OCCUPATION MEASURE FUNCTIONALS IN MERGING PHASE SPACE

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Abstract

This paper gives average and diffusion approximation results for occupation time of semi-Markov processes under split and merging of a general standard phase space. These results seems to be particularly interesting since the variance coefficient is obtained in a simple form.

1 Introduction

Occupation measures of stochastic processes are particular cases of integral functionals. As integral functionals, they are of particular interest in theory (compensating processes, ...) and applications, and statistics (performance evaluation, probability estimators, ...), [5, 6].

Let us consider a right continuous stochastic process $x(t), t \geq 0$, with values into the measurable space $(E, \mathcal{E})$, and for a fixed $A \in \mathcal{E}$, define the following process

$$\beta_A(t) := \int_0^t I\left(x(s) \in A\right) ds, \quad t \geq 0,$$

where $I(\cdot)$ is the indicator function of the measurable set A. The stochastic process $\beta(t), t \geq 0$, is well defined with values in $[0, +\infty)$.

The occupation measure functionals for a Markov continuous time process in merging phase space was investigated in the book of G. Yin and Q. Chang ([8], Chapter 7).

In this paper, we investigate average and diffusion approximation of the above process $\beta(t), t \geq 0$, in the case when the process $x(t), t \geq 0$ is a semi-Markov process.

The methods represented in the book [5] (chapter 4-5) allow to investigate OMF (occupation measure functionals) for semi-Markov processes in split and double merging phase space.

2 Semi-Markov processes

The semi-Markov process $\alpha(t)$, $t \geq 0$, on the standard (Polish) phase space $(E, \mathcal{E})$ is given by the semi-Markov kernel

$$Q(x, B, t) := P(x, B)F_x(t), \quad x \in E, \quad B \in \mathcal{E}, \quad t \geq 0. \tag{1}$$

The stochastic kernel $P(x, B)$ determines transition probabilities of the embedded Markov chain $\alpha_n = \alpha(\tau_n)$, $n \geq 0$:

$$P(x, B) := P\left(\alpha_{n+1} \in B \,|\, \alpha_n = x\right). \tag{2}$$

The renewal moments of jump

$$\tau_{n+1} := \tau_n + \theta_{n+1}, \quad n \geq 0, \quad \tau_0 = 0 \tag{3}$$

is determined by the distribution functions of sojourn times θ_{n+1}, $n \geq 0$:

$$F_x(t) = P\left(\theta_{n+1} \leq t \,|\alpha_n = x\right) =: P(\theta_x \leq t). \tag{4}$$

In particular case of exponential distributions

$$F_x(t) = 1 - \exp\left(-q(x)t\right), \quad x \in E,$$

the corresponding process $\alpha(t)$, $t \geq 0$, is Markovian and can be defined by the generator

$$Q\varphi(x) = q(x) \int_E P(x, dy)\left[\varphi(y) - \varphi(x)\right]. \tag{5}$$

In semi-Markov case we will consider the auxiliary Markov process $\alpha^0(t)$, $t \geq 0$, given by the generator (5) with the intensity function

$$q(x) = 1/m(x), \; m(x) := E\theta_x = \int_0^\infty \bar{F}_x(t)dt, \; \bar{F}_x(t) := 1 - F_x(t). \tag{6}$$

3 Occupation measure functionals

Let us consider the following two assumptions.
(A1): *The semi-Markov processes* $\alpha^\varepsilon(t)$, $t \geq 0$ *are considered in the split phase space*

$$E = \bigcup_{k=1}^{N} E_k, \quad E_k \bigcap E_{k'} = \emptyset, \quad k \neq k', \tag{7}$$

and given by the semi-Markov kernels

$$Q^\varepsilon(x, B, t) := P^\varepsilon(x, B) F_x(t).$$

The stochastic kernel

$$P^\varepsilon(x, B) = P(x, B) + \varepsilon P_1(x, B) \tag{8}$$

is coordinated with the split (7) as follows:

$$P(x, E_k) := 1\!\!1_k(x) := \begin{cases} 1, & x \in E_k, \\ 0, & \textit{otherwise.} \end{cases}$$

Under the assumptions of Theorem 4.1 (see [4], chapter 4) the weak convergence

$$V(\alpha^\varepsilon(t/\varepsilon)) \to \hat{\alpha}(t), \quad t \to 0, \tag{9}$$

with the merging function

$$V(x) = k, \quad x \in E_k, \tag{10}$$

takes place.

