℓ as in Lemma D.1, Λ a bounded $\mathcal{D}^{(\ell)}$-measurable region and $\rho \in N^{\pm}_{\zeta,\ell,\Lambda}$, then $T^\Lambda_t(\rho) \in N^{\pm}_{\zeta,\ell,\Lambda}$ for all $t \ge 0$.*

Proof. We just consider the + case. Let $\tau > 0$, $v_0 \in N^{+}_{\zeta,\ell,\Lambda}$,

$$\mathcal{S}_{\tau,v_0} = \Big\{ v \in C\big([0,\tau]; L^\infty(\mathbb{R}^d,\mathbb{R}^+)\big) : v(\cdot,0) = v_0(\cdot) \;\text{ and } v(r,t) = v_0(r) \text{ on } \Lambda^c \times [0,\tau] \Big\}$$

Let then Ψ be the map on $\mathcal{S}_{\tau,v_0}$ defined by setting for $t \in [0,\tau]$ and $r \in \Lambda$:

$$\Psi(v)(r,t) = e^{-t} v_0(r) + \int_0^t ds\, e^{-(t-s)} \Phi_\beta(v(\cdot,s);r)$$

As in (C.24) we have that $\Psi(v) : [0,\tau] \to L^\infty(\mathbb{R}^d,\mathbb{R}^+)$ is continuous and

$$e^{-t} v_0(r) \le \Psi(v)(r,t) \le e^{-t} v_0(r) + (1-e^{-t}) e^{-\beta b}$$

Hence Ψ maps $\mathcal{S}_{\tau,v_0}$ onto itself. If $\tau > 0$ is small enough, Ψ is a contraction on $\mathcal{S}_{\tau,v_0}$ w.r.t. the sup norm, because $0 \le \Phi_\beta(v(\cdot,s)) \le e^{-\beta b}$. The fixed point of Ψ is the orbit $T^\Lambda_t(v_0)$, $0 \le t \le \tau$.

If $v \in N^{+}_{\zeta,\ell,\Lambda}$, by (D.2),

$$\rho^+_\beta - (1-\epsilon_0)\zeta \le \Phi_\beta(v;r) \le \rho^+_\beta + (1-\epsilon_0)\zeta \tag{D.11}$$

Hence the set

$$Y = \big\{ v \in \mathcal{S}_{\tau,v_0} : v(\cdot,t) \in N^{+}_{\zeta,\ell,\Lambda} \;\text{ for all } t \in [0,\tau] \big\}$$

is left invariant by Ψ. Since Y is closed, the fixed point of Ψ is in Y, hence $T^\Lambda_t(v_0) \in N^{+}_{\zeta,\ell,\Lambda}$, $0 \le t \le \tau$. By induction the statement remains valid for all t. □

Remark. The same conclusion holds for the semigroup $T^{\Lambda,\pm,u}_t$ defined below (C.32), namely $T^{\Lambda,\pm,u}_t(\rho) \in N^{\pm}_{\zeta,\ell,\Lambda}$ for all $t \ge 0$. The proof is completely analogous and when we get to (D.11) we observe that $\Phi^{\pm}_{\beta,u}(\rho;r)$, which is defined in (C.22), verifies

$$\Phi^{+}_{\beta,u}(v;r) = \big(\Phi_\beta(v;r)\big)^u \big(\rho^+_\beta\big)^{1-u}$$

hence, if $v \in N^{+}_{\zeta,\ell,\Lambda}$,

$$\Phi^{\pm}_{\beta,u}(v;r) \le \big(\rho^+_\beta\big)^{1-u} \big(\rho^+_\beta + (1-\epsilon_0)\zeta\big)^u \le \rho^+_\beta + (1-\epsilon_0)\zeta$$

with equality only if $u = 1$. The lower bound is proved analogously, hence also $T^{\Lambda,\pm,u}_t(\rho) \in N^{\pm}_{\zeta,\ell,\Lambda}$ for all $t \ge 0$.

With the above notation, we set for any $\rho \in N^{\pm}_{\zeta,\ell,\Lambda}$,

$$\mathcal{X}^{\pm}_{\Lambda,\rho} = \Big\{ v \in N^{\pm}_{\zeta,\ell,\Lambda} : v_{\Lambda^c} = \rho_{\Lambda^c} \Big\} \tag{D.12}$$

where ψ_Δ stands for the restriction of a function ψ to a set Δ.

Theorem D.3. *There are ζ_0 ($\zeta_0 < \zeta_0'$, ζ_0' as in Lemma D.1) ω and c_ω all positive, so that the following holds. Let $\zeta < \zeta_0$, $\ell < d_0\zeta$ (d_0 as in Lemma D.1) , Λ a bounded, $\mathcal{D}^{(\ell)}$-measurable region, ρ and $\hat\rho$ in $N^{\pm}_{\zeta,\ell,\Lambda}$. Then:*

- *In $\mathcal{X}^{\pm}_{\Lambda,\rho}$, there is a unique minimizer, $\psi^{\pm}$, of $F_{\beta,\lambda_\beta}(\cdot|\rho_{\Lambda^c})$; $\psi^{\pm}(r) = \Phi_\beta(\psi^{\pm}; r)$, $r \in \Lambda$, and it is the unique solution of this equation in $\mathcal{X}^{\pm}_{\Lambda,\rho}$. Moreover*

$$\psi^{\pm}(r) \in [\rho^{\pm}_\beta - \zeta + \epsilon_0\zeta, \rho^{\pm}_\beta + \zeta - \epsilon_0\zeta], \quad r \in \Lambda \tag{D.13}$$

$$|\psi_\Lambda(r) - \rho^{\pm}_\beta| \le c_\omega e^{-\omega\, dist(r,\Lambda^c_{\neq})} \tag{D.14}$$

  where $\Lambda^c_{\neq} = \{r \in \Lambda^c : dist(r,\Lambda) \le 2; \rho_{\Lambda^c}(r) \neq \rho^{\pm}_\beta)\}$ (ϵ_0 as in Lemma D.1).
- *Finally, if $\hat\psi$ is the minimizer of $F_{\beta,\lambda_\beta}(\cdot|\hat\rho_{\Lambda^c})$ in $\mathcal{X}_{\Lambda,\hat\rho}$, and calling, by an abuse of notation, $\Lambda^c_{\neq} = \{r \in \Lambda^c : dist(r,\Lambda) \le 2; \hat\rho_{\Lambda^c}(r) \neq \rho_{\Lambda^c}(r)\}$, then*

$$|\hat\psi_\Lambda(r) - \psi_\Lambda(r)| \le c_\omega e^{-\omega\, dist(r,\Lambda^c_{\neq})} \tag{D.15}$$

Proof. To simplify notation we consider only the + case. By Lemma D.2, the following is known: T^Λ_t leaves $\mathcal{X}^+_{\Lambda,\rho}$ invariant and since $\mathcal{X}^+_{\Lambda,\rho}$ is closed under uniform convergence on the compacts, for any $u \in \mathcal{X}^+_{\Lambda,\rho}$, $T^\Lambda_t u$ converges by subsequences to an element ψ of

$$X^0_{\Lambda,\rho} = \Big\{ \psi \in \mathcal{X}^+_{\Lambda,\rho} :\ \psi \text{ solves (C.28) in } \Lambda \Big\}$$

and $F_{\beta,\lambda_\beta}(v_\Lambda|\rho_{\Lambda^c}) \ge F_{\beta,\lambda_\beta}(\psi_\Lambda|\rho_{\Lambda^c})$, the inequality being strict unless $v \in X^0_{\Lambda,\rho}$:

$$F_{\beta,\lambda_\beta}(v_\Lambda|\rho_{\Lambda^c}) > \inf_{\psi \in X^0_{\Lambda,\rho}} F_{\beta,\lambda_\beta}(\psi_\Lambda|\rho_{\Lambda^c}), \quad \text{for any } v \in \mathcal{X}_{\Lambda,\rho} \setminus X^0_{\Lambda,\rho} \tag{D.16}$$

By Lemma D.1, any $\psi \in X^0_{\Lambda,\rho}$ satisfies the first condition (D.13).

We will next show that for ζ small enough, $X^0_{\Lambda,\rho}$ consists of only one element, ψ^+, which is therefore the strict minimizer of $F_{\beta,\lambda_\beta}(v_\Lambda|\rho_{\Lambda^c})$. Suppose that ψ and ψ' are in $\mathcal{X}^0_{\Lambda,\rho}$, then, for any $r \in \Lambda$, and writing $\psi_s = s\psi + (1-s)\psi'$, $0 \le s \le 1$,

$$\begin{aligned} \psi(r) - \psi'(r) &= \Phi_\beta(\psi; r) - \Phi_\beta(\psi'; r) \\ &= -\beta \int_0^1 e^{-\beta j * e'_{\lambda_\beta}(j*\psi_s)} j * \Big(e''_{\lambda_\beta}(j*\psi_s) j * (\psi - \psi') \Big) ds \end{aligned} \tag{D.17}$$

Since $|e^{-\beta e'_{\lambda_\beta}(\rho^+_\beta)} e''_{\lambda_\beta}(\rho^+_\beta)| < 1$ and by (D.3), $j*\psi(r)$ and $j*\psi'(r)$ are both in $[\rho^+_\beta - 2\zeta, \rho^+_\beta + 2\zeta]$ for all $\mathrm{dist}(r,\Lambda) \le 2$, it then follows that if ζ_0 is small enough,

$$e^{-\omega} := \sup_{|a|,|b| \le 2\zeta_0} \beta e^{-\beta e'_{\lambda_\beta}(\rho^+_\beta + a)} e''_{\lambda_\beta}(\rho^+_\beta + b) < 1 \tag{D.18}$$

Therefore

$$\big|\psi(r) - \psi'(r)\big| \;\; \le \;\; e^{-\omega} j * j * \big|\psi - \psi'\big|$$

and since $\psi(r) = \psi'(r) = \rho(r)$ for $r \notin \Lambda$, it then follows that $\psi = \psi'$.

Finally, let $\psi \in X^0_{\Lambda,\rho}$ and $\hat\psi \in X^0_{\Lambda,\hat\rho}$, then, for $r \in \Lambda$,

$$|\psi(r) - \hat\psi(r)| \le e^{-\omega} j * j * (|\psi - \hat\psi|\mathbf{1}_\Lambda) + cj * j * \mathbf{1}_{\Lambda^c_{\neq}} \tag{D.19}$$

By iterating (D.19) and calling n_0 the largest integer such that $2n_0 \le \mathrm{dist}(r, \Lambda^c_{\neq})$, we get

$$|\psi(r) - \hat\psi(r)| \le c \sum_{n \ge n_0} e^{-\omega n}$$

which yields (D.15) with

$$c_\omega := \frac{c\, e^{\omega}}{1 - e^{-\omega}} \tag{D.20}$$

By choosing $\hat\psi(r) = \rho^+_\beta$, we have $\hat\psi = \rho^+_\beta$ and (D.14) follows from (D.15). □

Remark. The same conclusions hold for the interpolating free energy functionals $F^{\pm}_{\beta,u}(\rho_\Lambda|\rho_{\Lambda^c})$. The r.h.s. of (D.17) becomes, for the interpolating functional,

$$-\beta u \int_0^1 e^{-\beta\left(uj*e'_{\lambda_\beta}(j*\psi_s)+(1-u)e'_{\lambda_\beta}(\rho^+_\beta)\right)} j * \Big(e''_{\lambda_\beta}(j*\psi_s) j * (\psi - \psi')\Big) ds$$

for which the same bounds used in the previous proof apply.