The limit merged Markov process $\hat{\alpha}(t)$, $t \geq 0$, is defined on the merged phase space $\hat{E} = \{1, 2, \ldots, N\}$ by the generating matrix

$$\hat{Q} = \left[\hat{q}_{kr} \,|\, k, \, r \in \hat{E}\right],$$

determined by the relation (see [4], chapter 4)

$$\hat{Q} = \Pi Q_1 \Pi, \quad Q_1\varphi(x) := q(x) \int_E P_1(x, dy)\varphi(y). \tag{11}$$

The projector Π is defined by the stationary distributions $\pi_k(dx)$, $k \in \hat{E}$, of the auxiliary Markov process $\alpha^0(t)$, $t \geq 0$, given by the generator (5):

$$\Pi\varphi(x) = \sum_{k \in \hat{E}} \hat{\varphi} 1\!\!1_k(x), \quad \hat{\varphi}_k := \int_{E_k} \varphi(x)\pi_k(dx).$$

(A2): *The limit merged Markov process $\hat{\alpha}(t)$, $t \geq 0$, on the merged phase space $\hat{E}$, given by the generating matrix (11) is ergodic with the stationary distribution $\hat{\pi} = \left(\hat{\pi}_k,\ k \in \hat{E}\right)$, that is the vector $\hat{\pi}$ is unique solution of the equation*

$$\hat{\pi}\hat{Q} = 0.$$

The occupation measure functionals (OMF) on the split phase space (7) in the series scheme with the small parameter series $\varepsilon > 0$ is determined by the relation

$$\eta_k^\varepsilon(t) = \int_0^t I\left(\alpha^\varepsilon(s/\varepsilon^2) \in E_k\right) ds, \quad k \in \hat{E}, \tag{12}$$

where $I(A)$ is the indicator of event A. Introduce the merged process

$$\hat{\alpha^\varepsilon}(t) = V\left(\alpha^\varepsilon(t/\varepsilon^2)\right).$$

It is worth noticing that

$$I\left(\alpha^\varepsilon(s/\varepsilon^2) \in E_k\right) = I\left(\hat{\alpha}^\varepsilon(s) = k\right).$$

The convergence (9) allow expect the following convergence

$$I\left(\alpha^\varepsilon(s/\varepsilon^2) \in E_k\right) \to \hat{\pi}_k, \quad \varepsilon \to 0, \ k \in \hat{E}.$$

Theorem 3.1 *(Average) Under the assumptions (A1) and (A2), the OMF defined by (12) converges weakly*

$$\eta_k^\varepsilon(t) \to \hat{\pi}_k t, \quad \varepsilon \to 0, \ k \in \hat{E}. \tag{13}$$

This result provides the investigation of the normalized OMF in the following form:

$$\zeta_k^\varepsilon(t) = \varepsilon^{-1} \int_0^t \left[I\left(\alpha^\varepsilon(s/\varepsilon^3) \in E_k\right) - \hat{\pi}_k\right] b_k ds \tag{14}$$

where $b = \left(b_k,\ k \in \hat{E}\right)$ is the scaling vector.

Theorem 3.2 *(Asymptotic normality) Under the assumptions (A1) and (A2), the weak convergence*

$$\zeta^\varepsilon(t) \to W_\sigma(t), \quad \varepsilon \to 0,$$

takes place.

The limit Wiener process $W_\sigma(t)$, $t \geq 0$, is defined by the zero mean value and the covariance matrix $\hat{B} = \sigma\sigma^\star = \left[\hat{B}_{kr} \,|\, k,\, r \in \hat{E}\right]$

$$\hat{B}_{kr} = \hat{\pi}_k b_k \hat{R}_{kr} b_r + \hat{\pi}_r b_r \hat{R}_{rk} b_k + \hat{\pi}_k(\hat{\delta}_{kr} - \hat{\pi}_r),$$

here the potential matrix $\hat{R}_0 = \left[\hat{R}_{kr}\,;\, k,\, r \in \hat{E}\right]$ is defined by the equation

$$\hat{Q}\hat{R}_0 = \hat{R}_0\hat{Q} = \hat{\Pi} - I.$$

The projector $\hat{\Pi}$ is defined by the stationary distribution of the merged Markov process $\hat{\alpha}(t)$, $t \geq 0$:

$$\hat{\Pi}\hat{\varphi}(k) = \sum_{k \in \hat{E}} \hat{\pi}_k \hat{\varphi}(k) =: \hat{\hat{\varphi}}$$

is constant in $\hat{E}$.