Appendix E. Non local functionals, large deviations

Lemma E.1. *For any $\rho \in L^\infty(\mathbb{R}^d; [0, X_0])$, $X_0 > 0$ (e.g. as in* (C.6)*),*

$$s(j * \rho(r)) - j * s(\rho)(r) \;\ge\; \frac{1}{2X_0} \int j(r, r')\Big(\rho(r') - j * \rho(r)\Big)^2 dr' \tag{E.1}$$

Proof. According to the integral form of the Taylor reminder one can express for $x, y \in \mathbb{R}$

$$s(y) - s(x) = s'(y)(y - x) + \int_x^y s''(z)(x - z) dz$$

For $x, y \in [0, X_0]$ one can bound this integral below by

$$\geq s'(y)(y-x) - \frac{1}{X_0}\int_x^y (x-z)dz = s'(y)(y-x) + \frac{1}{2X_0}(x-y)^2$$

Hence putting $y = j * \rho(r),\ x = \rho(r')$ one obtains the required result having in mind that

$$\int j(r,r')s'(j*\rho(r))\,(j*\rho(r) - \rho(r')) = 0.$$

□

In the next theorem we will use the following notation.

• Given ζ and ℓ, Λ denotes a bounded, $\mathcal{D}^{(\ell)}$-measurable set, such that the maximal connected components of Λ^c are at mutual distance > 2. By $\chi_{\Lambda^c}(r)$, we will denote a function on Λ^c which, on each one of the maximal connected components of Λ^c, is constantly equal either to ρ_β^+ or to ρ_β^-. We will then call $\Lambda^c_\pm$ the regions where $\chi_{\Lambda^c}(r) = \rho_\beta^\pm$

• Given $\rho_\Lambda \in L^\infty(\Lambda;\mathbb{R}_+)$, we define the families $\{C_i^0, i \leq N_0\}$ and $\{C_j^\pm, j \leq N_\pm\}$, $N_0, N_\pm \geq 0$, as

$$\{C_i^0\} = \Big\{C_i^0 \in \mathcal{D}^{(\ell)} : C_i^0 \sqsubset \Lambda;\ \int_\Lambda j(r,r')\,dr = 1 \ \text{ for all } r' \in C_i^0; \\ \eta^{(\zeta,\ell)}(\rho_\Lambda; r) = 0,\, r \in C_i^0\Big\} \tag{E.2}$$

$$\begin{aligned}\{C_j^\pm\} \;=\; \Big\{&(C_j^+, C_j^-) \in \mathcal{D}^{(\ell)} \times \mathcal{D}^{(\ell)} :\ [C_j^+ \sqcup C_j^-] \sqsubset \Lambda;\ C_j^- \sqsubset \delta^\ell_{\text{out}}[C_j^+] \\ &\eta^{(\zeta,\ell)}(\rho_\Lambda; r) = \pm 1 \text{ and } \int_\Lambda j(r',r)\,dr' = 1, \text{ for all } r \in C_j^\pm; \\ &[C_j^+ \sqcup C_j^-] \sqcap [C_k^+ \sqcup C_k^-] = \emptyset \text{ if } j \neq k\Big\}\end{aligned} \tag{E.3}$$

Theorem E.2. *There are ζ_1, d_1 and c all positive so that for any $\zeta < \zeta_1$ and $\ell \leq d_1\zeta^2$ the following holds. Let Λ be a bounded, $\mathcal{D}^{(\ell)}$-measurable region, $\rho_\Lambda \in L^\infty(\Lambda; [0, X_0])$, then*

$$F_{\beta,\lambda_\beta}(\rho_\Lambda|\chi_{\Lambda^c}) \geq f_{\beta,\lambda_\beta}(\rho_\beta^+)|\Lambda| + I^-_{\Lambda,\Lambda^c_-} + I^+_{\Lambda,\Lambda^c_+} + c\zeta^2\ell^d\big(N_0 + N_\pm\big) \tag{E.4}$$

(recall that $I^\pm_{\Lambda,\Delta}$ is defined in (C.18) *and that $f_{\beta,\lambda_\beta}(\rho_\beta^+) = f_{\beta,\lambda_\beta}(\rho_\beta^-)$).*

Proof. We will use the shorthand notation

$$\mathcal{R} = j*(\rho_\Lambda + \chi^+_{\Lambda^c}), \qquad D = \Big\{r \in \Lambda : |\mathcal{R}(r) - \rho_\beta^\pm| > \frac{\zeta}{2}\Big\} \tag{E.5}$$

Recalling that $F_{\beta,\lambda_\beta}(\rho_\Lambda|\chi_{\Lambda^c}) = F_{\beta,\lambda_\beta}(\rho_\Lambda+\chi_{\Lambda^c}) - F_{\beta,\lambda_\beta}(\chi_{\Lambda^c})$, we write

$$
\begin{aligned}
&F_{\beta,\lambda_\beta}(\rho_\Lambda|\chi_{\Lambda^c}) = F_1 + F_2 + F_3\\
&F_1 = \int_\Lambda f_{\beta,\lambda_\beta}(\mathcal{R})\,dr, \qquad F_2 = \frac{1}{\beta}\int_\Lambda s\big(\mathcal{R}\big) - j*s(\rho_\Lambda+\chi_{\Lambda^c})\,dr\\
&F_3 = \int_{\Lambda^c} f_{\beta,\lambda_\beta}(\mathcal{R}) - f_{\beta,\lambda_\beta}(j*\chi_{\Lambda^c})\,dr + \frac{1}{\beta}\int_{\Lambda^c} s\big(\mathcal{R}\big) - j*s(\rho_\Lambda+\chi_{\Lambda^c})\,dr\\
&\qquad - \int_\Lambda f_{\beta,\lambda_\beta}(j*\chi_{\Lambda^c})\,dr - \frac{1}{\beta}\int s(j*\chi_{\Lambda^c}) - j*s(\chi_{\Lambda^c})\,dr
\end{aligned}
$$

We claim that

$$F_3 \ge I^-_{\Lambda,\Lambda^c_-} + I^+_{\Lambda,\Lambda^c_+} \tag{E.6}$$

Since there is a positive constant c_1 so that

$$F_1 \ge f_{\beta,\lambda_\beta}(\rho^+_\beta)|\Lambda| + c_1\zeta^2|D| \tag{E.7}$$

(E.4) will then follow after proving that there are a and a' positive so that

$$F_2 \ge \frac{a}{16X_0}\zeta^2 N_0\ell^d - c_1\zeta^2|D| + \frac{a'}{16X_0}\zeta^2 N_\pm \ell^d \tag{E.8}$$

Proof of (E.6). Since • the second integral in the definition of F_3 is non negative; • $f_{\beta,\lambda_\beta}(\mathcal{R}) \ge f_{\beta,\lambda_\beta}(\rho^\pm_\beta)$; • $s(j*\chi_{\Lambda^c}) = s(j*\rho^+_\beta\mathbf{1}_{\Lambda^c_+}) + s(j*\rho^-_\beta\mathbf{1}_{\Lambda^c_-})$,

$$
\begin{aligned}
F_3 \;\ge\; & \int_{\Lambda^c_+} f_{\beta,\lambda_\beta}(\rho^+_\beta) - f_{\beta,\lambda_\beta}(j*\rho^+_\beta\mathbf{1}_{\Lambda^c_+})\,dr + \int_{\Lambda^c_-} f_{\beta,\lambda_\beta}(\rho^-_\beta) - f_{\beta,\lambda_\beta}(j*\rho^-_\beta\mathbf{1}_{\Lambda^c_-})\,dr\\
& - \int_\Lambda f_{\beta,\lambda_\beta}(j*\rho^+_\beta\mathbf{1}_{\Lambda^c_+})\,dr - \int_\Lambda f_{\beta,\lambda_\beta}(j*\rho^-_\beta\mathbf{1}_{\Lambda^c_-})\,dr\\
& -\frac{1}{\beta}\int s(j*\rho^+_\beta\mathbf{1}_{\Lambda^c_+}) - j*s(\rho^+_\beta\mathbf{1}_{\Lambda^c_+})\,dr - \frac{1}{\beta}\int s(j*\rho^-_\beta\mathbf{1}_{\Lambda^c_-}) - j*s(\rho^-_\beta\mathbf{1}_{\Lambda^c_-})\,dr
\end{aligned}
$$

hence (E.6), after recalling the definition (C.18).

Proof of (E.8). By Lemma E.1,

$$
\begin{aligned}
F_2 \;&\ge\; \sum_{i\le N_0} S(C^0_i) + \sum_{j\le N_\pm} S(C^\pm_j)\\
S(C^0_i) \;&=\; \frac{1}{2X_0}\int_\Lambda dr \int_{C^0_i} j(r,r')\Big(\rho(r') - \mathcal{R}(r)\Big)^2 dr'\\
S(C^\pm_j) \;&=\; \frac{1}{2X_0}\int_\Lambda dr \int_{C^+_j\sqcup C^-_j} j(r,r')\Big(\rho(r') - \mathcal{R}(r)\Big)^2
\end{aligned}
\tag{E.9}
$$

Using the notation (A.1), (A.3) and the assumption (A.2), $|j(r,r') - j^{(\ell)}(r,r')| \le A^{(\ell)}(r,r')$, for any $r' \in C^0_i$,

$$\int dr A^{(\ell)}(r,r') \le c_3\ell, \qquad r' \in C^0_i$$

Since $(\rho(r') - \mathcal{R}(r))^2 \le X_0^2$,

$$S(C_i^0) \ge \frac{1}{2X_0}\int_\Lambda dr \int_{C_i^0} j^{(\ell)}(r,r')\Big(\rho(r') - \mathcal{R}(r)\Big)^2 dr' - c_3\frac{X_0}{2}\ell\ell^d$$

Recalling that $j^{(\ell)}(r,r')$ is constant on $r' \in C_i^0$, by Cauchy-Schwartz and shorthanding C for C_i^0,

$$|C|\int_C \Big(\rho(r') - \mathcal{R}(r)\Big)^2 dr' \ge \Big(\int_C [\rho(r') - \mathcal{R}(r)]dr'\Big)^2$$

If $r \in D^c$,

$$\Big(\int_C \rho(r') - \mathcal{R}(r)\Big)^2 dr' \ge \big(|C|\frac{\zeta}{2}\big)^2$$

hence, calling r_i the center of C_i^0,

$$S(C_i^0) \ge \frac{\zeta^2}{8X_0}\ell^d \int_{D^c \sqcap \Lambda} j^{(\ell)}(r,r_i)dr - c_3\frac{X_0}{2}\ell\ell^d \tag{E.10}$$

Let $a \in (0,1)$ be such that

$$\frac{a}{8X_0} \le \frac{c_1}{2}, \qquad c_1 \text{ as in (E.7)} \tag{E.11}$$

then

$$\begin{aligned}\frac{\zeta^2}{8X_0}\ell^d \int_{D^c\sqcap\Lambda} j^{(\ell)}(r,r_i)dr &\ge a\frac{\zeta^2}{8X_0}\ell^d \int_{D^c\sqcap\Lambda} j^{(\ell)}(r,r_i)dr \\ &\ge a\frac{\zeta^2}{8X_0}\ell^d - a\frac{\zeta^2}{8X_0}\int_D dr \int_{C_i^0} j^{(\ell)}(r,r')dr'\end{aligned}$$

having used that $\int_\Lambda j^{(\ell)}(r,r')\,dr = 1$, $r' \in C_i^0$, which holds by the definition of the family $\{C_i^0\}$. Hence

$$\sum_{i=1}^{N_0} S(C_i^0) \ge \Big(\frac{a}{8X_0}\zeta^2 - c_3\frac{X_0}{2}\ell\Big)N_0\ell^d - \frac{c_1}{2}\zeta^2|D| \tag{E.12}$$