Corollary 3.1 *The asymptotic normality of the normalized OMF (14) coincides with the asymptotic normality OMF for the limit merged Markov process $\hat{\alpha}(t)$, $t \geq 0$:*

$$\hat{\zeta}^\varepsilon(t) = \varepsilon^{-1} \int_0^t \left[I\left(\hat{\alpha}(s/\varepsilon^2) = k\right) - \hat{\pi}_k\right] b_k ds \to W_\sigma(t),\ \varepsilon \to 0.$$

4 Algorithms of phase merging for OMF

4.1 Average scheme

The random evolution representation for OMF (12) is considered in the following form:

$$\eta^\varepsilon(t) = u + \int_0^t \delta\left(\alpha^\varepsilon(s/\varepsilon^2)\right) ds, \quad u \in \mathbb{R}^N \tag{15}$$

$$\delta(x) = \left(\mathbb{1}_k(x),\, k \in \hat{E}\right), \quad x \in E,$$

Lemma 4.1 *The extended Markov renewal process*

$$\eta_n^\varepsilon = \eta^\varepsilon(\tau_n^\varepsilon),\ \alpha_n^\varepsilon = \alpha^\varepsilon(\tau_n),\ \tau_n^\varepsilon = \varepsilon^2\tau_n,\ n \geq 0$$

can be characterized by the compensating operator

$$\mathbb{L}^\varepsilon\varphi(u,\,x) = \varepsilon^{-2}q(x)\left[\int_0^\infty F_x(dt)A_{\varepsilon^2 t}(x)P^\varepsilon\varphi(u,\,x) - \varphi(u,\,x)\right] \tag{16}$$

where the family of semigroups $A_t(x)$, $t \geq 0$, $x \in E$ is defined by the generators

$$A(x)\varphi(u) = \delta(x)\varphi'(u) = \sum_{k\in\hat{E}} \mathbb{1}_k(x)\varphi'_k(u), \tag{17}$$

$$\varphi'_k(u) := \partial\varphi(u)/\partial u_k.$$

Corollary 4.1 *Compensating operator (16) admits the asymptotic extension on the test-functions $\varphi(u) \in C^k(\mathbb{R}^N)$, $k \geq 3$:*

$$\mathbb{L}^\varepsilon\varphi(u,\,x) = \left[\varepsilon^{-2}Q + \varepsilon^{-1}Q_1 + A(x)P + \theta^\varepsilon(x)\right]\varphi(u,\,x) \tag{18}$$

with the negligible term

$$||\theta^\varepsilon(x)\varphi(u)|| \to 0, \;\; \varepsilon \to 0, \;\; \varphi(u) \in C^k(\mathbb{R}^N).$$

Corollary 4.2 *The limit operator $\mathbb{L}$ in Theorem 3.1 is determined by a solution to the singular perturbation problem for the operator (18),*

$$\mathbb{L}^\varepsilon\left[\varphi(u) + \varepsilon\varphi_1(u,\,x) + \varepsilon^2\varphi_2(u,\,x)\right] = \mathbb{L}\varphi(u) + \theta^\varepsilon(x)\varphi(u), \tag{19}$$

by the formula (see [5], Proposition 5.3)

$$\mathbb{L} = \hat{\Pi}\Pi A(x)P\Pi\hat{\Pi} = \hat{\Pi}\Pi A(x)\Pi\hat{\Pi}. \tag{20}$$

Conclusion 4.1 *Calculation in (20) gives*

$$\mathbb{L}\varphi(u) = \hat{\pi}\varphi'(u) = \sum_{k\in\hat{E}} \hat{\pi}_k\varphi'_k(u). \tag{21}$$

The limit operator (21) defines the evolution

$$\eta(t) = u + \hat{\pi}t, \;\; t \geq 0, \tag{22}$$

that is result of Theorem 3.1.