We choose d_1 so that

$$c_3\frac{X_0}{2}d_1\zeta^2 \le \frac{a}{16X_0}\zeta^2$$

so that (E.8) will follow from the proof that for a suitable $a' > 0$

$$\sum_{j\le N_\pm} S(C_j^\pm) \;\ge\; \frac{a'}{16X_0}\zeta^2 N_\pm \ell^d - \frac{c_1}{2}\zeta^2|D| \tag{E.13}$$

To bound $S(C_j^\pm)$ we proceed as before. By an abuse of notation, we denote by $j^{(\ell)}(r,r')$ the quantity defined by averaging $j(r,r'')$ over $r'' \in C_j^+ \sqcup C_j^-$ instead of a single cube. By assumption (A.2), there is a constant c_4 so that

$$\int dr A^{(\ell)}(r,r') \le c_4\ell, \qquad r' \in C_j^+ \sqcup C_j^-$$

hence

$$S(C_j^{\pm}) \geq \frac{1}{2X_0} \int dr \int_{C_j^+ \sqcup C_j^-} j^{(\ell)}(r,r') \Big(\rho(r') - \mathcal{R}(r)\Big)^2 dr' - c_4 \frac{X_0}{2} \ell 2 \ell^d$$

$j^{(\ell)}(r,r')$ drops from the integral, as it is constant when r' varies in $C_j^+ \sqcup C_j^-$; then, denoting by $r_j^{\pm}$ the centers of $C_j^{\pm}$ and calling $r^* = r_j^- - r_j^+$,

$$\int_{C_j^+ \sqcup C_j^-} \Big(\rho(r') - \mathcal{R}(r)\Big)^2 dr' \geq \frac{1}{2} \int_{C_j^+} \Big(\rho(r') - \rho(r'+r^*)\Big)^2 dr'$$

By Cauchy-Schwartz

$$|C_j^+| \int_{C_j^+} \Big(\rho(r') - \rho(r'+r^*)\Big)^2 dr' \geq \Big(\int_{C_j^+} \rho(r') dr' - \int_{C_j^-} \rho(r') dr' \Big)^2$$

The analysis of $S(C_j^{\pm})$ proceeds hereafter as for $S(C_i^0)$, till we end up with (E.13), thus (E.8) and consequently the theorem are proved.

The theorem is proved.

□

Appendix F. Reduction to variational problems

Following the classical proof by Lebowitz and Penrose, [**9**], we will derive here upper and lower bounds on the partition function in terms of a variational problem which involves a non local free energy functional.

We fix $\beta \in (\beta_c, \beta_0)$ and $\lambda \in (\lambda_\beta - 1, \lambda_\beta + 1)$, we will often drop β from the notation. We consider constrained partition functions generically denoted by

$$Z_{\gamma,\lambda}(\Lambda; \mathcal{A} | \bar{\underline{q}}) \tag{F.1}$$

where Λ is a bounded $\mathcal{D}^{(\ell_-,\gamma)}$-measurable region and $\bar{\underline{q}}$ a boundary condition. We will suppose that all loops in $\bar{\underline{q}}$ are short, that $\bar{\underline{q}}(0) \sqsubset \Lambda^c$ and

$$\ell_{-,\gamma}^{-d} \Big| \bar{\underline{q}}(0) \sqcap C_r^{(\ell_-,\gamma)} \Big| = \mathrm{Av}^{(\ell_-,\gamma)}(\bar{\underline{q}}(0); \cdot) \leq X_0 \tag{F.2}$$

The constraint is determined by a $\mathcal{D}^{(\ell_-,\gamma)}$-measurable, $\{0, \pm 1\}$-valued function $\eta_{\mathrm{cs}}(r)$, $r \in \Lambda$, as

$$\mathcal{A} = \Big\{ \underline{q}(0) : \eta^{(\zeta,\ell_-,\gamma)}(\underline{q}(0); r) = \eta_{\mathrm{cs}}(r); \;\; \mathrm{Av}^{(\ell_-,\gamma)}(\underline{q}(0); r) \leq X_0, \;\; r \in \Lambda \Big\} \tag{F.3}$$

The transition to continuum involves two spatial scales, whose lengths are $\gamma^{-1/2}$ and γ^{-1}. We set

$$\bar{\rho}(r) := \mathrm{Av}^{(\gamma^{-1/2})}(\bar{q}(0); \gamma^{-1} r), \qquad r \in (\gamma\Lambda)^c \tag{F.4}$$

and

$$\mathcal{A}^* := \Big\{\rho \in L^\infty(\gamma\Lambda, \mathbb{R}_+) : \eta^{(\zeta,\gamma\ell_{-,\gamma})}(\rho; r) = \eta_{\rm cs}(\gamma^{-1}r),$$
$$\mathrm{Av}^{(\gamma\ell_{-,\gamma})}(\rho; r) \le X_0,\ r \in \gamma\Lambda\Big\} \tag{F.5}$$

Proposition F.1. *There is c so that,*

$$\log Z_{\gamma,\lambda}(\Lambda; \mathcal{A}|\bar{\underline{q}}) \le -\inf_{\rho\in\mathcal{A}^*} \beta\gamma^{-d} F_{\beta,\lambda,\gamma\Lambda}(\rho|\bar\rho) + c\gamma^{1/2}\big(|\Lambda| + |\partial_{4\gamma^{-1}}\Lambda|\big) \tag{F.6}$$

where $\partial_R\Lambda$ is defined in (A.11)*; while, for any $\rho \in \mathcal{A}^*$,*

$$\log Z_{\gamma,\lambda}(\Lambda; \mathcal{A}|\bar{\underline{q}}) \ge -\beta\gamma^{-d} F_{\beta,\lambda,\gamma\Lambda}(\rho|\bar\rho) - c\gamma^{1/2}(|\Lambda| + |\partial_{4\gamma^{-1}}\Lambda|) \tag{F.7}$$

Proof. Upper bound. By Proposition A.2 and calling $\Delta = \Lambda \sqcup \partial_{4\gamma^{-1}}\Lambda$,

$$\log Z_{\gamma,\lambda}(\Lambda; \mathcal{A}|\bar{\underline{q}}) \le \log Z^{(\gamma^{-1/2})}_{\gamma,\lambda}(\Lambda; \mathcal{A}|\bar{\underline{q}}(0)) + c\gamma^{1/2}|\Delta|$$

where $Z^{(\gamma^{-1/2})}_{\gamma,\lambda}$ is the partition function for the classical model, i.e. with particle and not loop configurations and with hamiltonian $h^{(\gamma^{-1/2})}_{\gamma,\lambda}(\underline{q}(0)|\bar{\underline{q}}(0))$, which is defined in terms of the interaction kernel $j^{(\gamma^{-1/2})}_\gamma(r, r')$, namely the one obtained from $j_\gamma(r, r')$ by averaging the second variable over the cubes of $\mathcal{D}^{(\gamma^{-1/2})}$. Then $h^{(\gamma^{-1/2})}_{\gamma,\lambda}(q|\bar{\underline{q}}(0))$ depends on q and $\bar{\underline{q}}(0)$ only via the number of particles in each one of the cubes of $\mathcal{D}^{(\gamma^{-1/2})}$. We then integrate over the positions of the particles $\underline{q}(0)$ keeping fixed the above particles numbers. By the Stirling formula,

$$\big|\log n! - n(\log n - 1)\big| \le c\log n$$

with c a suitable constant. The number of particles in a cube of $\mathcal{D}^{(\gamma^{-1/2})}$ in Δ is bounded by $X_0\ell^d_{-,\gamma}$, due to the constraint $\mathcal{A}$ and the assumptions on $\bar{\underline{q}}(0)$. Thus, calling $\mathcal{A}^*_d$ the collection of all $\rho \in \mathcal{A}^*$ which are $\mathcal{D}^{(\gamma^{1/2})}$-measurable and such that for all r, $\rho(r)\gamma^{-d/2}$ is an integer, there is a constant c so that,

$$Z_{\gamma,\lambda}(\Lambda; \mathcal{A}|\bar{\underline{q}}) \le \sum_{\rho\in\mathcal{A}^*_d} \exp\Big\{ -\beta\gamma^{-d} F^{(\gamma^{1/2})}_{\beta,\lambda,\gamma\Lambda}(\rho|\bar\rho) + c\Big(\gamma^{1/2} + \gamma^{d/2}\log\gamma^{-1}\Big)|\Delta|\Big\}$$

where $F^{(\gamma^{1/2})}$ is the functional with kernel $j^{(\gamma^{1/2})}(r, r')$. Since $F^{(\gamma^{1/2})}_{\beta,\lambda,\gamma\Lambda} = F_{\beta,\lambda,\gamma\Lambda}$ on $\mathcal{D}^{(\gamma^{1/2})}$-measurable functions, the last inequality holds with $F_{\beta,\lambda,\gamma\Lambda}$ as well.

Since

$$\mathrm{Card}\ (\mathcal{A}^*_d) \le [X_0\ell^d_{-,\gamma}]^{|\Lambda|/\gamma^{-d/2}}$$

we have, for a suitable constant c,

$$\log Z_{\gamma,\lambda}(\Lambda; \mathcal{A}|\bar{\underline{q}}) \le -\beta\gamma^{-d} \inf_{\rho\in\mathcal{A}^*} F_{\beta,\lambda,\gamma\Lambda}(\rho|\bar\rho) + c\Big(\gamma^{1/2} + \gamma^{d/2}\log\gamma^{-1}\Big)|\Delta|$$

which proves (F.6).

Lower bound. Let $\rho \in \mathcal{A}^*$, $\rho \le X_0$ and $\hat\rho \in \mathcal{A}^*_d$, $\hat\rho(r) = n\gamma^{d/2}$ whenever $\rho(r) \in [(n-1/2)\gamma^{d/2}, (n+1/2)\gamma^{d/2})$. Then, as before

$$\log Z_{\gamma,\lambda}(\Lambda; \mathcal{A}|\bar{q}) \ge -\beta\gamma^{-d} F_{\beta,\lambda,\gamma\Lambda}(\hat\rho|\bar\rho) - c\Big(\gamma^{1/2} + \gamma^{d/2}\log\gamma^{-1}\Big)|\Delta|$$

(F.7) then follows from

$$\gamma^{-d}\Big|F_{\beta,\lambda,\gamma\Lambda}(\hat\rho|\bar\rho) - F_{\beta,\lambda,\gamma\Lambda}(\rho|\bar\rho)\Big| \le c\gamma^{d/2}\log\gamma^{-1}|\Delta|$$

□

Remark. Proposition F.1 extends to the case of the interpolating hamiltonian of Section 10 because the reference hamiltonian is one-body and for it, the transition to continuum is trivial.