5 Diffusion approximation of OMF

The random evolution representation for OMF (14) is considered in the following form:

$$\zeta^\varepsilon(t) = u + \varepsilon^{-1} \int_0^t b\left(\alpha^\varepsilon(s/\varepsilon^3)\right) ds \tag{23}$$

where

$$b(x) = \left(b_k(x),\, k \in \hat{E}\right),\ x \in E \tag{24}$$

$$b_k(x) := (\mathbb{1}_k(x) - \hat{\pi}_k)\, b_k,\ k \in \hat{E}.$$

Lemma 5.1 *The extended Markov renewal process*

$$\zeta_n^\varepsilon = \zeta^\varepsilon(\tau_n^\varepsilon),\ \alpha_n^\varepsilon = \alpha^\varepsilon(\tau_n),\ \tau_n^\varepsilon = \varepsilon^3\tau_n,\ n \geq 0,$$

can be characterized by the compensating operator

$$\mathbb{L}^\varepsilon\varphi(u,\, x) = \varepsilon^{-3}q(x)\left[\int_0^\infty F_x(dt)B_{\varepsilon^2 t}(x)P^\varepsilon\varphi(u,\, x) - \varphi(u,\, x)\right], \tag{25}$$

where B_t, $t \geq 0$, $x \in E$ is the family of semigroups characterized by the generators

$$B(x)\varphi(u) = b(x)\varphi'(u) = \sum_{k\in\hat{E}} b_k(x)\varphi_k'(u) \tag{26}$$

Corollary 5.1 *The compensating operator (25) admits the asymptotic extension on the test-functions $\varphi(u) \in C^k(\mathbb{R}^N)$, $k \geq 4$, as follows:*

$$\mathbb{L}^\varepsilon\varphi(u,\, x) = \left[\varepsilon^{-3}Q + \varepsilon^{-2}Q_1 + \varepsilon^{-1}B(x)P + B(x)P_1 + \theta^\varepsilon(x)P\right]\varphi(u,\, x) \tag{27}$$

with the negligible term

$$||\theta^\varepsilon(x)\varphi(u)|| \to 0,\ \ \varepsilon \to 0,\ \ \varphi(u) \in C^k(\mathbb{R}^N).$$

Corollary 5.2 *The limit operator $\mathbb{L}$ in Theorem 3.2 is determined by a solution to the singular perturbation problem for the operator (27):*

$$\mathbb{L}^\varepsilon\left[\varphi(u) + \varepsilon\varphi_1(u,\, x) + \varepsilon^2\varphi_2(u,\, x) + \varepsilon^3\varphi_3(u,\, x)\right] = \mathbb{L}\varphi(u) + \theta^\varepsilon(x)\varphi(u). \tag{28}$$

It is given by the following formula (see [5], Proposition 5.4)

$$\mathbb{L} = \hat{\Pi}\hat{B}(r)\hat{R}_0\hat{B}(r)\hat{\Pi}, \tag{29}$$

$$\hat{B}(r) := \Pi B(x)\Pi.$$

Remark 5.1 *The stochastic kernel* P_1 *in (8) satisfies the coordination condition* $P_1(x, E) = 0$*. Hence*

$$P_1 \Pi \varphi(u) = P_1(x, E)\varphi(u) = 0,$$

and

$$\Pi B(x) P_1 \Pi \varphi(u) = 0$$

also.

Conclusion 5.1 *The limit generator* $\mathbb{L}$ *in Theorem 3.2 is determined by*

$$\mathbb{L}\varphi(u) = \frac{1}{2}\hat{B}\varphi''(u) = \frac{1}{2}\sum_{k,\, r \in \hat{E}} \hat{B}_{kr}\varphi''_{kr}(u) \tag{30}$$

The elements of coveriance matrix are determined by

$$\hat{B}_{kr} = \hat{\pi}_k b_k \hat{R}_{kr} b_r + \hat{\pi}_r b_r \hat{R}_{rk} b_k \tag{31}$$

which coincide with an assertion of the Theorem 3.2.

The representations (19) and (28) can be used to prove the convergence for the finite dimensional distributions (see, for example [7]). To prove the weak convergence in Theorem 3.2. we have to use additional compact containment conditions (see [5], [2]).

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