Using instead of Proposition A.2 either Corollary A.3 or Corollary A.4 we obtain the following variants of Proposition F.1:

Corollary F.2. *There is c so that,*

$$\log Z_{\gamma,\lambda}(\Lambda; \mathcal{A}|\bar{q}) \le -\inf_{\rho\in\mathcal{A}^*} \beta\gamma^{-d} F_{\beta,\lambda,\gamma\Lambda}(\rho|\bar{q}) + c\gamma^{1/2}\big(|\Lambda| + |\partial_{4\gamma^{-1}}\Lambda|\big) \tag{F.8}$$

where $\partial_R\Lambda$ is defined in (A.11)*; while, for any $\rho \in \mathcal{A}^*$,*

$$\log Z_{\gamma,\lambda}(\Lambda; \mathcal{A}|\bar{q}) \ge -\beta\gamma^{-d} F_{\beta,\lambda,\gamma\Lambda}(\rho|\bar{q}) - c\gamma^{1/2}(|\Lambda| + |\partial_{4\gamma^{-1}}\Lambda|) \tag{F.9}$$

Corollary F.3. *There is c so that,*

$$\log Z_{\gamma,\lambda}(\Lambda; \mathcal{A}|\rho^{\pm}_{\beta,\lambda}) \le -\inf_{\rho\in\mathcal{A}^*} \beta\gamma^{-d} F_{\beta,\lambda,\gamma\Lambda}(\rho|\rho^{\pm}_{\beta,\lambda}) + c\gamma^{1/2}\big(|\Lambda| + |\partial_{4\gamma^{-1}}\Lambda|\big) \tag{F.10}$$

where $\partial_R\Lambda$ is defined in (A.11)*; while, for any $\rho \in \mathcal{A}^*$,*

$$\log Z_{\gamma,\lambda}(\Lambda; \mathcal{A}|\rho^{\pm}_{\beta,\lambda}) \ge -\beta\gamma^{-d} F_{\beta,\lambda,\gamma\Lambda}(\rho|\rho^{\pm}_{\beta,\lambda}) - c\gamma^{1/2}(|\Lambda| + |\partial_{4\gamma^{-1}}\Lambda|) \tag{F.11}$$

Appendix G. Cluster expansion

In this appendix we recall some basic facts of cluster expansion in the context of the contour models we have used in this paper. For simplicity we just refer to the plus case, and consider the family $\{\Gamma\}^+$ of all possible bounded plus contours and non negative weights $w^+(\Gamma)$, $\Gamma \in \{\Gamma\}^+$, bounded by $e^{-2c_f\gamma^{-1}N_\Gamma}$. For instance $w^+(\Gamma) = \hat{W}^{\pm}_{\gamma,\lambda}(\Gamma_i; \underset{\sim}{q})$, $\underset{\sim}{q} \in \underset{\sim}{Q}_+$.

G.1. The abstract setup. We regard $\{\Gamma\}^+$ as a graph, calling connected two elements, Γ and Γ' of $\{\Gamma\}^+$ if $\mathrm{sp}(\Gamma) \sqcap \mathrm{sp}(\Gamma') \neq \emptyset$. We then define for any finite set $\mathcal{C} \sqsubset \{\Gamma\}^+$:

$$Z(\mathcal{C}, w) := \sum_{\{\Gamma_i\} \sqsubset \mathcal{C} : \Gamma_i \text{is disconnected from } \Gamma_j, i \neq j} \prod_i w(\Gamma_i) \tag{G.1}$$

which we regard as a function of the variables $w(\Gamma)$.

Denote by $\mathcal{I}$ the collection of all multi-indices I on $\{\Gamma\}^+$, i.e. I is a $\mathbb{N}_+$-valued function on $\{\Gamma\}^+$ such that $|I| := \sum_{\Gamma \in \{\Gamma\}^+} I(\Gamma) < \infty$. Let $I \in \mathcal{I}$, call $\mathcal{C} = \{\Gamma : I(\Gamma) > 0\}$, and define

$$\omega_I = \frac{1}{I!} D^I \log Z(\mathcal{C}, w)\Big|_{w=0} \prod_{\Gamma \in \mathcal{C}} w(\Gamma)^{I(\Gamma)} \tag{G.2}$$

where D^I is a partial derivative with $w(\Gamma)$ differentiated $I(\Gamma)$-times, while $I! = \prod I(\Gamma)!$.

G.2. The main result. Using the above notation, $\omega_I = 0$ unless $\mathcal{C} = \{\Gamma : I(\Gamma) > 0\}$ is connected. Moreover, calling $R(I) := \bigsqcup_{\Gamma : I(\Gamma) > 0} \mathrm{sp}(\Gamma)$, then if γ is small enough,

$$\sum_{I : R(I) \ni r} |\,\omega_I| e^{c_f \gamma^{-1} |I|} \le 1 \tag{G.3}$$

and, for any finite set $\mathcal{C} \sqsubset \{\Gamma\}^+$:

$$\log Z(\mathcal{C}, w) := \sum_{I : R(I) \sqsubset \mathcal{C}} \omega_I \tag{G.4}$$

After noticing that the number $\#(\Delta)$ of contours Γ with same $\mathrm{sp}(\Gamma) = \Delta$ is bounded by $\left(3^{\gamma^{-2\alpha d}} 2\right)^{N_\Delta}$ and since $2\alpha \ll 1$, we have $\#(\Delta) e^{-2c_f\gamma^{-1}N_\Delta} \ll 1$ for γ small enough and (G.3) follows from the general theory, see for instance [**2**].

G.3. Effective hamiltonian. Let Λ be a bounded $\mathcal{D}^{(\ell_-,\gamma)}$-measurable region and $\underset{\sim}{q} \in \underset{\sim}{Q}_+$. Then

$$\sum_{\{\Gamma_i\} \sqsubset \mathcal{C}^+_\Lambda} \prod_i \hat{W}^{+}_{\gamma,\lambda}(\Gamma_i; \underset{\sim}{q}) = e^{-K^{\pm}_\Lambda(\underset{\sim}{q}_\Lambda)} \tag{G.5}$$

where $K_\Lambda^\pm$ is an hamiltonian whose potentials $U_\Delta^\pm$ are given by

$$U_\Delta^\pm(\underline{q}_\Delta) = -\sum_{I:R(I)=\Delta} \omega_I \tag{G.6}$$

if Δ is connected, otherwise $U_\Delta^\pm = 0$. and, by (G.3),

$$\|U_\Delta^\pm(\underline{q}_\Delta)\|_\infty = e^{-c_f\gamma^{-1}N_\Delta} 2^{N_\Delta} \tag{G.7}$$

In the remaining part of the appendix we state and prove results used in the text.

Lemma G.1. *There is $\gamma^* > 0$ so that for any $\gamma \le \gamma^*$ and any bounded, $\mathcal{D}^{(\ell_{+,\gamma})}$-measurable set Δ*

$$\log\left(\sum_{\{\Gamma_i\},\mathrm{sp}(\Gamma_i)\sqcap\Delta\neq\emptyset} \prod_i \left\|\hat W_{\gamma,\lambda}^\pm(\Gamma_i;\cdot)\right\|_\infty\right) \le N_\Delta e^{-c_f\gamma^{-1}} \tag{G.8}$$

where N_Δ is the number of $\mathcal{D}^{(\ell_{+,\gamma})}$ cubes in Δ, i.e. $N_\Delta \ell_{+,\gamma}^d = |\Delta|$.

Proof. By (4.25),

$$\left\|\hat W_{\gamma,\lambda}^\pm(\Gamma_i;\cdot)\right\|_\infty \le e^{-2c_f\gamma^{-1}N_{\Gamma_i}} \tag{G.9}$$

Moreover if D is a bounded, $\mathcal{D}^{\ell_{+,\gamma}}$-measurable set, and since $\ell_{+,\gamma}/\ell_{-,\gamma} = \gamma^{-2\alpha}$,

$$\text{cardinality of } \{\Gamma : \mathrm{sp}(\Gamma) = D\} \le \left(3^{\gamma^{-2\alpha d}} 2\right)^{N_D} \tag{G.10}$$

Thus

$$\text{exp of l.h.s. of (G.8)} \le \left(1 + \sum_{D\ni 0} \left(3^{\gamma^{-2\alpha d}}2\right)^{N_D} e^{-2c_f\gamma^{-1}N_D}\right)^{N_\Delta} \tag{G.11}$$

Since $2\alpha d < 1$ and if γ is small enough, we have

$$\sum_{D\ni 0} 3^{N_D\gamma^{-2\alpha d}} 2^{N_D} e^{-2c_f\gamma^{-1}N_D} \le e^{-c_f\gamma^{-1}} \tag{G.12}$$

and

$$\text{exp of l.h.s. of (G.8)} \le \left(1 + e^{-c_f\gamma^{-1}}\right)^{N_\Delta} \le \exp\{N_\Delta e^{-c_f\gamma^{-1}}\} \tag{G.13}$$

The lemma is proved. □

Several times in the text we have used the following corollary of the above lemma. It is convenient to give here a general formulation which covers different cases. Λ is a bounded, $\mathcal{D}^{(\ell_{-,\gamma})}$-measurable region, and considering, for the sake of definiteness, the plus restricted ensemble, let $\bar q$ be a plus boundary condition outside Λ ($\bar{\underline{q}}$ may as well be replaced by $\rho_\beta^+ \mathbf{1}_{\Lambda^c}$). $\mathcal{A}$ below is a "decreasing constraint" on the contours, meaning that if $\{\Gamma_i\} \in \mathcal{A}$

and $\{\Gamma_{i_k}\}$ a subset of $\{\Gamma_i\}$ then $\{\Gamma_{i_k}\} \in \mathcal{A}$; we also suppose that the cardinality of $\mathcal{A}$ and of its elements are finite. Finally, Δ is a bounded, $\mathcal{D}^{(\ell_+,\gamma)}$-measurable subset of Λ.

Lemma G.2. *Let $\gamma^* > 0$ as in Lemma G.1, $\gamma \le \gamma^*$ and Λ, $\bar{\underline{q}}$, Δ and $\mathcal{A}$ as above. Then*

$$\frac{\hat{Z}^+_{\gamma,\lambda}\big(\Lambda; \{\mathcal{A}, \mathrm{sp}(\underline{\Gamma}) \sqcap \Delta = \emptyset\} \big| \bar{\underline{q}}\big)}{\hat{Z}^+_{\gamma,\lambda}\big(\Lambda; \mathcal{A} \big| \bar{\underline{q}}\big)} \ge e^{-N_\Delta e^{-c_f \gamma^{-1}}} \tag{G.14}$$

Proof. Call Ξ the denominator on the l.h.s. of (G.14). Its expression involves a sum over $\{\Gamma'_j\} \in \mathcal{A}$. For each element $\{\Gamma'_j\} \in \mathcal{A}$, call $\{\Gamma_i\}$ the subset of contours such that $\mathrm{sp}(\Gamma_i) \sqcap \Delta \neq \emptyset$. Since $\mathcal{A}$ is decreasing, the configuration obtained from $\{\Gamma'_j\}$ erasing $\{\Gamma_i\}$ is still in $\mathcal{A}$ and, by construction, verifies the constraint in the argument of the partition function in the numerator of (G.14). Thus

$$\Xi \le \Big(\sum_{\{\Gamma_i\}, \mathrm{sp}(\Gamma_i) \sqcap \Delta \neq \emptyset} \prod_i \Big\| \hat{W}^{\pm}_{\gamma,\lambda}(\Gamma_i; \cdot) \Big\|_\infty \Big) \, \hat{Z}^+_{\gamma,\lambda}\big(\Lambda; \{\mathcal{A}, \mathrm{sp}(\underline{\Gamma}) \sqcap \Delta = \emptyset\} \big| \bar{\underline{q}}\big)$$

which, by (G.8), yields (G.14). The lemma is proved. □

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